The Convergence Results of Differential Variational Inequality Problems

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Abstract: In this work, we suggest a differential variational inequality in reflexive Banach spaces and construct a sequence with a set of constraints and a penalty parameter. We use the penalty method to prove a unique solution to the problem and make suitable assumptions to prove the convergence of the sequence. The proof is based on arguments for compactness, symmetry, pseudomonotonicity, Mosco convergence, inverse strong monotonicity and Lipschitz continuity. Finally, we discuss the boundary value problem for the differential variational inequality problem as an application.

Keywords: differential variational inequality; unilateral constraint; penalty method; Mosco convergence; initial and boundary value problem; inverse strongly monotonicity; Lipschitz continuity

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1. Introduction

Pang and Stewart [1] pioneered differential variational inequality in the context of engineering challenges, capturing the mathematical literature. The differential variational inequality is a strong tool for studying numerous models, such as ideal diode electrical circuits, contact of deformable bodies, traffic networks and hybrid engineering systems with changing topologies. In [2], the topological degree theory was used to examine the periodic solution of a differential variational inequality, and in the study of two parameters, periodic solutions for a class of differential variational inequalities. Significant results in the field are contained in [3–8].

A differential variational inequality is a system that combines a variational inequality with an evolution equation and concludes the attributes of the solution set that are derived through the suitable conditions of compactness, convexity and set of constraints. In [9], further results were gained by relaxing the set’s compactness. The existence results for differential variational inequalities involving non-compact sets of constraints and non-local boundary conditions are given in the citation [10–12].

The penalty techniques are a type of mathematical instrument that can be used to solve a wide range of issues, including constrained problem analysis and numerical solutions. The purpose of penalty techniques is to create a sequence of unconstrained problems that converge to the solution as the penalty parameter approaches zero. Many authors have looked into penalty approaches for variational inequalities, primarily for numerical purposes; see [13–15] and the references therein for more information. However, the majority of references only employ penalty techniques to analyze a variational inequality, and few studies deal with penalty methods for differential variational inequalities. The authors demonstrate that a penalty approach to analyzing differential variational inequalities yields existence, uniqueness and convergence results; see [16–20].
The goal of this paper is to suggest a penalty method for differential variational inequality problems in Banach spaces. We construct a sequence of problems by utilizing the set of constraints, symmetry and penalty parameters and prove the convergence of sequences to the solution of the problem in the sense of Mosco. Finally, we discuss the boundary value problem for the differential variational inequality problem with unilateral constraints.

2. Preliminaries

Throughout this paper, we denote by \((E, \| \cdot \|_E)\) and \((V, \| \cdot \|_V)\) the reflexible Banach spaces. Let \(\mathcal{L}(E)\) be the space of linear continuous operators on \(E\) endowed with the norm \(\| \cdot \|_{\mathcal{L}(E)}\). In addition, we denote by \(V^*\) the dual of \(V\), by \(0_{V^*}\) the zero element of \(V^*\) and by \(\langle \cdot, \cdot \rangle\) the duality pairing between \(V^*\) and \(V\). Furthermore, we use \(E \times V\) for the product of the spaces \(E\) and \(V\) with the canonical product topology. Moreover, for \(\gamma > 0\), we denote by \(C([0, \gamma], E)\) the space of continuous functions defined on \([0, \gamma]\) with values in \(E\), and we use \(C([0, \gamma], U)\) for the set of continuous functions defined on \([0, \gamma]\) with values in \(U \subset V\). Even if we do not explicitly state it, all limits, lower limits and upper limits are assumed to be \(n \to \infty\). We need to recall the following definitions.

**Definition 1** ([21,22]). Let \(Q : V \to V^*\) be the function. Then \(Q\) is said to be

(i) monotone, if

\[ \langle Q(u) - Q(v), u - v \rangle \geq 0, \forall u, v \in V; \]

(ii) strongly monotone, if there exists \(a_Q > 0\) such that

\[ \langle Q(u) - Q(v), u - v \rangle \geq a_Q \| u - v \|^2, \forall u, v \in V; \]

(iii) inverse strong monotone, if there exists \(a_Q > 0\) such that

\[ \langle Q(u) - Q(v), u - v \rangle \geq a_Q \| Q(u) - Q(v) \|^2, \forall u, v \in V; \]

(iv) Lipschitz continuous, if there exists \(\beta_Q \geq 0\) such that

\[ \| Q(u) - Q(v) \| \leq \beta_Q \| u - v \|, \forall u, v \in V; \]

(v) bounded, if there are maps bounded sets in \(V\) into bounded sets of \(V^*\);

(vi) pseudomonotone, if \(Q\) is bounded and for every sequence \(\{u_n\} \subseteq V\) converging weakly to \(u \in V\) such that

\[ \limsup \langle Q(u_n), u_n - u \rangle \leq 0, \]

we have

\[ \langle Q(u), u - v \rangle \leq \liminf \langle Q(u_n), u_n - v \rangle \forall v \in V; \]

(vii) hemicontinuous, if for all \(u, v, w \in V\), the function

\[ \zeta \mapsto \langle Q(u + \zeta v), w \rangle \]

is continuous on \([0,1]\);

(vii) demicontinuous, if \(u_n \to u \in V\) implies

\[ Q(u_n) \xrightarrow{\text{weakly}} Q(u) \in V^*. \]

**Definition 2.** An operator \(P : V \times V \to V^*\) is said to be a penalty operator of the set \(\Omega \subset V\) if \(P\) is bounded, demicontinuous, monotone and

\[ \Omega = \{ u \in V \mid P(u, u) = 0_{V^*} \}. \]
Definition 3. Let \( \varphi : \mathcal{V} \to \mathbb{R} \) be the function. The \( \varphi \) is said to be lower semicontinuous, if
\[
\liminf_{n \to \infty} \varphi(u_n) \geq \varphi(u)
\]
and for any sequence \( \{u_n\} \subset \mathcal{V} \),
\[
u_n \to u \in \mathcal{V}.
\]

Definition 4. Let \( \{\Omega_n\} \) be a sequence of non-empty subsets of \( \mathcal{V} \) and let \( \phi \neq \Omega \subset \mathcal{V} \). If the sequence
\[
\Omega_n \xrightarrow{\text{Mosco}} \Omega
\]
then the following holds:

(i) For each \( u \in \Omega \), there exists a sequence \( \{u_n\} \) such that for \( n \in \mathbb{N} \), \( u_n \in \Omega_n \) and
\[
u_n \to u \in \mathcal{V}.
\]

(ii) For each sequence \( \{u_n\} \) such that for \( n \in \mathbb{N} \), \( u_n \in \Omega_n \) and \( u_n \xrightarrow{\text{weakly}} u \in \mathcal{V} \), we have
\[
u \in \Omega.
\]

We shall denote the Mosco convergence by \( \Omega_n \xrightarrow{M} \Omega \) which is proposed in [23].

Assume that \( A : D(A) \subset E \to E \) and \( x_0 \in E \) and \( \Omega \subset \mathcal{V} \). In addition, let
\[
p : [0, \gamma] \times E \times \mathcal{V} \to E, q : [0, \gamma] \times E \times \mathcal{V} \to \mathcal{V}^* \text{ and } \varphi : \mathcal{V} \to \mathbb{R}.
\]
From now on we note that \( p(\zeta, \cdot, \cdot) = p_\zeta(\cdot, \cdot) \) and \( q(\zeta, \cdot, \cdot) = q_\zeta(\cdot, \cdot) \) unless otherwise specified. With these data, we consider the following differential variational inequality problem for finding two functions \( x : [0, \gamma] \to \mathcal{E} \) and \( u : [0, \gamma] \to \mathcal{V} \) such that
\[
\begin{cases}
x'(\zeta) &= Ax(\zeta) + p_\zeta(x(\zeta), u(\zeta)) \text{ a.e. } \zeta \in [0, \gamma], \\
u(\zeta) &\in \text{Sol}(\Omega, q_\zeta(x(\zeta), u(\zeta)), \varphi), \text{ for all } \zeta \in [0, \gamma], \\
x(0) &= x_0,
\end{cases}
\tag{1}
\]
where \( x' \) is a derivative of \( x \) with respect to the time variable \( \zeta \in [0, \gamma] \) and the inclusion
\[
u(\zeta) \in \text{Sol}(\Omega, q_\zeta(x(\zeta), u(\zeta)), \varphi)
\]
is a notation which means that \( u(\zeta) \) satisfies the variational inequality problem
\[
u(\zeta) \in \Omega, \quad (q_\zeta(x(\zeta), u(\zeta)), v - u(\zeta)) + \varphi(v) - \varphi(u(\zeta)) \geq 0, \forall v \in \Omega.
\tag{2}
\]

We evaluate the following assumptions on the data in the research of (1).
\[
\begin{align*}
A &\colon D(A) \subset E \to E \text{ is the generator of a } \theta_0\text{-semigroup of linear continuous operators } \{\mathcal{N}(\zeta)\}_{\zeta \geq 0} \text{ on the space } \mathcal{E}.
\tag{3}
\end{align*}
\]
\[
\begin{align*}
p &\colon [0, \gamma] \times E \times \mathcal{V} \to E \text{ is such that:} \\
(a) &\text{ For all } (x, u) \in E \times \mathcal{V}, \text{ the function } \zeta \mapsto p_\zeta(x, u) \text{ is measurable on } [0, \gamma];
\tag{4}
(b) &\text{ The function } \zeta \mapsto p_\zeta(0, 0) \text{ in } L^1([0, \gamma], E);
\text{(c) } &\text{ There exists a positive function } \psi \in L^1([0, \gamma], \mathbb{R}^+) \text{ such that}
\|p_\zeta(x_1, u_1) - p_\zeta(x_2, u_2)\| \leq \psi_\zeta(\|x_1 - x_2\| + \|u_1 - u_2\|)
\text{ for a.e. } \zeta \in [0, \gamma] \text{ and all } (x_1, u_1), (x_2, u_2) \in E \times \mathcal{V};
\end{align*}
\tag{5}
\]
\( \Omega \neq \emptyset \) is a closed convex subset of \( \mathcal{V} \).
Assume that Theorem 1 was demonstrated in [16] under the following additional assumption:

\[ \|q(x_1, u) - q(x_2, u)\|_{V^*} \leq L_q(|x_1 - x_2| + \|x_1 - x_2\|_{E}) \]

\( \forall x_1, x_2 \in [0, \gamma], \; u \in V \) and \( x_1, x_2 \in E \).

The function \( \varphi : V \to \mathbb{R} \) is convex and lower semicontinuous.

Therefore, we recall that

\[ x_0 \in E. \]

**Definition 5 ([24,25]).** A pair of functions \((x, u)\) is said to be a solution of (1) if \(x \in C([0, \gamma], E)\), \(u \in C([0, \gamma], \Omega)\),

\[ x(\xi) = \mathcal{N}(\xi)x_0 + \int_0^\xi \mathcal{N}(\xi - \tau)p_\tau(x(\tau), u(\tau)) \, d\tau \; \forall \xi \in [0, \gamma] \]  

and

\[ u(\xi) \in \text{Sol}(\Omega, q_\xi(x(\xi), u(\xi)), \varphi) \; \forall \xi \in [0, \gamma]. \]

Assume that if \((x, u)\) is a solution of (1), then \(x\) is called the mild trajectory and \(u\) is the variational control trajectory.

We are now able to establish the following existence and uniqueness results.

**Theorem 1.** Assume that (3), (8) are satisfied. Then there exists a unique solution \((x, u) \in C([0, \gamma], E) \times C([0, \gamma], \Omega)\) to (1).

**Proof.** Theorem 1 was demonstrated in [16] under the following additional assumption: either \(\Omega\) is a bounded subset in \(V\) or there exists an element \(v^* \in \Omega\) such that

\[ \liminf_{u \in \Omega, \|u\|_{V} \to \infty} \frac{\langle q_\xi(x, u), u - v^* \rangle + \varphi(u) - \varphi(v^*)}{\|u\|_{V}} = +\infty, \; \forall (\xi, x) \in [0, \gamma] \times E. \]  

(10)

However, if \(\Omega\) is unbounded, then conditions (6)(a) and (7) guarantee the legitimacy of (10) with any \(v \in \Omega\).

Let \(u, v \in \Omega\) and \((\xi, x) \in [0, \gamma] \times E\) be fixed. We compose

\[ \langle q_\xi(x, u), u - v \rangle = \langle q_\xi(x, u) - q_\xi(x, v), u - v \rangle + \langle q_\xi(x, v), u - v \rangle. \]

By inverse monotonicity and Lipschitz continuity of \(q(\xi, x, \cdot)\), we obtain

\[ \langle q_\xi(x, u), u - v \rangle \geq a_q\beta_q^2\|u - v\|_{V^*}^2 - \|q_\xi(x, v)\|_{V^*}\|u - v\|_V \]

\[ \geq a_q\beta_q^2(\|u\|_V - \|v\|_V)^2 - \|q_\xi(x, v)\|_{V^*}\|u\|_V + \|v\|_V) \]

\[ = a_q\beta_q^2\|u\|_V^2 - (2a_q\beta_q^2\|v\|_V + \|q_\xi(x, v)\|_{V^*})\|u\|_V + a_q\beta_q^2\|v\|_V^2 \]

\[ - \|q_\xi(x, v)\|_{V^*}\|v\|_V. \]  

(11)

We know that \(\varphi\) is convex and lower semicontinuous, and there exists an element \(\ell \in V^*\) and a constant \(\rho \in \mathbb{R}\) such that

\[ \varphi(\mu) \geq \langle \ell, \mu \rangle + \rho, \; \forall \mu \in V, \]

(12)
see ([26], Proposition 5.2.25). Hence, from (11) and (12), we infer that
\[
\langle q_c(x,u), u-v \rangle + \varphi(u) - \varphi(v) \geq a_q b_q^2 \|u\|^2 - \left(2a_q b_q^2 \|v\| \|u\| + \|q_c(x,v)\| \|v\| + \|\ell\| \|v\| \right) \|u\| \|v\| + a_q b_q^2 \|v\|^2 - \|q_c(x,v)\| \|v\| - \varphi(v) - \beta.
\]
(13)

Therefore, (13) shows that when \( \Omega \) is unbounded, we have
\[
\lim \inf_{u \in \Omega, \|u\| \to \infty} \frac{\langle q_c(x,u), u-v \rangle + \varphi(u) - \varphi(v)}{\|u\| \|v\|} = +\infty.
\]

As a result, condition (10) is true for any \( v \in \Omega \), as claimed. To finish the proof, we will employ ([16], Theorem 3.1).

3. Main Results

In this section, we suggest the penalty approach for (1). It entails identifying that the approximation problem has a unique solution, and the sequence of the problem converges to the unique solution of (1). To that end, we consider an operator \( \mathcal{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^\ast \), two sequences \( \{\Omega_n\} \subset \mathcal{V}, \{\lambda_n\} \subset \mathbb{R} \), and for each \( n \in \mathbb{N} \) the differential variational inequality problem for finding a pair of functions \( (x_n, u_n) \) with \( x_n : [0, \gamma] \to \mathbb{E} \) and \( u_n : [0, \gamma] \to \Omega_n \) such that
\[
\begin{cases}
  x_n' = Ax_n + p_c(x_n, u_n) & \text{a.e. } \xi \in [0, \gamma], \\
  u_n \in \text{Sol}(\Omega_n, q_c(x_n, u_n)) + \frac{1}{\lambda_n} \mathcal{P}(u_n, u_n, \varphi), \text{ for all } \xi \in [0, \gamma], \\
  x_n(0) = x_0,
\end{cases}
\]

(14)

where
\[
  u_n(\xi) \in \text{Sol}(\Omega_n, q_c(x_n(\xi), u_n(\xi))) + \frac{1}{\lambda_n} \mathcal{P}(u_n(\xi), u_n(\xi), \varphi)
\]

is a notation and \( u_n(\xi) \) is satisfying the variational inequality problem:
\[
  u_n(\xi) \in \Omega_n, \quad \langle q_c(x_n(\xi), u_n(\xi)), v - u_n(\xi) \rangle + \frac{1}{\lambda_n} \langle \mathcal{P}(u_n(\xi), u_n(\xi)), v - u_n(\xi) \rangle + \varphi(v) - \varphi(u_n(\xi)) \geq 0, \text{ for all } v \in \Omega_n.
\]

(15)

We evaluate the following hypotheses on the data in the research of (14).
\[
\begin{cases}
  \text{For } n \in \mathbb{N}, \mathcal{V} \neq \Omega_n \text{ is a closed convex subset of } \mathcal{V} \text{ and } \Omega_n \supset \Omega. \\
  \text{For } n \in \mathbb{N}, \lambda_n > 0.
\end{cases}
\]

(16)

For \( n \in \mathbb{N}, \lambda_n > 0. \)

(17)

The operator \( \mathcal{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^\ast \) is a bounded, demicontinuous and monotone operator.

(18)

There exists a set \( \hat{\Omega} \) such that
\[
\begin{cases}
  (a) \quad \Omega_n \subset \hat{\Omega} \subset \mathcal{V} \text{ for } n \in \mathbb{N}; \\
  (b) \quad \Omega_n \uparrow \hat{\Omega} \text{ as } n \to \infty; \\
  (c) \quad \langle \mathcal{P}(u, v - u) \rangle \leq 0 \forall u \in \hat{\Omega}, v \in \Omega; \\
  (d) \quad \text{if } u \in \Omega, \langle \mathcal{P}(u, v - u) \rangle = 0, \forall v \in \Omega \text{ then } u \in \Omega.
\end{cases}
\]

\[ \lambda_n \to 0 \text{ as } n \to \infty. \]

(19)

We recall from Definition 5 that the functions \( (x_n, u_n) \) are a solution of (14); if \( x_n \in C([0, \gamma], \mathbb{E}), u_n \in C([0, \gamma], \Omega_n) \), then
\[
  x_n(\xi) = \mathcal{N}(\xi)x_0 + \int_0^\xi \mathcal{N}(\xi - \tau)p_\tau(x_n(\tau), u_n(\tau)) \, d\tau \quad \forall \xi \in [0, \gamma]
\]

(21)
Theorem 2. Assume that (3), (4), (6), (8), (16) and (18) are satisfied. Then for \( n \in \mathbb{N} \):

1. There exists a unique solution \((x_n, u_n) \in C([0, \gamma], \mathbb{E}) \times C([0, \gamma], \Omega_n)\) of (14).

2. If (5), (19) and (20) hold as well, then the functions \((x_n, u_n)\) of (14) converge to the unique solution \((x, u)\) of (1) obtained in Theorem 1, i.e., for all \( \xi \in [0, \gamma] \)

   \[
   (x_n(\xi), u_n(\xi)) \to (x(\xi), u(\xi)) \quad \text{in} \\ E \times V, \text{ as } n \to \infty.
   \]

Proof.

1. For \( n \in \mathbb{N} \), the function \( q_n : [0, \gamma] \times \mathbb{E} \times V \to V^\ast \) is defined by

   \[
   q_{\xi u}(x, u) = q_{\xi}(x, u) + \frac{1}{\lambda_n} P(u, u).
   \]

Under the hypotheses (6), (17) and (18), it is simple to see that \( q_n \) meets condition (6) with the constants \( a_n, \beta_n^2 \) and \( \mathcal{L}_n \). As a result, since (16) holds, we can utilize the Theorem 1 with \( \Omega = \Omega_n \). Hence, the first part of the proof concludes with the deduction that there exists a unique solution \((x_n, u_n) \in C([0, \gamma], \mathbb{E}) \times C([0, \gamma], \Omega_n)\) to the system (14).

2. In the second part, we suppose that (5) and (19)–(20) are satisfied. For \( n \in \mathbb{N} \), the auxiliary problem for finding a function \( \tilde{u}_n \in C([0, \gamma], \Omega_n) \) is:

   \[
   \langle q_{\xi}(x(\xi), \tilde{u}_n(\xi)), \nu - \tilde{u}_n(\xi) \rangle + \frac{1}{\lambda_n} \langle P(\tilde{u}_n(\xi), \tilde{u}_n(\xi)), \nu - \tilde{u}_n(\xi) \rangle + \varphi(\nu) \\
   - \varphi(\tilde{u}_n(\xi)) \geq 0, \forall \nu \in \Omega_n \text{ and } \xi \in [0, \gamma].
   \]

Take note of the fact that \( x \) is the mild trajectory of (1). We utilize a standard result on time-dependent variational inequalities to see that the problem (23) has a unique solution \( \tilde{u}_n \in C([0, \gamma], \Omega_n) \).

The rest of the proof is now divided into four steps.

Step (i) Assert that for \( \xi \in [0, \gamma] \), there exists a subsequence of the sequence \( \{\tilde{u}_n(\xi)\} \), again denoted by \( \{\tilde{u}_n(\xi)\} \) which converges weakly to \( \tilde{u}(\xi) \in V \), in other words,

   \[
   \tilde{u}_n(\xi) \quad \text{weakly} \quad \tilde{u}(\xi) \in V \quad \text{as } n \to \infty.
   \]

To prove this assertion, we fix \( \xi \in [0, \gamma] \) and \( u_0 \in \Omega \). Using the inverse strong monotonicity and Lipschitz continuity of \( q \), (23) with \( v = u_0 \in \Omega \subset \Omega_n \) and (19)(c), we obtain

\[
\alpha \beta_n^2 \|\tilde{u}_n(\xi) - u_0\|^2_V \leq \langle q_{\xi}(x(\xi), \tilde{u}_n(\xi)) - q_{\xi}(x(\xi), u_0), \tilde{u}_n(\xi) - u_0 \rangle \\
\leq \frac{1}{\lambda_n} \langle P(\tilde{u}_n(\xi), \tilde{u}_n(\xi)), u_0 - \tilde{u}_n(\xi) \rangle + \varphi(u_0) - \varphi(\tilde{u}_n(\xi)) \\
+ \langle q_{\xi}(x(\xi), u_0), u_0 - \tilde{u}_n(\xi) \rangle \\
\leq \varphi(u_0) - \varphi(\tilde{u}_n(\xi)) + \langle q_{\xi}(x(\xi), u_0), u_0 - \tilde{u}_n(\xi) \rangle.
\]

Now, from (12), we have

\[
\alpha \beta_n^2 \|\tilde{u}_n(\xi) - u_0\|^2_V \leq \varphi(u_0) - \langle \xi, \tilde{u}_n(\xi) \rangle - \rho + \|q_{\xi}(x(\xi), u_0)\|_V \|\tilde{u}_n(\xi) - u_0\|_V \\
\leq \varphi(u_0) + q_{\xi}(\xi) \|\tilde{u}_n(\xi) - u_0\|_V + \|\xi\|_V \|u_0\|_V + |\rho|,
\]
Step (ii) For all \( \varsigma \), according to Definition 4(i), let \( \left\{ v \right\} \) be bounded in \( V \), then there exists a sequence \( \tilde{\varsigma} \) such that, passing to a subsequence if necessary, denoted by \( \left\{ \tilde{\varsigma} \right\} \), \( \tilde{\varsigma} \) is weakly convergent. Therefore, from assumption (6)(b) and Definition 4(ii), \( \tilde{\varsigma} \) is weakly convergent. Thus, the sequence \( \tilde{\varsigma} \) is also bounded in \( V \). Since \( V \) is a reflexive Banach space, there exists \( \tilde{\varsigma} \in V \) such that, passing to a subsequence if necessary, denoted by \( \left\{ \tilde{\varsigma} \right\} \), \( \tilde{\varsigma} \) is weakly convergent. Therefore, from assumption (6)(b) and Definition 4(ii), \( \tilde{\varsigma} \in \Omega_n \) implies that

\[ \tilde{\varsigma} \in \Omega, \]

and the statement is proved.

Step (ii) For all \( \varsigma \in [0, \gamma] \), we claim that

\[ \tilde{\varsigma} = u(\varsigma). \]

According to Definition 4(i), let \( \varsigma \in [0, \gamma] \) and \( v \in \tilde{\Omega} \), then there exists a sequence \( \{v_n\} \) such that \( v_n \in \Omega_n \) for \( n \in \mathbb{N} \) and

\[ v_n \to v \in V \text{ as } n \to \infty. \]

From (23), we have

\[
\frac{1}{\lambda_n} \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), \tilde{\varsigma} - v_n \rangle \leq \langle \tilde{\varsigma}, \mathcal{P}(\tilde{\varsigma}, v_n), \tilde{\varsigma} - v_n \rangle + \phi(v_n) - \phi(\tilde{\varsigma}) \leq \| \tilde{\varsigma} \| \| v_n - \tilde{\varsigma} \| + \phi(v_n) + \| v_n - \tilde{\varsigma} \| \| \tilde{\varsigma} \| + \| v_n - \tilde{\varsigma} \| \| v_n - \tilde{\varsigma} \| + \| v_n - \tilde{\varsigma} \|.\]

From (6) and (7), we combine the functions \( q, \varphi \) with the convergence of \( \{v_n\} \) and boundedness \( \{\tilde{\varsigma}\} \) to show that there exists a positive constant \( \vartheta \) which does not depend on \( n \), such that

\[
\frac{1}{\lambda_n} \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), \tilde{\varsigma} - v_n \rangle \leq \vartheta.
\]

Since \( \lambda_n \to 0 \), we have

\[
\limsup \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), \tilde{\varsigma} - v_n \rangle \leq 0. \tag{25}
\]

Since the sequence \( \{v_n\} \) converges in \( V \) and the sequence \( \{\mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma})\} \) is bounded in \( V^* \), we have

\[
\limsup \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), \tilde{\varsigma} - v \rangle \leq \limsup \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), \tilde{\varsigma} - v_0 \rangle + \limsup \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), v_n - v \rangle = \limsup \langle \mathcal{P}(\tilde{\varsigma}, \tilde{\varsigma}), v_n - v \rangle.
\]
Therefore, (25) yields
\[
\limsup \langle P(\bar{u}_n(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - v \rangle \leq 0, \forall v \in \Omega. \tag{26}
\]
Furthermore, the regularity \( \bar{u}(\xi) \in \Omega \) allows us to take \( v = \bar{u}(\xi) \) in (26) to obtain
\[
\limsup \langle P(\bar{u}_n(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - \bar{u}(\xi) \rangle \leq 0. \tag{27}
\]
However, keep in mind that assumption (18) ensures the operator \( P : \mathbb{V} \times \mathbb{V} \to \mathbb{V}^* \) is bounded, demicontinuous and monotone. From \([27], \text{Theorem 3.69}\), we can derive that \( P \) is pseudomonotone. Thus, the pseudomonotonicity of \( P \) together with (27) is taken to imply
\[
\langle P(\bar{u}(\xi), \bar{u}(\xi)), \bar{u}(\xi) - v \rangle \leq \liminf \langle P(\bar{u}_n(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - v \rangle \leq \limsup \langle P(\bar{u}_n(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - v \rangle, \forall v \in \mathbb{V}.
\]
Hence (26), we have
\[
\langle P(\bar{u}(\xi), \bar{u}(\xi)), \bar{u}(\xi) - v \rangle \leq 0, \forall v \in \Omega. \tag{28}
\]
The Equations (16) and (19)(a) ensure that \( \Omega \subset \Omega \), and from (28) derive that
\[
\langle P(\bar{u}(\xi), \bar{u}(\xi)), \bar{u}(\xi) - v \rangle \leq 0, \forall v \in \Omega. \tag{29}
\]
Now adding the inequality (29) with assumption (19)(c), we find that
\[
\langle P(\bar{u}(\xi), \bar{u}(\xi)), \bar{u}(\xi) - v \rangle = 0, \forall v \in \Omega.
\]
Then from assumption (19)(d), we obtain the regularity
\[
\bar{u}(\xi) \in \Omega. \tag{30}
\]
Assume that \( \omega \in \Omega \). Then from the inequality (23), we obtain
\[
\langle q_c(x(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - \omega \rangle \leq \frac{1}{\lambda_n} \langle P(\bar{u}_n(\xi), \bar{u}_n(\xi)), \omega - \bar{u}_n(\xi) \rangle + \varphi(\omega) - \varphi(\bar{u}_n(\xi)).
\]
Therefore, from assumption (19)(c) we have
\[
\langle q_c(x(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - \omega \rangle \leq \varphi(\omega) - \varphi(\bar{u}_n(\xi)). \tag{31}
\]
Again, from the monotonicity of \( q \) we have
\[
\langle q_c(x(\xi), \omega), \bar{u}_n(\xi) - \omega \rangle \leq \langle q_c(x(\xi), \bar{u}_n(\xi)), \bar{u}_n(\xi) - \omega \rangle
\]
and from (31), we find that
\[
\langle q_c(x(\xi), \omega), \bar{u}_n(\xi) - \omega \rangle \leq \varphi(\omega) - \varphi(\bar{u}_n(\xi)). \tag{32}
\]
Using the upper limit of the inequality, the assumption (7) and the convergence (24), we obtain
\[
\langle q_c(x(\xi), \omega), \omega - \bar{u}_n(\xi) \rangle + \varphi(\omega) - \varphi(\bar{u}_n(\xi)) \geq 0.
\]
For \( v \in \Omega \) and \( \omega \in (0, 1) \), we take \( \omega = (1 - \omega)\bar{u}(\xi) + \omega v \). Then, from (30) and (5), we have
\[
\omega \in \Omega.
Hence, with the convexity $\varphi$ together with the previous inequality, we have
\[ \langle q_\xi(x(\xi)), (1 - \omega)\tilde{u}(\xi) + \omega v), v - \tilde{u}(\xi) \rangle + \varphi(v) - \varphi(\tilde{u}(\xi)) \geq 0. \]

Taking $\lambda \to 0$ and (6)(a), we obtain
\[ \langle q_\xi(x(\xi), \tilde{u}(\xi)), v - \tilde{u}(\xi) \rangle + \varphi(v) - \varphi(\tilde{u}(\xi)) \geq 0, \forall v \in \Omega. \]

We point out that the assumption (6)(a) ensures that the solution of this inequality is unique. Hence, from the uniqueness, $u(t) \in \Omega$ is a unique solution of the above inequality. Therefore, we derive that
\[ \tilde{u}(\xi) = u(\xi), \]

and our claim is proved.

Step (iii) For $\xi \in [0, \gamma]$, we show that $\tilde{u}_n(\xi) \to u(\xi) \in \mathcal{V}$.

Let $\xi \in [0, \gamma]$. To begin a careful observation of the proofs in steps (i) and (ii) shows that any weakly convergent subsequence of the sequence $\{\tilde{u}_n(\xi)\}$ converges weakly to $u(\xi)$ in $\mathcal{V}$ as $n \to \infty$. Furthermore, the sequence $\{\tilde{u}_n(\xi)\}$ is bounded and the whole sequence $\{\tilde{u}_n(\xi)\}$ weakly converges to $u(\xi)$ in $\mathcal{V}$.

Taking $\omega = u(\xi)$ in (31) and going to the upper limit, we can observe that
\[ \limsup \langle q_\xi(x(\xi), \tilde{u}_n(\xi)), \tilde{u}_n(\xi) - u(\xi) \rangle \leq 0. \tag{33} \]

Taking the inequality (33) together with (24) and the pseudomonotonicity of the function $q$, ensured by assumption (6)(a), we have
\[ \langle q_\xi(x(\xi), \tilde{u}(\xi)), \tilde{u}(\xi) - v \rangle \leq \liminf \langle q_\xi(x(\xi), \tilde{u}_n(\xi)), \tilde{u}_n(\xi) - v \rangle, \forall v \in \mathcal{X}. \]

We put $v = u(t)$ in the above inequality to obtain
\[ \liminf \langle q_\xi(x(\xi), \tilde{u}_n(\xi)), \tilde{u}_n(\xi) - u(\xi) \rangle \geq 0. \tag{34} \]

Using now inequalities the (33) and (34), it follows that
\[ \langle q_\xi(x(\xi), \tilde{u}_n(\xi)), \tilde{u}_n(\xi) - u(\xi) \rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{35} \]

Using the inverse strong monotonicity and Lipschitz continuity of $q$ together with
\[ \tilde{u}_n(\xi) \xrightarrow{\text{weakly}} u(\xi) \in \mathcal{V} \]

and (35), we have
\[ a_\xi \beta^2 \| \tilde{u}_n(\xi) - u(\xi) \|_X^2 \leq \langle q_\xi(x(\xi), \tilde{u}_n(\xi)) - q_\xi(x(\xi), u(\xi)), \tilde{u}_n(\xi) - u(\xi) \rangle \]
\[ = \langle q_\xi(x(\xi), \tilde{u}_n(\xi)), \tilde{u}_n(\xi) - u(\xi) \rangle - \langle q_\xi(x(\xi), u(\xi)), \tilde{u}_n(\xi) - u(\xi) \rangle \]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]

The proof of Step (iii) is completed.

Step (iv) Eventually, we prove
\[ (x_n(\xi), u_n(\xi)) \to (x(\xi), u(\xi)) \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad \xi \in [0, \gamma] \]

where $(x_n, u_n) \in C([0, \gamma], \mathcal{X}) \times C([0, \gamma], \mathcal{Y})$ is a unique solution of (1).

For $n \in \mathbb{N}, \xi \in [0, \gamma]$ and $v \in \Omega_n$, we have
\[ \langle q_\xi(x_n(\xi), u_n(\xi)), v - u_n(\xi) \rangle + \frac{1}{\lambda_n} \langle P(u_n(\xi), u_n(\xi)), v - u_n(\xi) \rangle + \varphi(v) - \varphi(u_n(\xi)) \geq 0. \tag{36} \]
Put \( v = \hat{u}_n(\xi) \) in (36) and \( v = u_n(\xi) \) in (36), and adding these inequalities with the monotonicity of \( P \), we obtain
\[
\langle q_\xi(x_n(\xi), u_n(\xi)) - q_\xi(x(\xi), \hat{u}_n(\xi)), u_n(\xi) - \hat{u}_n(\xi) \rangle \leq 0.
\]

Combining the above inequality with (6) leads to
\[
\alpha q_\xi^2 \| u_n(\xi) - \hat{u}_n(\xi) \|^2 \leq \langle q_\xi(x_n(\xi), u_n(\xi)) - q_\xi(x_n(\xi), \hat{u}_n(\xi)), u_n(\xi) - \hat{u}_n(\xi) \rangle \\
\leq \langle q_\xi(x(\xi), \hat{u}_n(\xi)) - q_\xi(x_n(\xi), \hat{u}_n(\xi)), u_n(\xi) - \hat{u}_n(\xi) \rangle \\
\leq C \| x(\xi) - x_n(\xi) \|_E \| u_n(\xi) - \hat{u}_n(\xi) \|_V.
\]
Consequently,
\[
\| u_n(\xi) - u(\xi) \|_V \leq \| u_n(\xi) - \hat{u}_n(\xi) \|_V + \| \hat{u}_n(\xi) - u(\xi) \|_V
\]
yields
\[
\| u_n(\xi) - u(\xi) \|_V \leq \frac{C}{\alpha q_\xi^2} \| x(\xi) - x_n(\xi) \|_E + \| \hat{u}_n(\xi) - u(\xi) \|_V. \tag{37}
\]

On the other hand, when we use (9) and (21), we obtain
\[
\| x_n(\xi) - x(\xi) \|_E \leq \varrho_A \int_0^\xi \| p_\tau(x_n(\tau), u_n(\tau)) - p_\tau(x(\tau), u(\tau)) \|_E \, d\tau
\]
where \( \varrho_A \) is a positive constant, such that
\[
\| N'(\tau) \|_{L(E)} \leq \varrho_A, \quad \forall \tau \in [0, \gamma].
\]
From (4) and (37), we obtain
\[
\| x_n(\xi) - x(\xi) \|_E \leq \varrho_A \int_0^\xi \psi_\tau \| \hat{u}_n(\tau) - u(\tau) \|_V \, d\tau \\
+ \varrho_A \int_0^\xi \psi_\tau \left( 1 + \frac{C}{\alpha q_\xi^2} \right) \| x_n(\tau) - x(\tau) \|_E \, d\tau.
\]
From the Gronwall inequality, there exists a positive constant \( \theta_0 \) (does not depend on \( n \)), so we have
\[
\| x_n(\xi) - x(\xi) \|_E \leq \theta_0 \int_0^\xi \psi_\tau \| \hat{u}_n(\tau) - u(\tau) \|_V \, d\tau.
\]
This inequality combined with the \( \hat{u}_n(\tau) \to u(\tau) \in V, \quad \forall \tau \in [0, \gamma], \) the boundedness result in the proof of Step (i) and the Lebesgue convergence theorem implies that
\[
\limsup \| x_n(\xi) - x(\xi) \|_E \leq \theta_0 \int_0^\xi \limsup \| \hat{u}_n(\tau) - u(\tau) \|_V \, d\tau = 0.
\]
Hence, we have
\[
x_n(\xi) \to x(\xi) \in E_v \quad \text{as} \quad n \to \infty. \tag{38}
\]
Again, we use inequality (37), \( \hat{u}_n(\xi) \to u(\tau) \in V \) obtained in Step (iii) and (38) to show that
\[
u_n(\xi) \to u(\xi) \in V \quad \text{as} \quad n \to \infty. \tag{39}
\]
Therefore, we say that the convergence (22) is a direct consequence of (38) and (39), and the proof is completed.
Now, we discuss the special cases of Theorem 3.1. To this end, we assume that (3), (8) hold and \((x, u)\) is a solution of (1) provided by Theorem 1. Ensure that this solution satisfies

\[ x(\xi) = X(\xi)_{\xi} + \int_{0}^{\xi} N(\xi - \tau)p_{\tau}(x(\tau), u(\tau)) \, d\tau, \quad \forall \xi \in [0, \gamma], \quad (40) \]

\[ u(\xi) \in \Omega, \quad \langle q_{\xi}(x(\xi), u(\xi)), v - u(\xi) \rangle + \phi(v) - \phi(u(\xi)) \geq 0, \quad \forall v \in \Omega, \xi \in [0, \gamma]. \quad (41) \]

Assuming that \(\Omega = \Omega_{n}\) and \(\Omega_{n}\) do not depend on \(n\), then we have the following corollary of Theorem 2.

**Corollary 1.** Assume that (3), (8), (17), (18), (20) hold, and assume that there exists a convex closed subset \(\Omega\) of \(V\) such that \(\Omega \subset \tilde{\Omega}\) and (19)(c), (d) hold. For each \(n \in \mathbb{N}\), there exists a unique pair of functions \((x_{n}, u_{n}) \in C([0, \gamma], \mathbb{E}) \times C([0, \gamma], \tilde{\Omega})\) such that

\[ x_{n}(\xi) = X(\xi)_{\xi} + \int_{0}^{\xi} N(\xi - \tau)p_{\tau}(x(\tau), u(\tau)) \, d\tau, \quad \forall \xi \in [0, \gamma], \quad (42) \]

\[ u_{n}(\xi) \in \tilde{\Omega}, \quad \langle q_{\xi}(x_{n}(\xi), u_{n}(\xi)), v - u_{n}(\xi) \rangle + \frac{1}{\lambda_{n}} \langle \mathcal{P}(u_{n}(\xi), u_{n}(\xi)), v - u_{n}(\xi) \rangle + \phi(v) - \phi(u_{n}(\xi)) \geq 0, \quad \forall v \in \Omega, \xi \in [0, \gamma]. \quad (43) \]

Moreover, the convergence (22) holds, for any \(\xi \in [0, \gamma]\).

**Remark 1.**

(i) If \(\tilde{\Omega} = V\), then Corollary 1 represents the main convergence result studied in [16] and inequality (42) is an unconstrained variational inequality.

(ii) If set \(\tilde{\Omega}\) is different from the whole space \(V\), then inequality (42) may be a time-dependent variational inequality with constraints.

(iii) In general, case Corollary 1 will be the solution of the differential variational inequality (42), (43) governed by the set of constraints \(\Omega\) and can be approached by the solution of the differential variational inequality (42), (43) governed by a different set of constraints \(\tilde{\Omega}\) as the penalty parameter \(\lambda_{n}\) is small enough.

### 4. Parabolic–Elliptic Equations

Suppose that \(\Omega\) is a bounded domain of \(\mathbb{R}^{i} (i \in \mathbb{N})\) with a smooth boundary \(\partial\mathbb{R}\), separated in two measurable portions \(\partial_{1}\) and \(\partial_{2}\), such that

\[ \text{meas} \partial_{1} > 0. \]

Assume that \(\gamma > 0\) is a finite interval of time. The spatial variable denoted by \(z \in \Omega \cup \partial\mathbb{R}\) and the time variable by \(\xi \in [0, \gamma]\), and the \(v\) is the outward unit normal at \(\partial\mathbb{R}\). We investigate the following parabolic–elliptic problem with these notations for finding \(x : \Omega \times [0, \gamma] \to \mathbb{R}\) and \(u : \Omega \times [0, \gamma] \to \mathbb{R}\) such that

\[ x'(z, \xi) - \Delta x(z, \xi) = \epsilon_{\xi}(z, x(z, \xi), u(z, \xi)) \quad \text{in} \quad \Omega \times [0, \gamma], \quad (44) \]

\[ x(z, \xi) = 0 \quad \text{on} \quad \partial \times [0, \gamma], \quad (45) \]

\[ x(z, 0) = x_{0}(z) \quad \text{in} \quad \Omega, \quad (46) \]

\[ \begin{align*}
 u(z, \xi) & \geq \zeta, \\
 -\Delta u(z, \xi) + \kappa(z)u(z, \xi) & \geq \gamma_{\xi}(z, x(z, \xi)), \\
 (u(z, \xi) - \zeta)(\Delta u(z, \xi) - \kappa(z)u(z, \xi) + g_{\xi}(z, x(z, \xi))) &= 0
\end{align*} \quad \text{in} \quad \Omega \times [0, \gamma]. \quad (47) \]

\[ u(z, \xi) = 0 \quad \text{on} \quad \partial_{1} \times [0, \gamma], \quad (48) \]
\[
\frac{\partial u(z, \zeta)}{\partial v} \leq \phi(z), \\
\frac{\partial u(z, \zeta)}{\partial v} = \phi(z) \frac{u(z, \zeta)}{|u(z, \zeta)|} \text{ if } u(z, \zeta) \neq 0
\] on \( \Omega \times [0, \gamma] \). \tag{49}

We utilize the Lebesgue and Sobolev spaces to study the problem (44)–(49). Furthermore, we define the space \( V = \{ v \in H^1(\Omega) : v(z) = 0 \text{ a.e. } z \in \Omega \} \) with the inner product

\[
(u, v)_V = \int_{\Omega} \nabla u \cdot \nabla v \, dx \tag{51}
\]
and norm \( \| \cdot \|_V \). Let \((V, \| \cdot \|_V) \) be a Hilbert space. \( V^* \) is used for the dual of \( V \) and \( \langle \cdot, \cdot \rangle \) is used for the duality pairing mapping between \( V^* \) and \( V \). Now, we consider the following assumptions about the data.

\[
\epsilon : \tilde{\Omega} \times [0, \gamma] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is such that:}
\]

\[
\{(a) \text{ For all } (\xi, \tau) \in \mathbb{R}^2, \text{ the function } (z, \zeta) \mapsto \epsilon(z, \tau, r) \text{ is measurable on } \tilde{\Omega} \times [0, \gamma] ;
\]

\[
(b) \text{ There exist two positive functions } \theta \in L^1(\tilde{\Omega}) \times (0, \gamma) \text{ and } \psi \in L^1(0, \gamma) \text{ such that}
\]

\[
|\epsilon(z, 0, 0)| \leq \theta(z, \zeta),
\]

\[
|\epsilon(z, \tau_1, 1) - \epsilon(z, \tau_2, 2)| \leq \psi(z) (|\tau_1 - \tau_2| + |r_1 - r_2|), \text{ for a.e. } z \in \tilde{\Omega}, \xi \in [0, \gamma]
\]

and all \((\tau_1, \tau_1), (\tau_2, \tau_2) \in \mathbb{R}^2\);

\[
g : \tilde{\Omega} \times [0, \gamma] \times \mathbb{R} \to \mathbb{R} \text{ is such that:}
\]

\[
(a) \text{ The function } z \mapsto g(z, 0, 0) \text{ is belongs to } L^2(\tilde{\Omega}) ;
\]

\[
(b) \text{ There exists a constant } \mathcal{L}_g > 0 \text{ such that}
\]

\[
|g_{\xi_1}(z, \tau_1) - g_{\xi_2}(z, \tau_2)| \leq \mathcal{L}_g (|\xi_1 - \xi_2| + |\tau_1 - \tau_2|), \text{ for all } \xi_1, \xi_2 \in [0, \gamma], \tau_1, \tau_2 \in \mathbb{R}, \text{ a.e. } z \in \tilde{\Omega}.
\]

\[
\kappa \in L^\infty(\tilde{\Omega}), \kappa(z) \geq 0 \text{ a.e. } z \in \tilde{\Omega},
\]

\[
\phi \in L^2(\Omega), \phi(z) \geq 0 \text{ a.e. } z \in \Omega,
\]

\[
x_0 \in L^2(\tilde{\Omega}),
\]

\[
\zeta > 0.
\]

Now, we assume that the functions \( A : D(A) \subset L^2(\tilde{\Omega}) \to L^2(\tilde{\Omega}), p : [0, \gamma] \times L^2(\tilde{\Omega}) \times V \to L^2(\tilde{\Omega}), \Omega \subset V, q : [0, \gamma] \times L^2(\tilde{\Omega}) \times V \to V^* \) and \( \varphi : V \to \mathbb{R} \) are defined by

\[
\begin{align*}
D(A) &= H^2(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega}) \subset L^2(\tilde{\Omega}), \\
Ax &= \Delta x \quad \forall x \in D(A),
\end{align*}
\tag{58}
\]

\[
p_{\xi}(x, u)(z) = \epsilon(z, x(z), u(z)), \quad \forall \xi \in [0, \gamma], x \in L^2(\tilde{\Omega}), u \in V, \text{ a.e. } z \in \tilde{\Omega},
\]

\[
\Omega = \{ u \in V : u(z) \geq \zeta \text{ a.e. } z \in \tilde{\Omega} \},
\]

\[
\langle q_{\xi}(x, u), v \rangle = \int_{\tilde{\Omega}} \nabla u \cdot \nabla v \, dz + \int_{\tilde{\Omega}} k uv \, dz - \int_{\tilde{\Omega}} g_{\xi}(x) v \, dz, \quad \forall v \in [0, \gamma], x \in L^2(\tilde{\Omega}), u, v \in V,
\]

\[
\varphi(u) = \int_{\tilde{\Omega}} \zeta |u| \, da \quad \forall u \in V.
\]

\[
\tag{59}
\tag{60}
\tag{61}
\tag{62}
\]
Using the above notation, we can conclude the following variational formulation of problems for finding \( x : [0, \gamma] \to \mathbb{R} \) and \( u : [0, \gamma] \to \mathbb{V} \), such that
\[
x'(\xi) = Ax(\xi) + p_c(x(\xi), u(\xi)), \quad \text{a.e. } \xi \in [0, \gamma], \tag{63}
\]
\[
u(\xi) \in \Omega, \quad \langle q_c(x(\xi), u(\xi)), v - u(\xi) \rangle + \varphi(v) - \varphi(u(\xi)) \geq 0, \quad \forall v \in \Omega, \quad \xi \in [0, \gamma], \tag{64}
\]
\[
x(0) = x_0. \tag{65}
\]

**Theorem 3.** Assume (52), (57) holds true. Then the problem (63)–(65) has a unique solution \((x, u) \in C([0, \gamma], L^2(\Omega)) \times C([0, \gamma], \Omega)\).

**Proof.** The proof of Theorem 3 may be found by applying the Theorem 1 with
\[
\Delta u = 0, \quad z > 0 \quad \text{such that} \quad \int_{\Omega} h(\xi, u(\xi), \nabla u(\xi)) d\xi = \mathcal{E}(u) \]
and the variational formulation of (66) is a problem for finding \( u_0 : \Omega \times [0, \gamma] \to \mathbb{R} \) that satisfies (44), (66), (48), (49) and
\[
u_0(\xi) \geq \zeta_n, \\
\Delta u_0(\xi, \zeta) + g_c(\xi, x_0(\xi, \zeta)) = k(\xi)u_0(\xi, \zeta) + \frac{1}{\lambda_n} h(\xi, u_0(\xi, \zeta) - \zeta_n) \}
\]
in \( \Omega \times [0, \gamma] \). \tag{66}

Now we revel in the problem (14) to replace the condition (47) with the condition (66). There, \( \lambda_n > 0, \zeta_n \in \mathbb{R} \) and \( h : \tilde{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a function that has the following properties:

\[
\begin{align*}
(a) & \quad |h(r, r) - h(r, \tau)| \leq \mathcal{L}_p|\tau - \tau|, \quad \forall r, \tau \in \mathbb{R}, \quad \text{a.e. } z \in \tilde{\Omega}, \quad \text{with } \mathcal{L}_h > 0; \\
(b) & \quad (h(r, r) - h(r, \tau))|\tau - \tau| \geq 0, \quad \forall r, \tau \in \mathbb{R}, \quad \text{a.e. } z \in \tilde{\Omega}; \\
(c) & \quad z \mapsto h(z, r) \text{ is measurable on } \tilde{\Omega}, \quad \forall r \in \mathbb{R}; \\
(d) & \quad h(z, r) = 0 \quad \text{if and only if } r \geq 0 \text{ a.e. } z \in \tilde{\Omega}.
\end{align*}
\]

The example of such a function is given by
\[
h_0(z, r) = -\vartheta r^-, \quad \forall r \in \mathbb{R}, \quad z \in \tilde{\Omega}
\]
where \( \vartheta > 0 \) and \( r^- \) is a negative part of \( r \), i.e.,
\[
r^- = \max\{-r, 0\}.
\]

Let \( \zeta \in \mathbb{R} \) and
\[
\zeta \geq \zeta_n \geq \zeta, \quad \forall n \in \mathbb{N}, \tag{68}
\]
\[
\zeta_n \to \zeta \quad \text{as } n \to \infty.
\]

Again, we define the sets \( \tilde{\Omega}, \Omega_n \) and \( \mathcal{P} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}^+ \) by
\[
\tilde{\Omega} = \{ u \in \mathbb{V} : u(z) \geq \zeta \text{ a.e. } z \in \tilde{\Omega} \}, \tag{70}
\]
\[
\Omega_n = \{ u \in \mathbb{V} : u(z) \geq \zeta_n \text{ a.e. } z \in \tilde{\Omega} \}, \quad \forall n \in \mathbb{N}, \tag{71}
\]
\[
\langle \mathcal{P}(u, u), v \rangle = \int_{\tilde{\Omega}} h(u - \zeta) v dz, \quad \forall u, v \in \mathbb{V}. \tag{72}
\]

Then, the variational formulation of (66) is a problem for finding \( x : [0, \gamma] \to \mathbb{R} \) and \( u : [0, \gamma] \to \mathbb{V} \), such that
\[
x'_n(\xi) = Ax_n(\xi) + p_c(x_n(\xi), u_n(\xi)), \quad \text{a.e. } \xi \in [0, \gamma], \tag{73}
\]
\( u_n(\zeta) \in \Omega_n, \langle q_\zeta(x_n(\zeta), u_n(\zeta)), v - u_n(\zeta) \rangle + \frac{1}{h_n} \langle P(u_n(\zeta), u_n(\zeta)), v - u_n(\zeta) \rangle + \varphi(v) \)
\[ - \varphi(u_n(\zeta)) \geq 0, \quad \forall v \in \Omega_n, \zeta \in [0, \gamma], \] (74)

\[ x_n(0) = x_0. \] (75)

The following is the critical feature of this section.

**Theorem 4.** Assume that (52), (57), (67), (69), (17) and (20) hold. Then for \( n \in \mathbb{N} \), there exists a unique solution \((x_n, u_n) \in \mathbb{C}([0, \gamma], \mathbb{E}) \times \mathbb{C}([0, \gamma], \Omega_n)\) to (73)–(75). Furthermore, the solution converges to \((x, u)\) of (66) obtained in Theorem 3, i.e., for all \( \zeta \in [0, \gamma] \),
\[ (x_n(\zeta), u_n(\zeta)) \to (x(\zeta), u(\zeta)) \in L^2(\Omega) \times V \text{ as } n \to \infty. \] (76)

**Proof.** We use Theorem 2 on the spaces \( \mathbb{E} = L^2(\Omega) \) and \( V \) suggested in (50), with (58), (62), (70) and (72). To begin, we should observe that assumption (67) implies that condition (16) and (19)(a) are satisfied. Second, it is straightforward to demonstrate that the operator (72) meets condition (18) using the attributes (67) of the function \( h \). Using the assumption (68), we may derive that
\[ \Omega_n = \frac{\zeta u}{\zeta} \tilde{\Omega} \]
implies that (19)(b) holds as well.

Next, suppose that \( u \in \tilde{\Omega} \) and \( v \in \Omega \). Then from (67), we have
\[ \begin{align*}
    h(u - \zeta)(v - \zeta) &\leq 0 \\
    \text{and} \\
    h(u - \zeta)(\zeta - u) &\leq 0
\end{align*} \]
\text{a.e. in } \tilde{\Omega} \] (77)

which implies
\[ h(u - \zeta)(v - u) \leq 0 \text{ a.e. } \in \tilde{\Omega}. \]

Therefore,
\[ \int_{\tilde{\Omega}} h(u - \zeta)(v - u) dz \leq 0 \]
and (19)(c) holds. Next, from \( u \in \tilde{\Omega} \) and
\[ \langle P(u, u), v - u \rangle = 0, \quad \forall v \in \Omega, \]
which argues that
\[ \int_{\tilde{\Omega}} h(u - \zeta)(v - \zeta) dz = \int_{\tilde{\Omega}} h(u - \zeta)(v - \zeta) dz, \quad \forall v \in \Omega. \] (78)

Now from (77), we have
\[ \int_{\tilde{\Omega}} h(u - \zeta)(v - \zeta) dz = 0. \] (79)

Therefore, the connotation
\[ g \geq 0, \quad \int_{\tilde{\Omega}} g dz = 0 \Rightarrow g = 0 \text{ a.e. on } \tilde{\Omega} \] (80)

together with (77) and (79) give
\[ h(u - \zeta)(u - \zeta) = 0 \text{ a.e. in } \tilde{\Omega}. \]
Combining the above equality with assumption (67)(d) leads to
\[ u \geq \zeta \text{ a.e. in } \Omega. \]

Eventually, we see that
\[ u \in \Omega, \]
and the condition (19)(d) holds.

Now, the assumptions (17) and (20) are true. Furthermore, the proof of Theorem 3 establishes the validity of the remaining criteria in Theorem 2. We can now employ Theorem 2 in the investigation of (1) and (14) to finish the proof. 

5. Conclusions

Differential variational inequality problems can be viewed as natural and innovative generalizations of differential variational inclusion problems. Two of the most difficult and important problems related to these inequalities are the establishment of the sequences of the problem with a set of constraints and a penalty parameter. In this article, a differential variational inequality problem is suggested and studied, which is more general than many existing problems in the literature. The discussion of differential variational inequality problem depends on the concepts of compactness, symmetry, pseudomonotonicity, Mosco convergence, inverse strong monotonicity and Lipschitz continuous mapping. Finally a parabolic–elliptic equation of the initial and boundary values problem is also discussed as an illustration.

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