Article

Spectrum of Primordial Gravitational Waves in Modified Gravities: A Short Overview

Sergei D. Odintsov [1,2,3,4], **Vasilis K. Oikonomou** [5,*] **and Ratbay Myrzakulov** [4,6,7]

1   ICREA, Passeig Luis Companys, 23, 08010 Barcelona, Spain; odintsov@ice.cat
2   Institute of Space Sciences (ICE, CSIC), C. Can Magrans s/n, 08193 Barcelona, Spain
3   Institute of Space Sciences of Catalonia (IEEC), 08193 Barcelona, Spain
4   Ratbay Myrzakulov Eurasian International Centre for Theoretical Physics, Nur-Sultan 010009, Kazakhstan; rmyrzakulov@csufresno.edu
5   Department of Physics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece
6   Faculty of Natural Sciences, Eurasian National University, Nur-Sultan 010008, Kazakhstan
7   Laboratory for Theoretical Cosmology, International Centre of Gravity and Cosmos, (TUSUR) , Tomsk 634050, Russia
*   Correspondence: v.k.oikonomou1979@gmail.com or voikonomou@auth.gr or voiko@physics.auth.gr

Abstract: In this work, we shall exhaustively study the effects of modified gravity on the energy spectrum of the primordial gravitational waves background. S. Weinberg has also produced significant works related to the primordial gravitational waves, with the most important one being the effects of neutrinos on primordial gravitational waves. With this short review, our main aim is to gather all the necessary information for studying the effects of modified gravity on primordial gravitational waves in a concrete and quantitative way and in a single paper. After reviewing all the necessary techniques for extracting the general relativistic energy spectrum, and how to obtain, in a WKB way, the modified gravity damping or amplifying factor, we concentrate on specific forms of modified gravity of interest. The most important parameter involved for the calculation of the effects of modified gravity on the energy spectrum is the parameter a_M, which we calculate for the cases of $f(R, \phi)$ gravity, Chern–Simons-corrected $f(R, \phi)$ gravity, Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity, and higher derivative extended Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity. The exact form of a_M is presented explicitly for the first time in the literature. With regard to Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity, and higher derivative extended Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity theories, we focus on the case in which the gravitational wave propagating speed is equal to that of light in a vacuum. We provide expressions for a_M expressed in terms of the cosmic time and in terms of the redshift, which can be used directly for the numerical calculation of the effect of modified gravity on the primordial gravitational wave energy spectrum.

Keywords: neutron stars; scalar–tensor gravity; Higgs inflationary model

1. Introduction

To date, the current perception of our Universe indicates that the Universe went through four distinct evolutionary eras, the inflationary era [1–4], the radiation domination era, the matter domination era, and the dark energy eras. Our knowledge is limited, though, since we know very well only the physics beyond the recombination era, nearly at redshift $z \sim 1100$, where the Cosmic Microwave Background (CMB) modes exited the Hubble horizon. Concerning the dark energy era, we do not know the origin of this late-time acceleration era, and the same applies for the inflationary era and the most mysterious of all, the reheating and the subsequent radiation domination eras. With regard to the inflationary era, we do not know whether it even occurred. Modified gravity in its various forms [5–10] can play a prominent role toward consistently describing inflation and the dark energy eras with or without the need for scalar fields. In some cases, it is possible to

describe inflation and the dark energy eras within the same theoretical framework; see [11] for the first attempt toward this research line, and also Refs. [12–20] for some more recent works. However, admittedly, the radiation domination era, and specifically its early stages, remains quite mysterious and, to date, inaccessible by any currently undergoing experiment. Hopefully, though, all the future interferometric gravitational wave experiments, such as the LISA laser interferometer space antenna [21,22], the dHz probing DECIGO [23,24], the Hz-kHz frequencies probing Einstein Telescope [25], and the future BBO (Big Bang Observatory) [26,27], are expected to probe the frequency range corresponding to the reheating and radiation domination era. Specifically, the frequency range of the future gravitational wave experiments corresponds to much higher frequencies compared to the CMB ones, and will probe modes that became subhorizon shortly after the inflationary era, during the early stages of the reheating era. At intermediate frequencies, very promising results may be obtained by the Square Kilometer Array (SKA) [28] and the NANOGrav collaboration [29,30], which are based on measurements of pulsar timing arrays. In the literature, there exist several theoretical attempts to efficiently predict the energy spectrum of the primordial gravitational waves; see Refs. [31–83] and references therein. Notable is also the work of S. Weinberg in the field; see, for example, Refs. [84,85]; especially notable is the effect of neutrinos on the primordial gravitational waves, which was first discussed in [84]. The primordial gravitational waves form a stochastic background, and on this background, important information during and after the inflationary era is imprinted. The stochastic primordial gravitational wave background is a unique tool that will probe directly the inflationary and short-post-inflationary era, since these gravitational waves are superadiabatic amplifications of the gravitational field's zero-point fluctuations. More importantly, the evolution of the primordial gravitational waves is described by linear differential equations since the coupling of the gravitational waves with matter is tiny, contrary to the CMB modes, which obey non-linear evolution equations for wavelengths larger than 10 Mpc. There are three vital effects on the primordial gravitational wave spectrum: first, the effects of the first horizon crossing during inflation; secondly, the post-inflationary second horizon crossing, where the primordial tensor modes become subhorizon modes again, and thirdly, post-inflationary effects on the spectrum, caused by several sources, such as the matter content of the Universe, supersymmetry breaking, or even modified gravity. All these effects are encoded on the stochastic background of primordial gravitational waves and will certainly offer insights into the physics that stretch back from the electroweak phase transition to the inflationary era.

If the outcome of future interferometric experiments is the discovery of a primordial gravitational wave stochastic background, modified gravity seems to be a compelling description of the primordial era of our Universe, and specifically of the inflationary and short post-inflationary ones. This is due to the fact that standard descriptions of the inflationary era, such as scalar field theories, fail to produce an observable energy spectrum of primordial gravitational waves [86], unless tachyons are used. In the context of modified gravity, on the other hand, blue-tilted inflation is predicted, and a blue tilt can even explain recent observations on pulsar timing arrays; see, for example, [48,52]. To be specific, a blue-tilted tensor spectral index or an abnormal reheating era can produce an observable gravitational wave spectrum. In view of the above, in this paper, we shall thoroughly study how to calculate the modified gravity effects on the energy spectrum of primordial gravitational waves. We shall use a WKB method firstly introduced in Ref. [54], which offers the possibility to quantify the overall effect of modified gravity on an integral of a single parameter a_M. After reviewing how to calculate the energy spectrum of the primordial gravitational waves, including the modified gravity effects, we shall calculate the parameter a_M for several modified gravity theories of interest. Our aim is to offer all the information needed for the calculation of modified gravity effects for redshifts stretching from the present time up to the end of the inflationary era.

This article is organized as follows: in Section 2, we present all the formalisms necessary for the extraction of the energy spectrum of the primordial gravitational waves. We

review standard features of primordial gravitational waves, and we extract the differential equation that governs the evolution of the primordial gravitational waves. We also show explicitly how the effects of modified gravity are encoded on a single parameter and we discuss how to calculate the overall effect of modified gravity on the primordial gravitational waves. In Section 3, we calculate and present formulas for the parameter a_M, which quantifies the effect of modified gravity on primordial gravitational waves. We shall calculate it for several modified gravities of phenomenological interest, and specifically for $f(R, \phi)$ gravity, for Chern–Simons-corrected $f(R, \phi)$, for Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity, and for higher derivative Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity. Finally, the conclusions of this work are presented at the end of the paper.

2. The Spectrum of Primordial Gravitational Waves in General Relativity and Modified Gravity Effects

In this section, we shall review the general features of primordial gravitational waves in the context of general relativity (GR) and we shall also quantify the way that modified gravity affects the spectrum. The analysis shall be based on Refs. [34,46,47,54,55,57,76] and references therein, and more details can be found in [34].

The primordial tensor perturbations are basically perturbations of a flat Friedmann–Robertson–Walker (FRW) metric,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2, \tag{1}$$

where $a(t)$ is the scale factor as usual. Using the conformal time τ, the perturbed FRW metric is

$$ds^2 = a^2[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \tag{2}$$

with x^i being the comoving spatial coordinates, and h_{ij} denotes the gauge-invariant metric tensor perturbation, which has symmetric $h_{ij} = h_{ji}$, traceless $h_{ii} = 0$, and transverse $\partial^j h_{ij} = 0$ conditions. The reason for the traceless, transverse, and symmetric characteristics is that every tensor mode describing a gravitational wave should actually have these features. The second-order Lagrangian corresponding to the tensor perturbation $h_{ij}(\tau, \mathbf{x})$ is

$$S = \int d\tau d\mathbf{x} \sqrt{-g} \left[\frac{-g^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \partial_\nu h_{ij} + \frac{1}{2} \Pi_{ij} h_{ij} \right], \tag{3}$$

where the tensor part of the anisotropic stress $\Pi_{\mu\nu}$ is

$$\Pi_j^i = T_j^i - p\delta_j^i \tag{4}$$

and satisfies $\Pi_{ii} = 0$, $\partial^i \Pi_{ij} = 0$, while it acts as an external source in the action (3). Upon varying the action (3) with respect to h_{ij}, we obtain

$$h_{ij}'' + 2\frac{a'(\tau)}{a(\tau)} h_{ij}' - \nabla^2 h_{ij} = 16\pi G a^2(\tau) \Pi_{ij}(\tau, \mathbf{x}), \tag{5}$$

with the "prime" denoting differentiation with respect to the conformal time. Upon Fourier transforming Equation (5), we get

$$h_{ij}(\tau, \mathbf{x}) = \sum_r \sqrt{16\pi G} \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \epsilon_{ij}^r(\mathbf{k}) h_{\mathbf{k}}^r(\tau) e^{i\mathbf{k}\mathbf{x}}, \tag{6a}$$

$$\Pi_{ij}(\tau, \mathbf{x}) = \sum_r \sqrt{16\pi G} \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \epsilon_{ij}^r(\mathbf{k}) \Pi_{\mathbf{k}}^r(\tau) e^{i\mathbf{k}\mathbf{x}}, \tag{6b}$$

with $r = $("+" or "×") denoting the polarization of the tensor perturbation, and the polarization tensors satisfy $[\epsilon_{ij}^r(\mathbf{k}) = \epsilon_{ji}^r(\mathbf{k})]$, and also $\epsilon_{ii}^r(\mathbf{k}) = 0$, and $k_i\epsilon_{ij}^r(\mathbf{k}) = 0$. Equation (6) in conjunction with Equation (3) yields

$$S = \sum_r \int d\tau d\mathbf{k} \frac{a^2}{2} \left[h_{\mathbf{k}}^{r\prime} h_{-\mathbf{k}}^{r\prime} - k^2 h_{\mathbf{k}}^r h_{-\mathbf{k}}^r + 32\pi G a^2 \Pi_{\mathbf{k}}^r h_{-\mathbf{k}}^r \right], \tag{7}$$

which is the action for the Fourier-transformed gravitational tensor perturbations. The resulting theory can easily be quantized, with $h_{\mathbf{k}}^r$ playing the role of the canonical variable, and the corresponding conjugate momentum is

$$\pi_{\mathbf{k}}^r(\tau) = a^2(\tau) h_{-\mathbf{k}}^{r\prime}(\tau), \tag{8}$$

and, hence, the theory is quantized if the following equal time commutation relations hold true:

$$\left[\hat{h}_{\mathbf{k}}^r(\tau), \hat{\pi}_{\mathbf{k'}}^s(\tau) \right] = i\delta^{rs}\delta^{(3)}(\mathbf{k} - \mathbf{k'}), \tag{9a}$$

$$\left[\hat{h}_{\mathbf{k}}^r(\tau), \hat{h}_{\mathbf{k'}}^s(\tau) \right] = [\hat{\pi}_{\mathbf{k}}^r(\tau), \hat{\pi}_{\mathbf{k'}}^s(\tau)] = 0. \tag{9b}$$

The Fourier components of $\hat{h}_{ij}(\tau, \mathbf{x})$ satisfy the relation $\hat{h}_{\mathbf{k}}^r = \hat{h}_{-\mathbf{k}}^{r\dagger}$, since $\hat{h}_{ij}(\tau, \mathbf{x})$ is a Hermitian operator. Therefore, we have

$$\hat{h}_{\mathbf{k}}^r(\tau) = h_k(\tau)\hat{a}_{\mathbf{k}}^r + h_k^*(\tau)\hat{a}_{-\mathbf{k}}^{r\dagger}, \tag{10}$$

where $\hat{a}_{\mathbf{k}}^{r\dagger}$ and $\hat{a}_{\mathbf{k}}^r$ are the creation and annihilation operators, respectively, which satisfy

$$\left[\hat{a}_{\mathbf{k}}^r, \hat{a}_{\mathbf{k'}}^{s\dagger} \right] = \delta^{rs}\delta^{(3)}(\mathbf{k} - \mathbf{k'}), \tag{11a}$$

$$\left[\hat{a}_{\mathbf{k}}^r, \hat{a}_{\mathbf{k'}}^s \right] = \left[\hat{a}_{\mathbf{k}}^{r\dagger}, \hat{a}_{\mathbf{k'}}^{s\dagger} \right] = 0. \tag{11b}$$

More importantly, each mode $h_k(\tau)$ satisfies the equation

$$h_k'' + 2\frac{a'(\tau)}{a(\tau)}h_k' + k^2 h_k = 16\pi G a^2(\tau)\Pi_k(\tau). \tag{12}$$

The modes $h_k(\tau)$ depend both on the conformal time and on the wavenumber $k = |\mathbf{k}|$, but do not depend on the polarization and on the direction, as is apparent from Equation (10). The Wronskian normalization condition

$$h_k(\tau)h_k^{*\prime}(\tau) - h_k^*(\tau)h_k'(\tau) = \frac{i}{a^2(\tau)}, \tag{13}$$

makes compatible the commutation relations (9) and (11), and also the initial condition for the modes is

$$h_k(\tau) \rightarrow \frac{\exp(-ik\tau)}{a(\tau)\sqrt{2k}} \qquad (\text{as } \tau \rightarrow -\infty), \tag{14}$$

which corresponds to the Bunch–Davies vacuum and describes modes which are initially at subhorizon scales and satisfy Equation (13).

Let us proceed to the spectrum of the primordial gravitational waves $\Omega_{gw}(k, \tau)$, and an inherent quantity to the energy spectrum, the tensor power spectrum $\Delta_h^2(k, \tau)$. The latter can be obtained by considering

$$\langle 0|\hat{h}_{ij}(\tau, \mathbf{x})\hat{h}_{ij}(\tau, \mathbf{x})|0\rangle = \int_0^\infty 64\pi G \frac{k^3}{2\pi^2}|h_k(\tau)|^2 \frac{dk}{k}, \tag{15}$$

and, thus, the inflationary tensor power spectrum is obtained:

$$\Delta_h^2(k,\tau) \equiv \frac{d\langle 0|\hat{h}_{ij}^2|0\rangle}{d\ln k} = 64\pi G \frac{k^3}{2\pi^2}|h_k(\tau)|^2. \tag{16}$$

The energy spectrum of the primordial gravitational waves evaluated at present day $\Omega_{gw}(k,\tau)$ is equal to

$$\Omega_{gw}(k,\tau) \equiv \frac{1}{\rho_{crit}(\tau)} \frac{d\langle 0|\hat{\rho}_{gw}(\tau)|0\rangle}{d\ln k}. \tag{17}$$

By definition, the energy spectrum of the primordial gravitational waves is the energy density of the gravitational waves, evaluated per logarithmic wavenumber interval. For the evaluation of $\Omega_{gw}(k,\tau)$, we can treat the tensor perturbation h_{ij} as a quantum field in an unperturbed FRW geometric background, the stress–energy tensor of which is

$$T_{\alpha\beta} = -2\frac{\delta L}{\delta \bar{g}^{\alpha\beta}} + \bar{g}_{\alpha\beta}L, \tag{18}$$

as is obtained by the action (3), with L denoting the Lagrangian function in Equation (3). Since the future laser interferometers will seek for primordial gravitational waves in high frequencies compared to the CMB ones, we can omit the anisotropic stress couplings, and thus the gravitational wave energy density is

$$\rho_{gw} = -T_0^0 = \frac{1}{64\pi G}\frac{(h_{ij}')^2 + (\vec{\nabla}h_{ij})^2}{a^2}, \tag{19}$$

and the corresponding vacuum expectation value of it is

$$\langle 0|\rho_{gw}|0\rangle = \int_0^\infty \frac{k^3}{2\pi^2}\frac{|h_k'|^2 + k^2|h_k|^2}{a^2}\frac{dk}{k}, \tag{20}$$

and hence the gravitational wave energy spectrum reads

$$\Omega_{gw}(k,\tau) = \frac{8\pi G}{3H^2(\tau)}\frac{k^3}{2\pi^2}\frac{|h_k'(\tau)|^2 + k^2|h_k(\tau)|^2}{a^2(\tau)}. \tag{21}$$

Moreover, using $|h_k'(\tau)|^2 = k^2|h_k(\tau)|^2$, the gravitational wave energy spectrum evaluated at present day can be rewritten as follows:

$$\Omega_{gw}(k,\tau) = \frac{1}{12}\frac{k^2\Delta_h^2(k,\tau)}{H_0^2(\tau)}, \tag{22}$$

where H_0 is the Hubble rate at present day, and we also assume that the present-day scale factor is equal to unity, in order for comoving quantities (frequencies and wavelengths) to be identical with physical quantities. The interest in primordial gravitational wave searches is for large frequency modes that became subhorizon during the dark era of the reheating and radiation domination era—thus, for modes which entered the Hubble horizon first after the end of the inflationary era. Let us now be more concrete on the calculation of the energy spectrum of the primordial gravity waves, and first we consider the effects on it, caused by the horizon re-entry of a mode k, in which case [34,46,47,54,55,57]

$$h_k^\lambda(\tau) = h_k^{\lambda(\mathrm{P})}\left(\frac{3j_1(k\tau)}{k\tau}\right), \tag{23}$$

where j_ℓ denotes the ℓ-th spherical Bessel function. The Fourier transformation of the primordial tensor perturbation during a power-law cosmological evolution $a(t) \propto t^p$ is

$$h_k(\tau) \propto a(t)^{\frac{1-3p}{2p}} J_{\frac{3p-1}{2(1-p)}}(k\tau), \tag{24}$$

with $J_n(x)$ being the Bessel function. Moreover, taking into account the damping caused by the relativistic degrees of freedom in the early Universe, which do not remain constant, the following factor for $h_k(\tau)$ is obtained [49]:

$$\left(\frac{g_*(T_{\text{in}})}{g_{*0}}\right)\left(\frac{g_{*s0}}{g_{*s}(T_{\text{in}})}\right)^{4/3}, \tag{25}$$

where the scale factor evolves as $a(t) \propto T^{-1}$ assuming adiabatic evolution, which follows by the fact that the entropy of the Universe S is constant:

$$S = \frac{2\pi^2}{4s}g_{*s}(aT)^3 = \text{const}. \tag{26}$$

Above, T_{in} denotes the temperature of the Universe at horizon re-entry,

$$T_{\text{in}} \simeq 5.8 \times 10^6 \text{ GeV}\left(\frac{g_{*s}(T_{\text{in}})}{106.75}\right)^{-1/6}\left(\frac{k}{10^{14} \text{ Mpc}^{-1}}\right). \tag{27}$$

Moreover, it is vital to take into account the damping factor caused by the current acceleration of the Universe $\sim (\Omega_m/\Omega_\Lambda)^2$ [34,35]. The full expression for the present-day measured primordial gravity waves energy spectrum per log frequency includes all the aforementioned damping effects and, in addition, and more importantly, it contains the transfer functions, which are calculated numerically by integrating the evolution differential equation of the Fourier-transformed tensor perturbation. The full expression for the energy density of the primordial gravitational waves is

$$\Omega_{\text{gw}}(f) = \frac{k^2}{12H_0^2}\Delta_h^2(k), \tag{28}$$

with the detailed form of $\Delta_h^2(k)$ being [34,46,47,54,55,57]

$$\Delta_h^2(k) = \Delta_h^{(\text{p})}(k)^2\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^2\left(\frac{g_*(T_{\text{in}})}{g_{*0}}\right)\left(\frac{g_{*s0}}{g_{*s}(T_{\text{in}})}\right)^{4/3}\left(\frac{3j_1(k\tau_0)}{k\tau_0}\right)^2 T_1^2(x_{\text{eq}})T_2^2(x_R), \tag{29}$$

where $g_*(T_{\text{in}}(k))$ in Equation (29) is [42]

$$g_*(T_{\text{in}}(k)) = g_{*0}\left(\frac{A + \tanh\left[-2.5\log_{10}\left(\frac{k/2\pi}{2.5\times10^{-12}\text{Hz}}\right)\right]}{A+1}\right)\left(\frac{B + \tanh\left[-2\log_{10}\left(\frac{k/2\pi}{6\times10^{-19}\text{Hz}}\right)\right]}{B+1}\right), \tag{30}$$

with the parameters A and B being equal to

$$A = \frac{-1 - 10.75/g_{*0}}{-1 + 10.75g_{*0}}, \tag{31}$$

$$B = \frac{-1 - g_{max}/10.75}{-1 + g_{max}/10.75}, \tag{32}$$

with $g_{max} = 106.75$ and $g_{*0} = 3.36$. Moreover, by replacing $g_{*0} = 3.36$ with $g_{*s} = 3.91$, we can calculate $g_{*0}(T_{\text{in}}(k))$, using the same formulas, namely Equations (30)–(32). Note

that the "bar" above the Bessel function in Equation (29) indicates averaging over many integration periods. In addition, the term $\Delta_h^{(p)}(k)^2$ in Equation (29) denotes the inflationary era's primordial tensor spectrum, the analytic form of which is [34,46,47,54,55,57]

$$\Delta_h^{(p)}(k)^2 = \mathcal{A}_T(k_{ref})\left(\frac{k}{k_{ref}}\right)^{n_T},\tag{33}$$

where the scale $k_{ref} = 0.002\,\mathrm{Mpc}^{-1}$ is the CMB pivot scale. Moreover, $\mathcal{A}_T(k_{ref})$ denotes the amplitude of the primordial tensor perturbations, and n_T denotes the tensor spectral index. Replacing

$$\mathcal{A}_T(k_{ref}) = r\mathcal{P}_\zeta(k_{ref}),\tag{34}$$

where $\mathcal{P}_\zeta(k_{ref})$ is the amplitude of the primordial scalar perturbations, we have, finally,

$$\Delta_h^{(p)}(k)^2 = r\mathcal{P}_\zeta(k_{ref})\left(\frac{k}{k_{ref}}\right)^{n_T}.\tag{35}$$

Regarding the transfer functions $T_1(x_{\mathrm{eq}})$ and $T_2(x_R)$, the analytic form of the first one is [34,46,47,54,55,57]

$$T_1^2(x_{\mathrm{eq}}) = \left[1 + 1.57x_{\mathrm{eq}} + 3.42x_{\mathrm{eq}}^2\right],\tag{36}$$

and it basically characterizes the modes that re-entered the Hubble horizon approximately during the matter–radiation equality with cosmic time instance $t = t_{\mathrm{eq}}$. Regarding the notation of several parameters in Equation (36), $x_{\mathrm{eq}} = k/k_{\mathrm{eq}}$ and $k_{\mathrm{eq}} \equiv a(t_{\mathrm{eq}})H(t_{\mathrm{eq}}) = 7.1 \times 10^{-2}\Omega_m h^2\,\mathrm{Mpc}^{-1}$. Now, regarding the transfer function $T_2(x_R)$ appearing in Equation (29), it is related to modes that entered the Hubble horizon during the reheating era, with the mode k being $k > k_R$. Its analytic form is

$$T_2^2(x_R) = \left(1 - 0.22x^{1.5} + 0.65x^2\right)^{-1},\tag{37}$$

where $x_R = \frac{k}{k_R}$, and also the k_R wavenumber is equal to

$$k_R \simeq 1.7 \times 10^{13}\,\mathrm{Mpc}^{-1}\left(\frac{g_{*s}(T_R)}{106.75}\right)^{1/6}\left(\frac{T_R}{10^6\,\mathrm{GeV}}\right),\tag{38}$$

with T_R being the reheating temperature. Moreover, the reheating frequency is

$$f_R \simeq 0.026\,\mathrm{Hz}\left(\frac{g_{*s}(T_R)}{106.75}\right)^{1/6}\left(\frac{T_R}{10^6\,\mathrm{GeV}}\right).\tag{39}$$

Having presented the GR primordial gravitational wave energy density, in the next section, we shall consider the effects caused by modified gravity of various forms on the GR waveform. Thus, we shall quantify the effects of modified gravity on the GR waveform in an explicit way.

The Modified Gravity Effect on the Energy Spectrum of the Primordial Gravitation Waves: A WKB Approach

In this section, we shall quantify the effect of an arbitrary modified gravity on the GR waveform of primordial gravitational waves. Let us recall for convenience at this point the differential equation that is obeyed by the Fourier transformation of the tensor perturbation h_{ij},

$$\ddot{h}(k) + (3 + a_M)H\dot{h}(k) + \frac{k^2}{a^2}h(k) = 0,\tag{40}$$

with α_M being defined as follows:

$$a_M = \frac{\dot{Q}_t}{Q_t H},\tag{41}$$

where Q_t is unique for every distinct modified gravity. The overall effect of modified gravity is encoded on the parameter a_M, and the functional form of this parameter is different for distinct modified gravities. In addition, the evolution differential Equation (40) characterizes all the distinct polarizations of the gravitational waves. In order to extract in a consistent way the overall modified gravity effect, we shall use Nishizawa's approach [54,55], which is basically a WKB approach. Expressed in terms of the conformal time, the differential Equation (40) becomes

$$h''(k) + (2 + a_M)\mathcal{H}h'(k) + k^2 h(k) = 0,\tag{42}$$

with the "prime" in the above equation indicating differentiation with respect to the conformal time τ, and also we define $\mathcal{H} = \frac{a'}{a}$. We shall extract the WKB solution by taking into account only subhorizon modes satisfying Equation (42) (see Figure 1). This is highly justified for primordial gravitational wave studies, since the high-frequency interferometers will probe modes that became subhorizon modes immediately after the inflationary era, so during the early reheating era.
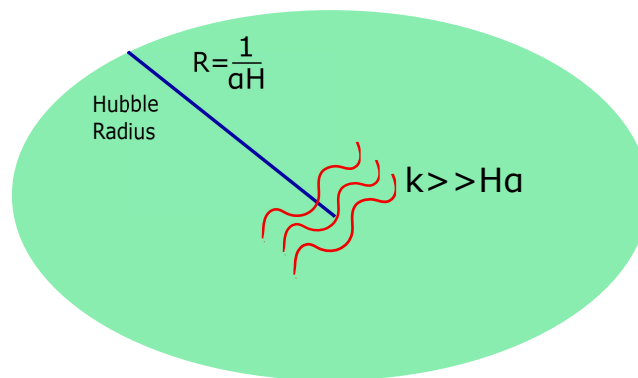
Figure 1. Post-inflationary subhorizon modes for which the WKB solution of the evolution Equation (42) is justified. These subhorizon modes became subhorizon immediately after inflation, so during the early stages of the dark ages—the reheating and the subsequent radiation domination eras. These early subhorizon modes will be probed by the future interferometer gravitational wave experiments.

Considering a solution of the form $h_{ij} = \mathcal{A}e^{i\mathcal{B}}h_{ij}^{GR}$, for theories in which the speed of the gravitational wave is equal to unity in natural units, the WKB solution for subhorizon modes is as given in [54,55], and (42) is of the form

$$h = e^{-\mathcal{D}}h_{GR},\tag{43}$$

where $h_{ij} = he_{ij}$, with h_{GR} denoting the GR waveform, which is the solution to the differential Equation (42) by taking $a_M = 0$. More importantly, the parameter $\mathcal{D}$ is equal to

$$\mathcal{D} = \frac{1}{2}\int^{\tau} a_M \mathcal{H}d\tau_1 = \frac{1}{2}\int_0^z \frac{a_M}{1+z'}dz',\tag{44}$$

and mainly quantifies the direct effect of modified gravity on the GR waveform of primordial gravitational waves. Now, if one wishes to calculate the energy spectrum of the primordial gravitational waves, it is vital to calculate the damping/amplification factor of Equation (44) starting from redshift $z = 0$, which corresponds to present time, up to redshifts corresponding deeply to the reheating era. The latter redshifts correspond to primordial modes that became subhorizon modes immediately after the inflationary era, so basically we are interested in extreme subhorizon modes at present day, with significantly

small wavelength compared to the CMB scale modes. These subhorizon modes will be probed fifteen years from now, through the LISA, BBO, DECIGO, and other gravitational wave experiments. Thus, by also taking into account the effects of modified gravity, the energy spectrum of the primordial gravitational waves at present day is

$$\Omega_{\text{gw}}(f) = e^{-2\mathcal{D}} \times \frac{k^2}{12H_0^2} r \mathcal{P}_\zeta(k_{ref}) \left(\frac{k}{k_{ref}} \right)^{n_T} \left(\frac{\Omega_m}{\Omega_\Lambda} \right)^2 \left(\frac{g_*(T_{\text{in}})}{g_{*0}} \right) \left(\frac{g_{*s0}}{g_{*s}(T_{\text{in}})} \right)^{4/3} \left(\frac{\overline{3j_1(k\tau_0)}}{k\tau_0} \right)^2 T_1^2(x_{\text{eq}}) T_2^2(x_R) \, .$$

Depending on the specific form of the modified gravity, the parameter $\mathcal{D}$ might be positive or negative. Therefore, the GR energy spectrum might be damped or amplified due to the overall modified gravity effects. In the next subsections, we shall calculate in detail all the possible forms of the parameter a_M appearing in Equations (41) and (42) corresponding to various modified gravities of interest. We shall give the expressions of a_M both with respect to the cosmic time and with respect to the redshift, for the calculational convenience of the reader. We shall also present the results in a table, also for the convenience of the reader.

3. Primordial Gravity Waves in Modified Gravity in its Various Forms

Let us first consider the calculation of the parameter a_M appearing in Equation (41) for the case of pure $f(R, \phi)$ gravity, in which case the gravitational action is

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right) . \tag{45}$$

For this form of modified gravity, the parameter Q_t is equal to $Q_t = \frac{1}{\kappa^2} \frac{\partial f(R, \phi)}{\partial R}$, where $\kappa = \frac{1}{M_p}$, with M_p being the reduced Planck mass. Hence, for pure $f(R, \phi)$ gravity, the parameter a_M reads

$$a_M = \frac{\frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{\partial^2 f}{\partial R^2} \dot{R}}{\frac{\partial f}{\partial R} H} \, . \tag{46}$$

We can express the above formula in terms of the redshift in order to have an expression ready for the integral in Equation (44) in terms of the redshift. Using the following formula,

$$\frac{\mathrm{d}}{\mathrm{d}t} = -H(1+z) \frac{\mathrm{d}}{\mathrm{d}z} \, , \tag{47}$$

the parameter a_M expressed in terms of the redshift reads

$$a_M = \frac{ -\frac{\partial^2 f}{\partial R \partial \phi} H(z)(1+z) \frac{\mathrm{d}\phi}{\mathrm{d}z} - \frac{\partial^2 f}{\partial R^2} H(z)(1+z) \frac{\mathrm{d}R}{\mathrm{d}z} }{\frac{\partial f}{\partial R} H} \, . \tag{48}$$

Let us consider the case of a Chern–Simons-corrected $f(R, \phi)$ gravity, in which case the gravitational action reads

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) + \frac{1}{8} \nu(\phi) R\tilde{R} \right) , \tag{49}$$

where $R\tilde{R} = \epsilon^{abcd} R_{ab}^{ef} R_{cdef}$ and ϵ^{abcd} stands for the totally antisymmetric Levi–Civita tensor. In the literature, terms of the form $\nu(\phi)\tilde{R}R$ are known as Chern–Simons terms. We need to note, though, that the term $\nu(\phi)\tilde{R}R$ is formally the Chern–Pontryagin density, which is connected to an actual three-dimensional Chern–Simons term via the exterior derivative $\nu(\phi)\tilde{R}R = d(Chern - Simons)$. Notably, the Chern–Pontryagin density is the analogue of the quantity $^*F_{\mu\nu}F^{\mu\nu}$, which is constructed by the curvature $F_{\mu\nu}$ on a principal bundle with connection A_μ, but in the literature it is called the Chern–Simons term, due to the analogy

that we have pointed out. In the Chern–Simons-corrected $f(R, \phi)$ gravity, the term Q_t is equal to [87]

$$Q_t = \frac{1}{\kappa^2} \frac{\partial f}{\partial R} + \frac{2\lambda_\ell \dot{v} k}{a}, \qquad (50)$$

where λ_ℓ denotes the polarization of the gravitational wave and takes the values $\lambda_R = 1$ for right-handed gravity waves and $\lambda_L = -1$ for left-handed gravity waves, while k is the wavenumber of the tensor mode. Thus, for the Chern–Simons-corrected $f(R, \phi)$ gravity, the a_M term in terms of the cosmic time reads

$$a_M = \frac{\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \dot{R} + \frac{2\lambda_\ell \ddot{v} k}{a} - \frac{2\lambda_\ell \dot{v} k H}{a}}{\left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} + \frac{2\lambda_\ell \dot{v} k}{a} \right) H}, \qquad (51)$$

where the last term numerator is found by differentiating Q_t in Equation (50). Expressing a_M in terms of the redshift, we have

$$a_M = \frac{-\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} H(z)(1+z) \frac{d\phi}{dz} - \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} H(z)(1+z) \frac{dR}{dz} + \frac{2\lambda_\ell H^2(1+z) \frac{dv}{dz} k}{a} + \frac{2\lambda_\ell v_{dd}(z) k}{a}}{\left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} - \frac{2\lambda_\ell (1+z) H \frac{dv}{dz}}{a} \right) H}, \quad (52)$$

where $v_{dd}(z)$ is equal to

$$v_{dd}(z) = -(1+z)H \left(\frac{dH}{dz}(1+z) \frac{dv}{dz} + \frac{dv}{dz} H + H(1+z) \frac{d^2 v}{dz^2} \right). \qquad (53)$$

Now, let us consider the case of $f(R, \phi)$ gravity with an Einstein–Gauss–Bonnet term, in which case the gravitational action reads

$$\mathcal{S} = \int d^4 x \sqrt{-g} \left(\frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) - \frac{1}{2} \xi(\phi) \mathcal{G} \right). \qquad (54)$$

For these theories, the speed of gravitational tensor perturbations is not equal to that of light, but it is equal to [87]

$$c_T^2 = 1 - \frac{4\left(\ddot{\xi} - \dot{\xi} H \right)}{\frac{\partial f}{\partial R} - 4\dot{\xi} H}. \qquad (55)$$

Since we shall consider only theories for which the gravitational wave speed is equal to that of light, the condition $c_T^2 = 1$ can be satisfied only if the Gauss–Bonnet scalar coupling function $\xi(\phi)$ satisfies the differential equation $\ddot{\xi} - \dot{\xi} H = 0$. The inflationary phenomenology of this class of Einstein–Gauss–Bonnet theories has been thoroughly studied in the literature; see [88–91]. For Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ theories, the parameter Q_t reads [87]

$$Q_t = \frac{\partial f}{\partial R} - 4\dot{\xi} H, \qquad (56)$$

and, therefore, the parameter a_M for the Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ theory reads

$$a_M = \frac{\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \dot{R} - 4\ddot{\xi} H - 4\dot{\xi} \dot{H}}{\left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} - 4\dot{\xi} H \right) H}, \qquad (57)$$

and $\ddot{\xi}$ must be replaced with $\ddot{\xi} = \dot{\xi} H$ due to the gravitational wave speed constraint $c_T^2 = 1$. In terms of the redshift, the parameter a_M has the following form:

$$a_M = \frac{-\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} H(z)(1+z) \frac{d\phi}{dz} - \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} H(z)(1+z) \frac{dR}{dz} + 4H^2(1+z) \frac{d\xi}{d\phi} \frac{d\phi}{dz} - 4H^2(1+z)^2 \frac{d\xi}{d\phi} \frac{d\phi}{dz} \frac{dH}{dz}}{\left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} + 4H^2(1+z) \frac{d\xi}{d\phi} \frac{d\phi}{dz} \right) H}, \qquad (58)$$

Finally, let us consider a generalized Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ theory with higher-order derivative couplings, in which case the gravitational action is

$$\mathcal{S} = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) - \frac{c_1}{2} \xi(\phi) \mathcal{G} - \frac{c_2}{2} \xi(\phi) G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right), \quad (59)$$

where $G^{\mu\nu}$ is the Einstein tensor and c_1, c_2 are dimensionless constants. For this theory, the gravitational wave speed is again non-trivial and different from that of light in a vacuum. Specifically, the gravitational wave speed is [87]

$$c_T^2 = 1 - \frac{8 c_1 (\ddot{\xi} - \dot{\xi} H) + 2 c_2 \xi \dot{\phi}^2}{\frac{\partial f}{\partial R} - 4 \dot{\xi} H}. \quad (60)$$

As in the in Einstein–Gauss–Bonnet-corrected theory we presented previously, we are interested in theories with gravitational wave speed equal to that of light in a vacuum, and for the case at hand, the theory can have $c_T^2 = 1$ when the Gauss–Bonnet scalar coupling $\xi(\phi)$ satisfies the differential equation

$$8 c_1 (\ddot{\xi} - \dot{\xi} H) + 2 c_2 \xi \dot{\phi}^2 = 0. \quad (61)$$

With this constraint satisfied, the only parameter that needs to be calculated for evaluating the energy spectrum of the primordial gravitational wave spectrum is a_M, given in Equation (41). In the case at hand, the parameter Q_t is equal to [87]

$$Q_t = \frac{\partial f}{\partial R} - 4 c_1 \dot{\xi} H - \frac{c_2}{2} \xi \dot{\phi}^2, \quad (62)$$

and, therefore, the parameter a_M (41) for the Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ theory with higher derivative coupling terms reads

$$a_M = \frac{\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \dot{R} - 4 c_1 \ddot{\xi} H - 4 c_1 \dot{\xi} \dot{H} + \frac{c_2}{2} \dot{\xi} \dot{\phi}^2 + c_2 \xi \dot{\phi} \ddot{\phi}}{\left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} - 4 c_1 \dot{\xi} H + \frac{c_2}{2} \xi \dot{\phi}^2 \right) H}. \quad (63)$$

The above concludes the most complicated extension of $f(R, \phi)$ modified gravity with $c_T^2 = 1$, for which the parameter a_M of Equation (41) can be calculated. It is always a computational challenge to calculate numerically the parameter $\mathcal{D}$ appearing in Equation (44) for redshifts corresponding to modes that became subhorizon after the inflationary era, during the reheating and the radiation domination era. In Table 1, we gather all the various forms of the parameter a_M for the various forms of modified gravity that were considered in this section.

Table 1. Forms of the parameter a_M for the various modified gravities.

Modified Gravity Type	a_M
Pure $f(R, \phi)$	$a_M = \left(\frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{\partial^2 f}{\partial R^2} \dot{R} \right) / \left(\frac{\partial f}{\partial R} H \right)$
C-S $f(R, \phi)$	$a_M = \left(\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \dot{R} + \frac{2 \lambda_\ell \dddot{v} k}{a} - \frac{2 \lambda_\ell \dot{v} k H}{a} \right) / \left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} + \frac{2 \lambda_\ell \dot{v} k}{a} \right) H$
EGB $f(R, \phi)$	$a_M = \left(\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \dot{R} - 4 \ddot{\xi} H - 4 \dot{\xi} \dot{H} \right) / \left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} - 4 \dot{\xi} H \right) H$
H-EGB $f(R, \phi)$	$a_M = \left(\frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \dot{R} - 4 c_1 \ddot{\xi} H - 4 c_1 \dot{\xi} \dot{H} + \frac{c_2}{2} \dot{\xi} \dot{\phi}^2 + c_2 \xi \dot{\phi} \ddot{\phi} \right) / \left(\frac{1}{\kappa^2} \frac{\partial f}{\partial R} - 4 c_1 \dot{\xi} H + \frac{c_2}{2} \xi \dot{\phi}^2 \right) H$

4. Conclusions

In this work, we studied the ways that various forms of modified gravity affect the energy spectrum of the primordial gravitational waves. We presented the standard features of the energy spectrum of the primordial gravitational waves in GR, and how modified gravity affects the spectrum in a quantitative way. The critical effect of modified gravity on the energy spectrum of the primordial gravitational waves is quantified on a single parameter denoted a_M and its integral for redshifts extending from the present day up to redshifts corresponding to the early post-inflationary era. We calculated the parameter a_M for several modified gravities of interest, and specifically for the $f(R, \phi)$ gravity, for Chern–Simons-corrected $f(R, \phi)$ gravity, for Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity, and for higher derivative extensions of Einstein–Gauss–Bonnet-corrected $f(R, \phi)$ gravity. The motivation for studying several modified gravity effects on primordial gravitational waves is based on the possible verification of the stochastic primordial gravitational waves background by future experiments. The actual verification will stir things up significantly in theoretical cosmology, since theorists must consider how such a signal is generated. This is due to the fact that standard theories of inflation, such as scalar field theory, do not produce a detectable signal of stochastic primordial gravity wave background. Thus, a possible detection will indicate either that some modified gravity controls the physics of inflation and post-inflation eras, or that some alternative reheating mechanism controls the post-inflationary physics. However, in the context of GR, the abnormal reheating effects could be minor and could not amplify the spectrum significantly enough to be detectable; see, for example, the w EoS post-inflationary parameter C_2 of [34]. Thus, it is rather compelling to study in detail and thoroughly all the effects of modified gravity in its various forms, on the energy spectrum of primordial gravitational waves. With this paper, we present a rigid overview of the method needed for calculating, in a formally correct way, the overall effect of several modified gravities of interest on the energy spectrum of the primordial gravitational waves.

Author Contributions: Conceptualization, S.D.O., V. K.O and R.M.; methodology, S.D.O., V. K.O. and R.M.; software, S.D.O., V. K.O. and R.M.; validation, S.D.O., V. K.O. and R.M.; formal analysis, S.D.O., V. K.O. and R.M.; investigation, S.D.O., V. K.O. and R.M.; resources, S.D.O., V. K.O. and R.M.; data curation, S.D.O., V. K.O. and R.M.; writing—original draft preparation, S.D.O., V. K.O. and R.M.; writing—review and editing, S.D.O., V. K.O. and R.M.; visualization, S.D.O., V. K.O. and R.M.; supervision, S.D.O., V. K.O. and R.M.; project administration, S.D.O., V. K.O. and R.M.; funding acquisition, S.D.O., V. K.O. and R.M. All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Linde, A. Inflationary Cosmology. In *Lecture Notes in Physics*; Springer: Berlin, Germany, 2008; Volume 738, pp. 1–54.
2. Gorbunov, D.S.; Rubakov, V.A. *Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory*; World Scientific: Hackensack, NJ, USA, 2011; p. 489.
3. Linde, A. *Inflationary Cosmology after Planck*; Oxford University Press: England, UK, 2013.
4. Lyth, D.H.; Riotto, A. Particle physics models of inflation and the cosmological density perturbation. *Phys. Rep.* **1999**, *314*, 1–146. [CrossRef]
5. Nojiri, S.; Odintsov, S.D.; Oikonomou, V.K. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. *Phys. Rep.* **2017**, *692*, 1–104. [CrossRef]
6. Capozziello, S.; De Laurentis, M. Extended Theories of Gravity. *Phys. Rep.* **2011**, *509*, 167–321. [CrossRef]

7.   Nojiri, S; Odintsov, S.D. Introduction to modified gravity and gravitational alternative for dark energy. *Int. J. Geom. Methods Mod. Phys.* **2007**, *4*, 115–145. [CrossRef]

8.   Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models. *Phys. Rep.* **2011**, *505*, 59–144. [CrossRef]

9.   De la Cruz-Dombriz, A.; Saez-Gomez, D. Black holes, cosmological solutions, future singularities, and their thermodynamical properties in modified gravity theories. *Entropy* **2012**, *14*, 1717–1770. [CrossRef]

10. Olmo, G.J.; Rubiera-Garcia, D. Some Recent Results on Ricci-Based Gravity Theories. *arXiv* **2022**, arXiv:2203.04116.

11. Nojiri, S.; Odintsov, S.D. Modified gravity with negative and positive powers of the curvature: Unification of the inflation and of the cosmic acceleration. *Phys. Rev. D* **2003**, *68*, 123512. [CrossRef]

12. Nojiri, S.; Odintsov, S.D. Unifying inflation with ΛCDM epoch in modified f(R) gravity consistent with Solar System tests. *Phys. Lett. B* **2007**, *657*, 238–245. [CrossRef]

13. Nojiri, S.; Odintsov, S.D. Modified f(R) gravity unifying R^m inflation with ΛCDM epoch, *Phys. Rev. D* **2008**, *77*, 026007. [CrossRef]

14. Cognola, G.; Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Sebastiani, L.; Zerbini, S. A Class of viable modified f(R) gravities describing inflation and the onset of accelerated expansion. *Phys. Rev. D* **2008**, *77*, 046009. [CrossRef]

15. Nojiri, S.; Odintsov, S.D. Modified f(R) gravity consistent with realistic cosmology: From matter dominated epoch to dark energy universe. *Phys. Rev. D* **2006**, *74*, 086005 [CrossRef]

16. Appleby, A.S.; Battye, A.R. Do consistent $F(R)$ models mimic General Relativity plus Λ? *Phys. Lett. B* **2007**, *654*, 7–12. [CrossRef]

17. Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Sebastiani, L; Zerbini, S. Non-singular exponential gravity: A simple theory for early- and late-time accelerated expansion. *Phys. Rev. D* **2011**, *83*, 086006. [CrossRef]

18. Odintsov, S.D.; Oikonomou, V.K. Geometric Inflation and Dark Energy with Axion $F(R)$ Gravity. *Phys. Rev. D* **2020**, *101*, 044009. [CrossRef]

19. Oikonomou, V.K. Unifying inflation with early and late dark energy epochs in axion $F(R)$ gravity. *Phys. Rev. D* **2021**, *103*, 044036. [CrossRef]

20. Oikonomou, V.K. Rescaled Einstein-Hilbert Gravity from $f(R)$ Gravity: Inflation, Dark Energy and the Swampland Criteria. *Phys. Rev. D* **2021**, *103*, 124028. [CrossRef]

21. Baker, J.; Bellovary, J.; Bender, P.L.; Berti, E.; Caldwell, R.; Camp, J.; Conklin, J.W.; Cornish, N; Cutler, C.; De Rosa, R.; et al. The Laser Interferometer Space Antenna: Unveiling the Millihertz Gravitational Wave Sky. *arXiv* **2019**, arXiv:1907.06482.

22. Smith, T.L.; Caldwell, R. LISA for Cosmologists: Calculating the Signal-to-Noise Ratio for Stochastic and Deterministic Sources. *Phys. Rev. D* **2019**, *100*, 104055. [CrossRef]

23. Seto, N.; Kawamura, S.; Nakamura, T. Possibility of direct measurement of the acceleration of the universe using 0.1-Hz band laser interferometer gravitational wave antenna in space. *Phys. Rev. Lett.* **2001**, *87*, 221103. [CrossRef]

24. Kawamura, S.; Ando, M.; Seto, N.; Sato, S.; Musha, M.; Kawano, I.; Yokoyama, J.; Tanaka, T.; Ioka, K.; Akutsu, T.; et al. Current status of space gravitational wave antenna DECIGO and B-DECIGO. *arXiv* **2020**, arXiv:2006.13545.

25. Hild, S.; Abernathy, M.; Acernese, F.; Amaro-Seoane, P.; Andersson, N.; Arun, K.; Barone, F.; Barr, B.; Barsuglia, M.; Beker, M.; et al. Sensitivity Studies for Third-Generation Gravitational Wave Observatories. *Class. Quant. Gravity* **2011**, *28*, 094013. [CrossRef]

26. Crowder, J.; Cornish, J.N. Beyond LISA: Exploring future gravitational wave missions. *Phys. Rev. D* **2005**, *72*, 083005. [CrossRef]

27. Smith, T.L.; Caldwell, R. Sensitivity to a Frequency-Dependent Circular Polarization in an Isotropic Stochastic Gravitational Wave Background. *Phys. Rev. D* **2017**, *95*, 044036. [CrossRef]

28. Weltman, A.; Bull, P.; Camera, S.; Kelley, K.; Padmanabhan, H Pritchard, J.; Raccanelli, A.; Riemer-Sørensen, S.; Shao, L.; Andrianomena, S.; et al. Fundamental physics with the Square Kilometre Array. *Publ. Astron. Soc. Austral.* **2020**, *37*, e002. [CrossRef]

29. Arzoumanian, Z.; Baker, P.T.; Blumer, H.; Bécsy, B.; Brazier, A.; Brook, P.R.; Burke-Spolaor, S.; Chatterjee, S.; Chen, S.; Cordes, J.M.; et al. The NANOGrav 12.5 yr Data Set: Search for an Isotropic Stochastic Gravitational-wave Background. *Astrophys. J. Lett.* **2020**, *905*, L34. [CrossRef]

30. Pol, N.S.; Taylor, S.R.; Kelley, L.Z.; Vigel, ; S.J.; Simon, J.; Chen, S.; Arzoumanian, Z.; Baker, P.T.; Bécsy, B.; Brazier, A.; et al. Astrophysics Milestones For Pulsar Timing Array Gravitational Wave Detection. *arXiv* **2010**, arXiv:2010.11950.

31. Kamionkowski, M.; Kovetz, E.D. The Quest for B Modes from Inflationary Gravitational Waves. *Ann. Rev. Astron. Astrophys.* **2016**, *54*, 227–269. [CrossRef]

32. Denissenya, M.; Linder, E.V. Gravity's Islands: Parametrizing Horndeski Stability. *J. Cosmol. Astropart. Phys.* **2018**, *2018*, 010. [CrossRef]

33. Turner, M.S.; White, M.J.; Lidsey, J.E. Tensor perturbations in inflationary models as a probe of cosmology. *Phys. Rev. D* **1993**, *48*, 4613–4622. [CrossRef]

34. Boyle, A.L.; Steinhardt, P.J. Probing the early universe with inflationary gravitational waves. *Phys. Rev. D* **2008**, *77*, 063504. [CrossRef]

35. Zhang, Y.; Yuan, Y.; Zhao, W.; Chen, Y.T. Relic gravitational waves in the accelerating Universe. *Class. Quant. Gravity* **2005**, *22*, 1383. [CrossRef]

36. Schutz, B.F.; Ricci, F. 'Gravitational Waves, Sources, and Detectors. *arXiv* **2010**, arXiv:1005.4735.

37. Sathyaprakash, S.B.; Schutz, F.B. Physics, Astrophysics and Cosmology with Gravitational Waves. *Living Rev. Relativ.* **2009**, *12*, 1–141. [CrossRef] [PubMed]

38. Caprini, C.; Figueroa, D.G. Cosmological Backgrounds of Gravitational Waves. *Class. Quant. Gravity* **2018**, *35*, 163001. [CrossRef]
39. Arutyunov, G.; Heinze, M.; Medina-Rincon, D. Superintegrability of Geodesic Motion on the Sausage Model. *J. Phys. A* **2017**, *50*, 244002. [CrossRef]
40. Kuroyanagi, S.; Chiba, T.; Sugiyama, N. Precision calculations of the gravitational wave background spectrum from inflation. *Phys. Rev. D* **2009**, *79*, 103501. [CrossRef]
41. Clarke, T.J.; Copel, E.J.; Moss, A. Constraints on primordial gravitational waves from the Cosmic Microwave Background. *J. Cosmol. Astropart. Phys.* **2020**, *2020*, 2. [CrossRef]
42. Kuroyanagi, S.; Takahashi, T.; Yokoyama, S. Blue-tilted Tensor Spectrum and Thermal History of the Universe. *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 3. [CrossRef]
43. Nakayama, K.; Yokoyama, J. Gravitational Wave Background and Non-Gaussianity as a Probe of the Curvaton Scenario. *J. Cosmol. Astropart. Phys.* **2010**, *2010*, 10. [CrossRef]
44. Smith, T.L.; Kamionkowski, M.; Cooray, A. Direct detection of the inflationary gravitational wave background. *Phys. Rev. D* **2006**, *73*, 023504. [CrossRef]
45. Giovannini, M. Thermal history of the plasma and high-frequency gravitons. *Class. Quant. Gravity* **2009**, *26*, 045004. [CrossRef]
46. Liu, X.J.; Zhao, W.; Zhang, Y.; Zhu, Z.H. Detecting Relic Gravitational Waves by Pulsar Timing Arrays: Effects of Cosmic Phase Transitions and Relativistic Free-Streaming Gases. *Phys. Rev. D* **2016**, *93*, 024031. [CrossRef]
47. Zhao, W.; Zhang, Y.; You, X.P.; Zhu, Z.H. Constraints of relic gravitational waves by pulsar timing arrays: Forecasts for the FAST and SKA projects. *Phys. Rev. D* **2013**, *87*, 124012. [CrossRef]
48. Vagnozzi, S. Implications of the NANOGrav results for inflation. *Mon. Not. R. Astron. Soc. Lett.* **2021**, *502*, L11–L15. [CrossRef]
49. Watanabe, Y.; Komatsu, E. Improved Calculation of the Primordial Gravitational Wave Spectrum in the Standard Model. *Phys. Rev. D* **2006**, *73*, 123515. [CrossRef]
50. Kamionkowski, M.; Kosowsky, A.; Turner, M.S. Gravitational radiation from first order phase transitions. *Phys. Rev. D* **1994**, *49*, 2837–2851. [CrossRef] [PubMed]
51. Giarè, W.; Renzi, F. Propagating speed of primordial gravitational waves. *Phys. Rev. D* **2020**, *102*, 083530. [CrossRef]
52. Kuroyanagi, S.; Takahashi, T.; Yokoyama, S. Blue-tilted inflationary tensor spectrum and reheating in the light of NANOGrav results. *J. Cosmol. Astropart. Phys.* **2021**, *2021*, 071. [CrossRef]
53. Zhao, W.; Zhang, Y. Relic gravitational waves and their detection. *Phys. Rev. D* **2006**, *74*, 043503. [CrossRef]
54. Nishizawa, A. Generalized framework for testing gravity with gravitational-wave propagation. I. Formulation. *Phys. Rev. D* **2018**, *97*, 104037. [CrossRef]
55. Arai, S.; Nishizawa, A. Generalized framework for testing gravity with gravitational-wave propagation. II. Constraints on Horndeski theory. *Phys. Rev. D* **2018**, *97*, 104038. [CrossRef]
56. Bellini, E.; Sawicki, I. Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity. *J. Cosmol. Astropart. Physic* **2014**, *2014*, 50. [CrossRef]
57. Nunes, C.R.; Alves, S.E.M.; de Araujo, N.C.J. Primordial gravitational waves in Horndeski gravity. *Phys. Rev. D* **2019**, *99*, 084022. [CrossRef]
58. D'Agostino, R.; Nunes, R.C. Probing observational bounds on scalar-tensor theories from standard sirens. *Phys. Rev. D* **2019**, *100*, 044041. [CrossRef]
59. Mitra, A.; Mifsud, J.; Mota, D.F.; Parkinson, D. Cosmology with the Einstein Telescope: No Slip Gravity Model and Redshift Specifications. *Mon. Not. R. Astron. Soc.* **2021**, *502*, 5563–5575. [CrossRef]
60. Kuroyanagi, S.; Nakayama, K.; Saito, S. Prospects for determination of thermal history after inflation with future gravitational wave detectors. *Phys. Rev. D* **2011**, *84*, 123513. [CrossRef]
61. Campeti, P.; Komatsu, E.; Poletti, D.; Baccigalupi, C. Measuring the spectrum of primordial gravitational waves with CMB, PTA and Laser Interferometers. *J. Cosmol. Astropart. Phys.* **2021**, *2021*, 012. [CrossRef]
62. Nishizawa, A.; Motohashi, H. Constraint on reheating after $f(R)$ inflation from gravitational waves. *Phys. Rev. D* **2014**, *89*, 063541. [CrossRef]
63. Zhao, W. Improved calculation of relic gravitational waves. *Chin. Phys.* **2007**, *16*, 2894–2902.
64. Cheng, W.; Qian, T.; Yu, Q.; Zhou, H.; Zhou, Y.R. Wave from Axion-like Particle Inflation. *arXiv,* **2021**, arXiv:2107.04242.
65. Nishizawa, A.; Yagi, K.; Taruya, A.; Tanaka, T. Cosmology with space-based gravitational-wave detectors—Dark energy and primordial gravitational waves. *Phys. Rev. D* **2012**, *85*, 044047. [CrossRef]
66. Chongchitnan, S.; Efstathiou, G. Prospects for direct detection of primordial gravitational waves. *Phys. Rev. D* **2006**, *73*, 083511. [CrossRef]
67. Lasky, P.D.; Mingarelli, C.M.F.; Smith, T.L.; Giblin, J.T.; Reardon D.J.; Caldwell, R.; Bailes, M.; Bhat, N.D.R.; Burke-Spolaor, S.; Coles W.; et al. Gravitational-wave cosmology across 29 decades in frequency. *Phys. Rev. X* **2016**, *6*, 011035. [CrossRef]
68. Guzzetti, M.C.; ; Bartolo, N.; Liguori, M.; Matarrese, S. Gravitational waves from inflation. *Riv. Nuovo Cim.* **2016**, *39*, 399–495.
69. Ben-Dayan, I.; Keating, B.; Leon, D.; Wolfson, I. Constraints on scalar and tensor spectra from N_{eff}. *J. Cosmol. Astropart. Phys.* **2019**, *2019*, 007. [CrossRef]
70. Nakayama, K.; Saito, S.; Suwa, Y.; Yokoyama, J. Probing reheating temperature of the universe with gravitational wave background. *J. Cosmol. Astropart. Phys.* **2008**, *2008*, 020. [CrossRef]

71. Capozziello, S.; De Laurentis, M.; Nojiri, S.; Odintsov, S.D. Evolution of gravitons in accelerating cosmologies: The case of extended gravity. *Phys. Rev. D* **2017**, *95*, 083524. [CrossRef]
72. Capozziello, S.; De Laurentis, M.; Nojiri, S.; Odintsov, S.D. f(R) gravity constrained by PPN parameters and stochastic background of gravitational waves. *Gen. Relativ. Gravit.* **2009**, *41*, 2313–2344. [CrossRef]
73. Capozziello, S.; Corda, C.; De Laurentis, F.M. Massive gravitational waves from f(R) theories of gravity: Potential detection with LISA. *Phys. Lett. B* **2008**, *669*, 255–259. [CrossRef]
74. Cai, R.G.; Fu, C.; Yu, W.W. Parity violation in stochastic gravitational wave background from inflation. *arXiv* **2021**, arXiv:2112.04794.
75. Cai, R.G.; Pi, S.; Sasaki, M. Gravitational Waves Induced by non-Gaussian Scalar Perturbations. *Phys. Rev. Lett.* **2019**, *122*, 201101. [CrossRef] [PubMed]
76. Odintsov, S.D.; Oikonomou, V.K.; Fronimos, F.P. Quantitative predictions for f(R) gravity primordial gravitational waves. *Phys. Dark Univ.* **2022**, *35*, 100950. [CrossRef]
77. Benetti, M.; Graef, L.L.; Vagnozzi, S. Primordial gravitational waves from NANOGrav: A broken power-law approach. *Phys. Rev. D* **2022**, *105*, 043520. [CrossRef]
78. Lin, J.; Gao, S.; Gong, Y.; Lu, Y.; Wang, Z.; Zhang, F. Primordial black holes and scalar induced secondary gravitational waves from Higgs inflation with non-canonical kinetic term. *arXiv* **2021**, arXiv:2111.01362.
79. Zhang, F.; Lin, J.; Lu, Y. Double-peaked inflation model: Scalar induced gravitational waves and primordial-black-hole suppression from primordial non-Gaussianity. *Phys. Rev. D* **2021**, *104*, 063515. [CrossRef]
80. Odintsov, S.D.; Oikonomou, V.K. Pre-inflationary bounce effects on primordial gravitational waves of f(R) gravity. *Phys. Lett. B* **2022**, *824*, 136817. [CrossRef]
81. Pritchard, J.R.; Kamionkowski, M. Cosmic microwave background fluctuations from gravitational waves: An analytic approach. *Ann. Phys.* **2005**, *318*, 2–36. [CrossRef]
82. Zhang, Y.; Zhao, W.; Xia, T.; Yuan, Y. Analytic approach to the CMB polarizations generated by relic gravitational waves. *Phys. Rev. D* **2006**, *74*, 083006. [CrossRef]
83. Baskaran, D.; Grishchuk, L.P.; Polnarev, A.G. Imprints of Relic Gravitational Waves in Cosmic Microwave Background Radiation. *Phys. Rev. D* **2006**,*74*, 083008. [CrossRef]
84. Weinberg, S. Damping of tensor modes in cosmology. *Phys. Rev. D* **2004**, *69*, 023503. [CrossRef]
85. Flauger, R.; Weinberg, S. Tensor Microwave Background Fluctuations for Large Multipole Order. *Phys. Rev. D* **2007**, *75*, 123505. [CrossRef]
86. Breitbach, M.; Kopp, J.; Madge, E.; Opferkuch, T.; Schwaller, P. Dark, Cold, and Noisy: Constraining Secluded Hidden Sectors with Gravitational Waves. *J. Cosmol. Astropart. Phys.* **2019**, *2019*, 007. [CrossRef]
87. Hwang, J.C.; Noh, H. Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyon: Unified analyses. *Phys. Rev. D* **2005**, *71*, 063536. [CrossRef]
88. Odintsov, S.D.; Oikonomou, V.K.; Fronimos, F.P. Rectifying Einstein-Gauss-Bonnet Inflation in View of GW170817. *Nucl. Phys. B* **2020**, *958*, 115135. [CrossRef]
89. Odintsov, S.D.; Oikonomou, V.K. Swampland implications of GW170817-compatible Einstein-Gauss-Bonnet gravity. *Phys. Lett. B* **2020**, *805*, 135437. [CrossRef]
90. Oikonomou, V.K.; Fronimos, F.P. A Nearly Massless Graviton in Einstein-Gauss-Bonnet Inflation with Linear Coupling Implies Constant-roll for the Scalar Field. *Europhys. Lett.* **2020**, *131*, 30001. [CrossRef]
91. Oikonomou, V.K. A refined Einstein–Gauss–Bonnet inflationary theoretical framework. *Class. Quant. Gravity* **2021**, *38*, 195025. [CrossRef]