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Interval Estimation for the Two-Parameter Exponential Distribution under Progressive Type II Censoring on the Bayesian Approach

Shu-Fei Wu

Department of Statistics, Tamkang University, Tamsui, Taipei 251301, Taiwan; 100665@mail.tku.edu.tw

Abstract: Under progressive type II censoring, the credible interval estimation and the credible region for parameters of two-parameter exponential distribution based on the Bayesian approach are presented in this paper. Two methods of Bayesian credible region are proposed under a given confidence level. We also presented the predictive interval of the future observation under this type of censoring. In order to compare the performance of our proposed Bayesian credible interval and region with the existing non-Bayesian methods, we conduct a simulation study by the Monte Carlo method to find the corresponding coverage probabilities. This research is related to the topic of asymmetrical probability distributions and applications across disciplines. Finally, one engineering example is used to demonstrate the Bayesian credible interval estimation methods proposed in this paper.

Keywords: progressive type II censored sample; two-parameter exponential distribution; Bayesian estimation; interval estimation; credible intervals; credible regions

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Citation: Wu, S.-F. Interval Estimation for the Two-Parameter Exponential Distribution under Progressive Type II Censoring on the Bayesian Approach. *Symmetry* **2022**, *14*, 808. https://doi.org/10.3390/sym14040808

Academic Editor: Jinyu Li

Received: 18 March 2022
Accepted: 11 April 2022
Published: 13 April 2022

1. Introduction

In many practical cases, the experimenters can only collect progressive type II censored sample instead of complete sample for some factors of experimental designs. The advantage of progressive type II censoring is to allow for the immediate removal of products in the lifetime test. See Balakrishnan and Aggarwala [1] for the details of this censoring scheme and the brief introduction of this censoring scheme is given as follows: Suppose that the experimenters put n units of item on the life test. Once the first failure time X_1 is collected, we remove r_1 units out of the remaining $n-1$ surviving units. When the second failure time X_2 is collected, we remove r_2 units out of the remaining $n-2-r_1$ surviving units. Repeat the same process until the mth failure time X_{m} is collected and the experiment is terminated. The remaining $r_m = n - r_1 - \ldots - r_{m-1} - m$ units will be automatically removed. When the experiment is ended, we have collected a progressive type II censored sample given by $X_1 < X_2 < \cdots < X_m$ under the progressive censoring scheme of $(r_1, \ldots, r_m)$. For this type of censored sample, Cohen [2] and Cohen and Norgaard [3] had conducted research on the statistical inference for the parameters of some lifetime distribution. Wu [4] proposed the interval estimation for scale parameter and the confidence region for the two parameters of two-parameter exponential distribution under doubly type II censoring. The necessity to find the confidence region for two parameters is because some characteristics of exponential distribution is related to these two parameters. Wu [5] developed the Bayesian interval estimation and predictive interval for the right type II censored sample. For Bayesian inferences, Wang et al. [6] presented the Bayesian infinite mixture models for wind speed distribution estimation. Mohammed [7] proposed the empirical E-Bayesian estimation for the parameter of Poisson distribution. Under type II censoring, Heidari et al. [8] investigated the E-Bayesian and hierarchical Bayesian estimation for Rayleigh distribution. Arekar [9] proposed the Bayesian estimation of reliability of system of n s-independent two-state component. Jana and Bera [10] considered the interval estimation for the stress–strength reliability with multicomponent for the inverse Weibull distribution. Under progressive type II censoring, Wu [11] proposed the interval estimation for two parameters of exponential distribution. In this research, we are planning to present the Bayesian credible interval for the scale parameter and the Bayesian credible region for two parameters of exponential distribution under progressive type II censoring in Section 2. Our Bayesian approach provides the methodology incorporating previous information with the current data. In addition to this, we proposed the Bayesian prediction interval for future observation when $r_m \neq 0$. The Bayesian concept of a credible interval is a more practical concept than the confidence interval. A credible interval is an interval in the domain of a posterior probability distribution. To multivariate problems, the credible interval is generalized to the credible region. Lee [12] found that credible intervals are analogous to confidence intervals in frequentist statistics. Jake [13] claimed that these two methods are different in philosophical basis. The credible interval treats the estimated parameter as a random variable and the corresponding bounds as fixed values. The frequentist confidence interval treats the parameter as a fixed value and the confidence bounds as random variables. If the prior information of the parameter is available, the Bayesian approach should be used. If it is informative, the confidence interval should be used. In Section 3, a simulation study is conducted to compare the performance of our proposed Bayesian credible interval and region with the existing methods in Wu [11] in terms of coverage probability. In Section 4, one engineering example is given to illustrate the proposed Bayesian interval estimation methods. At last, the conclusion is provided in Section 5.

2. Credible Interval Estimation of Parameters

Suppose that the lifetime distribution of a random variable X is a two-parameter exponential distribution with location parameter μ and scale parameter θ.

For the progressive type II censored sample $X_1 < X_2 < \cdots < X_m$ from the two-parameter exponential distribution with the variable transformation of $Y_i = X_i - \mu$, $i = 1, \ldots, m$, we obtain the progressive type II censored sample $Y_1 < Y_2 < \cdots < Y_m$ from the exponential distribution with zero location parameter and scale parameter θ. From Balakrishnan and Aggarwala [1], we transform $Y_1, Y_2, \cdots, Y_m$ to $Z_1 = nY_1$, $Z_2 = (n - r_1 - 1)(Y_2 - Y_1), \ldots, Z_m = (r_m + 1)(Y_m - Y_{m-1})$.

They indicated that $Z_1, \ldots, Z_m$ are all independent and identically distributed from the exponential distribution with scale parameter θ. The joint pdf is $f_{Z_1,\ldots,Z_m}(z_1, \cdots, z_m) = \theta^{-m} \exp\left\{-\sum_{i=1}^{m} z_i/\theta\right\}$.

We considered the Bayesian approach that provides the methodology incorporating previous information with the current data to find the credible interval and credible region in this section. Similar to Wu et al. [14], we let the random variable $\lambda = 1/\theta$. Suppose that the prior distribution of λ is gamma distribution denoted as $\Gamma(a, b)$. We can find the posterior pdf of λ as

$$\pi(\lambda|z_1, \cdots, z_m) \propto \lambda^{m+a-1} \exp\left\{-\lambda\left(\frac{1}{b} + \sum_{i=1}^{m} z_i\right)\right\}.$$

Apparently, the posterior pdf of λ is a gamma distribution denoted as *gamma* $(m + a, W^{-1})$, where $W = \frac{1}{b} + \sum_{i=1}^{m} z_i$.

Let $U = 2\lambda W$. From Casella and Berger [15], we can find that the pdf of U has a chi-squared distribution with $2\,(m + a)$ degrees of freedom.

Two sets of pivotal quantities are considered to build the interval estimation of two parameters. The first set is $h_1(\mu, \theta) = 2\lambda Z_1 = 2Z_1/\theta = 2n(X_1 - \mu)/\theta$ and $g_1(\theta) = 2\lambda W - 2\lambda Z_1 = 2\lambda\left(\frac{1}{b} + \sum_{i=2}^{m} z_i\right) = 2\left(\frac{1}{b} + \sum_{i=2}^{m} (r_i + 1)X_i - (n - r - 1)X_1\right)/\theta$.

These two pivotal quantities are independent. $h_1(\mu,\theta)$ has a chi-squared distribution with 2 degrees of freedom and $g_1(\theta)$ has a chi-squared distribution with $2m + 2a - 2$ degrees of freedom. The second set of two pivotal quantities is $h_2(\mu) = \frac{h_1(\mu,\theta)/2}{g_1(\theta)/(2m+2a-2)} = n(m + a - 1)(\frac{X_1-\mu}{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1})$ and $g_2(\mu,\theta) = 2\lambda W = 2\left\{\frac{1}{b} + \sum_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}/\theta$. These two quantities are independent where $h_2(\mu) \sim F(2, 2m+2a-2)$ and $g_2(\mu,\theta) \sim \chi^2(2m+2a)$. The distributions of all pivotal quantities are not related to parameters. Using the pivotal quantity $g_1(\theta)$, we can build the credible interval for the scale parameter θ as follows:

Theorem 1. *For the progressive type II censored sample $X_1, X_2, \cdots, X_m$, the $(1-\alpha)$100% credible interval of the scale parameter θ is given by*

$$\left(\frac{2\left\{\frac{1}{b} + \sum_{i=2}^{m}(r_i+1)X_i - (n-r_1-1)X_1\right\}}{\chi^2_{\frac{\alpha}{2}}(2m+2a-2)} < \theta < \frac{2\left\{\frac{1}{b} + \sum_{i=2}^{m}(r_i+1)X_i - (n-r_1-1)X_1\right\}}{\chi^2_{1-\frac{\alpha}{2}}(2m+2a-2)}\right),$$

where $\chi^2_q(2m+2a-2)$ is the right-tailed q percentile for chi-squared distribution with $2m + 2a - 2$ degrees of freedom.

Proof of Theorem 1. *Based on the distribution of the pivotal quantity* $g_1(\theta) \sim \chi^2(2m-2)$, *we have*

$$\begin{aligned}1-\alpha =& P(\chi^2_{1-\frac{\alpha}{2}}(2m+2a-2) < g_1(\theta) < \chi^2_{\frac{\alpha}{2}}(2m+2a-2)) \\ =& P\left(\frac{2\left\{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1\right\}}{\chi^2_{\frac{\alpha}{2}}(2m+2a-2)} < \theta < \frac{2\left\{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1\right\}}{\chi^2_{1-\frac{\alpha}{2}}(2m+2a-2)}\right).\end{aligned}$$

The proof is thus completed. □

We are going to present two methods of credible region for two parameters by using two sets of pivotal quantities. Using the first set of pivotal quantities $h_1(\mu,\theta)$ and $g_1(\theta)$, the credible region of two parameters is developed in the following theorem called Method 1:

Theorem 2. *For the progressive type II censored sample $X_1, X_2, \cdots, X_m$, the $(1-\alpha)$100% joint credible region for parameters θ andμ is presented as*

$$\left(X_1 - \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2)\theta/(2n) < \mu < X_1 - \chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2)\theta/(2n), L_1 < \theta < U_1\right),$$

where $L_1 = \frac{2\left\{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1\right\}}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a-2)}$, $U_1 = \frac{2\left\{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1\right\}}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a-2)}$ *and $\chi^2_q(\nu)$ is the right-tailed q percentile for the chi-squared distribution with ν degrees of freedom.*

Proof of Theorem 2. Use the distributions for the first set of pivotal quantities $h_1(\mu,\theta) \sim \chi^2(2)$ and $g_1(\theta) \sim \chi^2(2m+2a-2)$. Since they are independent, then we have

$$1-\alpha=\sqrt{1-\alpha}\sqrt{1-\alpha}$$
$$=P(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2)<h_1(\mu,\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2))\times P(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a-2)<g_1(\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a-2))$$

$$=P(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2)<h_1(\mu,\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2),\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a-2)<g_1(\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a-2))$$

$$=P\left(X_1-\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2)\theta/(2n)<\mu<X_1-\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2)\theta/(2n),L_1<\theta<U_1\right).$$

The proof is thus completed. □

To develop the second method for building the credible region of two parameters, the second set of pivotal quantities $h_2(\mu)$ and $g_2(\mu,\theta)$ are used and proposed in the following theorem called Method 2:

Theorem 3. *For the progressive type II censored sample $X_1, X_2, \cdots, X_m$, the $(1-\alpha)100\%$ joint credible region for parameters θ andμ is presented as*

$$\left(L_2<\mu<U_2,\ \frac{2\left\{\frac{1}{b}+\sum\limits_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a)}<\theta<\frac{2\left\{\frac{1}{b}+\sum\limits_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a)}\right),$$

where $L_2 = X_1-F_{\frac{1-\sqrt{1-\alpha}}{2}}(2,2m+2a-2)\frac{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}{n(m+a-1)}$, $U_2 = X_1-F_{\frac{1+\sqrt{1-\alpha}}{2}}(2,2m+2a-2)\frac{\frac{1}{b}+\sum_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}{n(m+a-1)}$ and $F_q(2,2m+2a-2)$ is the right-tailed q percentile for F distribution with 2 and 2m+ 2a−2 degrees of freedom.

Proof of Theorem 3. Use the distributions for the second set of pivotal quantities $h_2(\mu)\sim F(2,2m+2a-2)$ and $g_2(\mu,\theta)\sim\chi^2(2m+2a)$. Since they are independent, then we have

$$\begin{aligned}1-\alpha=\sqrt{1-\alpha}\sqrt{1-\alpha}=\\ P(F_{\frac{1+\sqrt{1-\alpha}}{2}}(2,2m+2a-2)<h_2(\mu)<F_{\frac{1-\sqrt{1-\alpha}}{2}}(2,2m+2a-2))\\ \times P(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a)<g_2(\mu,\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a))\\ =P(F_{\frac{1+\sqrt{1-\alpha}}{2}}(2,2m+2a-2)<h_2(\mu)<F_{\frac{1-\sqrt{1-\alpha}}{2}}(2,2m+2a-2),\\ \chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a)<g_2(\mu,\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a))\\ =P\left(L_2<\mu<U_2,\ \frac{2\left\{\frac{1}{b}+\sum_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m+2a)}<\theta<\frac{2\left\{\frac{1}{b}+\sum_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m+2a)}\right).\end{aligned}$$

The proof is thus completed. □

The Bayesian credible region using Theorem 2 is called Method 1 and the Bayesian credible region using Theorem 3 is called Method 2.

We are going to obtain the prediction interval for the future observation X_{m+1} based on the observed progressive type II censored sample $X_1,\ldots,X_m$. When $r_m\neq 0$, the statistic $k = \frac{(m+a-1)r_m(X_{m+1}-X_m)}{\frac{1}{b}+\sum\limits_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}$ which is following an F distribution with 2 and $2m + 2a - 2$ degrees of freedom is considered to build the prediction interval for X_{m+1} in the following theorem.

Theorem 4. *For the progressive type II censored sample $X_1, X_2, \cdots, X_m$,the $(1-\alpha)100\%$ prediction interval for the future observation X_{m+1} is given by*

$$\left(X_m + \frac{\frac{1}{b}+\sum\limits_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}{(m+a-1)r_m}F_{1-\frac{\alpha}{2}}(2, 2m+2a-2),\right.$$
$$\left. X_m + \frac{\frac{1}{b}+\sum\limits_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}{(m+a-1)r_m}F_{\frac{\alpha}{2}}(2, 2m+2a-2)\right)$$

Proof of Theorem 4. Using the distribution of the statistic $k \sim F(2, 2m+2a-2)$, we have

$$\begin{aligned}
&1-\alpha = \\
&P(F_{1-\frac{\alpha}{2}}(2,2m+2a-2) < k < F_{\frac{\alpha}{2}}(2,2m+2a-2)) \\
&= P(F_{1-\frac{\alpha}{2}}(2,2m+2a-2) < \frac{(m+a-1)r_m(X_{m+1}-X_m)}{\frac{1}{b}+\sum\limits_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1} < F_{\frac{\alpha}{2}}(2,2m+2a-2)) \\
&= P\left(X_m + \frac{\frac{1}{b}+\sum\limits_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}{(m+a-1)r_m}F_{1-\frac{\alpha}{2}}(2,2m+2a-2), < X_{m+1} < \right. \\
&\left. X_m + \frac{\frac{1}{b}+\sum\limits_{i=2}^{m}(r_i+1)X_i-(n-r_1-1)X_1}{(m+a-1)r_m}F_{\frac{\alpha}{2}}(2,2m+2a-2)\right).
\end{aligned}$$

The proof is thus obtained. □

3. Simulation Study

The simulated coverage probabilities for the Bayesian credible interval for the scale parameter in Theorem 1 and the non-Bayesian confidence interval in Wu [11] are found in this section. In addition, the simulated coverage probabilities for two methods of Bayesian credible region for two parameters by using Theorem 2 and 3 and the non-Bayesian confidence region in Wu [11] are also found under the confidence levels of 0.90 and 0.95 with m = 17, 18, and 19 when n = 20 and m = 27, 28, and 29 when n = 30. For each fixed m, three types of progressive censoring schemes are considered. The simulation algorithm is described in the following steps:

Step 1: Give the initial values of $1-\alpha$ =0.90, 0.95, m = 17, 18, and 19 for n = 20 and m = 27, 28, and 29 for n = 30, μ = 0, θ = 1 for non-Bayesian confidence intervals and $1/\theta \sim \Gamma(a, b)$ for Bayesian credible intervals, ci1 = 0, ciw1 = 0, cr1 = 0, crw1 = 0, cr2 = 0, crw2 = 0, the number of replication run = 100,000.

Step 2: Generate independent random sample of $Z_1, \ldots, Z_m$ from exp(0, θ) distribution.

Step 3: Generate a progressive type II censored sample $X_1, \ldots, X_m$ by $X_1 = Z_1/n + \mu$, $X_2 = X_1 + Z_1/(n - r_1 - 1), \ldots, X_{\mathrm{m}} = X_{m-1} + Z_{\mathrm{m}}/(n - r_1 - \ldots - r_{m-1} - m + 1)$.

Step 4: If the value of θ is within the credible interval proposed in Theorem 1, ci1 = ci1 + 1/run; if the value of θ is within the confidence interval proposed in Wu [11], ciw1 = ciw1 + 1/run; if the values of (μ, θ) fall into the Bayesian credible region proposed in Theorem 2 (Method 1) and Theorem 3 (Method 2), we have cr1 = cr1 + 1/run and cr2 = cr2 + 1/run, respectively; if the values of (μ, θ) fall into the non-Bayesian confidence region proposed in Wu [11], we have crw1 = crw1 + 1/run and crw2 = crw2 + 1/run, respectively.

Step 5: Output ci1 as the coverage probabilities for Bayesian credible interval; output ciw1 as the coverage probabilities for non-Bayesian confidence interval; output cr1 and cr2 as the coverage probabilities for Bayesian credible regions based on Method 1 and Method 2; output crw1 and crw2 as the coverage probabilities for non-Bayesian confidence regions in Wu [11].

The simulated coverage probabilities for these methods are listed in Table 1 after 100,000 simulation runs. From Table 1, we found that the effect of different censoring scheme is not significant. For the confidence interval of scale parameter, the Bayesian method with a = 1 and b = 1 has higher coverage probabilities than the Bayesian method with a = 0.5 and b = 0.5 and the non-Bayesian method. For confidence region of two parameters, we have the same conclusion. Comparing two Bayesian methods for confidence regions, Method 1 by using Theorem 2 has higher coverage probabilities than Method 2 by using Theorem 3. Therefore, Method 1 with a = 1 and b = 1 should be considered to construct the Bayesian credible intervals.

Table 1. The simulated coverage probabilities for interval estimation of θ and two methods of confidence region for two parameters. Bayesian (1) and (2) stand for the cases of (a, b) = (0.5, 0.5) and (a, b) = (1, 1).

						Confidence Region					
			Interval for θ			**Method 1**			**Method 2**		
			Bayesian		**non-Bayesian**	**Bayesian**		**non-Bayesian**	**Bayesian**		**non-Bayesian**
n	m	$1-\alpha$	**(1)**	**(2)**		**(1)**	**(2)**		**(1)**	**(2)**	
20	17	Censoring scheme $= (r_1 = 2, r_2 = 1, r_i = 0, i \geq 3)$									
		0.90	0.902	0.911	0.900	0.930	0.936	0.929	0.927	0.932	0.928
		0.95	0.951	0.957	0.950	0.967	0.971	0.967	0.967	0.968	0.967
		Censoring scheme $= (r_8 = \ldots = r_{10} = 1, r_i = 0, i \neq 8, 9, 10)$									
		0.90	0.912	0.889	0.899	0.930	0.937	0.928	0.927	0.933	0.927
		0.95	0.950	0.957	0.950	0.967	0.970	0.966	0.967	0.968	0.967
		Censoring scheme $= (r_{16} = 1, r_{17} = 2, r_i = 0, i \leq 15)$									
		0.90	0.899	0.911	0.899	0.930	0.937	0.929	0.928	0.932	0.928
		0.95	0.950	0.957	0.950	0.965	0.971	0.965	0.965	0.969	0.965
20	18	Censoring scheme $= (r_1 = r_2 = 1, r_i = 0, i \geq 3)$									
		0.90	0.902	0.909	0.900	0.932	0.933	0.930	0.928	0.930	0.928
		0.95	0.951	0.958	0.950	0.966	0.970	0.966	0.965	0.968	0.965
		Censoring scheme $= (r_9 = r_{10} = 1, r_i = 0, i \neq 9, 10)$									
		0.90	0.900	0.911	0.899	0.929	0.935	0.929	0.927	0.931	0.927
		0.95	0.950	0.957	0.950	0.965	0.970	0.965	0.964	0.967	0.964
		Censoring scheme $= (r_{17} = r_{18} = 1, r_i = 0, i \leq 16)$									
		0.90	0.902	0.910	0.900	0.930	0.935	0.929	0.927	0.932	0.927
		0.95	0.949	0.957	0.948	0.965	0.970	0.965	0.964	0.968	0.964
20	19	Censoring scheme $= (r_1 = 1, r_i = 0, i \geq 2)$									
		0.90	0.902	0.908	0.901	0.930	0.934	0.929	0.927	0.930	0.927
		0.95	0.951	0.958	0.951	0.966	0.970	0.966	0.966	0.968	0.966

Table 1. *Cont.*

			Confidence Region								
			Interval for θ			**Method 1**			**Method 2**		
			Bayesian		**non-Bayesian**	**Bayesian**		**non-Bayesian**	**Bayesian**		**non-Bayesian**
n	m	$1-\alpha$	**(1)**	**(2)**		**(1)**	**(2)**		**(1)**	**(2)**	
			Censoring scheme $= (r_{10} = 1, r_i = 0, i \neq 10)$								
		0.90	0.902	0.909	0.900	0.929	0.936	0.929	0.926	0.931	0.926
		0.95	0.950	0.957	0.950	0.965	0.970	0.965	0.964	0.967	0.964
			Censoring scheme $= (r_{19} = 1, r_i = 0, i \leq 18)$								
		0.90	0.901	0.911	0.901	0.930	0.936	0.929	0.927	0.932	0.927
		0.95	0.950	0.957	0.950	0.966	0.970	0.966	0.965	0.967	0.965
30	27		Censoring scheme $= (r_1 = 2, r_2 = 1, r_i = 0, i \geq 3)$								
		0.90	0.900	0.906	0.899	0.927	0.929	0.926	0.917	0.912	0.908
		0.95	0.950	0.950	0.950	0.965	0.968	0.964	0.960	0.960	0.955
			Censoring scheme $= (r_{18} = \ldots = r_{20} = 1, r_i = 0, i \neq 18, 19, 20)$								
		0.90	0.902	0.906	0.900	0.928	0.931	0.926	0.918	0.913	0.908
		0.95	0.950	0.953	0.950	0.965	0.967	0.964	0.960	0.960	0.954
			Censoring scheme $= (r_{26} = 1, r_{27} = 2, r_i = 0, i \leq 25)$								
		0.90	0.900	0.908	0.899	0.926	0.930	0.925	0.917	0.914	0.907
		0.95	0.952	0.955	0.951	0.965	0.968	0.965	0.960	0.960	0.955
30	28		Censoring scheme $= (r_1 = r_2 = 1, r_i = 0, i \geq 3)$								
		0.90	0.900	0.905	0. 899	0.925	0.929	0.924	0.915	0.912	0.906
		0.95	0.950	0.953	0.950	0.964	0.967	0.960	0.960	0.960	0.955
			Censoring scheme $= (r_{19} = r_{20} = 1, r_i = 0, i \neq 19, 20)$								
		0.90	0.902	0.907	0.902	0.927	0.928	0.926	0.916	0.915	0.909
		0.95	0.951	0.955	0.950	0.966	0.968	0.965	0.961	0.960	0.955
			Censoring scheme $= (r_{27} = r_{28} = 1, r_i = 0, i \leq 26)$								
		0.90	0.902	0.906	0.902	0.928	0.931	0.927	0.918	0.918	0.910
		0.95	0.951	0.954	0.950	0.965	0.966	0.965	0.960	0.960	0.955
30	29		Censoring scheme $= (r_1 = 1, r_i = 0, i \geq 2)$								
		0.90	0.901	0.905	0.899	0.926	0.929	0.925	0.915	0.915	0.908
		0.95	0.950	0.955	0.950	0.965	0.966	0.964	0.960	0.956	0.954
			Censoring scheme $= (r_{20} = 1, r_i = 0, i \neq 20)$								
		0.90	0.902	0.906	0.902	0.927	0.930	0.926	0.917	0.917	0.908
		0.95	0.950	0.954	0.950	0.965	0.967	0.965	0.960	0.960	0.954
			Censoring scheme $= (r_{29} = 1, r_i = 0, i \leq 28)$								
		0.90	0.900	0.905	0.898	0.926	0.930	0.924	0.916	0.919	0.907
		0.95	0.950	0.955	0.949	0.965	0.968	0.964	0.960	0.960	0.954

4. One Engineering Example

An example in Lawless [16] is considered in this section. The data of this example consists of the failure times (number of cycles in 1000 times) of 18 ball bearings and it is listed in Table A1 of Appendix A.

In Figure 1, we have plotted the empirical distribution function (ECDF) for this data set. We use the Kolmogorov–Smirnov test (KS test) to test if this data set fits the exponential distribution or not. The p-value for this test is 0.9715 > 0.05 and it shows that this data fits the exponential distribution. Considering the case of m = 15 and the prefixed censoring scheme $(r_1, r_2, r_3, \cdots, r_{15})$ = (0, 0, 0, ... , 1, 2), the progressive type II censored sample is given by $(X_1, \cdots, X_{15})$ = (0.1788, 0.2892, 0.3300, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584) (in years). Under $a = 1$ and $b = 1$, by Theorem 1, the 95% confidence interval for θ is obtained as (0.4607652, 1.289184) with confidence length 0.8284191. By Theorem 2 (Method 1), the 95% joint confidence region for θ and μ is given by

$$\begin{cases} \max(0, 0.1788 - 0.2912857 \times \theta) < \mu < 0.1788 - 0.008494071 \times \theta \\ 0.4336757 < \theta < 1.406493 \end{cases}$$

with area 0.2599622. Applying Theorem 3(Method 2), the confidence region is given by

$$\begin{cases} 0 < \mu < 0.1781869 \\ \frac{2\left\{\frac{1}{b}+\sum\limits_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}}{52.48478} < \theta < \frac{2\left\{\frac{1}{b}+\sum\limits_{i=1}^{m}(r_i+1)(X_i-\mu)\right\}}{16.82137} \end{cases}$$

with area 0.03188604. In this case, Method 2 has smaller area than Method 1. In Figure 2a,b, we plotted the confidence regions for Method 1 and Method 2.

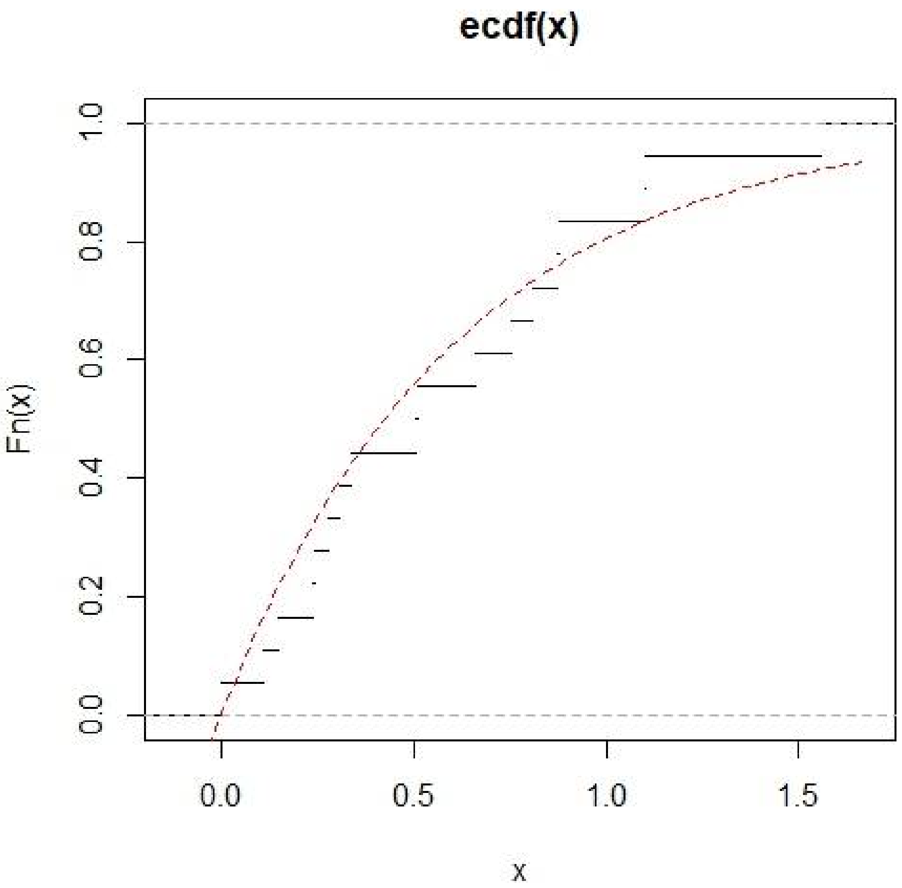
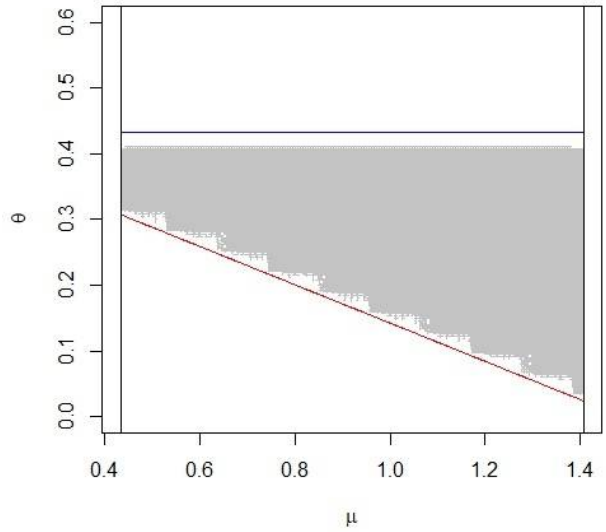
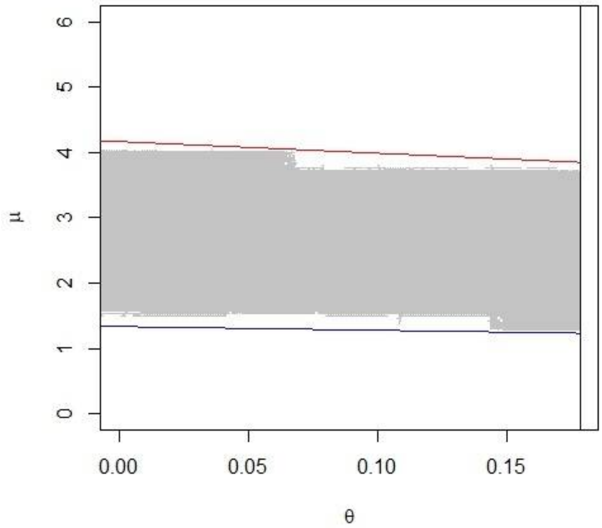


Figure 1. The ECDF for this example.



(**a**)



(**b**)

Figure 2. (**a**) The confidence region for Method 1. (**b**) The confidence region for Method 2. By Theorem 4, the prediction interval for X_{16} is obtained as (1.067542, 2.567176).

5. Conclusions

The Bayesian credible interval for the scale parameter θ and two Bayesian methods including Method 1 and Method 2 for obtaining the credible region of θ and μ for the two-parameter exponential distribution under progressive type II censoring are investigated in this research. Comparing two Bayesian credible regions with the confidence regions proposed in Wu [11] by the simulation study under different censoring schemes, we found that Method 1 with the parameters $a = 1$ and $b = 1$ has higher coverage probability than Method 2. In addition to building the credible interval and credible region for parameters, we proposed the predictive intervals for the future observation for this distribution under progressive type II censoring. In the end, we used an engineering example to illustrate all the proposed methods in this paper.

Funding: This research was funded by the Ministry of Science and Technology, Taiwan MOST 108-2118-M-032-001 and MOST 109-2118-M-032-001-MY2 and the APC was funded by MOST 109-2118-M-032-001-MY2.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data available in a publicly accessible repository. The data presented in this study are openly available in Lawless.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

Table A1. The data of the failure times (number of cycles in 1000 times) of 18 ball bearings.

0.1788	0.2892	0.3300	0.4152	0.4212	0.4560	0.4848	0.5184	0.6864
0.6888	0.8412	0.9312	0.9864	1.0512	1.0584	1.2792	1.2804	1.7840

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