

Article

Asymmetric Probability Mass Function for Count Data Based on the Binomial Technique: Synthesis and Analysis with Inference

Afrah Al-Bossly ^{1,*} and Mohamed S. Eliwa ^{2,3} 

¹ Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

² Department of Statistics and Operation TResearch, College of Science, Qassim University, P.O. Box 6644, Buraydah 51482, Saudi Arabia; mseliwa@mans.edu.eg or m.eliwa@qu.edu.sa

³ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

* Correspondence: a.basli@psau.edu.sa

Abstract: In this article, a new probability mass function for count data is proposed based on the binomial technique. After introducing the methodology of the newly model, some of its distributional characteristics are discussed in-detail. It is found that the newly model has explicit mathematical expressions for its statistical and reliability properties, which is not the case with many well-known discrete models. Moreover, it can be used as an effectively probability tool for modeling asymmetric over-dispersed data with leptokurtic shapes. The parameters estimation through the classical point of view have been done via utilizing the technique of maximum likelihood and Bayesian approaches. A MCMC simulation study is carried out to examine the performance of the estimators. Finally, two distinct real data sets are analyzed to prove the flexibility and notability of the newly model.

Keywords: binomial technique; count data; bayesian analysis; MCMC simulation

MSC: 60E05; 62E10; 62F10



Citation: Al-Bossly, A.; Eliwa, M.S.

Asymmetric Probability Mass Function for Count Data Based on the Binomial Technique: Synthesis and Analysis with Inference. *Symmetry* **2022**, *14*, 826. <https://doi.org/10.3390/sym14040826>

Academic Editor: Jinyu Li

Received: 22 March 2022

Accepted: 11 April 2022

Published: 15 April 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

In today's competitive era, the data generated from various fields such as engineering, economics, medical sciences, etc., is becoming more complex day by day. As a result, for modeling such data, we need distributions that are best suited for analytical studies of these multidimensional and complex data. For these reasons, over the past three decades, the development of new probability distribution has become the center of the statistical research. However, a large part of this research has been devoted to the development of continuous probability distributions. However, there may be situations where discrete distributions may be more appropriate for data modeling or the generated data may be naturally discrete; for example, in the field of reliability analysis, the lifetime of an on/off switching device is a discrete random variable, see [1], among others. These circumstances demand suitable discrete distributions that can adequately model such data. Therefore, in recent years, the derivation of discrete distributions has also received the attention of researchers.

The first name in this episode comes from [2], who gave a discretized version of the continuous Weibull distribution. Then afterward, ref. [3] obtained discrete inverse Weibull distribution. The paper by [4] provided an excellent review of the development of discrete distributions until 2014. Thereafter, many important discrete distributions have evolved in the literature. Ref. [5] derived discrete generalized Rayleigh distribution, ref. [6] suggested discrete Weibull geometric distribution, and discrete additive Perks–Weibull distribution was proposed by [7]. Recently, to fit discrete increasing failure and count data, ref. [8] developed a discrete Perks distribution, ref. [9] introduced a new two-parameter exponentiated discrete Lindley distribution with bathtub-shaped hazard rate characteristics,

ref. [10] developed a discrete Gompertz-G family for over and under-dispersed data, ref. [11] presented a discrete Burr–Hatke distribution with the associated count regression model, ref. [12] discussed discrete Bilal distribution with properties and applications on integer-valued autoregressive processes, and [13] proposed a new three-parameter discrete Lindley distribution with an associated INAR (1) process, among others.

An important notable thing about the above-cited articles is that most of them are discretized versions of the continuous probability distribution, and they have evolved from the techniques described in [4]. Apart from these methods, another important approach to obtaining discrete models is due to Hu et al. [14]. This approach allows us to generate new discrete distributions by compounding two probability functions. According to this methodology, if X and M are two discrete random variables with probability mass functions (PMFs) $f(x)$ and $w(m)$, respectively, then the PMFs of these two random variables are connected by the binomial decay transformation, i.e.,

$$\Pr[X = x] = \sum_{m=x}^{\infty} \binom{m}{x} p^x (1-p)^{m-x} w(m); \quad x = 0, 1, 2, \dots \quad (1)$$

Here, it is notable that the expression in Equation (1) is a proper PMF with an attenuating coefficient $p \in [0, 1]$. Ref. [14] considered M as a Poisson variate with parameter $\lambda > 0$, and then using Equation (1), they obtained $\Pr[X = x]$ as a Poisson distribution with parameter $\lambda p > 0$. Ref. [15] derived uniform Poisson distribution by applying a similar idea to that of [14]. Ref. [16] proposed uniform-geometric distribution by replacing binomial distribution with uniform distribution and setting $w(m)$ as geometric distribution in Equation (1). Recently, Ref. [17] introduced binomial-discrete Lindley distribution by substituting $w(m)$ as discrete Lindley distribution in Equation (1). In this paper, the authors propose a new three-parameter binomial new Poisson-weighted exponential (BNPWE) distribution for modeling count data by substituting $w(m)$ as a new Poisson-weighted exponential (NPWE) distribution. This procedure has been followed by a very large number of researchers in order to look for new distributions that allow capturing “certain” properties of “certain” data sets. Such an abundance of this type of procedure has made it banal, with the consequent obtaining of uninteresting results. However, in the case of this paper, this procedure is applied to discrete distributions, which is not at all frequent. Thus, there is still a way to go here, which favors the interest of the paper under analysis. The proposed model is created and developed by deducing formulas and studying properties for its parameters. From the properties of these parameters, they try to intuit and justify their suitability for application to certain types of data, in particular for medical applications.

2. The BNPWE Distribution

The NPWE distribution has recently been introduced by [18]. He obtained this discrete model by compounding Poisson and a new weighted exponential distribution of [19] and showed its applicability in a first-order integer-valued autoregressive process. The survival function (SF) of NPWE distribution can be expressed as

$$S(x; \alpha, \theta) = (1 + \alpha + \alpha\theta)^{-(x+1)}; \quad x \in \mathbb{N}_0, \quad (2)$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots, \nu\}$ for $0 < \nu < \infty$, $\alpha > 0$, and $\theta > 0$. The PMF corresponding to Equation (2) is

$$p(x; \alpha, \theta) = \alpha(1 + \theta)(1 + \alpha + \alpha\theta)^{-(x+1)}; \quad x \in \mathbb{N}_0. \quad (3)$$

The PMF in Equation (1) can be represented as

$$\Pr[X = x] = \sum_{m=x}^{\infty} \Pr(X = x | M = m) w(m), \quad (4)$$

where $X|M = m$ has the binomial (m, p) distribution. Now, let us consider that $X|M = m$ follows binomial (m, β) distribution and $w(m)$ has the NPWE distribution given in Equation (3). Then, using Equation (4), the PMF of X can be expressed as

$$\Pr[X = x; \alpha, \beta, \theta] = \frac{\alpha(1 + \theta)\beta^x}{(\alpha + \beta + \alpha\theta)^{x+1}}, \quad (5)$$

where $x \in \mathbb{N}_0$, $\alpha > 0$, $0 < \beta < 1$, and $\theta > 0$. If random variable X has the PMF in Equation (5), then it is known as BNPWE distribution and it is symbolized by $X \sim \text{BNPWE}(\alpha, \beta, \theta)$. The corresponding CDF to Equation (5) is

$$F(x; \alpha, \theta, \beta) = 1 - \left(\frac{\beta}{\alpha + \beta + \alpha\theta} \right)^{x+1}; \quad x \in \mathbb{N}_0. \quad (6)$$

Based on Equation (6), the quantile function can be formulated as

$$x_q = \frac{\ln(1 - q)}{\ln\left(\frac{\beta}{\alpha + \beta + \alpha\theta}\right)} - 1; \quad 0 < q < 1. \quad (7)$$

Figure 1 shows the PMF plots for various values of the BNPWE parameters.

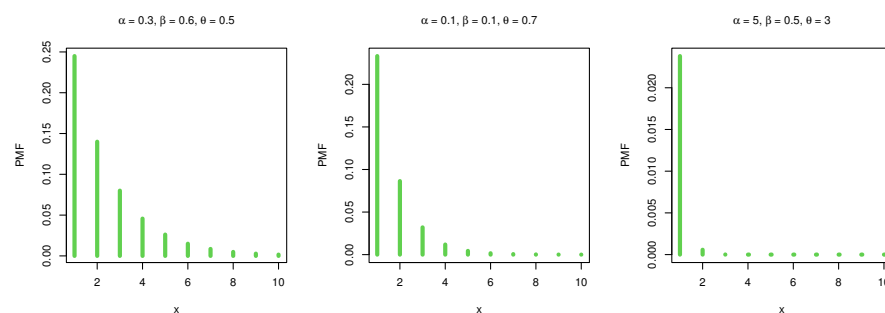


Figure 1. The PMF plots of the BNPWE model.

2.1. Moments, Skewness, Kurtosis and Index of Dispersion

Let X be a BNPWE random variable. Then, the probability-generating function (PrGF) can be proposed in closed form as

$$\Pi_X(s) = \frac{\alpha(1 + \theta)}{\alpha\theta - \beta s + \alpha + \beta}. \quad (8)$$

On replacing s by e^s in Equation (8), we get the moment-generating function (MGF). The first four derivatives of the MGF, with respect to s at $s = 0$, yield the first four moments (FFM) about origin. Thus, the FFM values of the BNPWE distribution, respectively, are

$$E(X) = \frac{\beta}{\alpha(1 + \theta)}, \quad (9)$$

$$E(X^2) = \frac{\beta(\alpha\theta + 2\beta + \alpha)}{\alpha^2(1 + \theta)^2}, \quad (10)$$

$$E(X^3) = \frac{6\beta\left(\frac{1}{6}\alpha^2(1 + \theta)^2 + \beta\alpha(1 + \theta) + \beta^2\right)}{\alpha^3(1 + \theta)^3} \quad (11)$$

and

$$E(X^4) = \frac{24\beta\left(\frac{1}{12}\alpha^2(1 + \theta)^2 + \beta\alpha(1 + \theta) + \beta^2\right)\left(\frac{1}{2}\alpha(1 + \theta) + \beta\right)}{\alpha^4(1 + \theta)^4}. \quad (12)$$

Based on the FFM of the BNPWE distribution about origin, the skewness, kurtosis, and index of dispersion (I_xOD) can be derived in explicit forms. The I_xOD is defined as variance to mean ratio; it indicates whether a certain distribution is suitable for over or (under)-dispersed data. If I_xOD > (<)1, the model is under (over)-dispersed. Table 1 lists some descriptive statistics for the BNPWE distribution for various values of the model parameters.

Table 1. Some descriptive measures of the BNPWE distribution for $\beta = 0.5$ and $\theta = 0.7$.

Parameter →	α							
Measure ↓	0.01	0.1	0.7	1.0	1.5	2.0	2.5	3.5
Mean	29.4117	2.9411	0.4201	0.2941	0.1960	0.1470	0.1176	0.0840
Variance	894.4636	11.5916	0.5967	0.3806	0.2345	0.1686	0.1314	0.0910
I _x OD	30.4117	3.9411	1.4201	1.2941	1.1960	1.1470	1.1176	1.0840
Skewness	2.0002	2.0214	2.3824	2.5743	2.8747	3.1509	3.4066	3.8700
Kurtosis	9.0011	9.0862	10.6758	11.6272	13.2639	14.9282	16.6052	19.9775

It is clear that the BNPWE distribution is appropriate for modeling over-dispersed data. Furthermore, it is capable of modeling leptokurtic and positively skewed data.

2.2. Mean Residual Life (MRL) and Mean Past Life (MPL)

The MRL is a helpful tool to analyze the burn-in and maintenance policies. In the discrete setting, the MRL is defined as

$$\Lambda(i) = E(X - i | X \geq i) = \frac{1}{1 - F(i - 1; x; \alpha, \theta, \beta)} \sum_{j=i+1}^{\infty} [1 - F(j - 1; x; \alpha, \theta, \beta)]; \quad i \in \mathbb{N}_0,$$

Let X be a BNPWE random variable, then the MRL can be reported in a closed form as

$$\Lambda(i) = \frac{\beta + \alpha(1 + \theta)}{\alpha(1 + \theta)} \left(\frac{\beta}{\alpha + \beta + \alpha\theta} \right). \quad (13)$$

The variance residual life (VRL) function can be defined in a closed form as

$$\begin{aligned} \Omega_{\text{VRL}}(i) &= E(X^2 | X \geq i) - [E(X | X \geq i)]^2 \\ &= \frac{2[\beta + i\alpha(1 + \theta)][\beta + \alpha(1 + \theta)]}{\alpha^2(1 + \theta)^2} \left(\frac{\beta}{\alpha + \beta + \alpha\theta} \right) - (2i - 1)\Lambda(i) - \Lambda^2(i). \end{aligned} \quad (14)$$

Thus, X is increasing (decreasing) VRL if

$$\Omega_{\text{VRL}}(i + 1) \geq (\leq) \Lambda(i)[1 + \Lambda(i + 1)]. \quad (15)$$

The residual coefficient of variation (RCV) can be listed in a closed form where $\text{RCV}(i) = \sqrt{\Omega_{\text{VRL}}(i)} / \Lambda(i)$.

Another measure of interest in reliability theory is MPL. It measures the time elapsed since the failure of X given that the system/component has failed sometime before i . In the discrete setting, the MPL is defined as

$$\delta(i) = E(i - X | X < i) = \frac{1}{F(i - 1; \alpha, \theta, \beta)} \sum_{m=1}^i F(m - 1; \alpha, \theta, \beta); \quad i \in \mathbb{N}_0 - \{0\},$$

where $\delta(0) = 0$. Let X be a BNPWE random variable; then, the MPL can be represented in a closed form as

$$\delta(i) = \left[1 - \left(\frac{\beta}{\alpha + \beta + \alpha\theta} \right)^i \right]^{-1} \left[i + \frac{\beta^{i+1}(\alpha + \beta + \alpha\theta)^{-i} - \beta}{\alpha(1 + \theta)} \right]. \quad (16)$$

For $i \in \mathbb{N}_0$, we get $\delta(i) \leq i$. The mean of the model can be listed as

$$\text{Mean} = i - \delta(i)F(i-1) + \Lambda(i)[1 - F(i-1)]; i \in \mathbb{N}_0 - \{0\}. \quad (17)$$

The reversed hazard rate function (RHRF) and the MPL are related as

$$r(i; \theta) = \frac{1 - \delta(i+1) + \delta(i)}{\delta(i)}; i \in \mathbb{N}_0 - \{0\}. \quad (18)$$

If X is a BNPWE random variable, then the CDF can be recovered by the MPL as

$$F(k; \alpha, \theta, \beta) = F(0)_{i=1}^k \left[\frac{\delta(i)}{\delta(i+1) - 1} \right]; k \in \mathbb{N}_0 - \{0\}, \quad (19)$$

where $F(0) = \left(\prod_{i=1}^{\infty} \left[\frac{\delta(i)}{\delta(i+1) - 1} \right] \right)^{-1}$. Tables 2–4 list some numerical computations of reliability concepts for different values of the model parameter α, β , and θ , respectively, at time $i = 10$ h.

Table 2. Some numerical computations of reliability concepts for various values of α .

Parameter $\rightarrow$	$\alpha \beta = 0.5, \theta = 0.7$							
Measure $\downarrow$	0.01	0.1	0.7	1.0	1.5	2.0	2.5	3.5
MRL	29.4117	2.9411	0.4201	0.2941	0.1960	0.1470	0.1176	0.0840
VRL	894.4636	11.5916	0.5967	0.3806	0.2345	0.1686	0.1314	0.0910
RCV	1.0168	1.1575	1.8384	2.0976	2.4698	2.7928	3.0822	3.5916
MPL	5.7753	7.6248	9.5798	9.7058	9.8039	9.8529	9.8823	9.9159

Table 3. Some numerical computations of reliability concepts for various values of β .

Parameter $\rightarrow$	$\beta \alpha = 0.5, \theta = 0.7$							
Measure $\downarrow$	0.01	0.1	0.2	0.3	0.5	0.7	0.9	0.99
MRL	0.0117	0.1176	0.2352	0.3529	0.5882	0.8235	1.0588	1.1647
VRL	0.0119	0.1314	0.2906	0.4775	0.9342	1.5017	2.1799	2.5212
RCV	9.2736	3.0822	2.2912	1.9578	1.6431	1.4880	1.3944	1.3632
MPL	9.9882	9.8823	9.7647	9.6470	9.4122	9.1800	8.9541	8.8556

Table 4. Some numerical computations of reliability concepts for various values of θ .

Parameter $\rightarrow$	$\theta \alpha = 0.5, \beta = 0.5$							
Measure $\downarrow$	0.01	0.1	0.7	1.0	1.5	2.0	2.5	3.5
MRL	0.9900	0.9090	0.5882	0.4999	0.4000	0.3333	0.2857	0.2222
VRL	1.9703	1.7355	0.9342	0.7500	0.5600	0.4444	0.3673	0.2716
RCV	1.4177	1.4491	1.6431	1.7320	1.8708	1.9999	2.1213	2.3452
MPL	9.0192	9.0969	9.4122	9.5001	9.6000	9.6666	9.7142	9.7777

From Tables 2–4, it is clear that:

1. The MRL and VRL decrease for (fixed β and θ with α grows) and (fixed α and β with θ grows), whereas for fixed α and θ with $\beta \rightarrow 1$, the MRL and VRL increase.
2. The RCV and MPL increase for (fixed β and θ with α grows) and (fixed α and β with θ grows), whereas for fixed α and θ with $\beta \rightarrow 1$, the RCV and MPL decrease.

3. Parameter Estimation

3.1. Classical Estimation Using Method of Maximum Likelihood

In this section, we determine the MLEs of the model parameter based on a complete sample. Assume that $X_1, X_2, \dots, X_n$ is a random sample of size n from the BNPWE distribution. The likelihood function (L) can be expressed as follows

$$L = \frac{\alpha^n (1 + \theta)^n \beta^{\sum_{i=1}^n x_i}}{(\alpha + \beta + \alpha\theta)^{\sum_{i=1}^n (x_i + 1)}}. \quad (20)$$

The log-likelihood function (LLF) corresponding to Equation (20) is

$$l = \log L = n \ln[\alpha(1 + \theta)] + \ln \beta \sum_{i=1}^n x_i - \ln(\alpha + \beta + \alpha\theta) \sum_{i=1}^n [x_i + 1]. \quad (21)$$

By differentiating Equation with respect to the parameters α , β , and θ , we get the non-linear likelihood equations. These equations cannot be solved analytically; therefore, an iterative procedure such as Newton–Raphson is required to solve these non-linear equations numerically.

3.2. Bayesian Estimation

The significance of Bayesian analysis has grown enormously over the last several decades not only because Bayesian estimators have become much simpler to compute, but also because it is one of the most acceptable methods of computing estimates for complicated models. Given this, the current section is dedicated to the Bayesian estimation of unknown parameters of the BNPWE distribution under informative priors (IPs) and non-informative priors (NIPs).

Suppose that the prior distributions of the parameters α , β , and θ are $\text{Gamma}(a_1, a_2)$, $\text{Beta}(b_1, b_2)$, and $\text{Gamma}(c_1, c_2)$, respectively. Then, the joint prior distribution of α , β , and θ is given by

$$g(\alpha, \beta, \theta) \propto \alpha^{a_1-1} \beta^{b_1-1} \theta^{c_1} \exp(-\alpha a_1 - \beta b_2 - \theta c_2), \quad (22)$$

where $(\alpha, \theta) > 0$, $0 < \beta < 1$, and $(a_1, b_1, a_2, b_2, c_1, c_2) > 0$ are the hyperparameters of the prior densities and can be fixed based on the amount of available prior information. If we fixed $a_1 = b_1 = a_2 = b_2 = c_1 = c_2 = 0$, the joint prior density becomes non-informative prior density.

Using the application of Bayes theorem, the unnormalized joint posterior distribution of the parameters α , β , and θ given data can be obtained via the likelihood function and the joint prior density as

$$P(\alpha, \beta, \theta | \underline{x}) \propto \frac{\alpha^{n+a_1-1} \beta^{\sum_{i=1}^n x_i + b_1 - 1} \theta^{c_1} (1 + \theta)^n \exp(-\alpha a_1 - \beta b_2 - \theta c_2)}{(\alpha + \beta + \alpha\theta)^{\sum_{i=1}^n (x_i + 1)}}. \quad (23)$$

Loss functions (LFs) play a vital role when researchers are not only interested in choosing the right decision but also consider the economic consequences that arise through a Bayes estimate of the unknown parameter. In our case, we use a well-known symmetric loss function called squared error function (SELF). In this loss function, positive and negative errors are equally penalized. Under SELF, the Bayes estimator of a parameter is simply the expectation of that parameter with respect to its posterior distribution.

Now, the Bayes estimator of a function of parameters α , β , and θ , say $w(\alpha, \beta, \theta)$ is

$$\hat{w}(\alpha, \beta, \theta) = \int_{\alpha} \int_{\beta} \int_{\theta} w(\alpha, \beta, \theta) P(\alpha, \beta, \theta | \underline{x}) d\alpha d\beta d\theta. \quad (24)$$

The integral is difficult to obtain in the closure form due to the complex form of the posterior distribution. Therefore, we use a mixture of two famous Monte Carlo simulation techniques under a hybrid algorithm. These techniques are Gibbs sampler, see [20] and the Metropolis–Hastings algorithm, see [21,22]. The first technique allows us to simulate posterior samples from full conditional marginal posterior density instead of the joint posterior distribution, whereas the latter one enables us to draw required samples when conventional methods fail to generate samples. For this purpose, the full conditional marginal posterior densities of the parameters α , β , and θ are, respectively as

$$P_{11}(\alpha | \beta, \theta, \underline{x}) \propto \frac{\alpha^{n+a_1-1} \exp(-\alpha a_1)}{(\alpha + \beta + \alpha\theta)^{\sum_{i=1}^n (x_i+1)}}, \quad (25)$$

$$P(\beta | \alpha, \theta, \underline{x}) \propto \frac{\beta^{\sum_{i=1}^n x_i + b_1 - 1} \exp(-\beta b_2)}{(\alpha + \beta + \alpha\theta)^{\sum_{i=1}^n (x_i+1)}}, \quad (26)$$

$$P(\theta | \alpha, \beta, \underline{x}) \propto \frac{\theta^{c_1} (1 + \theta)^n \exp(-\theta c_2)}{(\alpha + \beta + \alpha\theta)^{\sum_{i=1}^n (x_i+1)}}. \quad (27)$$

The steps of the above-mentioned hybrid algorithm are as follows:

1. Start with the initial values of α , β , and θ as $(\alpha_{(0)}, \beta_{(0)}, \theta_{(0)})$ and set $j = 0$.
2. Generate $\alpha_{(j)}$ from $P_{11}(\alpha | \beta_{(j-1)}, \theta_{(j-1)}, \underline{x})$ using the following steps:
 - (a) Generate the proposal point α^* from the proposal distribution $N(\alpha_{(j-1)}, \sigma_{\alpha}^2)$ and u_1 from Uniform (0,1) distribution. Here, the variance σ_{α}^2 is suitably chosen.
 - (b) Calculate the acceptance probability (AP) $\rho_{\alpha} = \min \left[1, \frac{P_{11}(\alpha^* | \beta_{(j-1)}, \theta_{(j-1)}, \underline{x})}{P_{11}(\alpha_{(j-1)} | \beta_{(j-1)}, \theta_{(j-1)}, \underline{x})} \right]$, and if $u_1 \leq \rho_{\alpha}$, then record $\alpha_{(j)} = \alpha^*$; otherwise, sustain $\alpha_{(j)} = \alpha_{(j-1)}$.
3. Generate $\beta_{(j)}$ from $P_{12}(\beta | \alpha_{(j)}, \theta_{(j-1)}, \underline{x})$ using the following steps:
 - (a) Generate the proposal value β^* from $N(\beta_{(j-1)}, \sigma_{\beta}^2)$ and u_2 from Uniform (0,1) distribution. The variance σ_{β}^2 is appropriately selected.
 - (b) Compute the AP $\rho_{\beta} = \min \left[1, \frac{P_{12}(\beta^* | \alpha_{(j)}, \theta_{(j-1)}, \underline{x})}{P_{12}(\beta_{(j-1)} | \alpha_{(j)}, \theta_{(j-1)}, \underline{x})} \right]$, and if $u_2 \leq \rho_{\beta}$, then record $\beta_{(j)} = \beta^*$; otherwise, store $\beta_{(j)} = \beta_{(j-1)}$.
4. Generate $\theta_{(j)}$ from $P_{13}(\theta | \alpha_{(j)}, \beta_{(j)}, \underline{x})$ using the following steps:
 - (a) Generate the proposal point θ^* from $N(\theta_{(j-1)}, \sigma_{\theta}^2)$, and u_3 from Uniform (0,1) distribution. The variance σ_{θ}^2 is well-chosen here.
 - (b) Calculate the AP $\rho_{\theta} = \min \left[1, \frac{P_{13}(\theta^* | \alpha_{(j)}, \beta_{(j)}, \underline{x})}{P_{13}(\theta_{(j-1)} | \alpha_{(j)}, \beta_{(j)}, \underline{x})} \right]$, and if $u_3 \leq \rho_{\theta}$, then record $\theta_{(j)} = \theta^*$; otherwise, sustain $\theta_{(j)} = \theta_{(j-1)}$.
5. Set $j = j + 1$.
6. Rerun the steps 2–5, a large number of times, say N times, and obtain $\alpha_{(j)}$, $\beta_{(j)}$, and $\theta_{(j)}$, $j = 1, 2, \dots, N$.

To avoid the effect of initial values, the first m draws are eliminated. After the convergence diagnostic of the generated chains through various graphical and statistical tests, the values $\alpha_{(j)}$, $\beta_{(j)}$, and $\theta_{(j)}$, $j = m + 1, m + 2, \dots, N$, represent the required posterior sam-

ples, which can be utilized to compute the Bayes estimators of the unknown parameters. Hence, the Bayes estimators of α , β , and θ under SELF are, respectively, obtained as

$$\hat{\alpha}_B = \frac{1}{N-m} \sum_{j=m+1}^N \alpha_{(j)}, \quad \hat{\beta}_B = \frac{1}{N-m} \sum_{j=m+1}^N \beta_{(j)}, \quad \text{and} \quad \hat{\theta}_B = \frac{1}{N-m} \sum_{j=m+1}^N \theta_{(j)}. \quad (28)$$

Notably, by fixing $a_1 = b_1 = a_2 = b_2 = c_1 = c_2 = 0$ and applying the above-mentioned procedure, we can obtain the Bayes estimators under NIPs.

4. A Monte Carlo Simulation Study

To discuss the behavior of the considered estimation methods based on various values of n and true parametric combination, we perform an MCMC simulation study. This assessment consists of the following steps: Draw 2000 samples of sizes $n = 25, 50, 100$, and 200 from the newly model with $(\alpha, \beta, \theta) = (0.5, 0.4, 0.5)$ and $(0.5, 0.4, 1.01)$. Calculate the MLE and BE (under IP and NIP with SELF) for the 2000 samples, say $\hat{\tau}_{\varphi}^j, \hat{\tau}_{\beta}^j, \hat{\tau}_{\theta}^j; \tau = \alpha, \beta, \theta; j = 1, 2, \dots, 2000$; $\varphi = \text{ML and Bayes}$. Compute the mean squared error (MSE) and average absolute error (ABE) for all point estimates, where

$$MSE = \frac{1}{2000} \sum_{j=1}^{2000} (\hat{\tau}_{\varphi}^j - \tau)^2 \quad \text{and} \quad ABE = \frac{1}{2000} \sum_{j=1}^{2000} |\hat{\tau}_{\varphi}^j - \tau|. \quad (29)$$

The empirical results are shown in Tables 5–7. To assess the convergence of generated chains, we observe the MCMC, histogram, and auto-correlation plots. These plots show that all chains reach their stationery distributions. Due to space constraints, we have included convergence diagnostic plots only for the parametric combination $(0.5, 0.4, 0.5)$ under IPs and NIPs, and they can be viewed in Figures 2 and 3, respectively.

Table 5. The MSE, ABE, and bootstrap CI for MLEs.

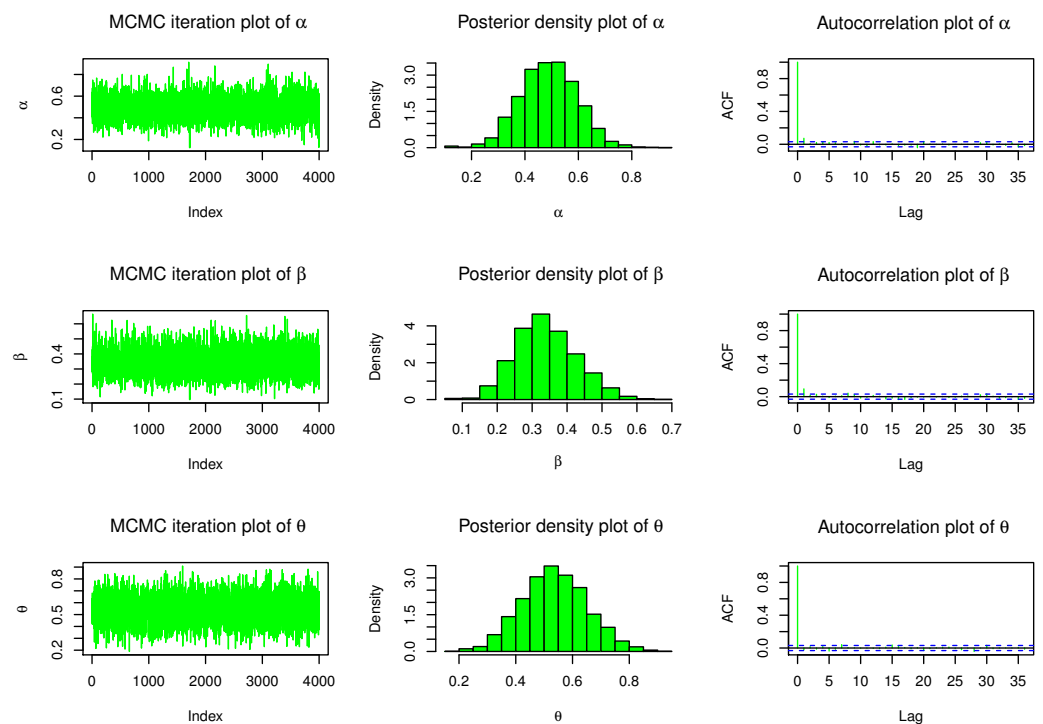
Schema	n	α		β		θ	
		MSE	ABE [bootstrap CI]	MSE	ABE [bootstrap CI]	MSE	ABE [bootstrap CI]
I	25	0.00481	0.05600 [0.363,0.566]	0.00604	0.05995 [0.310,0.487]	0.00189	0.02690 [0.455,0.591]
	50	0.00275	0.04192 [0.399,0.545]	0.00285	0.04099 [0.319,0.470]	0.00085	0.02495 [0.457,0.547]
	100	0.00155	0.03100 [0.413,0.531]	0.00132	0.02855 [0.354,0.451]	0.00078	0.02257 [0.461,0.536]
	200	0.00088	0.02215 [0.437,0.512]	0.00078	0.02069 [0.366,0.432]	0.00069	0.02069 [0.467,0.529]
II	25	0.00651	0.06331 [0.325,0.576]	0.00785	0.06791 [0.316,0.491]	0.00309	0.04407 [0.876,1.43]
	50	0.00377	0.04699 [0.377,0.558]	0.00405	0.04731 [0.324,0.486]	0.00215	0.03758 [0.894,1.39]
	100	0.00214	0.03464 [0.395,0.534]	0.00205	0.03328 [0.357,0.461]	0.00182	0.03740 [0.911,1.30]
	200	0.00149	0.02702 [0.428,0.510]	0.00111	0.02404 [0.364,0.435]	0.00171	0.03719 [0.951,1.25]

Table 6. The MSE, ABE, and bootstrap CI for Bayes estimates under IPs.

Schema	n	α		β		θ	
		MSE	ABE [bootstrap CI]	MSE	ABE [bootstrap CI]	MSE	ABE [bootstrap CI]
I	25	0.00121	0.02702 [0.311,0.568]	0.00222	0.03811 [0.326,0.507]	0.00029	0.01427 [0.387,0.583]
	50	0.00092	0.02570 [0.334,0.557]	0.00191	0.03623 [0.337,0.501]	0.00019	0.01121 [0.394,0.561]
	100	0.00077	0.02243 [0.356,0.543]	0.00140	0.03038 [0.342,0.487]	0.00018	0.01106 [0.398,0.553]
	200	0.00048	0.01760 [0.377,0.531]	0.00079	0.02311 [0.359,0.423]	0.00012	0.00948 [0.417,0.528]
II	25	0.00178	0.03197 [0.339,0.547]	0.00220	0.03847 [0.334,0.489]	0.00023	0.01297 [0.872,1.624]
	50	0.00127	0.02933 [0.350,0.541]	0.00217	0.03779 [0.351,0.478]	0.00018	0.01111 [0.911,1.426]
	100	0.00081	0.02334 [0.384,0.530]	0.00135	0.03057 [0.369,0.451]	0.00015	0.01009 [0.937,1.340]
	200	0.00052	0.01878 [0.390,0.522]	0.00099	0.02197 [0.388,0.430]	0.00012	0.00959 [0.970,1.229]

Table 7. The MSE, ABE, and bootstrap CI for Bayes estimates under NIPs.

Schema	n	α		β		θ	
		MSE	ABE [bootstrap CI]	MSE	ABE [bootstrap CI]	MSE	ABE [bootstrap CI]
I	25	0.00245	0.03993 [0.366,0.571]	0.00422	0.04865 [0.334,0.586]	0.00077	0.02617 [0.327,0.560]
	50	0.00191	0.03470 [0.387,0.562]	0.00266	0.04242 [0.373,0.553]	0.00082	0.02691 [0.335,0.551]
	100	0.00186	0.03332 [0.391,0.553]	0.00210	0.03745 [0.382,0.531]	0.00072	0.02604 [0.361,0.531]
	200	0.00159	0.03118 [0.399,0.532]	0.00120	0.02791 [0.389,0.511]	0.00065	0.02491 [0.380,0.510]
II	25	0.00232	0.0371 [0.353,0.560]	0.00524	0.05610 [0.321,0.558]	0.00022	0.01313 [0.901,1.141]
	50	0.00197	0.03627 [0.361,0.555]	0.00291	0.04235 [0.336,0.546]	0.00019	0.01228 [0.903,1.100]
	100	0.00180	0.03443 [0.374,0.536]	0.00206	0.03250 [0.371,0.523]	0.00017	0.01146 [0.924,1.090]
	200	0.00117	0.02694 [0.394,0.510]	0.00114	0.02308 [0.387,0.514]	0.00013	0.01032 [0.949,1.087]

**Figure 2.** The MCMC diagnosticplots under NIPs.

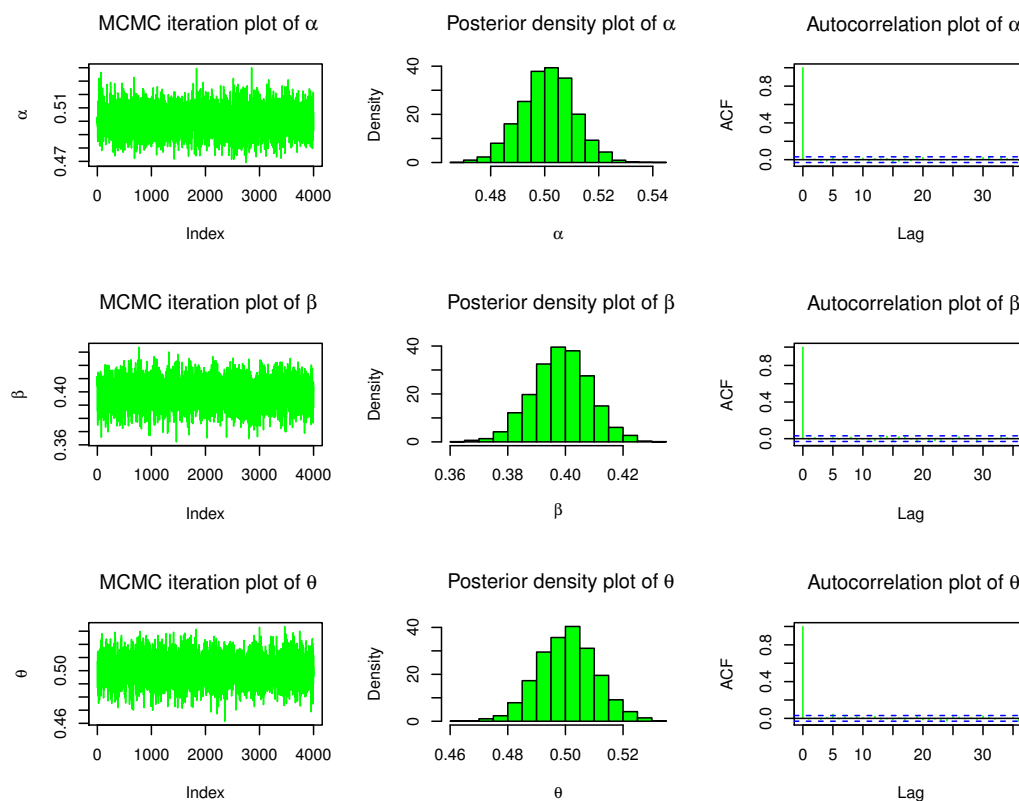


Figure 3. The MCMC diagnostic plots under IPs.

From Tables 5–7, it is noted that:

- Based on MSE, we observe that the parameter β is more sensitive as compared to α and θ .
- The MSE and ABE decrease to zero as n tends to infinity. This shows the consistency of the estimators.
- Both estimation procedures perform satisfactorily. However, in overall comparison, Bayes estimators perform better in comparison to MLE, especially in case of IPs.
- For higher values of the parameters α , β , and θ , the generated random samples produce a large number of 0s and 1s, due to which sometimes we face the convergence issue, and also the estimated error associated with an estimate becomes larger.

5. Data Analysis

In this section, we illustrate the importance of the BNPWE distribution by utilizing data from different areas. We shall compare the fits of the BNPWE distribution with some competitive models such as discrete Weibull inverse Weibull (DWIW), discrete inverse Weibull (DIW), discrete log-logistic (DLogL), discrete Burr type II (DB-II), discrete Rayleigh (DR), discrete Bilal (DBL), discrete Pareto (DPa), and discrete Burr–Hatke (DBH). Some statistical criteria have been used such as the negative log-likelihood ($-L$), corrected (Akaike information criterion) C(AIC), Bayesian IC (BIC), Hannan–Quinn IC (HQIC), and Chi-square (χ^2) test with its corresponding p -value.

5.1. Data Set I: COVID-19 Data

The data are reported in (<https://www.worldometers.info/coronavirus/country/southkorea/>) (accessed on 21 March 2022)) and represent the daily new deaths in South Korea from 15 February to 13 June 2020. The initial MS is reported using the nonparametric KME method in Figure 4, and it is noted that the mass function is asymmetric and multimodal. Moreover, Figure 4 shows that some extreme observations were listed.

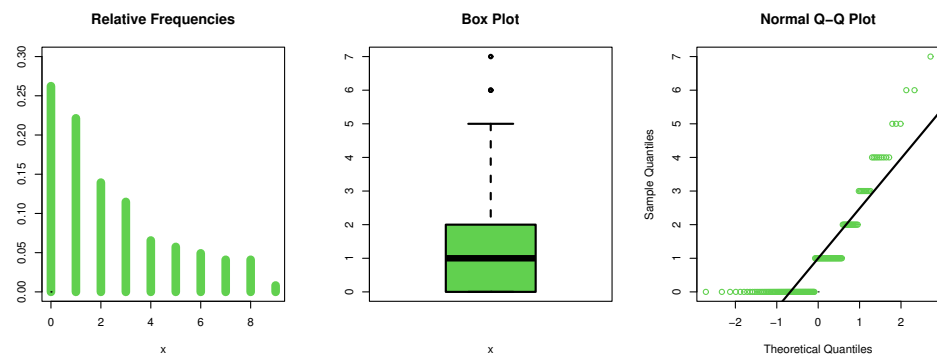


Figure 4. The KME, Q-Q, and box plots for data set I.

The MLEs and Std-er for the parameter(s) as well as GOF measures for data set I are reported in Tables 8 and 9.

Table 8. The MLEs and Std-er for data set I.

Parameter →	α		β		θ		λ	
Model ↓	MLE	Std-er	MLE	Std-er	MLE	Std-er	MLE	Std-er
BNPWE	0.1387	0.0148	0.4573	0.0347	0.3870	0.1488	—	—
DWIW	0.9524	0.0931	0.3953	0.4810	0.0112	0.0166	2.8785	3.5181
DIW	0.2338	0.0381	1.2658	0.1134	—	—	—	—
DLogL	2.0210	0.1890	1.7457	0.1523	—	—	—	—
DB-II	0.6225	0.0487	2.3359	0.3772	—	—	—	—
DR	0.9306	0.0061	—	—	—	—	—	—
DBL	0.7487	0.0143	—	—	—	—	—	—
DPa	0.4151	0.0332	—	—	—	—	—	—
DBH	0.9315	0.0269	—	—	—	—	—	—

Table 9. The GOF measures test for data set I.

X	OF	Expected Frequency								
		BNPWE	DWIW	DIW	DLogL	DB-II	DR	DBL	DBH	DPa
0	32	36.1276	30.4229	28.5245	27.6305	34.1655	8.4645	19.2478	65.1759	55.6679
1	27	25.4291	26.7554	38.1383	32.81328	35.8539	22.0300	30.7382	21.5347	19.8889
2	17	17.8988	20.0874	18.3045	20.7934	17.0802	27.6341	25.5994	10.6342	10.3779
3	14	12.5986	14.4039	9.9187	12.3449	9.0893	25.2606	17.8606	6.2813	6.4241
4	8	8.8678	10.0466	6.0537	7.6045	5.5023	18.3967	11.4844	4.1105	4.3897
5	7	6.2418	6.8713	4.0199	4.9354	3.6500	11.0489	7.0553	2.8746	3.2002
6	6	4.3934	4.6288	2.8368	3.3619	2.5837	5.5663	4.2138	2.1058	2.4424
7	5	3.0924	3.0797	2.0949	2.3864	1.9183	2.3750	2.4707	1.5963	1.9288
8	5	2.1767	2.0277	1.6026	1.7534	1.4770	0.8635	1.4305	1.2423	1.564
9	1	5.1736	3.6763	10.5061	8.37632	10.6798	0.3604	1.8993	6.4444	16.1161
Total	122	122	122	122	122	122	122	122	122	122
−L		250.3056	250.5345	262.3222	256.7394	263.5383	279.9239	255.5355	277.0495	279.8059
AIC		506.6112	509.0690	528.6444	517.4788	531.0766	561.8477	513.0710	556.099	561.6119
CAIC		506.8146	509.4109	528.7453	517.5796	531.1774	561.8811	513.1043	556.1323	561.6452
BIC		515.0232	520.28508	534.2525	523.0868	536.6846	564.6518	515.875	558.903	564.4159
HQIC		510.0279	513.6246	530.9222	519.7566	533.3544	562.9867	514.2099	557.2379	562.7508
χ^2		2.8444	2.4099	12.3012	5.5044	14.1272	89.7303	18.5571	43.5311	64.2454
d.f		4	3	4	4	4	5	5	4	5
p.value		0.5842	0.49179	0.0152	0.2393	0.0069	≤0.001	0.0023	≤0.001	≤0.001

It is clear that the DWIW and DLogL distributions work quite well besides the BNPWE distribution. However, the BNPWE distribution is the best model among all tested models.

Figures 5 and 6 support our empirical results where the BNPWE is more fit to analyze this data, whereas Figure 7 shows that the estimator is unique.

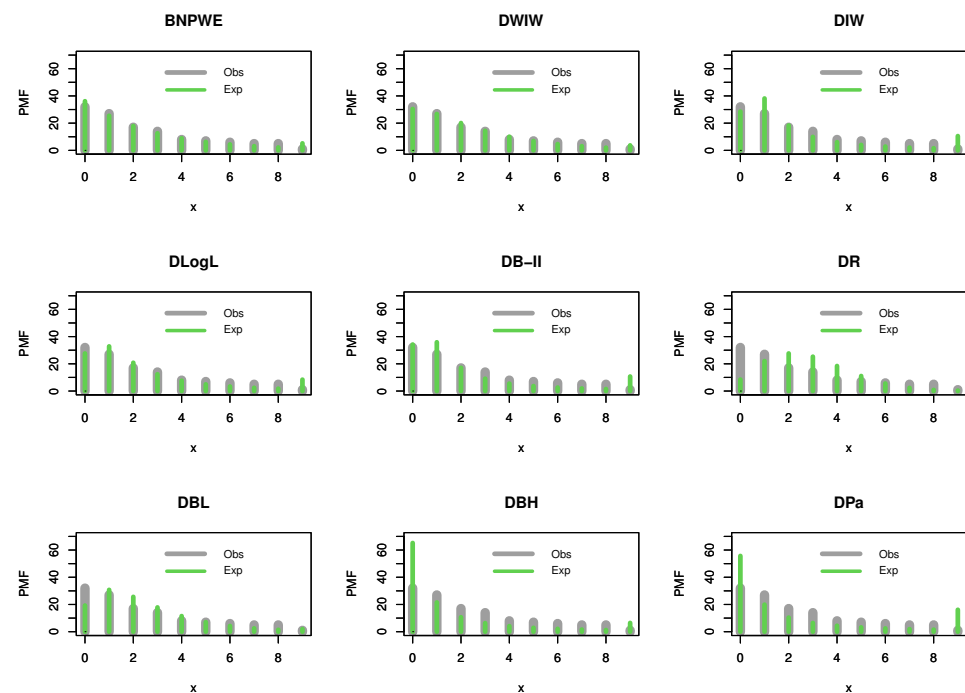


Figure 5. The estimated PMFs for data set I.

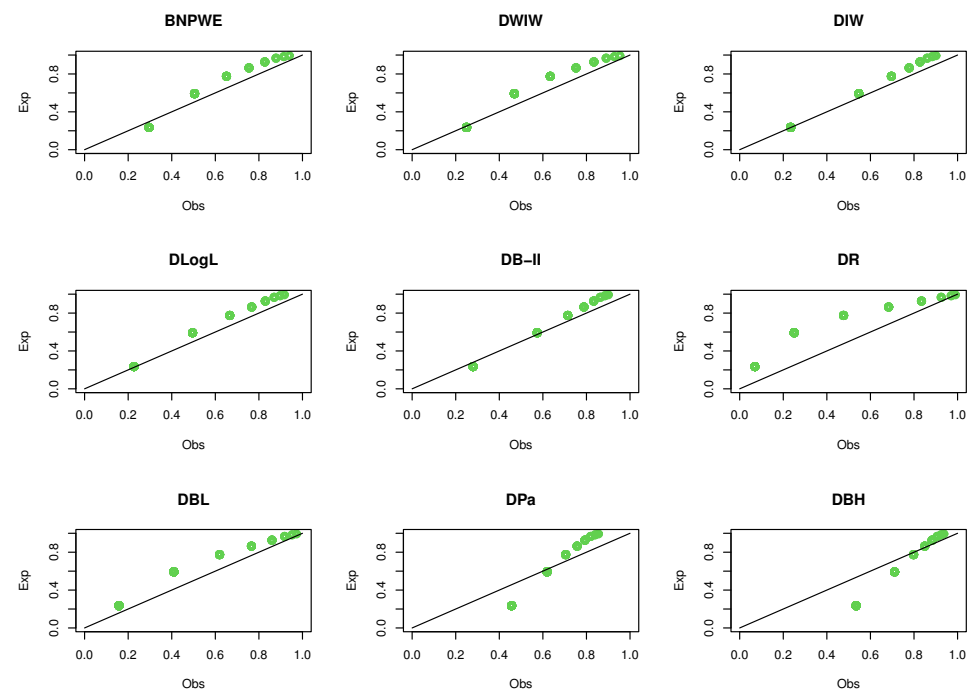


Figure 6. The P-P plot for data set I.

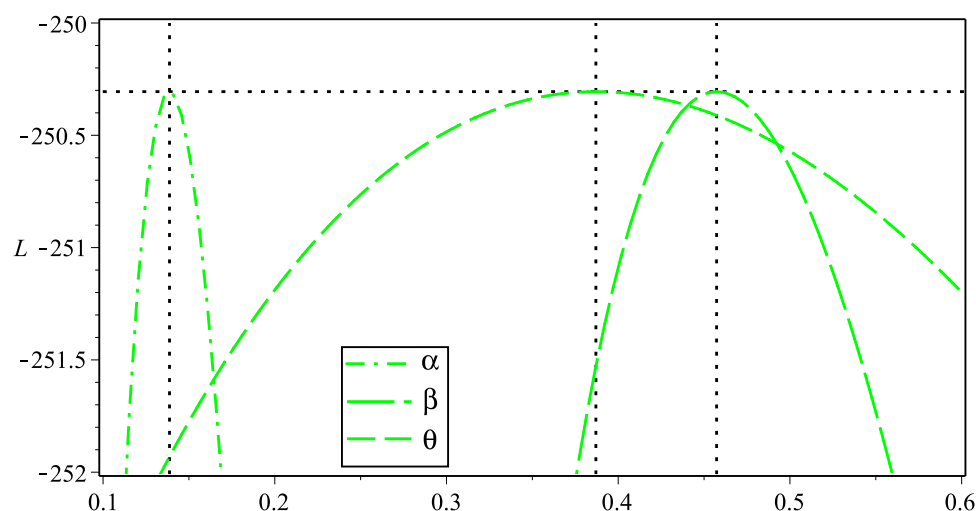


Figure 7. The L profiles for the model parameters for data set I.

According to the MLEs, the EDS for mean, variance, IxOD, skewness and kurtosis are 2.37710, 8.02773, 3.37710, 2.03090, and 9.12456, respectively. The data exhibit over dispersion. Moreover, it is moderately skewed to the right and leptokurtic.

5.2. Data Set II: Biological Data

This data set is the biological experiment data which represent the number of European corn borer (No. ECB) larvae pyrausta in the field, see [23]. It was an experiment conducted randomly on 8 hills in 15 replications, and the experimenter counts the number of borers per hill of corn. For these data, the mean, variance, skewness, and kurtosis equal 1.326, 3.669, 1.976, and 8.984, respectively. The initial MS is reported using the nonparametric KME method in Figure 8, and it is noted that the mass function is asymmetric and multimodal. Moreover, Figure 8 shows that some extreme observations were reported.

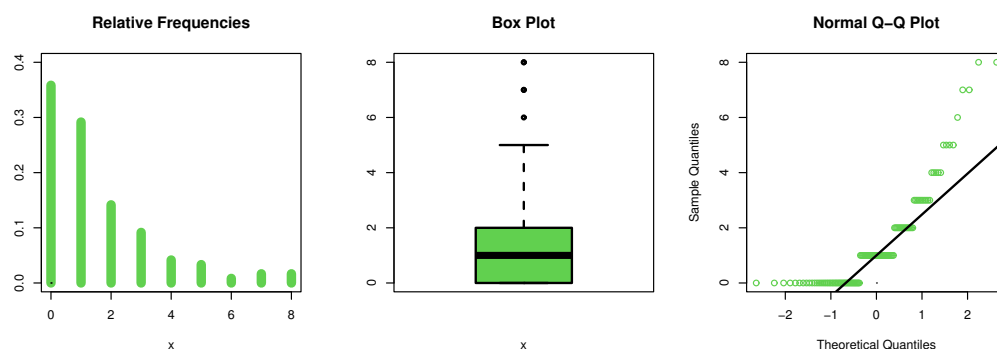


Figure 8. The KME, Q-Q, and box plots for data set II.

The MLEs and Std-er for the parameter(s) as well as GOF measures for data set II are reported in Tables 10 and 11.

It is clear that the DWIW and DLogL distributions work quite well besides the BNPWE distribution. However, the BNPWE distribution is the best model among all tested models. Figures 9 and 10 support our empirical results where the BNPWE more fit to analyze these data, whereas Figure 11 shows that the estimator is unique.

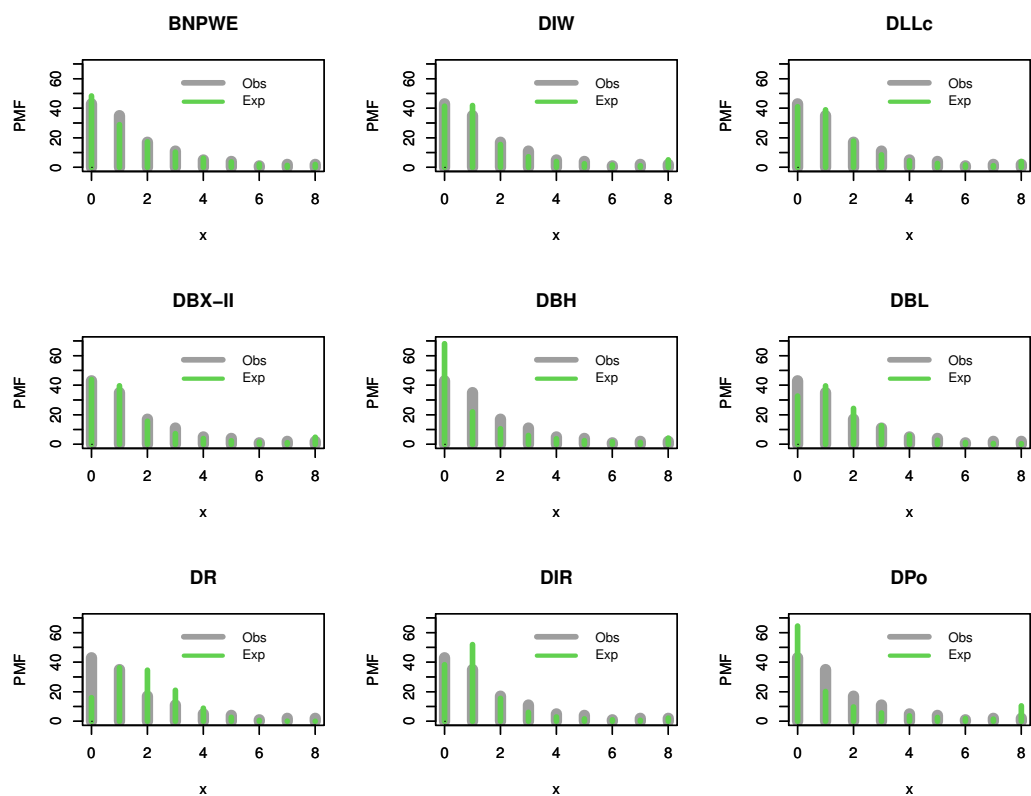


Figure 9. The estimated PMFs for data set II.

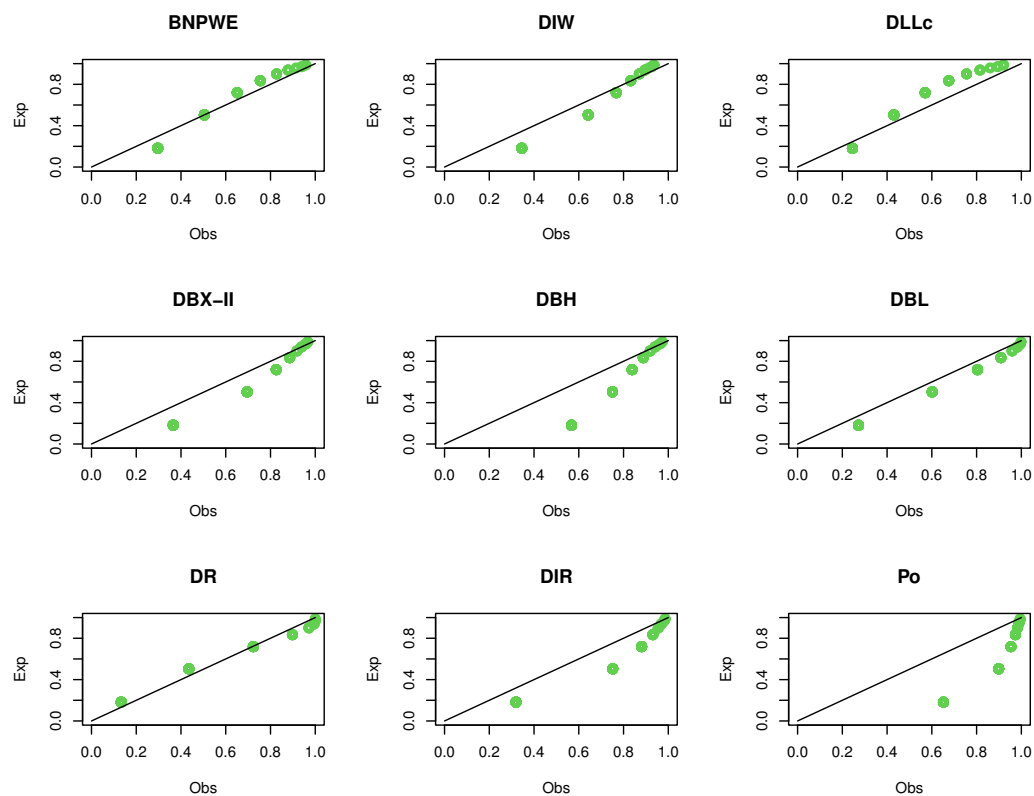


Figure 10. The P-P plot for data set II.

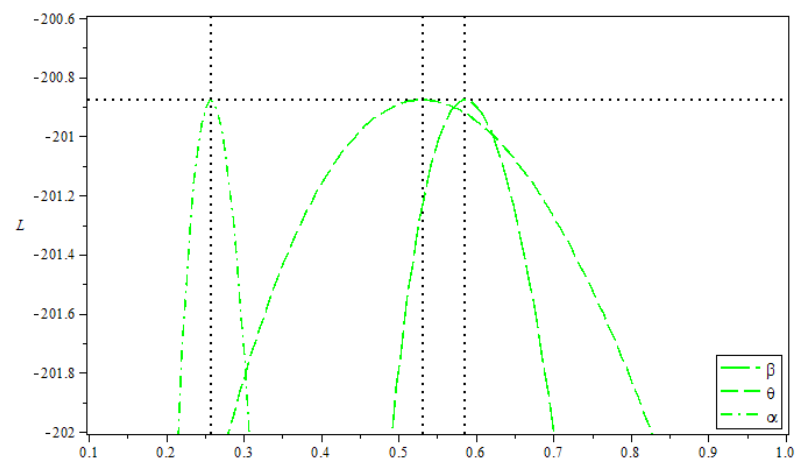


Figure 11. The L profiles for the model parameters for data set II.

Table 10. The MLEs and Std-er for data set II.

Model ↓ Parameter →	β		α		θ	
	MLE	Std-er	MLE	Std-er	MLE	Std-er
BNPWE	0.586	0.059	0.258	0.030	0.532	0.180
DIvW	0.345	0.043	1.541	0.156	—	—
DLLc	1.401	0.121	1.943	0.188	—	—
DBXII	0.519	0.051	2.358	0.366	—	—
DBHk	0.865	0.039	—	—	—	—
DBI	0.657	0.019	—	—	—	—
DRh	0.867	0.012	—	—	—	—
DIRh	0.319	0.042	—	—	—	—
DPo	0.329	0.034	—	—	—	—

Table 11. The GOF measures test for data set II.

No.		EF								
ECB	OF	BNPWE	DIW	DLLc	DBX-II	DBH	DBL	DR	DIR	DPo
0	43	48.318	41.400	41.019	43.836	68.099	32.669	15.960	38.280	64.470
1	35	28.863	41.844	38.940	39.614	21.971	39.558	36.236	51.904	20.149
2	17	17.241	15.418	17.779	15.622	10.513	24.294	34.588	15.509	9.684
3	11	10.299	7.166	8.434	7.204	5.980	12.534	20.985	6.036	5.645
4	5	6.152	3.942	4.486	3.908	3.751	5.991	8.846	2.909	3.679
5	4	3.675	2.422	2.631	2.374	2.504	2.751	2.681	1.612	2.578
6	1	2.195	1.607	1.664	1.561	1.746	1.234	0.594	0.983	1.903
7	2	1.311	1.128	1.116	1.088	1.256	0.546	0.097	0.642	1.459
8	2	1.946	5.073	3.931	4.793	4.18	0.423	0.013	2.125	10.433
Total	120	120	120	120	120	120	120	120	120	120
− L		200.877	204.812	202.630	204.293	214.053	204.675	235.227	208.439	220.618
AIC		407.755	413.624	409.261	412.587	430.106	411.351	472.453	418.878	443.236
CAIC		416.117	413.727	409.363	412.689	430.139	411.384	472.487	418.912	443.270
BIC		407.962	419.199	414.836	418.162	432.893	414.138	475.241	421.665	446.024
HQIC		411.151	415.888	411.525	414.851	431.238	412.483	473.585	420.010	444.368
χ^2		2.159	5.497	3.844	4.652	27.061	7.023	59.805	14.281	35.513
d.f		2	3	3	3	2	3	3	3	4
p.value		0.339	0.139	0.279	0.199	<0.001	0.071	<0.001	<0.001	<0.001

According to the MLEs, the EDS for mean, variance, IxOD, skewness, and kurtosis are 1.48258, 3.68063, 2.48258, 2.06680, and 9.27169, respectively. The data exhibit over dispersion. Furthermore, it is moderately skewed to the right and leptokurtic.

5.3. Bayesian Analysis of Real Data Sets I and II

In this section, we derive Bayes estimates for the unknown parameters of the proposed distribution with their posterior standard errors (PSEs). Here, it is worth noting that in this estimation process, we use NIPs because there is no prior knowledge available about the model parameters for the data sets under study. These estimates can be viewed in Table 12. From this table, we can conclude that Bayesian estimation captures real data sets more accurately than the considered classical estimation procedure in terms of estimation errors.

Table 12. Bayes estimates and PSE for real data sets.

Data Sets	Parameters	Bayes Estimates	PSE
I	α	0.1384	0.0125
	β	0.4562	0.0197
	θ	0.3874	0.0196
II	α	0.2427	0.0291
	β	0.5778	0.0564
	θ	0.5293	0.1170

6. Conclusions

In this article, a binomial new Poisson weighted exponential distribution is developed for analysis of count data. Its various impressive statistical properties, including moments, skewness, kurtosis, index of dispersion, mean residual life, and mean past life, are derived. One of the main features of the proposed model is that it has closed-form expressions for various important distributional characteristics, which is not the case with many well-known discrete distributions. Furthermore, it may be used to model over-dispersed, leptokurtic, and positively skewed data sets. Under classical estimation approach, the method of maximum likelihood is used, whereas in the Bayesian paradigm, we have used informative and non-informative priors with squared error loss function to estimate the unknown parameters. To evaluate the performance of the estimators with respect to the sample size, a thorough simulation study is performed. Finally, two different real data sets are examined to demonstrate the adaptability of the suggested distribution.

A future plan of action regarding the current study might be an examination of the censored data using the proposed model. We may investigate the load share model where the component failure time follows the BNPWE distribution. The stress-strength parameter may be examined using various censored data. In addition, a bivariate extension of the BNPWE distribution can be developed.

Author Contributions: A.A.-B. Software, Resources, Writing—review & editing, Funding; and M.S.E. Methodology, Supervision, Data curation, Formal analysis, Software, Writing—review & editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia grant number IF-PSAU-2021/01/18784.

Acknowledgments: The authors extend their appreciations to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF-PSAU-2021/01/18784).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Krishna, H.; Pundir, P.S. Discrete Burr and discrete Pareto distributions. *Stat. Methodol.* **2009**, *6*, 177–188. [[CrossRef](#)]
2. Nakagawa, T.; Osaki, S. The discrete Weibull distribution. *IEEE Trans. Reliab.* **1975**, *24*, 300–301. [[CrossRef](#)]
3. Stein, W.E.; Dattero, R. A new discrete Weibull distribution. *IEEE Trans. Reliab.* **1984**, *33*, 196–197. [[CrossRef](#)]
4. Chakraborty, S. Generating discrete analogues of continuous probability distributions: A survey of methods and constructions. *J. Stat. Distrib. Appl.* **2015**, *2*, 6. [[CrossRef](#)]

5. Alamatsaz, M.H.; Dey, S.; Dey, T.; Harandi, S.S. Discrete generalized Rayleigh distribution. *Pak. J. Stat.* **2016**, *32*, 1–20.
6. Jayakumar, K.; Babu, M.G. Discrete Weibull geometric distribution and its properties. *Commun. Stat. Theory Methods* **2018**, *47*, 1767–1783. [[CrossRef](#)]
7. Tyagi, A.; Choudhary, N.; Singh, B. Discrete additive Perks-Weibull distribution: Properties and applications. *Life Cycle Reliab. Saf. Eng.* **2019**, *8*, 183–199. [[CrossRef](#)]
8. Tyagi, A.; Choudhary, N.; Singh, B. A new discrete distribution: Theory and applications to discrete failure lifetime and count data. *J. Appl. Probab. Stat.* **2020**, *15*, 119–145.
9. El-Morshedy, M.; Eliwa, M.S.; Nagy, H. A new two-parameter exponentiated discrete Lindley distribution: Properties, estimation and applications. *J. Appl. Stat.* **2020**, *47*, 354–375. [[CrossRef](#)]
10. Eliwa, M.S.; Alhussain, Z.A.; El-Morshedy, M. Discrete Gompertz-G family of distributions for over-and under-dispersed data with properties, estimation, and applications. *Mathematics* **2020**, *8*, 358. [[CrossRef](#)]
11. El-Morshedy, M.; Eliwa, M.S.; Altun, E. Discrete Burr-Hatke distribution with properties, estimation methods and regression model. *IEEE Access* **2020**, *8*, 74359–74370. [[CrossRef](#)]
12. Altun, E.; El-Morshedy, M.; Eliwa, M.S. A study on discrete Bilal distribution with properties and applications on integer-valued autoregressive process. *Revstat Stat. J.* **2020**, *18*, 70–99.
13. Eliwa, M.S.; Altun, E.; El-Dawoody, M.; El-Morshedy, M. A new three-parameter discrete distribution with associated INAR (1) process and applications. *IEEE Access* **2020**, *8*, 91150–91162. [[CrossRef](#)]
14. Hu, Y.; Peng, X.; Li, T.; Guo, H. On the Poisson approximation to photon distribution for faint lasers. *Phys. Lett.* **2007**, *367*, 173–176. [[CrossRef](#)]
15. Déniz, E.G. A new discrete distribution: Properties and applications in medical care. *J. Appl. Stat.* **2013**, *40*, 2760–2770. [[CrossRef](#)]
16. Akdoğan, Y.; Kus, C.; Asgharzadeh, A.; Kinaci, I.; Sharafi, F. Uniform-geometric distribution. *J. Stat. Comput. Simul.* **2016**, *86*, 1754–1770. [[CrossRef](#)]
17. Coşkun, K.U.Ş.; Akdoğan, Y.; Asgharzadeh, A.; Kinaci, İ.; Karakaya, K. Binomial-discrete Lindley distribution. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2018**, *68*, 401–411.
18. Altun, E. A new generalization of geometric distribution with properties and applications. *Commun. Stat. Simul. Comput.* **2020**, *49*, 793–807. [[CrossRef](#)]
19. Oguntunde, P.E.; Owoloko, E.A.; Balogun, O.S. On a new weighted exponential distribution: Theory and application. *Asian J. Appl. Sci.* **2016**, *9*, 1–12. [[CrossRef](#)]
20. Geman, S.; Geman, D. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Mach. Intell.* **1984**, *6*, 721–741. [[CrossRef](#)]
21. Metropolis, N.; Ulam, S. The Monte Carlo method. *J. Am. Stat. Assoc.* **1949**, *44*, 335–341. [[CrossRef](#)] [[PubMed](#)]
22. Hastings, W.K. Monte Carlo sampling methods using Markov chains and their applications. *Biometrika* **1970**, *57*, 97–109. [[CrossRef](#)]
23. Bodhisuwan, W.; Sangpoom, S. The discrete weighted Lindley distribution. In Proceedings of the 2016 12th International Conference on Mathematics, Statistics, and Their Applications (ICMSA), Banda Aceh, Indonesia, 4–6 October 2016; pp. 99–103.