Topological Structure of Solution Set to a Fractional Differential Inclusion Problem with Delay

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Abstract: In this paper, we investigate the topological structure of the solution set to a fractional differential inclusion with delay defined on the half-line. We first prove that the solution set to the inclusion is an $R_δ$-set on compact intervals. Then, by means of the inverse limit method, we generalize our results to noncompact intervals. Moreover, under convex and nonconvex conditions, an $R_δ$-property solution set is obtained for some nonlocal problems, where the nonlocal function is set-valued. Further, we study the symmetry of the solution set under some conditions.

Keywords: topological structure; fractional differential inclusion; $R_δ$-set; inverse limit method; nonlocal problems; symmetry

MSC: 34B15; 34A08; 54F17

1. Introduction and Main Results

The study of the structure of solution sets began with Peano in 1890. He proposed the existence theorem of the solution set. Later, Kneser generalized Peano’s theorem that the solution set was not only nonempty, but also compact and connected. In 1942, Aronszajn further improved the theorem by showing that the solution set was even an $R_δ$-set. Today, the topological structure of the solution set continues to arouse research enthusiasm. It is important not only from the viewpoint of academic interest, but also has a wide range of applications in practice. As is known to all, the topological structure of the solution set to differential inclusions is linked to obstacle problems, processes of controlled heat transfer, describing hybrid systems with dry friction and others (see, e.g., [1–3] and references therein).

Some scholars have conducted numerous studies on the topological structure of the solution set to differential equations or inclusions, especially Gabor, who presented many results on the topological structure of the solution set and developed effective techniques for dealing with the structure of fixed point sets (see [4–6]). Concerning other related results, we refer readers to Wojciech [7], Zhou et al. [8], Cheng et al. [9], Grniewicz [10], Andres [11] and Djebali [12,13]. However, all the above results are for integer-order differential systems. It is known that fractional differential systems can better describe practical problems than integer differential systems, especially in describing the memory and hereditary nature of various processes and materials. Based on the background of the above research, are there similar conclusions for fractional differential systems?

In recent years, some results have been obtained on the structure of the solution set of fractional differential systems. For Riemann–Liouville fractional derivatives, Bugajwska-Kasprzak studied two Aronszajn-type theorems for some initial value problems; see [14]; Ziane [15] used a condensing map and Chalco-Cano et al. [16] used the extended Kneser’s theorem to research an initial value problem for nonlinear fractional differential equations.
For Caputo fractional derivatives, Wang et al. [17] illustrated that a fractional control problem was approximately controllable by studying the structure of its solution set; Hoa et al. [18] used Krasnosel’skii-type operators to prove the $R_δ$-property of the solution set of a fractional neutral evolution equation, and used the inverse limit method to obtain the same result on the half-line. There are also many studies on the stability of fractional differential systems. A. Singh et al. [19] discussed the asymptotic stability of stochastic differential equations of fractional order $1 < \alpha \leq 2$ in Banach spaces. For more details, see the research articles [20,21].

Nevertheless, most of the papers describe research on the fractional differential equation on finite intervals, with related results about fractional differential inclusion on infinite interval being very rare. Motivated by the above consideration, we study the following fractional differential inclusion problem:

\[
\begin{cases}
\frac{D^\alpha}{\alpha} x(t) + B(t)x(t) \in H(t, x, x_1), & \text{for a.e. } t \in [0, m], \\
x(t) \in \psi(t), & \text{for } t \in [-\tau, 0],
\end{cases}
\]

where $\frac{D^\alpha}{\alpha}$ denotes the Caputo fractional derivative with order $\alpha \in (0, 1)$. $B : [0, m] \to \mathbb{R}^{N \times \mathbb{N}}$ and is a matrix operator; $H : [0, m] \times \mathbb{R}^N \times C([-\tau, 0]; \mathbb{R}^N) \to 2^\mathbb{R}^N$ is a set-valued function, $x_1(s) = x(t + s)$, for $s \in [-\tau, 0]$, $m \leq \infty$.

Specifically, we consider the following three cases:

**Case 1:** $m$ is a determined constant; $\psi(x)$ is a single-valued function.

\[
\begin{cases}
\frac{D^\alpha}{\alpha} x(t) + B(t)x(t) \in H(t, x, x_1), & \text{for a.e. } t \in [0, m], \\
x(t) = \phi(t), & \text{for } t \in [-\tau, 0],
\end{cases}
\]

We use $\phi(t) \in C([-\tau, 0]; \mathbb{R}^N)$ to denote a single-valued function.

**Case 2:** $m$ is infinite; $\psi(x)$ is a single-valued function.

\[
\begin{cases}
\frac{D^\alpha}{\alpha} x(t) + B(t)x(t) \in H(t, x, x_1), & \text{for a.e. } t \in [0, \infty), \\
x(t) = \phi(t), & \text{for } t \in [-\tau, 0],
\end{cases}
\]

where $\phi(t)$ is a single-valued function.

**Case 3:** $m$ is infinite; $\psi(x)$ is a set-valued function.

\[
\begin{cases}
\frac{D^\alpha}{\alpha} x(t) + B(t)x(t) \in H(t, x, x_1), & \text{for a.e. } t \in [0, \infty), \\
x(t) \in \psi(x), & \text{for } t \in [-\tau, 0],
\end{cases}
\]

where $\psi(x) : C_\infty([-\tau, \infty); \mathbb{R}^N) \to 2^{C([-\tau, 0]; \mathbb{R}^N)}$ is a set-valued function, and $C_\infty([-\tau, \infty); \mathbb{R}^N)$ is defined in Section 4.

The first contribution of this paper is to investigate the $R_δ$-property of the solution set to a fractional differential inclusion with time delay defined on the infinite interval; it is embodied in the proof of the solution set to problem (2), which is an $R_δ$-set. Furthermore, to the best of our knowledge, there are many studies on the nonlocal problem concerning fractional differential inclusion, but few involve the nonlocal function being set-valued. Stimulated by this consideration, we study the topological structure of the solution set to the nonlocal problem of a fractional differential inclusion, for cases of the nonlocal set-valued function with convex or nonconvex value, and this is the second contribution of this paper. It is reflected in the proof of the $R_δ$-property of the solution set to problem (3).

Throughout the paper, $\| \cdot \|_m = \max_{t \in [-\tau, 0]} \| \cdot \|$ denotes the sup-norm of Banach space $C([-\tau, m]; \mathbb{R}^N)$, where $\| \cdot \|$ stands for the Euclidean norm in $\mathbb{R}^N$, and $\| \cdot \|_0$ denotes the norm of $C([-\tau, 0]; \mathbb{R}^N)$.

In order to put forward our main results, we will first present all the hypotheses that we need for this paper.
Hypothesis 1 (H1). \( B : [0,m] \to \mathbb{R}^{N\times N} \) is an integrable matrix function, which satisfies \( \langle B(t)x, x \rangle \geq 0 \), for a.e. \( t \in [0,m] \).

Hypothesis 2 (H2). \( H : [0,m] \times \mathbb{R}^N \times C([-\tau,0];\mathbb{R}^N) \to \mathbb{R}^N \) is an upper capthéodory function with compact and convex value:

(i) for every \( x_1, x_2 \in \mathbb{R}^N, y_1, y_2 \in C([-\tau,0];\mathbb{R}^N) \), there exists a function \( \mu(t) \in L^\infty(0,m) \), such that

\[
d_H(H(t,x_1,y_1), H(t,x_2,y_2)) \leq \mu(t)(\|x_1 - x_2\| + \|y_1 - y_2\|) \quad \text{for a.e.} \ t \in [0,m],
\]

where \( d_H(\cdot, \cdot) \) denotes the Hausdorff metric, \( \mu(t) \) satisfies \( \int_0^t (t-s)^{\alpha-1} \mu^2(s) ds < \frac{\Gamma(\alpha)}{5M_E} \), and \( M_E \) is a constant defined in Section 3.

(ii) there exists an integrable function \( \rho : [0,m] \to [0, +\infty) \), and a continuous function \( \gamma : \mathbb{R}^+ \to \mathbb{R} \) with \( |\gamma(\xi)| \leq \gamma(\eta) \) for all \( \|\xi\| \leq \eta \) such that

\[
|H(t,x,y)| \leq \rho(t)(1 + \gamma(\|x\|) + \gamma(\|y\|))
\]

for all \( (t,x,y) \in [0,m] \times \mathbb{R}^N \times C([-\tau,0];\mathbb{R}^N) \).

Hypothesis 3 (H3). The fractional equation \( \frac{D_t^\alpha x(t)}{x(t)} = 2\rho(t)\gamma(x(t)) + \rho(t) \) has a unique solution for the Cauchy problem, where \( x(0) = \eta \in \mathbb{R}^N \) for \( t \in [0,m] \).

Hypothesis 4 (H4). \( \psi : \mathcal{C}_\infty([-\tau,\infty);\mathbb{R}^N) \to 2^\mathcal{C}([-\tau,0];\mathbb{R}^N) \) is u.s.c with convex and compact value, and satisfies

(i) \( |\psi(x)| = \{\|v\|_0 : v \in \psi(x)\} \leq \|\psi\|_0 \), for every

\[
x \in \Delta_u := \{x \in \mathcal{C}_\infty([-\tau,\infty);\mathbb{R}^N), \|x\| \leq M_E \cdot u_\psi(t) \ \text{for} \ t \in [0,\infty),
\]

and \( \|x(t)\| \leq u_\psi(t) = \|\psi\|_0 \ \text{for} \ t \in [-\tau,0] \},
\]

(ii) if \( \Theta \subseteq \Delta_u \) is relatively compact in \( \mathcal{C}_\infty([-\tau,\infty);\mathbb{R}^N) \), then \( \psi(\Theta) \) is relatively compact in \( \mathcal{C}([-\tau,0);\mathbb{R}^N) \).

Hypothesis 5 (H5). For all \( x_1, x_2 \in \mathcal{C}_\infty([-\tau,\infty);\mathbb{R}^N) \), and \( v_1 \in \psi(x_1), v_2 \in \psi(x_2) \), then

\[
\|v_1 - v_2\| \leq \|x_1 - x_2\|, \ \text{where} \ 0 < I < \frac{1}{5M_E}.
\]

Hypothesis 6 (H6). \( \psi : \mathcal{C}_\infty([-\tau,\infty);\mathbb{R}^N) \to 2^\mathcal{C}([-\tau,0];\mathbb{R}^N) \) is l.s.c with closed value, and (H4)(i)(ii) holds.

Hypothesis 7 (H7). \( H : [0,\infty) \times \mathbb{R}^N \times C([-\tau,0];\mathbb{R}^N) \to \mathbb{R}^N \), and there exists a constant \( r > 0 \), such that \( \langle x, f \rangle < 0 \), where \( f \in H(t,x,x_t) \) in the case of \( \|x\| \geq r \).

The main results are stated as follows:

We first study the topological structure of the solution set to a fractional differential inclusion with time delay on compact intervals.

Theorem 1. If the hypotheses (H1)–(H3) are satisfied, the solution set to the inclusion problem (1) is an \( R_\phi \)-set.
Then, by means of the inverse limit method, we generalize the $R_\delta$-property of the solution set to the inclusion to noncompact intervals.

**Theorem 2.** If the hypotheses (H1)–(H3) are satisfied, the solution set to the inclusion problem (2) is an $R_\delta$-set.

When the nonlocal function is set-valued and with convex value, we study the existence theorem of a solution to the nonlocal problem.

**Theorem 3.** If the hypotheses (H1)–(H4) are satisfied, the nonlocal problem (3) has at least one solution.

Further, we investigate the topological structure of the solution set to the nonlocal problem.

**Theorem 4.** If the hypotheses (H1)–(H5) are satisfied, the solution set to the nonlocal problem (3) is an $R_\delta$-set.

Changing the convex condition of the nonlocal function to the unconvex condition, we still obtain the $R_\delta$ property of the solution set to the nonlocal problem.

**Theorem 5.** If the hypotheses (H1)–(H3), (H5)–(H7) are satisfied, the solution set to the nonlocal problem (3) is an $R_\delta$-set.

The rest of this paper is organized as follows. In Section 2, we present some definitions and lemmas for the fractional calculus and topological structure of the solution set. In Sections 3–7, we complete the proofs of Theorems 1–5, respectively, and Section 8 presents the conclusions.

2. Preliminaries

This section provides some properties of fractional calculus and some notions of the topological structure of the solution set that will be needed in our analysis. For more results on fractional calculus, we refer readers to [22–26], and for the topological structure of the solution set, we refer the interested readers to [4,6,27].

**Definition 1.** The Riemann–Liouville fractional integral of order $p > 0$ for a function $f$ is defined as

$$I_0^t f(t) = \frac{1}{\Gamma(p)} \int_0^t (t - \tau)^{p-1} f(\tau)d\tau, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

**Definition 2.** The Caputo fractional derivative of order $p > 0$ for a function $f$ can be written as

$$C^\alpha_0 D_0^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t - \tau)^{n-p-1} f^{(n)}(\tau)d\tau, \quad t > 0,$$

where $\forall n \in N, \ p \in (n-1, n)$.

There is an important property of the Caputo fractional derivative, which will be used in our following proof:

If $f(t) \in C^1[0,T]$ and $p \in (0,1)$, then

$$I_0^p C^\alpha_0 D_0^p f(t) = f(t) - f(0).$$

The following lemmas are crucial in our research:
Lemma 1. [28] Assume that \( f(t) = (f_1(t), f_2(t) \cdots f_N(t))^T \in \mathbb{R}^N \) is a vector, where \( f_i(t) \) are continuous differentiable functions for all \( i = 1, 2, \cdots, N \), and \( Q \in \mathbb{R}^{N \times N} \) is a positive definite matrix. Then, for any \( p \in (0, 1) \),
\[
\frac{1}{2} C \|f^T(t)Qf(t)\| \leq f^T(t)Q^p D^p f(t).
\]

Lemma 2. [29] (Gronwall inequality)

Suppose that \( p > 0, a(t) \) is a nondecreasing function on \( 0 \leq t < T \) and \( g(t) \) is a nonnegative, nondecreasing continuous function defined on \( 0 \leq t < T \), and suppose that \( u(t) \) is nonnegative and integrable on \( 0 \leq t < T \) with
\[
u(t) \leq a(t) + g(t) \int_0^t (t - s)^{p-1} u(s) ds,
\] on this interval. Then,
\[
u(t) \leq a(t) E_p[\Gamma(p)g(t)t^p], \quad 0 \leq t < T,
\]where \( E_p(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(pk+1)} \), \( \forall z \in \mathbb{R}, p > 0 \), is the single-parameter Mittag–Leffler function.

Let \( X \) be a Hausdorff topological space. For \( A, B \subseteq X \), the Hausdorff metric is obtained by
\[
d_H(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \}.
\]

According to the metric, let \( F : X \to 2^Y \) be a multivalued map with bounded value; if there exists a constant \( L > 0 \) such that
\[
d_H(F(x), F(y)) \leq Ld(x, y),
\]then \( F \) is called Lipschitzian, and if the constant \( L \in (0, 1) \), \( F \) is called a contraction.

Let \( U \) be a nonempty subset of \( X \), and suppose that there exists a retraction \( r : X \to U \) such that \( r|_U \) is the identity map; then, \( U \) is called a retract of \( X \). Clearly, a retract \( U \subset X \) is closed.

Definition 3. [30] Let \( X \) be a metric space and \( U \) be a closed subset of \( X \). For every metric space \( Y \) and a closed set \( Z \subset Y \),
(i) \( U \) is called an absolute retract (AR space), if each continuous map \( \omega : Z \to U \) can be extended to a continuous function \( \tilde{\omega} : Y \to U \).
(ii) \( U \) is called an absolute neighborhood retract (ANR space), if there exists a neighborhood \( Z \subset H \), such that the continuous map \( \omega : Z \to U \) can extend to be a continuous map \( \tilde{\omega} : H \to U \).

From the definition, it is easy to see that the AR space contains the ANR space. Furthermore, in a Fréchet space, a retract of the convex set must be an AR space (see [4]). In particular, each Banach space is an AR space. Space \( C(I, \mathbb{R}^N) \) is an AR space, where \( I \subset \mathbb{R} \) is an arbitrary interval.

Definition 4. Let \( X \) be a metric space and \( U \) be a subset of \( X \). \( U \) is said to be contractible, if there exist a continuous function \( \eta : [0, 1] \times U \to X \) and a point \( u \in U \), such that \( \eta(0, x) = u \) and \( \eta(1, x) = x \) for all \( x \in U \).

Definition 5. A subset \( U \) of a metric space \( X \) is called an \( R_\delta \)-set, if there exists a decreasing sequence \( \{U_n\}_{n \geq 1} \) of absolute retracts satisfying
\[
U = \bigcap_{n=1}^{\infty} U_n.
\]
Specifically, if every \( \{U_n\} \) is compact, \( U \) is called a compact \( R_\delta \) set. If every \( \{U_n\} \) is symmetric, \( U \) is called a symmetric \( R_\delta \)-set.

We can find that if a set \( U \) is a compact \( R_\delta \)-set, then it must be nonempty, compact and connected. Thus, the following hierarchy for nonempty subsets of a metric space is true:

\[
\text{compact} \subset \text{convex} \subset \text{compact AR space} \subset \text{compact + contractible} \subset R_\delta \text{ set}
\]

and all the above inclusions are proper.

Now, we present some useful facts connected with the semicontinuity of the set-valued map.

**Definition 6.** A set-valued map \( H(\cdot) : X \to Y \) is called upper semicontinuous (u.s.c.) provided that, for every open subset \( U \subseteq Y \), the set \( H^{-1}(U) = \{ x \in X : H(x) \subseteq U \} \) is open in \( X \).

**Definition 7.** A set-valued map \( H(\cdot) : X \to Y \) is called lower semicontinuous (l.s.c.) if, for every open subset \( U \subseteq Y \), the set \( H^{-1}_+(U) = \{ x \in X : H(x) \cap U \neq \emptyset \} \) is open in \( X \).

**Proposition 1.** [27] A set-valued map \( H(\cdot) : X \to 2^Y \) is u.s.c. if and only if, for every closed set \( A \subseteq Y \), the set \( H^{-1}(A) \) is a closed subset of \( X \).

**Proposition 2.** [27] Let \( X, Y \) be metric spaces and \( Z \) be a compact set, if a set-valued map \( H(\cdot) : X \to 2^Y \) is u.s.c. map with compact values; \( Z \) be two u.s.c. maps with compact values; \( \psi \) is an R\( _\delta \)-map, if \( H(\cdot) \) is u.s.c. and \( H(x) \) is an \( R_\delta \)-set for every \( x \in X \).

Clearly, if a set-valued map with contractible value is u.s.c., it can be seen as an \( R_\delta \)-map. Let \( \varphi : X \to 2^X \) and \( \psi : Y \to 2^Z \) be two set-valued maps; the composition \( \psi \circ \varphi : X \to 2^Z \) is defined in the following form:

\[
(\psi \circ \varphi)(x) = \bigcup \{ \psi(y) : y \in \varphi(x) \}
\]

for each \( x \in X \). We obtain the proposition as follows:

**Proposition 3.** [27] Let \( \varphi : X \to 2^X \) and \( \psi : Y \to 2^Z \) be two u.s.c. maps with compact values; then, the composition \( \psi \circ \varphi : X \to 2^Z \) is an u.s.c. map with compact values.

\( \text{Fix}(\alpha) \) denotes the fixed point set of \( \alpha \), and when \( \alpha \) is set-valued and contractional, \( \text{Fix}(\alpha) \) is more complex. Thus, the topological property of \( \text{Fix}(\alpha) \) is a question worth researching.

**Lemma 3.** [31] Let \( X \) be a Banach space, and \( Y \) be a closed, convex subset of \( X \); if set-valued map \( \alpha : Y \to 2^Y \) is a contraction with compact convex value, then \( \text{Fix}(\alpha) \) is a nonempty, compact AR space.

**Lemma 4.** [27] Let \( \alpha : X \to 2^Y \) be an u.s.c. map with compact value and \( A \) be a compact subset of \( X \). Then, \( \alpha(A) \) is compact.

What follows is a fixed point theorem due to Górniewicz and Lassonde [32] (Corollary 4.3), which plays an important role in our proof.

**Lemma 5.** Let \( Y \) be an ANR space. Suppose that \( \alpha : Y \to 2^Y \) can be factorized as \( \alpha = \alpha_N \circ \alpha_{N-1} \circ \cdots \circ \alpha_1 \), where \( \alpha_i : Y^i \to 2^{Y^i} \) (\( i = 1, 2, \cdots, N \)) are \( R_\delta \)-maps, \( Y_i (i = 1, 2, \cdots, N) \) are
ANR spaces, and \( Y^0 = Y^N = Y \) are AR spaces. If there exists a compact subset \( Z \subset Y \) satisfying \( a(Y) \subset Z \), then \( a \) admits a fixed point.

**Lemma 6.** (Bressan–Colombo continuous selection theorem) Let \( X \) be a measurable and separable Banach space, and \( (\Omega, \Sigma, \mu) \) be a finite measure space. Suppose that \( H : X \rightarrow L^p(\Omega, X) \) is a set-valued map with closed decomposable values and l.s.c. Then, \( H \) has a continuous selection.

**3. Proof of Theorem 1**

In this section, we will prove Theorem 1. In order to research the topological structure of the solution set to the inclusion problem (1), we will first consider the following fractional differential equation:

\[
\begin{cases}
C_0^\alpha D_t^\alpha x(t) + B(t)x(t) = f(t), & \text{for a.e. } t \in [0, m], \\
x(t) = \varphi(t), & \text{for } t \in [-\tau, 0],
\end{cases}
\]

(7)

where \( \forall f(t) \in L^p([0, m]; R^N) (p > \frac{1}{2}) \), and it is easy to check that the problem (7) has a unique solution \( x(t) \in C([-\tau, m]; R^N) \).

Thus, we define a solution map by

\[
\text{Integrating in time and noting that } \alpha(t) \in Z, \text{ then, }
\]

\[
\text{Applying Lemma 2, we obtain }
\]

Thus, we define a solution map by

\[
\text{Taking the inner product of } x_f - x_g \text{ with both sides of the above equation, we obtain }
\]

\[
\langle C_0^\alpha D_t^\alpha (x_f - x_g), x_f - x_g \rangle + \langle B(t)(x_f - x_g), x_f - x_g \rangle = \langle f(t) - g(t), x_f - x_g \rangle,
\]

and for hypothesis (H1) and Lemma 1, we yield

\[
\frac{1}{2} \int_0^s C_t^\alpha \| x_f - x_g \|^2 \leq \| f(t) - g(t) \| \| x_f - x_g \| \leq \frac{1}{2} (\| f(t) - g(t) \|^2 + \| x_f - x_g \|^2).
\]

Integrating in time and noting that

\[
L_t^\alpha C_t^\alpha \| x_f - x_g \|^2 \leq L_t^\alpha \| f(t) - g(t) \|^2 + L_t^\alpha \| x_f - x_g \|^2,
\]

then,

\[
\| x_f - x_g \|^2 \leq \| x_f(0) - x_g(0) \|^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f(s) - g(s) \|^2 ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| x_f - x_g \|^2 ds.
\]

Applying Lemma 2, we obtain

\[
\| x_f - x_g \|^2 \leq \left[ \| x_f(0) - x_g(0) \|^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f(s) - g(s) \|^2 ds \right] E_a(t^{\alpha})
\]

\[
\leq M_E \left[ \| x_f(0) - x_g(0) \|^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f(s) - g(s) \|^2 ds \right],
\]

(8)

where \( M_E = \sup_{t \in [-\tau, m]} E_a(t) \).

**Proof.** Let us define the Nemitsky operator \( N : C([-\tau, m]; R^N) \rightarrow L^p([0, m]; R^N) \) by

\[
N(x) = \left\{ f \in L^p([0, m]; R^N) : f(t) \in H(t, x, x_t), \text{ for a.e. } t \in [0, m] \right\}.
\]
Thanks to Theorem 3.2 in [33], it is easy to check that \( N(\cdot) \) is nonempty, closed, decomposable and l.s.c.

For given \( \varphi(t) \in C([-\tau,0]; R^N) \), we define the set

\[
Q^m_{\varphi} = \{ x \in C([-\tau,m]; R^N) : \| x(t) \| \leq M_E \cdot u_{\varphi}(t) \text{ for } t \in [0,m], \\
\text{ and } x(t) = \varphi(t) \text{ for } t \in [-\tau,0] \},
\]

where \( u_{\varphi} \) is the unique solution of (5). What follows is to seek for the solutions in \( Q^m_{\varphi} \). For this purpose, a set-valued map \( F^m_{\varphi} \) is defined on \( Q^m_{\varphi} \):

\[
F^m_{\varphi}(x) := P_m \circ N(x), x \in Q^m_{\varphi}.
\]

Next, we consider the fixed point problem \( x \in F^m_{\varphi}(x) \).

Let \( x \in Q^m_{\varphi} \); we assume a step function \( (x_n, y_n) : [0, m] \to R^N \times C([-\tau,0]; R^N) \), for every \( t \in [0,m], (x_n, y_n) \to (x, x_1) \) and \( \| x_n \| \leq \| x \|, \| y_n \| \leq \| x_1 \| \leq \| x \| \). Because of (H2), for each \( n \), \( H(\cdot, x_n(\cdot), y_n(\cdot)) \) has a measurable selection \( f_n(\cdot) \), and \( \{ f_n \} \) is integrably bounded in \( L^p([0,m]; R^N) \). Consequently, applying the Dunford–Pettis theorem, by passing to a subsequence if necessary, we may assume that \( f_n \to f \) weakly in \( L^p([0,m]; R^N) \). Through the same arguments as in the proof of Theorem 3.1.2 in [34], we find that \( f \in N(x) \), which means \( N(x) \neq \emptyset \). It can be seen from this that, for every \( x \in C([-\tau, m]; R^N) \), \( F^m_{\varphi}(x) \) is nonempty. Considering (H2) and (8), taking \( f \in N(x) \) with \( x \in Q^m_{\varphi}(x) \), for every \( t \in [0,m] \), we obtain

\[
\| P_m(f) \|^2 \leq M_E \left[ \| \varphi(0) \|^2 + \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} \| f(s) \|^2 ds \right]
\leq M_E \left[ \| \varphi(0) \|^2 + \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} \rho(s)(1+\gamma(\| x \|)) \gamma(\| x_1 \|) ds \right]^2
\leq \left\{ M_E \left[ \| \varphi(0) \|^2 + \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} \rho(s)(1+2\gamma(\| u_{\varphi}(s) \|)) ds \right] \right\}^2
= \left[ M_E \cdot u_{\varphi}(t) \right]^2.
\]

That is, \( \| P_m(f) \| \leq M_E \cdot u_{\varphi}(t) \) for \( t \in [0,m] \). Let \( u_{[-\tau,0]} \) denote the restriction of \( u \) on \([-\tau,0] \), noting that \( \{ u_{[-\tau,0]} : u \in F^m_{\varphi}(x) \} = \varphi \). Therefore, for \( t \in [-\tau, m] \), \( P_m(f) \in Q^m_{\varphi} \).

Hence, for every \( x \in Q^m_{\varphi} \), we have \( F^m_{\varphi}(x) \subset Q^m_{\varphi} \); that is, \( F^m_{\varphi}(x) \) is a nonempty subset of \( Q^m_{\varphi} \).

Then, there exists \( x \in Q^m_{\varphi} \) such that \( x \in F^m_{\varphi}(x) \). Namely, \( F^m_{\varphi}(x) \) has at least a fixed point.

It is well known that the solution set to inclusion problem (1) is equal to the set of fixed points of operator \( F^m_{\varphi}(\cdot) \).

Next, we will prove that \( \text{Fix} F^m_{\varphi} \) is a compact AR space in three steps:

First step. For every \( x \in Q^m_{\varphi} \), the map \( F^m_{\varphi}(\cdot) \) has convex value.

If \( v_1, v_2 \in F^m_{\varphi}(\cdot) \), there exist integrable selections \( f_1(t), f_2(t) \in H(\cdot, x(\cdot), x_1(\cdot)) \) such that \( v_1 = P_m(f_1), v_2 = P_m(f_2), \) i.e.,

\[
\int_0^1 D^\beta_0 v_1(t) + B(t)v_1(t) = f_1(t), \quad \int_0^1 D^\beta_0 v_2(t) + B(t)v_2(t) = f_2(t),
\]

for any \( \lambda \in [0,1] \), we gain

\[
\lambda v_1(t) + (1-\lambda)v_2(t) = P_m[\lambda f_1(t) + (1-\lambda)f_2(t)], \quad \lambda f_1(t) + (1-\lambda)f_2(t).
\]

As the set-valued map \( N(x) \) has convex value,

\[
\lambda v_1(t) + (1-\lambda)v_2(t) = P_m[\lambda f_1(t) + (1-\lambda)f_2(t)] \in P_m \circ N(x),
\]

Thus,

\[
\lambda v_1(t) + (1-\lambda)v_2(t) = P_m[\lambda f_1(t) + (1-\lambda)f_2(t)] \in P_m \circ N(x),
\]

for a.e. \( t \in [0,m] \).
that is,
\[ \lambda v_1(t) + (1 - \lambda)v_2(t) \in F^m_{\varphi}(x), \]
which means that the map \( F^m_{\varphi}(\cdot) \) has convex value.

Second step. For every \( x \in Q^m_{\varphi} \), the map \( F^m_{\varphi}(\cdot) \) has compact value.

We can obtain \( F^m_{\varphi}(x) \subset Q^m_{\varphi} \subset C([-\tau, m]; R^N) \) from the previous analysis, which implies that \( F^m_{\varphi}(x) \) is bounded and equicontinuous. From the Arzela–Ascoli theorem, \( F^m_{\varphi}(x) \) is a relatively compact set. By \( N(\cdot) : C([-\tau, m]; R^N) \to L^p([0, m]; R^N) \), \( P_m : L^p([0, m]; R^N) \to C([-\tau, m]; R^N) \), \( F^m_{\varphi}(x) \) can be represented as the closed graph composition of operators \( P_m \circ N(x) \). By the meaning of \([34]\), we derive the closedness of \( F^m_{\varphi}(x) \); thus, the map \( F^m_{\varphi}(\cdot) \) has compact value.

Third step. The map \( F^m_{\varphi}(\cdot) \) is a contraction.

By taking into account (8) and (H2), for any \( x, y \in C([-\tau, m]; R^N) \), there exist \( v_x \in F^m_{\varphi}(x) \), \( v_y \in F^m_{\varphi}(y) \) and integrable selections \( f_x(\cdot) \in H(\cdot, x, x_1), f_y(\cdot) \in H(\cdot, y, y_1) \) such that
\[
\begin{align*}
\|v_x - v_y\| & = \|F^m_{\varphi}(x) - F^m_{\varphi}(y)\|_2 \\
& \leq M_E \left\| v_x (0) - v_y (0) \right\|^2 + \frac{1}{T(a)} \int_0^T (t-s)^{a-1} \|f_x(s) - f_y(s)\|^2 ds \\
& \leq \frac{4M_E}{\Gamma(a)} \int_0^T (t-s)^{a-1} \mu^2(s) ds \|x - y\|^2 \\
& \leq L \|x - y\|^2
\end{align*}
\]
where \( L := \frac{4M_E}{\Gamma(a)} \int_0^T (t-s)^{a-1} \mu^2(s) ds \in (0, 1) \). Thus, the map \( F^m_{\varphi}(\cdot) \) is a contraction.

Since the map \( F^m_{\varphi}(\cdot) \) is a contraction with convex and compact value, using Lemma 3, \( \text{Fix} F^m_{\varphi} \) is a nonempty, compact AR space; that is, the solution set to inclusion problem (1) is a compact \( R_\delta \)-set. \( \square \)

**Remark 1.** Here, we study the Caputo fractional derivative; there is also related research for Riemann–Liouville fractional derivatives. However, for Hilfer or Antangana Baleanu derivatives, so far, there are not any relevant studies. This would be an interesting question worth studying.

**4. Proof of Theorem 2**

In this section, we will study the \( R_\delta \)-property of the solution set to the fractional differential inclusion defined on the half-line. In order to study the problem on an infinite interval, we recall some related knowledge of the inverse system.

The system \( S = \{X_a, \pi^b_a, \Sigma\} \) can be known as an inverse system, where \( \Sigma \) is a set denoted for the relation \( \leq \), for all \( a \in \Sigma \); \( X_a \) is a metric space, and for all \( a, b \in \Sigma \) with \( a \leq b \), \( \pi^b_a : X_b \to X_a \) is a continuous function. \( \lim S \) is defined as the limit of inverse systems \( S \), in the form of
\[
\liminf S = \left\{ (x_d) \in \prod_{a \in \Sigma} X_a : \pi^b_a(x_b) = x_a, \text{ for all } a \leq b \right\}.
\]

For more details, we refer readers to [4,5]. The following lemmas are useful for our research.

**Lemma 7.** ([4] Theorem 3.9) Let \( S = \{X_a, \pi^b_a, N\} \) be an inverse system, and \( \varphi : \liminf S \to \liminf S \) be a limit map derived by a family \{id, qa\}, where \( q_a : X_a \to X_d \), and if all \( a \in N \), \( \text{Fix}(q_a) \) are \( R_\delta \)-sets, then \( \text{Fix}(\varphi) \) is an \( R_\delta \)-set too.

**Lemma 8.** ([35]) Let \( S = \{X_a, \pi^b_a, N\} \) be an inverse system. For each \( a \in N \), if \( X_a \) is nonempty and compact (relatively compact), then \( \liminf S \) is also nonempty and compact (relatively compact).
Before the proof, we need to present some notations.

For any \( m, n \in N^+ \) and \( m \leq n \), we define a projection \( \pi^n_m : C([0,m];R^N) \to C([0,n];R^N) \), in the form of

\[
\pi^n_m(u) = u|_{[0,m]}, u \in C([0,n];R^N).
\]

Let \( S_C = \{ C([0,m];R^N), \pi^n_m, N^+ \} \), and it is easy to see that \( S_C \) is an inverse system, and its limit is isometrically homeomorphic to \( C_\infty([0,\infty);R^N) \); then, for convenience, we express

\[
\lim_{\tau} S_C = \lim_{\tau} \{ C([0,m];R^N), \pi^n_m, N^+ \} := C_\infty([0,\infty);R^N).
\]

Then, \( S_Q = \{ Q^m_n, \pi^n_m, N^+ \} \) is an inverse system and its limit can be represented as

\[
\lim_{\tau} S_Q = \lim_{\tau} \{ Q^m_n, \pi^n_m, N^+ \} := Q^\infty_n.
\]

Moreover, \( S_L = \{ L^p([0,m];R^N), \tilde{\pi}^n_m, N^+ \} \) is an inverse system with

\[
\tilde{\pi}^n_m(f) = f|_{[0,m]}, f \in L^p([0,n];R^N),
\]

and its limit is written as

\[
\lim_{\tau} S_L = \lim_{\tau} \{ L^p([0,m];R^N), \tilde{\pi}^n_m, N^+ \} := L_\infty([0,\infty);R^N),
\]

where \( L_\infty([0,\infty);R^N) \) is the separated locally convex space, which can be composed of all locally Bocher integrable components from \( R^+ \) to \( R^N \) endowed with a family of seminorms \( \{ \| \cdot \|_m, m \in N^+ \} \), defined by \( \| u \|_m = \int_0^m \| u(s) \| ds, m \in N^+ \).

It is easy to see that \( P_m(f)|_{[0,m]} = P_m(f) \) for \( f \in L^p([0,n];R^N) \) with \( m \leq n \); thus, the family \( \{ id, P_m \} \) is a map from \( S_L \) to \( S_C \). As a result, for all \( f \in L_\infty([0,\infty);R^N) \) with \( m \in N^+ \), the family \( \{ id, P_m \} \) induces a limit map \( P_\infty : L_\infty \to C_\infty \), such that \( P_\infty(f)|_{[0,m]} = P_m(f) \). In what follows, we demonstrate the topological structure of the solution set on the infinite intervals.

**Proof.** From the proof of Theorem 1, we see that the fixed point set of set-valued map \( F^m_q : Q^m_q \to 2^{\Omega_q^m} \) is the solution set to problem (1), and it is an \( R^q \)-set.

What follows is to indicate that the family \( \{ id, F^m_q \} \) is a map from \( S_Q \) to \( S_Q \). Since \( F^m_q(x) = P_m \circ N(x) \), we only need to show that, for \( x \in Q^m_q \),

\[
N\big(x|_{[-\tau,0]}\big)|_{[0,m]} = \left\{ f|_{[0,m]} : f \in N(x)|_{[0,m]} \right\},
\]

where \( Q^m_q \) is defined in (9) with \( n \) instead of \( m \). In the case of \( m = n \), it is obviously true. In the case of \( m < n \), it is clear that

\[
\left\{ f|_{[0,m]} : f \in N(x)|_{[0,m]} \right\} \subset N\big(x|_{[-\tau,m]}\big)|_{[0,m]}.
\]
Thus, we only need to explain the reverse inclusion. For \( f \in N\left( x|_{[-\tau,m]} \right)|_{[0,m]} \) and \( g \in N\left( x|_{[0,n]} \right) \) we set \( \tilde{f} = f(t)x_{[0,m]}(t) + g(t)x_{[m,n]}(t) \) for a.e. \( t \in [0,n] \), where \( \chi(t) \) is the characteristic function. It is obvious that \( \tilde{f} \in N\left( x|_{[0,n]} \right) \), which means that

\[
N\left( x|_{[-\tau,m]} \right)|_{[0,m]} \subset \left\{ f|_{[0,m]} : f \in N\left( x|_{[0,n]} \right) \right\}.
\]  

Combining (12) and (13), we have

\[
N\left( x|_{[-\tau,m]} \right)|_{[0,m]} = \left\{ f|_{[0,m]} : f \in N\left( x|_{[0,n]} \right) \right\}.
\]  

Therefore, the map \( \left\{ F^m_{\varphi} \right\}_{m=1}^\infty \) induces the limit map \( F^\infty_\varphi : Q^\infty_\varphi \to Q^\infty_\varphi \), such that 

\[
F^\infty_\varphi(x)|_{[0,m]} = F^\infty_\varphi(x)|_{[0,m]}.
\]  

Consequently, the fixed point set of the map \( F^\infty_\varphi \) is the solution set to inclusion problem (2). As \( \text{Fix} F^\infty_\varphi \) is an \( \mathcal{R}_\varphi \)-set, invoking Lemma 7, for each \( m \in N^+ \), \( \text{Fix} F^\infty_\varphi \) is an \( \mathcal{R}_\varphi \)-set too. The proof is completed. \( \square \)

5. Proof of Theorem 3

In this section, we will research the existence theorem of a solution to the nonlocal problem of a fractional differential inclusion, where the nonlocal function is set-valued and with convex value.

**Proof.** For given \( \varphi(t) \in C\left([-\tau,0];\mathbb{R}^N\right) \), let \( \theta = \|\varphi\|_0 \), and it is noted that

\[
\Psi_\theta = \left\{ v \in C\left([-\tau,0];\mathbb{R}^N\right) : \|v\|_0 \leq \theta \right\}.
\]

For each \( \tilde{\varphi} \in \Psi_\theta \), the set-valued map \( \Gamma : \Psi_\theta \to 2^\mathbb{B}_u \) is defined by \( \Gamma(\tilde{\varphi}) = \text{Fix} F^\infty_\tilde{\varphi} \). Then, the solution set to problem (2) can be expressed as \( \text{Fix} F^\infty_\varphi = \Gamma(\varphi) \).

Next, the proof of the existence theorem will be divided into three steps.

First step. \( \Gamma \) is an \( \mathcal{R}_\varphi \)-map.

Firstly, we claim that \( \Gamma \) is u.s.c. Let \( Q \) be a nonempty closed subset of \( C_\infty\left([-\tau,\infty);\mathbb{R}^N\right) \); in order to use Proposition 1, we need to illustrate the closedness of

\[
\Gamma^{-1}(Q) = \{ \varphi \in C\left([-\tau,0];\mathbb{R}^N\right) : \Gamma(\varphi) \cap Q \neq \emptyset \}.
\]

Let \( \{\varphi_n\}_{n \geq 1} \subseteq \Gamma^{-1}(Q) \) and suppose that \( \varphi_n \to \varphi \) in \( \Psi_\theta \). For \( n \geq 1 \), let \( x_n \in \Gamma(\varphi) \cap Q \); exploiting the prior estimation of the solution in Theorem 1, we obtain that \( \{x_n\}_{n \geq 1} \) is uniformly bounded in \( C_\infty\left([-\tau,\infty);\mathbb{R}^N\right) \). Thanks to the Arzela–Ascoli theorem, a convergent subsequence exists, and without loss of generality, we suppose that \( x_n \to x \) in \( Q \). In view of Theorem 3.1 in [30], we know that

\[
\overset{\in}{D}_t^\alpha x(t) + B(t)x(t) \in \limsup\left\{ \overset{\in}{D}_t^\alpha x_n(t) + B(t)x_n(t) \right\}_{n \geq 1}
\]

\[
\in \liminf\left\{ N(x_n) \right\}_{n \geq 1}
\]

\[
N(x) \text{ for a.e. } t \in [0,\infty).
\]

This implies that \( x \in \Gamma(\varphi) \), so \( x \in \Gamma(\varphi) \cap Q \), and then \( \Gamma^{-1}(Q) \) is closed in \( C_\infty\left([-\tau,\infty);\mathbb{R}^N\right) \). With Proposition 1 in mind, we see that \( \Gamma \) is u.s.c. For Theorem 2, \( \Gamma(\varphi) \) is an \( \mathcal{R}_\varphi \)-set, and by Definition 8, \( \Gamma : \Psi_\theta \to 2^\mathbb{B}_u \) is an \( \mathcal{R}_\varphi \)-map.

Second step. \( \Gamma \circ \psi \) is an \( \mathcal{R}_\varphi \)-map. According to (H4), it follows that \( \psi : \Delta_u \to 2^\mathbb{B}_u \) is u.s.c with compact and convex value; by (6), we gain that, for each \( x \in \Delta_u \), \( \psi(x) \) is an \( \mathcal{R}_\varphi \)-set. Again, owing to Definition 8, we obtain that \( \psi \) is an \( \mathcal{R}_\varphi \)-map. Noting \( \psi(\Delta_u) \subseteq \Psi_\theta, \Gamma(\Psi_\theta) \subseteq \Delta_u \), so
\( \Gamma(\psi(\Delta_u)) \subseteq \Delta_u \). As a result, the composite map is well defined by \( \Gamma \circ \psi : \Delta_u \to \Delta_u \); from the fact of Proposition 3, \( \Gamma \circ \psi \) is an \( R_\beta \)-map from \( \Delta_u \) to \( \Delta_u \).

Third step. \( \Gamma \circ \psi : \Delta_u \to \Delta_u \) has a fixed point.

It can be seen from the above analysis that \( \Delta_u \) and \( \Psi_\beta \) are, respectively, a convex subset of \( C_\infty([−\tau, \infty); R^N) \) and \( C([−\tau, 0); R^N) \); then, they are AR spaces. Next, we wish to prove that \( \Delta_u \) is relatively compact in \( C_\infty([−\tau, \infty); R^N) \). Notice that

\[
\Gamma(\Psi_\beta)[−\tau,m] \subset [x \in C([−\tau,m]; R^N); x(t) = P_m(f), \|x\| \leq \bar{u},
\text{for a.e. } t \in [−\tau,m], f \in N(Q_0^\infty)|_{[0,m]} \}
\]

(15)

where \( \bar{u} = \left\{ \begin{array}{ll}
\theta M_E \cdot u_0(t) & t \in [−\tau,0], \\
M_E \cdot u_0(t) & t \in [0,m],
\end{array} \right. \) then \( \Gamma(\Psi_\beta)[−\tau,m] \) is relatively compact in \( C([−\tau,m]; R^N) \). From \( \Gamma(\Psi_\beta)[−\tau,m] \) is the subset of \( \Delta_u[−\tau,m] \), and we can infer that \( \Delta_u[−\tau,m] \) is relatively compact in \( C([−\tau,m]; R^N) \). According to Lemma 8 and the arbitrariness of \( m, \Delta_u \) is relatively compact in \( C_\infty([−\tau, \infty); R^N) \). For each \( m > 0 \), due to (H4) (ii), \( \psi(\Delta_u) \) is relatively compact in \( C([−\tau,0); R^N) \). Let \( \Psi = \overline{\text{conv}}(\psi(\Delta_u)) \), and we can see that \( \Psi \) is compact in \( C([−\tau,0); R^N) \), and \( \Gamma(\Psi) \subset \Delta_u \). By Lemma 4, because \( \Gamma \) is u.s.c. with compact and convex value, it is easy to see that \( \Gamma(\Psi) \) is compact. Consequently, \( \Gamma \circ \psi(\Delta_u) \subset \Gamma(\Psi) \).

Taking account of Lemma 5, we derive that \( \Gamma \circ \psi \) has a fixed point in \( \Delta_u \). Moreover, it is easy to prove that \( x(t) \in \Gamma \circ \psi(x) \), and \( \max_{t \in [−\tau,\infty)} \{\|x(t)\|, \|x(t)\| \} \subset \Delta_u \), which means that \( x(t) \) is a continuous solution to the problem (3). The proof is complete.

6. Proof of Theorem 4

The last section deals with the existence of a solution to problem (3), and this section will demonstrate the topological structure of the solution set to problem (3).

**Proof.** Thanks to Theorem 3, we obtain that \( \text{Fix}(\Gamma \circ \psi) \) is the nonempty solution set to problem (3), and the composite map \( \Gamma \circ \psi \) is an \( R_\beta \) map from \( \Delta_u \) to \( \Delta_u \). What follows is to prove that

\[
\text{Fix}(\Gamma \circ \psi)[−\tau,m] = \left\{ x \in C\left([−\tau,m]; R^N\right); x \in \Gamma \circ \psi(x)[−\tau,m] \right\}
\]

is an \( R_\beta \)-set. For \( \Gamma \circ \psi \)[−\tau,m] is an \( R_\beta \)-map; then, we only need to show that \( \Gamma \circ \psi \)[−\tau,m] is a contraction.

For each \( u, v \in C([−\tau,m]; R^N) \), there exist \( x_u \in \Gamma \circ \psi(u)[−\tau,m], x_v \in \Gamma \circ \psi(v)[−\tau,m] \); that is, for \( t \in [−\tau,0] \), there exist \( x_u = \psi_u(t) \in \psi(u), x_v = \psi_v(t) \in \psi(v) \), and integrable selections \( f_u(\cdot) \in H(t, u, u_t), f_v(\cdot) \in H(t, v, v_t) \) such that

\[
d^2_H(\Gamma \circ \psi(u)[−\tau,m], \Gamma \circ \psi(v)[−\tau,m])
\]
\[
\leq \|x_u - x_v\|^2
\]
\[
\leq M_E \left[ \|\psi_u(t) - \psi_v(t)\|^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1}\|f_u(s) - f_v(s)\|^2 ds \right]
\]
\[
\leq M_E \left[ \frac{4}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1}4\mu^2(s)ds\|u - v\|^2 \right]
\]
\[
\leq \hat{L}\|u - v\|^2,
\]

where \( \hat{L} := M_E \left[ \frac{4}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1}\mu^2(s)ds \right] \); by the meaning of (H2) and (H5), it is easy to obtain \( 0 < \hat{L} < 1 \). Thus, \( \Gamma \circ \psi \)[−\tau,m] is a contraction with a Lipschitz constant \( \hat{L} \in (0,1) \). Since \( \Gamma \circ \psi \)[−\tau,m] is a contraction with compact and convex value, by Lemma 3,
7. Proof of Theorem 5

In this section, we change the convex condition of the nonlocal function $\psi$ to the unconvex condition, but we obtain the similar result that the solution set of problem (3) is still an $R_{0}$-set.

Proof. Firstly, we prove that the problem (3) has at least one solution; the proof is similar to Theorem 3. The set-valued map $\psi : C_{0}([-\tau, \infty); R^{N}) \to 2^{C([-\tau,0]; R^{N})}$ is nonempty, closed, decomposability valued and l.s.c. in $C([-\tau,0]; R^{N})$; Bressan–Colombo continuous selection theorem (Lemma 6) provides a continuous map $g : C_{0}([-\tau, \infty); R^{N}) \to C([-\tau,0]; R^{N})$ satisfying $g(x) \in \psi(x)$. For the sake of finishing our proof, we need to solve the fixed point problem: $x \in \Gamma \circ g(x)$.

It is clear that $g$ satisfies (H4) (i). For every $\Theta \subseteq \Delta_{r}$, where $\Delta$ is defined in (4) with $r$ instead of $u(t)$, and $r$ is the constant in (H7). If $\Theta$ is relatively compact in $C_{0}([-\tau, \infty); R^{N})$, as the continuous form of $g$, $g(\Theta)$ is relatively compact in $C_{0}([-\tau, \infty); R^{N})$. Thus, $g$ satisfies (H4) (ii). Therefore, we infer that $f : \Delta_{r} \to \Psi_{\delta}$ is an $R_{0}$-map. As $\Gamma$ is an $R_{0}$-map from $\Psi_{\delta}$ to $\Delta_{r}$, by Proposition 3, $\Gamma \circ g : \Delta_{r} \to \Delta_{r}$ is an $R_{0}$-map.

We claim that $\Gamma(\Psi_{\delta}) \subset \Delta_{r}$, to illustrate this, we argue by contradiction. Assume that $\Gamma(\Psi_{\delta}) \subset \Delta_{r}$ is not true, and then we can suppose that there exist $\varphi \in \Psi_{\delta}, x \in \Gamma(\varphi), t_{0} > 0$ such that $\|x(t_{0})\| > r$. Taking $f \in H(t, x, x_{i})$, we have

$$\frac{C}{0} D^{\alpha}_{t} x(t) + B(t) x(t) = f(t).$$

Taking an inner product above with $x(t)$, we can deduce that

$$\langle \frac{C}{0} D^{\alpha}_{t} x(t), x(t) \rangle + \langle B(t) x(t), x(t) \rangle = \langle f(t), x(t) \rangle.$$

By (H1) and Lemma 2, we gain

$$\frac{1}{2} \int_{0}^{C} D^{\alpha}_{t} x(t) \|x(t)\|^{2} \leq \langle f(t), x(t) \rangle,$$

when $t = t_{0}$,

$$0 < \frac{1}{2} \int_{0}^{C} D^{\alpha}_{t} \|x(t_{0})\|^{2} \leq \langle f(t_{0}), x(t_{0}) \rangle,$$

which contradicts (H7). Therefore, $\Gamma(\Psi_{\delta}) \subset \Delta_{r}$. Similarly to Theorem 3, let $\Lambda = \overline{\text{conv}}(g(\Delta_{r}))$, and we see that $\Lambda$ is compact in $C([-\tau,0], R^{N})$. For $\Gamma$ that is u.s.c. with compact value, we see that $\Gamma(\Lambda)$ is compact. Thus, $\Gamma \circ g(\Delta_{r}) \subset \Gamma(\Lambda)$. Applying Lemma 5, we gain that $\Gamma \circ g$ has a fixed point in $\Delta_{r}$, which means that there exists a solution to the problem (3).

Under hypothesis (H5), we research the topological structure of the solution set to problem (3), which is to prove that $\text{Fix} \Gamma \circ g$ is an $R_{0}$-set. The process of proof is similar to Theorem 4; we do not repeat it here. The proof is complete.

We further investigate the property of symmetry of the solution set to problem (3). If the set-valued functions $H$ and $\psi$ are symmetric, the solution set is also symmetric. That is, when $x \in \text{Fix} \Gamma \circ \psi$, we can obtain $-x \in \text{Fix} \Gamma \circ \psi$. Therefore, the solution set to problem (3) is a symmetric $R_{0}$-set. □

8. Conclusions

In this paper, we firstly demonstrate the $R_{0}$-property of the solution set to a fractional differential inclusion with time delay on finite intervals. Next, using the inverse limit method, we extend our results to infinite intervals. Furthermore, when the nonlocal function is set-valued, we also conclude that the solution set to nonlocal problem of the fractional differential inclusion is an $R_{0}$-set.
In the future, we will further research the application of the topological structure of the solution set. Vijayakumar et al. [36] discussed the approximate controllability of fractional semilinear integro-differential equations using resolvent operators; Anurag Shukla [37] considered the optimal control of second-order semilinear systems in a reflexive Hilbert space. Inspired by these papers, we will focus our study on the application of the topological structure of the solution set to different fractional systems in the control field.

Author Contributions: Writing—original draft, writing—review editing, methodology, S.G.; conceptualization, S.G. and B.G.; investigation, R.W.; supervision, B.G. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by Natural Science Foundation of Jilin Province (No. 20200201274JC); National Natural Science Foundation of China (No. 11201095); Postdoctoral research startup foundation Heilongjiang (No. LBH-Q14044); Science Research Funds for Overseas Returned Chinese Scholars of Heilongjiang (No. LC201502).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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