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Solving a System of Sylvester-like Quaternion Matrix Equations

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Abstract: Using the ranks and Moore-Penrose inverses of involved matrices, in this paper we establish some necessary and sufficient solvability conditions for a system of Sylvester-type quaternion matrix equations, and give an expression of the general solution to the system when it is solvable. As an application of the system, we consider a special symmetry solution, named the $\eta$-Hermitian solution, for a system of quaternion matrix equations. Moreover, we present an algorithm and a numerical example to verify the main results of this paper.

Keywords: Sylvester-type matrix equation; quaternion matrix; rank; Moore–Penrose inverse; $\eta$-Hermitian matrix

1. Introduction

In 1952, Roth [1] studied the following one-sided generalized Sylvester matrix equation for the first time

$$A_1 X + Y B_1 = C_1,$$  

(1)

which is widely used in system and control theory. Since then, many researches have paid attention to Sylvester-type matrix equations (e.g., [2–5]) because of their wide range of applications, such as in descriptor system control theory [6], neural networks [7], robust, feedback [8], graph theory [9] and other areas. For instance, Baksalary and Kala [10] established a necessary and sufficient condition for Equation (1) to have a solution and gave an expression of its general solution. In [11], Baksalary and Kala give a solvability condition for the equation

$$AXB + CYD = E.$$  

(2)

Wang investigated Equation (2) over arbitrary regular rings with identity [12].

In 1843, the very famous mathematician Hamilton discovered the quaternion. It is well known that quaternion algebra, denoted by $\mathbb{H}$, is an associative and non-commutative division algebra over the real number field $\mathbb{R}$, where

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k | i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$ 

Since the 1970s, quaternions and the quaternion matrix have been studied a lot (e.g., [13–16]). The widespread applications of quaternions and the quaternion matrix include theoretical mechanics, optics, computer graphics, flight mechanics and aerospace technology, quantum physics, signal processing and so on (e.g., [17–20]).
In the last decade, the study of Sylvester-type matrix equations was extended to $\mathbb{H}$ (e.g., [21–28]). In 2012, Wang and He [29] presented the necessary and sufficient conditions for the Sylvester-type matrix equation
\[ A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \] (3)
to be consistent and derived the expression of its general solution, which can be easily generalized to $\mathbb{H}$. For the Sylvester-type matrix equations with multiple variables and multiple equations, Wang [4] gave a solvability condition and the general solution to the system of Sylvester-type matrix equations
\[ A_3W = B_3, \quad ZC_3 = D_3, \]
\[ A_3W + ZB_5 = D_4. \] (4)

Zhang [30] investigated the necessary and sufficient conditions for the solvability of the following system of Sylvester-like matrix equations
\[ A_1X = B_1, \quad XC_1 = D_1, \]
\[ A_2Y = B_2, \quad YC_2 = D_2, \]
\[ ZC_3 = D_3, \quad A_4V = B_4, \]
\[ A_6V + ZB_6 + A_7XB_7 + A_8YB_8 = D_5, \] (5)
and presented a formula of its general solution. We note that Equations (1)–(5) are the special cases of the following Sylvester-type quaternion matrix equations
\[ A_1X = B_1, \quad XC_1 = D_1, \]
\[ A_2Y = B_2, \quad YC_2 = D_2, \]
\[ A_3W = B_3, \quad ZC_3 = D_3, \]
\[ A_4V = B_4, \]
\[ A_6V + ZB_6 + A_7XB_7 + A_8YB_8 = D_5, \] (6)
where $A_i, B_i, C_j, D_k$ ($i = i, j, k, l, m, n, o, p$) are given matrices over $\mathbb{H}$; $X, Y, Z, V, W$ are unknown.

Motivated by the work mentioned above, in this paper we aim to investigate the solvability conditions and the general solutions to a more general system of a Sylvester-type quaternion matrix equation, Equation (6). In 2011, Took et al. [31] defined a special class of symmetric matrices, named $\eta$-Hermitian. For $\eta \in \{i, j, k\}$, a quaternion matrix $A$ is called $\eta$-Hermitian if $A = A^\eta$, where $A^\eta = -\eta A^*\eta$, $A^*$ is the conjugate and transpose matrix of $A$. It is well known that $\eta$-Hermitian matrices have some applications in linear modeling (e.g., [32–34]) and so on.

As an application of (6), we derive the solvability conditions and an expression of the $\eta$-Hermitian solution to the system of matrix equations
\[ A_4V = B_4, \]
\[ A_1X = B_1, \quad X = X^\eta, \]
\[ A_2Y = B_2, \quad Y = Y^\eta, \]
\[ A_6V + (A_6V)^\eta + A_7XA_7^\eta + A_8YA_8^\eta = D_5, \quad D_5 = D_5^\eta, \] (7)
where $A_i (i = 1, 2, 4, 6, 8)$, $B_1, B_2, B_4, D_5$ are given matrices over $\mathbb{H}$; $X$ and $Y$ are $\eta$-Hermitian matrices over $\mathbb{H}$.

We organize the rest of this article as follows: In Section 2, we introduce the basic knowledge of quaternions and Moore–Penrose inverse of a quaternion matrix, and review some matrix equations. In Section 3, we establish the solvability conditions for the system of (6) in terms of the Moore–Penrose inverses and the ranks of the coefficients' quaternion matrices in (6). In Section 4, we give an expression of the general solution to the system of (6), and illustrate the main results using a numerical example. In Section 5, we give some solvability conditions and an expression of the $\eta$-Hermitian solution to the system (7). Finally, we present a brief conclusion in Section 6 to end this paper.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{H}^{m \times n}$ stand for the real number field and the set of all $m \times n$ matrix spaces over the quaternion algebra, respectively. The symbols $r(A)$, $A^*$, $I$ and 0 are denoted by the rank of a given quaternion matrix $A$, the conjugate transpose of $A$, an identity matrix, and a zero matrix with appropriate sizes, respectively. The Moore–Penrose inverse of $A \in \mathbb{H}^{l \times k}$ is defined to be the unique matrix, denoted by $A^\dagger$, satisfying

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, (A^\dagger A)^* = A^\dagger A.$$  

Moreover, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ represent two projectors. Clearly, $(L_A)^* = R_A$ and $(R_A)^* = L_A$ of $A$.

The following lemma was given by Marsaglia and Stynan [35], which is also available over $\mathbb{H}$.

**Lemma 1** ([35]). Let $A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times k}, C \in \mathbb{H}^{l \times n}, D \in \mathbb{H}^{j \times k}$ and $E \in \mathbb{H}^{l \times i}$. Then,

$$r \left( \begin{array}{cc} A & BLD \\ R \bar{E}C & 0 \end{array} \right) = r \left( \begin{array}{ccc} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{array} \right) - r(D) - r(E).$$

**Lemma 2** ([36]). Let $A_1$ and $C_1$ be known matrices with feasible dimensions over $\mathbb{H}$. Then, the matrix equation $A_1 X = C_1$ has a solution if and only if $R_{A_1} C_1 = 0$. In this case, its general solution is expressed as

$$X = A_1^\dagger C_1 + L_{A_1} T_1,$$

where $T_1$ is an arbitrary matrix of an appropriate size.

**Lemma 3** ([36]). Let $B_1$ and $D_1$ be known matrices with allowable dimensions over $\mathbb{H}$. Then, the matrix equation $Y B_1 = D_1$ has a solution if and only if $D_1 L_{B_1} = 0$. In this case, its general solution is

$$Y = D_1 B_1^\dagger + T_2 R_{B_1},$$

where $T_2$ is an arbitrary matrix of an appropriate size.

**Lemma 4** ([37]). Let $A_1, B_1, C_1$ and $C_2$ be the given matrices. Then, the system of matrix equations

$$A_1 Y = C_1, Y B_1 = C_2$$
is consistent if and only if

\[ R_{A_1}C_1 = 0, \ C_2L_{B_1} = 0, \ A_1C_2 = C_1B_1. \]

In this case, its general solution is

\[ Y = A_1^tC_1 + L_{A_1}C_2B_1^t + L_{A_1}T_3R_{B_1}, \]

where \( T_3 \) is an arbitrary matrix of an appropriate size.

**Lemma 5** ([10]). Let \( A, B \) and \( C \) be given over \( \mathbb{H} \). Then, the Equation (1) is solvable if and only if \( R_A CL_B = 0 \). Under this condition, the general solution to Equation (1) can be expressed as

\[ X = A^tC - U_1B + L_AU_2, \]

\[ Y = R_ACB^t + AU_1 + U_3R_B, \]

where \( U_1, U_2 \) and \( U_3 \) are arbitrary matrices with appropriate sizes over \( \mathbb{H} \).

**Lemma 6** ([38]). Consider the following matrix equation over \( \mathbb{H} \)

\[ A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B, \]  \hspace{1cm} (8)

where \( A_i, B_i \) (\( i = 1, 4 \)), \( B \) are given and the others are unknown. Let

\[
\begin{align*}
R_{A_1}A_2 = A_{11}, & \quad R_{A_1}A_3 = A_{22}, & \quad R_{A_1}A_4 = A_{33}, & \quad B_2L_{B_1} = B_{11}, & \quad B_{22}L_{B_1} = N_{11}, \\
B_3L_{B_1} = B_{22}, & \quad B_4L_{B_1} = B_{33}, & \quad R_{A_1}A_{22} = M_1, & \quad S_1 = A_{22}L_{M_1}, & \quad R_{A_1}B_1L_{B_1} = T_1, \\
C &= R_{A_1}A_{11}, & \quad C_1 = CA_{33}, & \quad C_2 = R_{A_1}A_{33}, & \quad C_3 = R_{A_2}A_{33}, & \quad C_4 = A_{33}, \\
D &= L_{B_1}L_{N_1}, & \quad D_1 = B_{33}, & \quad D_2 = B_{33}L_{B_2}, & \quad D_3 = B_{33}L_{B_1}, & \quad D_4 = B_{33}D, \\
E_1 = CT_1, & \quad E_2 = R_{A_1}T_1L_{B_2}, & \quad E_3 = R_{A_2}T_1L_{B_1}, & \quad E_4 = T_1D, \\
C_{11} &= (L_{C_2}, L_{C_4}), & \quad D_{11} &= \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, & \quad C_{22} &= L_{C_1}, & \quad D_{22} &= R_{D_2}, \quad C_{33} &= L_{C_3}, \\
D_{33} &= R_{D_4}, & \quad E_{11} &= R_{C_{11}}C_{22}, & \quad E_{22} &= R_{C_{11}}C_{33}, & \quad E_{35} &= D_{22}L_{D_{11}}, & \quad E_{44} &= D_{33}L_{D_{11}}, \\
M &= R_{E_1}E_{22}, & \quad N &= E_{44}L_{E_{33}}, & \quad F &= F_2 - F_1, & \quad E &= R_{C_{11}}FL_{D_{11}}, & \quad S &= E_{22}L_{M}, \\
F_{11} &= C_2L_{C_1}, & \quad G_1 &= E_2 - C_2C_1^tE_1D_1^tD_2, & \quad F_{22} &= C_4L_{C_3}, & \quad G_2 &= E_4 - C_4C_3^tE_3D_3^tD_4, \\
F_1 &= C_1^tE_1D_1^t + L_{C_1}C_2^tE_2D_2^t, & \quad F_2 &= C_3^tE_3D_3^t + L_{C_3}C_4^tE_4D_4^t.
\end{align*}
\]

Then, the following statements are equivalent:

(1) Equation (8) is consistent.

(2) \( R_{C_i}E_i = 0, \ E_iL_{D_i} = 0 \) (\( i = 1, 4 \)), \( R_{E_{22}}EL_{E_{33}} = 0 \).
\[
\begin{align*}
  r \left( \begin{array}{ccccc}
    B & A_2 & A_3 & A_4 & A_1 \\
    B_1 & 0 & 0 & 0 & 0
  \end{array} \right) &= r(B_1) + r \left( \begin{array}{ccc}
    A_2 & A_3 & A_4 \\
  \end{array} A_1 \right), \\
  r \left( \begin{array}{cccc}
    B & A_2 & A_4 & A_1 \\
    B_3 & 0 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    A_2 & A_4 & A_1 \\
  \end{array} B_3 \right), \\
  r \left( \begin{array}{cccc}
    B & A_3 & A_4 & A_1 \\
    B_2 & 0 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    A_3 & A_4 & A_1 \\
  \end{array} B_2 \right), \\
  r \left( \begin{array}{cccc}
    B & A_4 & A_1 \\
    B_2 & 0 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    B_2 \\
  \end{array} B_3 \right) + r \left( \begin{array}{ccc}
    A_4 & A_1 \\
  \end{array} B_1 \right), \\
  r \left( \begin{array}{ccccc}
    B & A_2 & A_3 & A_1 \\
    B_4 & 0 & 0 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    A_2 & A_3 & A_1 \\
  \end{array} B_4 \right), \\
  r \left( \begin{array}{cccc}
    B & A_3 & A_1 \\
    B_2 & 0 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    B_2 \\
  \end{array} B_3 \right) + r \left( \begin{array}{ccc}
    A_3 & A_1 \\
  \end{array} B_1 \right), \\
  r \left( \begin{array}{ccc}
    B & A_1 \\
    B_2 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    B_2 \\
  \end{array} B_3 \right) + r(A_1), \\
  r \left( \begin{array}{cccc}
    B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\
    B_3 & 0 & 0 & 0 & 0 & 0 & 0
  \end{array} \right) &= r \left( \begin{array}{ccc}
    B_3 & 0 \\
  \end{array} B_1 \right) + r \left( \begin{array}{cccc}
    A_2 & A_1 & 0 & 0 & A_4 \\
    0 & 0 & A_3 & A_1 & A_4 \\
  \end{array} B_4 \right).
\end{align*}
\]

In this case, the general solution to Equation (8) can be expressed as
\[ X_1 = A_1^t (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) - A_1^t U_1 B_1 + L_{A_1} U_2, \]
\[ X_2 = R_{A_1} (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) B_1^t + A_1 A_1^t U_1 + U_3 R_{B_1}, \]
\[ Y_1 = A_1^t T B_1^t - A_1^t A_2 M_1^t T B_1^t - A_1^t S_1 A_2^t T N_1^t B_2 B_1^t - A_1^t S_1 U_4 R_{N_1} B_2 B_1^t + L_{A_1} U_5 + U_6 R_{B_1}, \]
\[ Y_2 = M_1^t T B_2^t + S_1^t S_1^t A_2^t T N_1^t + L_{M_1} L_{S_1} U_7 + U_8 R_{B_2} + L_{M_1} U_4 R_{N_1}, \]
\[ Y_3 = F_1 + L_{C_1} V_1 + V_2 R_{D_1} + L_{C_1} V_3 R_{D_2}, \text{ or } Y_3 = F_2 - L_{C_1} W_1 - W_2 R_{D_3} - L_{C_3} W_3 R_{D_4}. \]

where \( T = T_1 - A_{33} Y_3 B_{33}; U_i(i = \overline{1,8}) \) represents any matrix with appropriate dimensions over \( \mathbb{H} \),
\[ V_1 = (I_m 0) \left[ C_{11}^t (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) - C_{11}^t U_{11 D_1} + L_{C_1 U_{12}} \right], \]
\[ W_1 = (0 I_n) \left[ C_{11}^t (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) - C_{11}^t U_{11 D_1} + L_{C_1 U_{12}} \right], \]
\[ W_2 = [ R_{C_1} (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) D_{11}^t + C_{11} U_{11} + U_{21} R_{D_1} ] \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \]
\[ V_2 = [ R_{C_1} (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) D_{11}^t + C_{11} U_{11} + U_{21} R_{D_1} ] \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \]
\[ V_3 = E_{11}^t F E_{33}^t - E_{11}^t E_{22}^t M^t F E_{33}^t - E_{11}^t S E_{22}^t N^t E_{44}^t F_{33}^t - E_{11}^t S U_{31} R_{N} E_{44}^t F_{33}^t + L_{E_1} U_{32} + U_{33} R_{F_{33}}, \]
\[ W_3 = M^t F E_{44}^t + S^t U_{22} F E_{33}^t + L_{M} L_{S} U_{41} + L_{M} U_{41} R_{N} - U_{42} R_{E_{44}}. \]

where \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, U_{42} \) are any matrix with appropriate dimensions over \( \mathbb{H} \).

3. Solvability Conditions to the System (6)

The goal of this section is to give the necessary and sufficient conditions for the existence of a solution to system (6).

**Theorem 1.** Let \( A_i \in \mathbb{H}^{m_i \times n_i} (i = \overline{1,4}), A_5 \in \mathbb{H}^{m \times m}, A_6 \in \mathbb{H}^{m \times n}, A_7 \in \mathbb{H}^{m \times n}, A_8 \in \mathbb{H}^{m \times n}, B_j \in \mathbb{H}^{m \times n} (j = \overline{1,2}), B_3 \in \mathbb{H}^{m_3 \times q}, B_4 \in \mathbb{H}^{m_4 \times l}, B_5 \in \mathbb{H}^{l_3 \times q}, B_6 \in \mathbb{H}^{l_3 \times l}, B_7 \in \mathbb{H}^{l_3 \times l}, B_8 \in \mathbb{H}^{l_3 \times l}, C_k \in \mathbb{H}^{l_k \times l}) (k = \overline{1,3}), D_j \in \mathbb{H}^{m_j \times p_j} (j = \overline{1,2}), D_3 \in \mathbb{H}^{m_3 \times l}, D_4 \in \mathbb{H}^{m_4 \times q} \) and \( D_5 \in \mathbb{H}^{m_5 \times l}. \) Set
\[ A_{11} = A_2 L_{A_3}, B_{11} = R_{C_3} B_5, C_{11} = D_4 - A_4 A_3^t B_3 - D_3 C_3^t B_5, A_{22} = A_6 L_{A_4}, \]
\[ B_{22} = R_{B_1} R_{C_3} B_5, A_{33} = A_7 L_{A_7}, B_{33} = R_{C_1} B_7, A_{44} = A_8 L_{A_2}, B_{44} = R_{C_2} B_8, \]
\[ A_{55} = A_{11}, B_{55} = R_{C_3} B_6, M_1 = R_{A_{22}} A_{33}, M_2 = R_{A_{22}} A_{44}, M_3 = R_{A_{22}} A_{55}, \]
\[ C_{22} = D_3 - A_6 A_3^t B_4 - D_3 C_3^t B_6 - R_{A_{11}} C_{11} B_3^t R_{C_3} B_6, \]
\[ - A_7 (A_1^t B_1 + L_{A_1} D_1 C_1^t) B_7 - A_8 (A_1^t B_2 + L_{A_2} D_2 C_1^t) B_8, \]
\[ N_1 = B_{33} L_{B_{22}}, N_2 = B_{44} L_{B_{22}}, N_3 = B_{55} L_{B_{22}}, G_1 = N_2 L_{N_1}, H_1 = R_{M_1} M_2, \]
\[ S_1 = M_2 L_{H_1}, T = R_{A_{22}} C_{22} L_{B_{22}}, P = R_{H_1} R_{M_1}, P_1 = P M_3, P_2 = R_{M_1} M_3, \]
\[ P_3 = R_{M_2} M_3, P_4 = M_3, Q = L_{N_1} L_{G_1}, Q_1 = N_3, Q_2 = N_3 L_{N_2}, Q_3 = N_3 L_{N_1}, \]
\[ Q_4 = N_3 Q, E_1 = P T, E_2 = R_{M_1} T L_{N_2}, E_3 = R_{M_2} T L_{N_1}, E_4 = T Q, \]
\[ E_{11} = (L_{P_1}, L_{P_2}), F_{11} = \begin{pmatrix} R_{Q_1} \\ R_{Q_3} \end{pmatrix}, E_{22} = L_{P_1}, F_{22} = R_{Q_4}, E_{33} = L_{P_1}, F_{33} = R_{Q_4}, \]
\[ M_{11} = R_{E_1} E_{22}, \quad M_{22} = R_{E_{11}} E_{33}, \quad M_{33} = F_{22} L_{F_{11}}, \quad M_{44} = F_{33} L_{F_{11}}, \quad M = R_{M_{11}} M_{22}, \]  
\[ N = M_{44} L_{M_{33}}, \quad F = F_2 - F_1, \quad E = R_{E_{11}} F L_{F_{11}}, \quad S = M_{22} L_{M_{11}}, \quad G_{11} = P_2 L_{P_1}, \]  
\[ H_{11} = E_2 - P_2 P_1^t E_1 Q_1^t Q_2, \quad G_{22} = P_4 L_{P_3}, \quad H_{22} = E_4 - P_4 P_3^t E_3 Q_3^t Q_4, \]  
\[ F_2 = P_1^t E_1 Q_1^t + L_{P_1} P_2^t E_2 Q_2, \quad F_1 = P_3^t E_3 Q_3^t + L_{P_3} P_4^t E_4 Q_4^t. \]

Then, the following statements are equivalent:

1. System (6) has a solution.

2.
\[ A_1 D_{1} = B_1 C_1, \quad A_2 D_{2} = B_2 C_2 \]  

and

\[ R_{A_1} B_1 = 0, \quad D_1 L_{C_1} = 0, \quad R_{A_2} B_2 = 0, \quad D_2 L_{C_2} = 0, \]
\[ R_{A_3} B_3 = 0, \quad D_3 L_{C_3} = 0, \quad R_{A_4} B_4 = 0, \quad R_{A_1} C_{11} L_{B_{11}} = 0, \]
\[ R_{P_1} E_i = 0, \quad E_i L_{Q_i} = 0 \quad (i = 1, 4), \quad R_{M_{22}} E L_{M_{33}} = 0. \]

3. (22) holds and

\[ r(A_1 B_1) = r(A_1) r \left( \begin{array}{c} C_1 \\ D_1 \end{array} \right) = r(C_1), \quad r(A_2 B_2) = r(A_2) r \left( \begin{array}{c} C_2 \\ D_2 \end{array} \right) = r(C_2), \]
\[ r(A_3 B_3) = r(A_3), \quad r \left( \begin{array}{c} C_3 \\ D_3 \end{array} \right) = r(C_3), \quad r(A_4 B_4) = r(A_4), \]
\[ r \left( \begin{array}{ccc} D_4 & A_3 & D_3 \\ B_2 & 0 & C_3 \\ B_3 & A_3 & 0 \end{array} \right) = r \left( \begin{array}{c} A_5 \\ A_3 \end{array} \right) + r \left( \begin{array}{c} B_5 \\ C_3 \end{array} \right), \]

\[ r \left( \begin{array}{ccc} D_5 & A_7 & A_8 \\ B_6 & 0 & 0 \\ B_1 B_7 & A_1 & 0 \\ B_2 B_8 & 0 & A_2 \\ B_4 & 0 & 0 \\ 0 & 0 & 0 \\ B_3 & A_3 & 0 \end{array} \right) \]
\[ r \left( \begin{array}{ccc} D_5 & A_7 & A_6 \\ B_8 & 0 & 0 \\ B_1 B_7 & A_1 & 0 \\ B_4 & 0 & 0 \\ 0 & 0 & 0 \\ B_3 & A_3 & 0 \end{array} \right) \]
\[ r \left( \begin{array}{ccc} D_5 & A_8 & A_6 \\ B_7 & 0 & 0 \\ B_2 B_8 & A_2 & 0 \\ B_4 & 0 & 0 \\ 0 & 0 & 0 \\ B_3 & A_3 & 0 \end{array} \right) \]
\[
\begin{align*}
&\frac{D_5 A_6 A_7 D_1 A_8 D_2 D_4 A_5 D_3}{B_7 0 \ C_1 0 0 0 0}
\begin{pmatrix}
B_7 & 0 & C_1 & 0 & 0 & 0 & 0 \\
B_8 & 0 & 0 & C_2 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & C_3 & 0 \\
B_4 & A_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & B_3 & A_3 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&= r \begin{pmatrix}
B_7 & C_1 & 0 & 0 & 0 \\
B_8 & 0 & C_2 & 0 & 0 \\
B_6 & 0 & 0 & 0 & C_3 \\
B_4 & A_4 & 0 & 0 & 0 \\
\end{pmatrix}
+ r \begin{pmatrix}
A_6 & A_5 \\
A_4 & 0 \\
0 & A_3 \\
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
&\frac{D_5 A_7 A_8 A_6 D_3}{B_8 0 \ C_2 0 0}
\begin{pmatrix}
B_7 & 0 & 0 & 0 & C_3 \\
B_1 B_7 & A_1 & 0 & 0 & 0 \\
B_2 B_8 & 0 & A_2 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&= r \begin{pmatrix}
D_5 & A_7 & A_8 & A_6 & D_3 \\
B_8 & 0 & 0 & 0 & C_3 \\
B_6 & 0 & 0 & 0 & C_3 \\
B_4 & 0 & A_4 & 0 & 0 \\
\end{pmatrix}
+ r \begin{pmatrix}
A_7 & A_8 & A_6 \\
A_1 & 0 & 0 \\
0 & A_2 & 0 \\
0 & 0 & A_4 \\
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
&\frac{D_5 A_8 A_7 D_1 A_6 D_2 D_3}{B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0}
\begin{pmatrix}
B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
B_2 B_8 & A_2 & 0 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&= r \begin{pmatrix}
D_5 & A_8 & A_7 & D_1 & A_6 & D_2 & D_3 \\
B_7 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
+ r \begin{pmatrix}
A_8 & A_6 \\
A_7 & A_6 \\
A_1 & 0 \\
0 & A_4 \\
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
&\frac{D_5 A_6 A_7 D_1 A_8 D_2 D_3}{B_7 & 0 & C_1 & 0 & 0}
\begin{pmatrix}
B_7 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&= r \begin{pmatrix}
D_5 & A_7 & A_8 & A_6 & D_2 & D_4 & A_5 & D_3 \\
B_7 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_8 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
B_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_7 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 \\
B_8 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
B_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\]
\[
\begin{pmatrix}
B_6 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 \\
B_5 & 0 & 0 & B_5 & 0 & 0 & C_3 & 0 \\
0 & B_6 & 0 & 0 & 0 & B_5 & 0 & 0 \\
B_6 & B_6 & 0 & 0 & 0 & 0 & C_3 & 0
\end{pmatrix}
= r
\begin{pmatrix}
A_7 & A_6 & 0 & 0 & A_5 & 0 \\
0 & 0 & A_8 & A_6 & 0 & A_5 \\
A_1 & 0 & 0 & 0 & 0 & 0 \\
0 & A_4 & 0 & 0 & 0 & 0 \\
0 & 0 & A_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_4 & 0 \\
0 & 0 & 0 & 0 & 0 & A_3 \\
0 & 0 & 0 & 0 & 0 & A_3
\end{pmatrix}
+ r. \tag{35}
\]

**Proof.** (1) ⇔ (2)

The proof is divided into three parts:

- Firstly, we divide the system (6) into the following:

\[
A_3 W = B_3, \quad Z C_3 = D_3, \quad A_4 V = B_4,
\]

\[
A_1 X = B_1, \quad X C_1 = D_1, \quad A_2 Y = B_2, \quad Y C_2 = D_2,
\]

\[
A_5 Z + W B_5 = D_4,
\]

\[
A_6 V + Z B_6 + A_7 X B_7 + A_8 Y B_8 = D_5,
\]

and consider the solvability conditions and the general solution to the system of matrices of Equation (36). For more information, see Step 1.

- Secondly, substituting the W and Z obtained in the first step into Equation (37) yields

\[
A_{11} T_3 + T_4 B_{11} = C_{11}, \tag{39}
\]

where \(A_{11}, B_{11}\) and \(C_{11}\) are defined by (9); \(T_3\) and \(T_4\) are unknowns. For more information, see Step 2.

- Finally, by substituting the \(X, Y, Z,\) and \(V\) obtained from the above two steps into Equation (38), we obtain a matrix equation with the following form

\[
A_{22} T_3 + U_3 B_{22} + A_{33} T_1 B_{33} + A_{44} T_2 B_{44} + A_{55} U_1 B_{55} = C_{22}, \tag{40}
\]

where \(A_{ii}, B_{ii}\) (\(i = 2, 5\)) and \(C_{22}\) are given by (9)–(12); \(T_1, T_2, T_5, U_1\) and \(U_3\) are unknowns. For more information, see Step 3.

We can obtain the results from the following steps: First, we consider the solvability conditions and the expression of the general solutions to the system of the matrix Equation (36).

**Step 1.** It follows from Lemmas 2–4 that system (36) has a solution if and only if (22) holds and

\[
R_{A_1} B_1 = 0, \quad D_1 L_{C_1} = 0, \quad R_{A_2} B_2 = 0, \quad D_2 L_{C_2} = 0,
\]

\[
R_{A_3} B_3 = 0, \quad D_3 L_{C_3} = 0, \quad R_{A_4} B_4 = 0. \tag{41}
\]

In this case, the general solution to system (36) can be written as

\[
X = A_1^L B_1 + L_{A_1} D_1 C_1^L + L_{A_1} T_1 R_{C_1},
\]

\[
Y = A_2^L B_2 + L_{A_2} D_2 C_2^L + L_{A_2} T_2 R_{C_2},
\]

\[
W = A_3^L B_3 + L_{A_3} T_3, \quad Z = D_3 C_3^L + T_4 R_{C_3}, \quad V = A_4^L B_4 + L_{A_4} T_5,
\]

where \(T_i (i = 1, 5)\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.
Step 2. Substituting \( W, Z \) in (42) into (37) yields (39). According to Lemma 5, it follows that Equation (39) has a solution if and only if
\[
R_{A_{11}}C_{11}L_{B_{11}} = 0.
\] (43)

In this case, the general solution to Equation (39) can be expressed as
\[
T_3 = A_{11}^t C_1 - U_1 B_{11} + L_{A_{11}} U_2,
\] (44)
\[
T_4 = R_{A_{11}} C_{11} B_{11}^t + A_{11} U_1 + U_3 R_{B_{11}},
\] (45)
where \( U_1, U_2 \) and \( U_3 \) are any matrix with appropriate sizes over \( \mathbb{H} \).

Substituting (45) into \( Z = D_3 C_3^t + T_4 R_{C_3} \) yields
\[
Z = D_3 C_3^t + R_{A_{11}} C_{11} B_{11}^t R_{C_3} + A_{11} U_1 R_{C_3} + U_3 R_{B_{11}} R_{C_3}.
\] (46)

Step 3. By substituting \( X, Y, V \) in (42) and \( Z \) in (46) into (38), we obtain Equation (40). By using Lemma 6, Equation (40) is consistent if and only if
\[
R_{P_i} E_i = 0, \ E_i L_{Q_i} = 0 (i = 1, 4), \ R_{M_{22}} E L_{M_{33}} = 0,
\] (47)

namely,
\[
\begin{pmatrix}
C_{22} & A_{33} & A_{44} & A_{55} & A_{22} \\
B_{22} & 0 & 0 & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
B_{22}
\end{pmatrix} + r \begin{pmatrix}
A_{33} & A_{44} & A_{55} & A_{22}
\end{pmatrix},
\] (48)
\[
\begin{pmatrix}
C_{22} & A_{33} & A_{55} & A_{22} \\
B_{44} & 0 & 0 & 0 \\
B_{22} & 0 & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
A_{33} & A_{55} & A_{22}
\end{pmatrix} + r \begin{pmatrix}
B_{44} & B_{22}
\end{pmatrix},
\] (49)
\[
\begin{pmatrix}
C_{22} & A_{44} & A_{55} & A_{22} \\
B_{33} & 0 & 0 & 0 \\
B_{22} & 0 & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
A_{44} & A_{55} & A_{22}
\end{pmatrix} + r \begin{pmatrix}
B_{33} & B_{22}
\end{pmatrix},
\] (50)
\[
\begin{pmatrix}
C_{22} & A_{55} & A_{22} \\
B_{33} & 0 & 0 \\
B_{44} & 0 & 0 \\
B_{22} & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
B_{33} & B_{44} & B_{22}
\end{pmatrix} + r \begin{pmatrix}
A_{55} & A_{22}
\end{pmatrix},
\] (51)
\[
\begin{pmatrix}
C_{22} & A_{33} & A_{44} & A_{22} \\
B_{55} & 0 & 0 & 0 \\
B_{22} & 0 & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
A_{33} & A_{44} & A_{22}
\end{pmatrix} + r \begin{pmatrix}
B_{55} & B_{22}
\end{pmatrix},
\] (52)
\[
\begin{pmatrix}
C_{22} & A_{33} & A_{22} \\
B_{44} & 0 & 0 \\
B_{55} & 0 & 0 \\
B_{22} & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
B_{44} & B_{55} & B_{22}
\end{pmatrix} + r \begin{pmatrix}
A_{33} & A_{22}
\end{pmatrix},
\] (53)
\[
\begin{pmatrix}
C_{22} & A_{44} & A_{22} \\
B_{33} & 0 & 0 \\
B_{44} & 0 & 0 \\
B_{22} & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
B_{33} \\
B_{55} \\
B_{44} \\
B_{22}
\end{pmatrix} + r \begin{pmatrix}
A_{44} & A_{22}
\end{pmatrix},
\]
(54)

\[
\begin{pmatrix}
C_{22} & A_{22} \\
B_{33} & 0 \\
B_{44} & 0 \\
B_{22} & 0
\end{pmatrix} = r \begin{pmatrix}
B_{33} \\
B_{44} \\
B_{22}
\end{pmatrix} + r(A_{22}),
\]
(55)

\[
\begin{pmatrix}
C_{22} & A_{33} & A_{22} \\
B_{44} & 0 & 0 \\
B_{22} & 0 & 0 \\
B_{55} & 0 & 0
\end{pmatrix} = r \begin{pmatrix}
B_{44} & 0 & 0 \\
B_{22} & 0 & 0 \\
B_{33} & 0 & 0
\end{pmatrix} + r \begin{pmatrix}
A_{33} & A_{22} & A_{55} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
(56)

In this case, the general solution to Equation (40) can be expressed as

\[T_3 = A_{22}^T(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55}) + A_{22}^TV_1B_{22} + L_{A_{22}}V_2,
\]
\[U_3 = R_{A_{22}}(C_{22} - A_{33}T_1B_{33} - A_{44}T_2B_{44} - A_{55}U_1B_{55})B_{22} + A_{22}^TA_{22}V_1 + V_3R_{B_{22}},
\]
\[T_1 = M_1^T T_{11} N_1^T - M_1^T M_2 H_1^T T_{11} N_1^T - M_1^T S_1^T M_1^T G_1^T N_2 N_1^T - M_1^T S_1^T V_4 R_G N_2 N_1^T + L_{M_1} V_5 + V_6 R_N,
\]
\[T_2 = H_1^T T_{11} N_2^T + S_1^T S_1^T M_2^T T_{11} G_1^T + L_{H_1} L_{S_1} V_7 + V_8 R_N + L_{H_1} V_4 R_G,
\]
\[U_1 = F_1 + L_{P_1} W_1 + W_2 R_{Q_1} + L_{P_1} W_3 R_{Q_2}, \text{ or } U_1 = F_2 - L_{P_1} W_4 - W_5 R_{Q_3} - L_{P_2} W_6 R_{Q_4},
\]

where \(T_{11} = T - M_3 U_1 N_3, V_i(i = 1, 8)\) are any matrix with suitable dimensions over \(\mathbb{H}\),

\[W_1 = \begin{bmatrix}
I_m & 0 \\
0 & I_m
\end{bmatrix} \begin{bmatrix}
E_{11}(F - E_{22} W_{33} W_{55} W_{33}) - E_{11}^T U_1 F_{11} + L E_{11} U_{12},
E_{11}(F - E_{22} W_{33} W_{55} W_{33}) - E_{11}^T U_1 F_{11} + L E_{11} U_{12},
\end{bmatrix},
\]
\[W_4 = \begin{bmatrix}
0 & I_m \\
I_m & 0
\end{bmatrix} \begin{bmatrix}
E_{11}(F - E_{22} W_{33} W_{55} W_{33}) - E_{11}^T U_1 F_{11} + L E_{11} U_{12},
E_{11}(F - E_{22} W_{33} W_{55} W_{33}) - E_{11}^T U_1 F_{11} + L E_{11} U_{12},
\end{bmatrix},
\]
\[W_2 = \begin{bmatrix}
R_{E_{11}}(F - E_{22} W_{33} W_{55} W_{33}) F_{11} + E_{11}^T U_1 F_{11} + U_{21} R F_{11},
0
\end{bmatrix},
\]
\[W_5 = \begin{bmatrix}
R_{E_{11}}(F - E_{22} W_{33} W_{55} W_{33}) F_{11} + E_{11}^T U_1 F_{11} + U_{21} R F_{11},
0
\end{bmatrix},
\]
\[W_3 = M_1^T F M_{33} - M_1^T M_2 M_2^T F M_{33} - M_1^T S M_2^T F N^T M_{44} M_{33} - M_1^T S U_3 R_N M_{44} M_{33} + L_{M_1} U_{32} + U_{33} R_{M_3},
\]
\[W_6 = M_1^T F M_{44} + S^T S M_2^T F N^T + L_{M_4} L_{U_4} + L_{M_5} U_3 R_N - U_{42} R_{M_4},
\]

where \(U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41} \) and \(U_{42}\) are any matrix with suitable dimensions over \(\mathbb{H}\).

To sum up, the system of matrices of Equation (6) has a solution if and only if (41), (43) and (47) hold.

(2) \(\Leftrightarrow\) (3) We divide it into three parts to prove its equivalence.
Part 1. In this part, we prove that (41) holds if and only if (24) and (25) hold. According to Lemma 1, it is easy to show that (41) holds if and only if (24) and (25) hold.

Part 2. In this part, we prove that (43) ⇐⇒ (26). It follows from Lemma 1 and elementary operations that

\[
(43) \iff r \begin{pmatrix} C_{11} & A_{11} \\ B_{11} & 0 \end{pmatrix} = r(A_{11}) + r(B_{11})
\]

\[
\iff r \begin{pmatrix} C_{11} & A_3L_{A_3} \\ R_{C_3}B_5 & 0 \end{pmatrix} = r(A_3L_{A_3}) + r(R_{C_3}B_5)
\]

\[
\iff r \begin{pmatrix} D_4 - A_3A_3^tB_3 & D_3C_3^tB_3 & A_3 & 0 \\ B_3 & 0 & C_3 & A_3 \\ 0 & A_3 & 0 \\ B_3 & A_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_3 \\ A_3 \\ A_3 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_5 \\ C_3 \end{pmatrix} \iff (26).
\]

Part 3. In this part, we show that (47) holds if and only if (27) to (35) hold. By using Lemma 6, (47) holds if and only if (48) to (56) hold. Hence, we only show that (48) to (56)

\[
\iff (43) \iff (26).
\]

Note that

\[
X_0 = A_1^tB_1 + L_{A_1}D_1C_1^t, \quad Y_0 = A_2^tB_2 + L_{A_2}D_2C_2^t, \quad Z_0 = D_3C_3^t, \quad V_0 = A_4^tB_4, \quad W_0 = A_3^tB_3
\]

are the special solution to the equations

\[
A_1X = B_1, \quad XC_1 = D_1,
A_2Y = B_2, \quad YC_2 = D_2,
A_3W = B_3, \quad ZC_3 = D_3, \quad A_4V = B_4,
\]

respectively. Then, we have that

\[
C_{11} = D_4 - A_3W_0 - Z_0B_5, \quad (57)
\]

\[
C_{22} = D_5 - A_6V_0 - Z_0B_6 - R_{A_1}C_{11}B_1^tR_{C_3}B_6 - A_7X_0B_7 - A_8Y_0B_6. \quad (58)
\]

It follows from Lemma 1 and elementary operations to (47) that

\[
(48) \iff r \begin{pmatrix} C_{22} & A_7 & A_8 & A_{11} & A_6 & 0 \\ R_{C_3}B_6 & 0 & 0 & 0 & 0 & B_{11} \\ 0 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 \end{pmatrix} = r \begin{pmatrix} C_{22}B_6 & B_{11} \end{pmatrix} + r \begin{pmatrix} A_7 & A_8 & A_{11} & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_4 & 0 \end{pmatrix}
\]

\[
\iff r \begin{pmatrix} D_5 - Z_0B_6 & A_7 & A_8 & A_6 & C_{11} & A_{11} \\ R_{C_3}B_6 & 0 & 0 & 0 & 0 & B_{11} \\ B_1B_7 & A_1 & 0 & 0 & 0 & 0 \\ B_2B_8 & A_2 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & A_4 & 0 \end{pmatrix} = r \begin{pmatrix} A_7 & A_8 & A_{11} & A_6 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_4 & 0 \end{pmatrix} + r \begin{pmatrix} R_{C_3}B_6 & B_{11} \end{pmatrix}
\]
\[
\begin{bmatrix}
D_5 & A_7 & A_8 & A_6 & D_4 & A_5 & D_3 \\
B_6 & 0 & 0 & 0 & B_5 & 0 & C_3 \\
B_1B_7 & A_1 & 0 & 0 & 0 & 0 & 0 \\
B_2B_6 & 0 & A_2 & 0 & 0 & 0 & 0 \\
B_4 & 0 & 0 & A_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & B_3 & A_3 & 0
\end{bmatrix}
\equiv r
\begin{bmatrix}
A_7 & A_8 & A_5 & A_6 \\
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & 0 & A_4 \\
0 & 0 & A_3 & 0
\end{bmatrix}
+ r( B_6, B_5, C_3 ) \equiv (27).
\]

Similarly, we can prove that \( R_{R_2} E_2 = 0 \Leftrightarrow (28) \), \( R_{R_3} E_3 = 0 \Leftrightarrow (29) \), \( R_{R_4} E_4 = 0 \Leftrightarrow (30) \) and \( E_i L_{Q_i} = 0 \ (i = \overline{1,4}) \) hold if and only if (31) to (34) hold, respectively. Next, we show that \( R_{M_{22}} E L_{M_{33}} = 0 \Leftrightarrow (35) \). According to Lemma 1 and elementary operations, we have that

\[
R_{M_{22}} E L_{M_{33}} = 0 \Leftrightarrow r \begin{bmatrix}
E & D_{22} \\
D_{33} & 0
\end{bmatrix} = r(D_{22}) + r(D_{33})
\]

\[
\begin{bmatrix}
C_{22} & A_3 & A_2 & 0 & 0 & 0 & A_{55} \\
B_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
B_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -C_{22} & A_4 & A_2 & A_{55} \\
0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & B_{22} & 0 & 0 & 0 \\
B_{33} & 0 & 0 & B_{55} & 0 & 0 & 0
\end{bmatrix}
\equiv r
\begin{bmatrix}
B_{44} & 0 \\
B_{22} & 0 \\
0 & B_{33} \\
0 & B_{22} \\
B_{33} & B_{55}
\end{bmatrix}
+ r \begin{bmatrix}
A_{33} & A_2 & 0 & 0 & A_{55} \\
0 & 0 & A_4 & A_2 & A_{55}
\end{bmatrix}
\]
\[
\begin{pmatrix}
D_5 - Z_6 B_6 & A_7 & A_6 & 0 & 0 & 0 & A_6 D_2 & D_4 & 0 & 0 & 0 & A_5 & D_3 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_6 & 0 & 0 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Z_5 B_6 - D_5 & A_8 & A_6 & 0 & 0 & -A_7 D_1 & -C_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_1 B_7 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_4 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_2 B_6 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
A_7 & A_6 & 0 & 0 & A_5 & 0 \\
0 & 0 & A_8 & A_6 & 0 & A_5 \\
A_1 & 0 & 0 & 0 & 0 & 0 \\
0 & A_4 & 0 & 0 & 0 & 0 \\
0 & 0 & A_2 & 0 & 0 & 0 \\
0 & 0 & 0 & A_3 & 0 & 0 \\
0 & 0 & 0 & 0 & A_3 & 0 \\
\end{pmatrix}
\]

\[\Leftrightarrow r = D_5 - Z_6 B_6 = A_7 + A_6 + A_5 = A_7 + A_6 + A_5 \Rightarrow (35).\]

4. The General Solution to the System (6)

In this section, we give an expression for the general solution of Equation (6) by using the Moore–Penrose inverse. According to the proof of Theorem 1, we obtain the following theorem:

**Theorem 2.** The general solution to system (6) can be expressed as follows when the solvability conditions are met:

\[
X = A_1^T B_1 + L_{A_1} D_1 C_1^T + L_{A_1} T_1 R_{C_1}, \quad Y = A_2^T B_2 + L_{A_2} D_2 C_2^T + L_{A_2} T_2 R_{C_2},
\]

\[
Z = D_3 C_3^T + R_{A_1} C_1 B_{11}^T R_{C_1} + A_{11} U_1 R_{C_1} + U_3 R_{B_{11}} R_{C_3},
\]

\[
W = A_3^T B_3 + L_{A_3} A_{11} C_1 - L_{A_3} A_{11} U_1 B_{11} + L_{A_3} L_{A_{11}} U_2,
\]

\[
V = A_4^T B_4 + L_{A_4} A_{22}^T (C_{22} - A_{33} T_1 B_{33} - A_{44} T_2 B_{44} - A_{55} U_1 B_{55}) + L_{A_4} A_{22} V_1 B_{22} + L_{A_4} L_{A_{22}} V_2,
\]
where \( T_{11} = T - M_3 U_1 N_3 \), \( V_i(i = \bar{1}, \bar{8}) \) are arbitrary matrices with appropriate sizes.

\[
T_1 = M_1^t T_{11} N_1^t - M_1^t M_2 H_1^t T_{11} N_1^t - M_1^t S_1 M_2^t T_{11} G_1^t N_2 N_3^t - M_1^t S_1 V_4 R G_1 N_2 N_3^t + L M_3 V_5 + V_6 N_i,
\]

\[
T_2 = H_1^t T_{11} N_1^t + S_1 M_2^t T_{11} G_1^t + L H_1 L S_1 V_7 + V_8 R N_5 + L H_1 V_4 R G_i,
\]

\[
U_3 = R A_{i2} (C_{22} - A_{33} T_1 B_{33} - A_{44} T_2 B_{44} - A_{55} U_1 B_{55}) B_{i2}^t + A_{42} A_{42} V_1 + V_3 R B_{22},
\]

\[
U_1 = F_1 + L P_1 W_1 + W_2 R Q_1 + L P_1 W_3 R Q_2, \text{ or } U_1 = F_2 - L P_2 W_4 - W_5 R Q_3 - L P_2 W_6 R Q_4,
\]

\[
W_1 = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} E_{11}^t (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) - E_{11} E_{11}^t U_1 + U_21 R_{F1} \end{bmatrix} \begin{bmatrix} I_n \end{bmatrix},
\]

\[
W_4 = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} E_{11}^t (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) - E_{11} E_{11}^t U_1 + U_21 R_{F1} \end{bmatrix} \begin{bmatrix} I_n \end{bmatrix},
\]

\[
W_5 = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} E_{11}^t (F - E_{22} W_3 F_{22} - E_{33} W_6 F_{33}) - E_{11} E_{11}^t U_1 + U_21 R_{F1} \end{bmatrix} \begin{bmatrix} I_n \end{bmatrix},
\]

\[
W_3 = M_{11}^T F M_{33} + M_{11}^T M_{22}^T F M_{33}^t - M_{11}^T S M_{22}^T F N^T M_{44} M_{33} - M_{11}^T S U_3 R N M_{44} M_{33}^t + L M_1 U_2 + U_3 R M_{33},
\]

\[
W_6 = M_{11}^T F M_{44} + S M_{22}^T F N^t + L M_5 U_4 + U_3 R M_{44},
\]

where \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41} \) and \( U_{42} \) are arbitrary matrices over \( \mathbb{H} \) of appropriate sizes.

Next, we discuss the special cases of the system of matrices of Equation (6). Letting \( A_3, B_3, A_5, B_5 \) and \( D_4 \) vanish yields the following:

**Corollary 1.** Suppose that \( A_j, B_j, C_j, D_j \) \((i = \bar{1}, \bar{3}, j = \bar{1}, \bar{5})\) and \( E_i \) are given, denote

\[
A_6 = A_4 L A_i, \quad B_6 = R_3 B_4, \quad C_6 = C_4 L A_i, \quad D_6 = R_3 B_4, \quad C_7 = C_3 L A_i, \quad D_7 = R_3 B_5,
\]

\[
E_6 = E_1 - A_4 A_1^t C_1 - D_1 B_1^t B_4 - C_4 (A_2 C_2 + L A_2 D_2 B_2^t) D_4 - C_5 (A_4 C_3 + L A_3 D_3 B_3^t) D_5,
\]

\[
A = R A_6 C_6, \quad B = D_6 L B_6, \quad C = R A_6 C_7, \quad D = D_7 L B_6,
\]

\[
E = R A_6 E_5 L B_6, \quad M = R A C, \quad N = D L B, \quad S = C L M.
\]

Then, the following statements are equivalent:

1. System (5) is consistent.
2. \( R A C_i = 0 \), \( D_i L B_i = 0 \) \((i = 1, 2, 3)\), \( A_2 D_2 = C_2 B_2\), \( A_3 D_3 = C_3 B_3\),

\[
R A E = MM^t E, \quad E L B = EN^t N, \quad R A E L D = 0, \quad R C E L B = 0.
\]
where $V$ are equivalent:

The above corollary is from the important findings of [30].

\[
\begin{align*}
\begin{bmatrix} E_1 & A_4 & D_1 & C_4D_2 & C_5D_3 \\ B_4 & 0 & B_1 & 0 & 0 \\ D_4 & 0 & 0 & B_2 & 0 \\ D_5 & 0 & 0 & 0 & B_3 \\ C_1 & A_1 & 0 & 0 & 0 
\end{bmatrix} = \begin{bmatrix} A_1 \\ A_4 \\ D_1 \\ C_4D_2 \\ C_5D_3 
\end{bmatrix} = r\left(\begin{bmatrix} B_1 \\ B_4 \\ D_4 \\ D_5 \\ C_1 
\end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 \\ B_4 \\ D_4 \\ D_5 \\ C_1 
\end{bmatrix}\right) + r\left(\begin{bmatrix} A_1 \\ A_4 \\ D_1 \\ C_4D_2 \\ C_5D_3 
\end{bmatrix}\right).
\end{align*}
\]

In this case, the general solution to system (5) can be expressed as

\[
\begin{align*}
X_1 &= A_1^2C_1 + L_{A_1}U_1, & X_2 &= D_1B_1^1 + U_2R_{B_1}, \\
X_3 &= A_2^2C_2 + L_{A_2}D_2B_1^1 + L_{A_2}U_3R_{B_2}, \\
X_4 &= A_3^2C_3 + L_{A_3}D_3B_1^1 + L_{A_3}U_4R_{B_3}, \\
U_1 &= A_1^0\left(E_6 - C_6U_3D_6 - C_7U_4D_7\right) - A_1^0W_2B_6 + L_{A_1}W_1, \\
U_2 &= R_{A_2}\left(E_6 - C_6U_3D_6 - C_7U_4D_7\right)B_1^1 + A_2^0\lambda_3W_2 + W_3R_{B_3}, \\
U_3 &= A^1EB^t - A^tCM^tEB^t - A^tSC^tEN^tDB^t - A^tSV_4RNDN^t + L_AV_1 + V_2R_{B}, \\
U_4 &= M^tED^t + S^tSC^tEN^t + L_ML_SV_3 + L_MV_4R_N + V_5R_{D},
\end{align*}
\]

where $V_i, W_i (i = 1, 3, j = 1, 3)$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Remark 1. The above corollary is from the important findings of [30].

Letting $A_i, B_i, C_j, D_j (i = 1, 2, 4, 6, 7, 8, j = 1, 2)$ and $D_5$ vanish, we have the following:

Corollary 2. Given $A_3, B_3, C_3, D_3, A_5, B_5$ and $D_4$ of feasible dimensions over $\mathbb{H}$. Set $A_{11} = A_5L_{A_3}, B_{11} = R_{C_3C}$ and $E_{11} = D_4 - A_5A_3^1B_3 - D_3C_3^1B_5$. Then, the following statements are equivalent:

1. System (4) is consistent.
(2) \( r(A_3 \ B_3) = r(A_3), r \left( \begin{array}{c} C_3 \\ D_3 \end{array} \right) = r(C_3), r \left( \begin{array}{c} D_4 \\ A_5 \\ D_3 \\ B_3 \\ A_3 \\ 0 \\ 3 \\ 0 \end{array} \right) = r(A_5) + r(B_5 \ C_3). \)

In this case, the general solution to system (4) can be expressed as

\[
W = A_1^t B_3 + L_{A_3}(A_1^t E_{11} - A_1^t W_2 B_1 + L_{A_3} W_1),
\]

\[
Z = D_3 C_3^t + (R_{A_3} E_{11} B_3^t + A_1^t A_1^t W_2 + W_3 B_{11}) R_{C_3},
\]

where \( W_1, W_2 \) and \( W_3 \) are arbitrary matrices over \( \mathbb{H} \) of appropriate sizes.

**Remark 2.** The above corollary is from the vital investigation of [4].

Finally, we give Algorithm 1 and an example to illustrate the main results of this paper.

**Algorithm 1:** Algorithm for solving Equation (6)

1. Feed the values of \( A_i, B_i, C_j, D_j \) \((i = 1, 8, j = 1, 3, k = 1, 5)\) with congruent shapes over \( \mathbb{H} \).
2. Compute the symbols in (9) to (21).
3. Check (22), (23) or rank equalities in (24) to (35) hold or not. If no, then return “inconsistent.”
4. Otherwise, compute \( X, Y, Z, V, W \).

**Example 1.** Consider the matrix of Equation (6). Assume

\[
A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} 1+j \\ i+k \end{pmatrix}, A_3 = \begin{pmatrix} j+k \\ -i+k \end{pmatrix}, A_4 = \begin{pmatrix} 1+j \\ i+k \end{pmatrix},
\]

\[
A_5 = \begin{pmatrix} i-j/5 \\ -1+i \\ 1-i+j \end{pmatrix}, A_6 = \begin{pmatrix} 1+i+j+k \\ 1 \\ 1+i \\ 0 \end{pmatrix}, A_7 = \begin{pmatrix} i+j \\ 1+i+j+k \\ 1+i+j \end{pmatrix}, A_8 = \begin{pmatrix} i+j \\ 1+i+j+k \\ 1+i+j \end{pmatrix},
\]

\[
B_1 = \begin{pmatrix} 2+i & -i+j & 3 \\ 0 & 1 & 2+i \\ j & i & -1 \\ k & i & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1+i+j+k \\ -1+i+j+k \\ 1-i+j \end{pmatrix}, B_3 = \begin{pmatrix} i+j \\ 1+i+j+k \\ 1+i+j \end{pmatrix}, B_4 = \begin{pmatrix} 2+i+j+k \\ 1+i+j+k \\ 1+i+j+k \end{pmatrix}, B_5 = \begin{pmatrix} i+j \\ 1+i+j+k \\ 1+i+j \end{pmatrix}, B_6 = \begin{pmatrix} i+j \\ 1+i+j+k \\ 1+i+j+k \end{pmatrix},
\]

Computing directly yields

\[
r(A_i \ B_i) = r(A_i) = 2, \ r \left( \begin{array}{c} C_j \\ D_j \end{array} \right) = r(C_j) = 2 \quad (i = \Gamma, 4, j = \Gamma, 3),
\]

\[
(26) = 4, (27) = 10, (28) = 10, (29) = 10, (30) = 10,
\]

\[
(31) = 8, (32) = 8, (33) = 8, (34) = 8, (35) = 24.
\]

All rank equations hold. Thus, according to Theorem 1, the system of matrix equations has a solution, and the general solution to the system can be expressed as
X = \begin{pmatrix} 1 + j & -1 \\ 2 + k & 1 + j \end{pmatrix}, \quad Y = \begin{pmatrix} 1 + i - j & 0 \\ 1 & 2i - j + k \end{pmatrix}, \quad Z = \begin{pmatrix} 1 + i & 1 + k \\ 1 + j + k & i + k \end{pmatrix},
V = \begin{pmatrix} 1 & 2 + i - k \\ 1 - j + k & i + j \end{pmatrix}, \quad W = \begin{pmatrix} j & 3i + j + k \\ 1 + i & i + j - i + 2j \end{pmatrix}.

5. The General Solution to the System (7) with \(\eta\)-Hermicity

As an application of the results of system (6), we study the necessary and sufficient conditions for system (7) to have a solution involving \(\eta\)-Hermicity and derive a formula of its general solution, where \(X, Y\) are \(\eta\)-Hermitian matrices.

**Theorem 3.** Given \(A_i, B_j\) \((i = 1, 2, 7, 8, j = 1, 2, 5, 8)\), \(C_3, D_3, D_4\) of appropriate dimensions over \(\mathbb{C}\). Set
\[
A_{22} = A_6 L_{A_1}, \quad A_{33} = A_7 L_{A_1}, \quad A_{44} = A_8 L_{A_1},
\]
\[
C_{22} = D_3 - A_6^T A_4 B_4 - A_7 A_1 (A_1^T)^{\eta^*} + L_{A_1} C_1^{\eta^*} (C_1^T)^{\eta^*} A_7^* - A_8 A_2 (A_2^T)^{\eta^*} + L_{A_2} C_2^{\eta^*} (C_2^T)^{\eta^*} A_8^*,
\]
\[
M_1 = R_{A_2} A_{33}, \quad M_2 = R_{A_2} A_{44}, \quad T = R_{A_2} C_{22} R_{A_2}^T, \quad M = R M_1 R, \quad S = M_2 L_{A_1}.
\]

Then, the following statements are equivalent:

1. System (7) has a solution.
2. \(R_{A_1} B_1 = 0, \ R_{A_2} B_2 = 0, \ R_{A_3} B_4 = 0, \ R_{M_1} R_M T = 0, \ R_{A_2} T (R_{A_4})^{\eta^*} = 0.\)

3. \(r(A_1) = r(A_3), \ r(A_2) = r(A_4), \ r(A_4) = r(A_4), \ r(A_4) = r(A_4), \)
\[
\begin{pmatrix} D_5 & A_6 & B_4^{\eta^*} & A_7 B_1^{\eta^*} & A_8 B_2^{\eta^*} \\ A_6^* & A_4 & 0 & 0 & 0 \\ A_7^* B_1^{\eta^*} & 0 & A_1^{\eta^*} & 0 & 0 \\ A_8^* B_2^{\eta^*} & 0 & 0 & A_2^{\eta^*} & 0 \\ B_4 & A_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_6^{\eta^*} & A_7^{\eta^*} & 0 & 0 & 0 \\ A_4^{\eta^*} & A_2^{\eta^*} & 0 & 0 & 0 \\ A_1^{\eta^*} & A_8^{\eta^*} & 0 & 0 & 0 \\ A_2^{\eta^*} & 0 & A_4^{\eta^*} & 0 & 0 \\ B_4 & A_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_6 & A_7 \\ A_4 & 0 \\ 0 & A_1 \end{pmatrix} + \begin{pmatrix} A_6 & A_7 \\ A_4 & 0 \\ 0 & A_1 \end{pmatrix}.
\]
Under these conditions, the general solution with \( \eta \)-Hermicity to the system (7) can be stated as

\[
V = A_1^4 B_4 + L_{A_1} U_1,
X = A_1^4 B_1 + L_{A_1} B_1^\eta (A_1^4)^\eta + L_{A_1} U_2 L_{A_1} Y,
Y = A_2^4 B_2 + L_{A_2} B_2^\eta (A_2^4)^\eta + L_{A_2} U_3 L_{A_2} Y,
\]

where \( V_i (i = 1, 3) \) and \( W_j (j = 1, 3) \) are arbitrary matrices with appropriate sizes over \( \mathbb{H} \).

**Proof.** Since the solvability of the system (7) is equivalent to the system

\[
A_4 V_1 = B_4, \quad V_2 (A_4)^\eta = (B_4)^\eta, \quad V_2 = (V_1)^\eta,
A_1 X_1 = B_1, \quad X_1 A_1^\eta = B_1^\eta, \quad X_1 = X_1^\eta,
A_2 Y_1 = B_2, \quad Y_1 A_2^\eta = B_2^\eta, \quad Y_1 = Y_1^\eta,
A_6 V_1 + V_2 A_6^\eta + A_7 X_1 A_7^\eta + A_8 Y_1 A_8^\eta = D_5, \quad D_5 = D_3^\eta.
\]

(59)

If system (7) has a solution, say, \((V, X, Y)\), then system (59) has a solution, \((V_1, V_2, X_1, Y_1) = (V, V^\eta, X, Y)\). Conversely, if system (59) has a solution \((V_1, V_2, X_1, Y_1)\), then

\[
(V, X, Y) = \left( \frac{V_1 + V_2^\eta}{2}, \frac{X_1 + X_1^\eta}{2}, \frac{X_2 + X_2^\eta}{2} \right)
\]

is the solution of (7). It follows from Corollary 1 that this proof can be completed. \(\square\)

6. Conclusions

We established the solvability conditions for system (6) by using the Moore–Penrose inverses and ranks of the coefficient quaternion matrices in (6), and derived a formula of its general solution when it is solvable. In terms of applications, we derived the necessary and sufficient conditions for system (7) to have an \( \eta \)-Hermitian solution as well as the expression of the general solution. In addition, we used an algorithm and a numerical example to verify the main results of this paper. It is worth noting that the main results of (6) are available not only for \( \mathbb{R} \) and \( \mathbb{C} \), but also any division ring. Moreover, inspired by [39], we can investigate the system (6) tensor equations over the quaternion algebra.

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