An Efficient Method for Split Quaternion Matrix Equation

X − Af(X)B = C

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1. Introduction

In this paper, we adopt the following notations.
- \( \mathbb{R} / \mathbb{Q} \): The real number field/the set of split quaternions.
- \( \mathbb{R}^{m \times n} / \mathbb{Q}^{m \times n} \): The set of all \( m \times n \) real matrices/split quaternion matrices.
- \( \mathbb{B} \mathbb{R}^{m \times n} / \mathbb{S} \mathbb{B} \mathbb{R}^{m \times n} / \mathbb{B} \mathbb{Q}^{m \times n} / \mathbb{S} \mathbb{B} \mathbb{Q}^{m \times n} \): The set of all \( n \times n \) real bisymmetric matrices/real skew bisymmetric matrices/split quaternion bisymmetric matrices/split quaternion skew bisymmetric matrices.
- \( I_n \): The unit matrix with order \( n \).
- \( \delta_{ik} \): The \( k \)-th column of \( I_n \).
- \( A^\dagger / A^T / A^\dagger \): The Moore–Penrose inverse/transpose/conjugate/conjugate transpose of \( A \), where \( A \in \mathbb{Q}^{m \times n} \).
- \( || \cdot || \): The Frobenius norm of a matrix or Euclidean norm of a vector.
- \( \times / \odot \): The semi-tensor product of matrices/Kronecker product of matrices.

In 1849, the British mathematician, Jame Cockle, introduced the split quaternion. As one of the emerging research topics, the split quaternion has important applications in the fields of classical mechanics, quantum mechanics and so on [1,2]. On the geometric theory, the rotation of a three-dimensional Minkowski space can be represented by a split quaternion [3,4]. A split quaternion \( q \in \mathbb{Q}_3 \) is represented as \( q = q_1 + q_2i + q_3j + q_4k \), in which \( q_1, q_2, q_3, q_4 \in \mathbb{R} \), and three imaginary units \( i, j, k \) satisfy \( -i^2 = j^2 = k^2 = ijk = 1, \ ij = -ji = k, \ jk = -kj = -i \) and \( ki = -ik = j \). As a generalization to complex numbers, the split quaternion do not satisfy the commutative law, which makes many problems associated with the split quaternion more complicated.

For any matrix \( X \in \mathbb{Q}^{m \times n}_n \), \( X \) can be uniquely expressed as \( X = X_1 + X_2i + X_3j + X_4k \), in which \( X_1, X_2, X_3, X_4 \in \mathbb{R}^{m \times n} \). Let \( X = X_1 - X_2i - X_3j - X_4k \) be the conjugate of \( X \), \( X^H = X_1 - X_2i - X_3j - X_4k \).
$X_1^T - X_2^T i - X_3^T j - X_4^T k$ be the conjugate transpose of $X$, and the $\eta$-conjugate transpose of $X$, $X^{\eta H}(\eta = i, j, k)$ is as follows:

$X^{i H} = -i(X^H)i = X_1^T - X_2^T i + X_3^T j + X_4^T k,$

$X^{j H} = j(X^H)j = X_1^T + X_2^T i - X_3^T j + X_4^T k,$

$X^{k H} = k(X^H)k = X_1^T + X_2^T i + X_3^T j - X_4^T k.$

As a branch of matrix theory, the matrix equation plays an important role in neural network [5], control theory [6], color image processing [7], stability theory and so on. The matrix equation solving problem has important practical value and theoretical significance, which many scholars have been widely concerned by, and which has obtained many valuable results. Liu et al. [8] and Mehany et al. [9] established the solvability conditions of the proposed method. Finally, some conclusions are put in Section 6.

Now, we turn our attention to the split quaternion matrix equation. Some research extended the results of quaternion matrix equations to the split quaternion equations [15–17]. Ling et al. [10] employed matrix LSQR algorithm to deal with quaternionic least squares problem. For the Stein matrix equation $X - AXB = C$, owing to their important applications in control theory, communication theory, neural network and image restoration, there has been an increased interest in solving them in recent years. In the real field, Zhou et al. [11] studied the iterative solution to the Stein matrix equation. In the complex field, Jiang et al. [12] derived the explicit solution of the complex matrix equation $X - AXB = C$ by means of the characteristic polynomial. By using the complex representation and real representation matrices of quaternion matrices, Yuan et al. [13] and Zhang et al. [14] derived the minimal norm least squares solution of the quaternion matrix equation $X - AXB = C$ ($\hat{X}$, which is the $j$-conjugate matrix of quaternion matrix $X$), respectively.

This paper is organized as follows. In Section 2, we recall and obtain some preliminary results that will be used in the paper. In Section 3, we introduce the $H$ representation and study its properties in the bisymmetric and skew bisymmetric split quaternion matrix. In Section 4, we derive the solution of Problems 1 and 2 by using a real representation and $H$ representation method. In Section 5, we provide numerical algorithms for solving Problems 1 and 2. Then, we present two numerical examples to illustrate the efficiency and accuracy of the proposed method. Finally, some conclusions are put in Section 6.
2. Preliminaries

In this section, the real representation of split quaternion, the swap matrix in semi-tensor product of matrix, the related theorems and basic knowledge of this paper are introduced. Different from the quaternion matrix, the norm of split quaternion is defined as follows.

**Definition 1 ([18]).** Let \( q = q_1 + q_2 i + q_3 j + q_4 k \in \mathbb{Q}_s \), in which \( q_1, q_2, q_3, q_4 \in \mathbb{R} \), the conjugate of \( q \) is defined as \( \bar{q} = q_1 - q_2 i - q_3 j - q_4 k \). The norm \( |q| \) of a split quaternion \( q \) is defined as

\[
|q| = \sqrt{|q\bar{q}|} = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}.
\]

For any \( A = (a_{ij}) \in \mathbb{Q}_s^{m \times n} \), \( a_{ij} = a_{i}^1 + a_{i}^2 i + a_{i}^3 j + a_{i}^4 k \in \mathbb{Q}_s \), the Frobenius norm of the split quaternion matrix is defined as

\[
\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{i}^1)^2 + (a_{i}^2)^2 - (a_{i}^3)^2 - (a_{i}^4)^2}.
\]

**Definition 2 ([19]).** Let \( A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{Q}_s^{n \times n} \), in which \( A_1, A_2, A_3, A_4 \in \mathbb{R}^{m \times n} \), the real representation matrix of the split quaternion matrix \( A \) is defined as

\[
A^R = \begin{bmatrix}
A_1 & -A_2 & A_3 & A_4 \\
A_2 & A_1 & -A_4 & A_3 \\
A_3 & A_4 & A_1 & -A_2 \\
A_4 & -A_3 & A_2 & A_1
\end{bmatrix}.
\]

For real representation matrix \( A^R \), denote the first column block as \( \vec{A} \), \( \vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \).

\( A^R \) and \( \vec{A} \) have the following properties.

**Proposition 1 ([19]).** Suppose \( A, B \in \mathbb{Q}_s^{m \times n} \), \( C \in \mathbb{R}^{n \times s} \) and \( a \in \mathbb{R} \), then

(i) \( A = B \Leftrightarrow A^R = B^R \);
(ii) \( (A + B)^R = A^R + B^R \), \( (AC)^R = A^R C^R \), \( (aA)^R = aA^R \).

**Proposition 2.** Suppose \( A, B \in \mathbb{Q}_s^{m \times n} \), \( C \in \mathbb{Q}_s^{n \times s} \) and \( a \in \mathbb{R} \), then

(i) \( A = B \Leftrightarrow \vec{A} = \vec{B} \);
(ii) \( (A + B) = \vec{A} + \vec{B} \), \( (aA) = a\vec{A} \);
(iii) \( (AC)^R = A^R \vec{C} \).

The Kronecker product of the matrices is defined as follows.

**Definition 3 ([20]).** Let \( A = (a_{ij}) \in \mathbb{Q}_s^{m \times n} \), \( B = (b_{ij}) \in \mathbb{Q}_s^{p \times q} \), the Kronecker product of \( A \) and \( B \) is defined as

\[
A \otimes B = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{mn}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix}.
\]
If $A = \begin{bmatrix} C & D \\ E & F \end{bmatrix}$, in which $C, D, E, F$ with appropriate size, then

$$A \otimes B = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \otimes B = \begin{bmatrix} C \otimes B & D \otimes B \\ E \otimes B & F \otimes B \end{bmatrix}.$$ 

Now we introduce the definition of the semi-tensor product of matrices.

**Definition 4** ([21]). Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, $t = \text{lcm}(n, p)$ is the least common multiple of $n$ and $p$. Then, the semi-tensor product of $A$ and $B$ is defined as

$$A \times B = (A \otimes I_{t/n})(B \otimes I_{t/p}).$$

When $n = p$, $A \times B = AB$. The semi-tensor product of matrices is a generalization of the conventional matrix product. The exchange of two vector factors in a semi-tensor product can be realized by means of a swap matrix.

**Theorem 1** ([22]). Let $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$, then

$$W_{[m,n]}^T \times X \times Y = Y \times X,$$

$$X^T \times Y^T \times W_{[m,n]} = Y^T \times X^T,$$

in which $W_{[m,n]} = [I_n \otimes \delta_{1m}^1, I_n \otimes \delta_{1m}^2, \ldots, I_n \otimes \delta_{1m}^m] \in \mathbb{R}^{mn \times mn}$ is called swap matrix.

For matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, the column stacking form (column straightening operator) of $A$ is $V_c(A) = (a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn})^T$, the row stacking form (row straightening operator) of $A$ is $V_r(A) = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{mn})^T$. The swap matrix and straightening operator have the following properties.

**Proposition 3** ([22]). (i) The swap matrix is invertible and $W_{[m,n]}^T = W_{[n,m]}^{-1} = W_{[n,m]}$.

(ii) Let $A \in \mathbb{R}^{m \times n}$, then $W_{[m,n]}V_r(A) = V_r(A)$, $W_{[n,m]}V_c(A) = V_c(A)$.

(iii) $V_r(A^T) = V_c(A)$, $V_c(A^T) = V_r(A)$.

Next, we study some lemmas, which are needed later. Similar to [14], the first column block of the real representation matrix has the following properties.

**Lemma 1.** Suppose $X \in \mathbb{Q}^{m \times n}$, then

$$V_r(X^R) = G V_r(X^C),$$

where $G = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} \in \mathbb{R}^{16mn \times 4mn}$, $G_i = \begin{pmatrix} G_{i1} \\ \vdots \\ G_{ik} \\ \vdots \\ G_{im} \end{pmatrix} \otimes I_n$, $(i = 1, 2, 3, 4, k = 1, \ldots, m)$, and

$$G_{1k} = \begin{bmatrix} (\delta_{1m}^k)^T & 0 & 0 & 0 \\ 0 & -(\delta_{2m}^k)^T & 0 & 0 \\ 0 & 0 & (\delta_{3m}^k)^T & 0 \\ 0 & 0 & 0 & (\delta_{4m}^k)^T \end{bmatrix}, \quad G_{2k} = \begin{bmatrix} 0 & (\delta_{1m}^k)^T & 0 & 0 \\ (\delta_{m}^k)^T & 0 & 0 & 0 \\ 0 & 0 & 0 & (\delta_{3m}^k)^T \\ 0 & 0 & - (\delta_{4m}^k)^T & 0 \end{bmatrix}, \quad G_{3k} = \begin{bmatrix} 0 & 0 & (\delta_{1m}^k)^T & 0 \\ 0 & 0 & 0 & (\delta_{2m}^k)^T \\ (\delta_{3m}^k)^T & 0 & 0 & 0 \\ 0 & (\delta_{4m}^k)^T & 0 & 0 \end{bmatrix}, \quad G_{4k} = \begin{bmatrix} 0 & 0 & 0 & (\delta_{1m}^k)^T \\ 0 & 0 & (\delta_{2m}^k)^T & 0 \\ (\delta_{3m}^k)^T & 0 & 0 & 0 \\ (\delta_{4m}^k)^T & 0 & 0 & 0 \end{bmatrix}.$


\[ G_{3k} = \begin{bmatrix} 0 & 0 & (\delta^k_m)^T & 0 \\ 0 & 0 & 0 & (\delta^k_m)^T \\ (\delta^k_m)^T & 0 & 0 & 0 \\ 0 & -(\delta^k_m)^T & 0 & 0 \end{bmatrix}, \quad G_{4k} = \begin{bmatrix} 0 & 0 & 0 & (\delta^k_m)^T \\ 0 & 0 & -(\delta^k_m)^T & 0 \\ (\delta^k_m)^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

**Lemma 2.** Suppose \( X = X_1 + X_2i + X_3j + X_4k \in \mathbb{Q}_s^{m \times n} \), then
\[ V_r(X^H) = M_1 V_r(X), \]
where \( M_1 = \text{diag}(1, -1, -1, -1) \otimes W_{[m,n]}. \)

**Proof of Lemma 2.** For any \( X = X_1 + X_2i + X_3j + X_4k \in \mathbb{Q}_s^{m \times n}, X_i \in \mathbb{R}^{m \times n}, (i = 1, 2, 3, 4). \)

\[ V_r(X^H) = V_r \left( \begin{bmatrix} X^T_1 \\ -X^T_2 \\ -X^T_3 \\ -X^T_4 \end{bmatrix} \right) = \left( \begin{bmatrix} V_r(X^T_1) \\ -V_r(X^T_2) \\ -V_r(X^T_3) \\ -V_r(X^T_4) \end{bmatrix} \right). \]

According to Proposition 3, we have
\[ V_r(X^T_i) = V_r(X_i) = W_{[m,n]} V_r(X_i), \]
in which \( W_{[m,n]} = [I_n \otimes \delta^1_m, I_n \otimes \delta^2_m, \ldots, I_n \otimes \delta^m_m]. \) Thus
\[ V_r(X^H) = \left( \begin{bmatrix} V_r(X^T_1) \\ -V_r(X^T_2) \\ -V_r(X^T_3) \\ -V_r(X^T_4) \end{bmatrix} \right) = \left( \begin{bmatrix} W_{[m,n]} \\ -W_{[m,n]} \\ 0 \\ -W_{[m,n]} \end{bmatrix} \right) \left( \begin{bmatrix} V_r(X_1) \\ V_r(X_2) \\ V_r(X_3) \\ V_r(X_4) \end{bmatrix} \right). \]

Let \( M_1 = \text{diag}(1, -1, -1, -1) \otimes W_{[m,n]}, \) then we have (2). \( \square \)

**Lemma 3.** Suppose \( X = X_1 + X_2i + X_3j + X_4k \in \mathbb{Q}_s^{m \times n} \), then
\[ V_r(X^{\eta H}) = M_\eta V_r(X), \quad \eta = i, j, k, \]
in which \( M_\eta = E_\eta \otimes W_{[m,n]}, E_i = \text{diag}(1, -1, 1, 1), E_j = \text{diag}(1, 1, -1, 1), E_k = \text{diag}(1, 1, 1, -1). \)

Similar to Lemma 2, it can be proved that Lemma 3 holds.

**Definition 5 ([23]).** Let \( A = (a_{ij}) \in \mathbb{Q}_s^{m \times n} \), if \( a_{ij} = \pi_{ij} = a_{n-i+1,n-j+1} \), the matrix \( A \) is called bisymmetric. If \( a_{ij} = -\pi_{ij} = a_{n-i+1,n-j+1} \), the matrix \( A \) is called skew bisymmetric.

In solving real linear matrix equation, the following lemma is obtained by using the Moore–Penrose inverse.

**Lemma 4 ([20]).** Suppose \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \), the linear matrix equation \( Ax = b \) has a solution if and only if
\[ AA^\dagger b = b. \]

The general solution is
\[ x = A^\dagger b + (I - A^\dagger A)y, \quad \forall y \in \mathbb{R}^n. \]

When \( \text{rank}(A) = n \), the matrix equation \( Ax = b \) has a unique solution. The unique solution is
\[ x = A^\dagger b. \]
3. \( \mathcal{H} \) Representation

\( \mathcal{H} \) representation was proposed by researcher Weihai Zhang. As a fixed extraction method for extracting independent elements from matrices with special structures, it can transform a matrix-valued equation into a standard vector-valued equation with independent coordinates. Therefore, in this section we apply it to split quaternion bisymmetric and skew bisymmetric matrices to simplify the operation. Firstly, we give the definition of \( \mathcal{H} \) representation.

**Definition 6** ([24]). Consider a \( p \)-dimensional matrix subspace \( \mathbb{X} \subset \mathbb{R}^{n \times n} \). Let \( \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p \) \( (p \leq n^2) \) be a set of basis of \( \mathbb{X} \), for any \( X = (x_{ij})_{n \times n} \in \mathbb{X} \), there exist \( x_{11}, x_{22}, \cdots, x_{pp} \in \mathbb{R} \), such that \( X = \sum_{i=1}^{p} x_i \varepsilon_i \). For mapping \( \psi : \mathbb{X} \to V_r(\mathbb{X}) \), the \( \mathcal{H} \) representation of \( \psi(X) \) can be expressed as follows:

\[
\psi(X) = V_r(X) = \mathcal{H}V_5(X),
\]

where \( \mathcal{H} \) is called an \( \mathcal{H} \) representation matrix of \( \psi(X) \).

**Remark 1.** (1) For \( X \in \mathbb{X} \), owing to the different selection of basis in \( \mathbb{X} \), the matrix \( \mathcal{H} \) is also different. That is to say, the \( \mathcal{H} \) representation of \( \psi(X) \) is not unique.

(2) In Definition 6, \( \psi(X) \) is a column vector formed by all elements of \( X \), \( V_5(X) \) is a column vector formed by different nonzero elements of \( X \). If the basis of \( \mathbb{X} \) is fixed, \( \mathcal{H} \) and \( V_5(X) \) will be uniquely determined.

The following example will illustrate the method of \( \mathcal{H} \) representation.

**Example 1.** Let \( \mathbb{X} = \mathbb{BR}^{3 \times 3} \), \( A = (a_{ij})_{3 \times 3} \in \mathbb{X} \), then \( \dim(\mathbb{X}) = 4 \). Selecting the basis of \( \mathbb{X} \) as

\[
\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \varepsilon_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Obviously, \( \psi(A) = V_r(A) = [a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}]^T \). According to the symmetry of bisymmetric matrix, \( V_5(A) = [a_{11}, a_{21}, a_{22}, a_{31}]^T \). It is easy to calculate

\[
\mathcal{H} = [V_r(\varepsilon_1), V_r(\varepsilon_2), V_r(\varepsilon_3), V_r(\varepsilon_4)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.
\]

**Definition 7.** (1) For \( A = (a_{ij}) \in \mathbb{BR}^{n \times n} \), let \( n = 2m \) or \( 2m + 1 \). Extracting the independent elements in matrix \( A \), when \( i \leq m \), let \( a_1 = (a_{11}, \cdots, a_{1i}, \cdots, a_{ii}, \cdots, a_{mm}) \), when \( m + 1 \leq i \leq n \), let \( a_2 = (a_{m+1,1}, \cdots, a_{m+1,n-m}, \cdots, a_{i1}, \cdots, a_{jn-i+1}, \cdots, a_{an}) \), denote \( V_a(A) \) as below:

\[
V_a(A) = (a_1, a_2)^T.
\]

(2) For \( A = (a_{ij}) \in \mathbb{BR}^{n \times n} \), let \( n = 2m \) or \( 2m + 1 \). Extracting the independent elements in matrix \( A \), when \( 2 \leq i \leq m \), let \( a_3 = (a_{21}, a_{31}, \cdots, a_{i1}, \cdots, a_{i1}, \cdots, a_{m1}, \cdots, a_{mm}) \), when \( m + 1 \leq i \leq n - 1 \), let \( a_4 = (a_{m+1,1}, \cdots, a_{m+1,n-m}, \cdots, a_{i1}, \cdots, a_{jn-i}, \cdots, a_{n-1}) \), denote \( V_b(A) \) as below:

\[
V_b(A) = (a_3, a_4)^T.
\]

The \( \mathcal{H} \) representations of real bisymmetric and skew bisymmetric matrices are as follows.
Lemma 5. Let $A = (a_{ij}) \in \mathbb{B}^{n \times n}$, then

$$V_r(A) = \mathcal{H}_1 V_n(A),$$

(7)

when $n = 2m$, $\mathcal{H}_1 \in \mathbb{R}^{n^2 \times (m^2 + m)}$, and

$$\mathcal{H}_1 = \begin{bmatrix}
\delta_{ij} & 0 & 0 & \cdots & 0 \\
0 & \delta_{ij} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \delta_{ij} & 0 \\
0 & 0 & \cdots & 0 & \delta_{ij}
\end{bmatrix},$$

when $n = 2m + 1$, $\mathcal{H}_1 \in \mathbb{R}^{n^2 \times (m+1)^2}$, and

$$\mathcal{H}_1 = \begin{bmatrix}
\delta_{ij} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \delta_{ij} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \delta_{ij} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \delta_{ij} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \delta_{ij} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \delta_{ij} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \delta_{ij} & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \delta_{ij}
\end{bmatrix}.$$ 

Proof of Lemma 5. $A = (a_{ij}) \in \mathbb{B}^{n \times n}$, namely $a_{ij} = a_{ji} = a_{n+1-i,n+1-j}$. According to the symmetry of bisymmetric matrix and Definition 7 (1), we select $V_5(A) = V_n(A)$. For $X = \mathbb{B}^{n \times n} \subset \mathbb{R}^{n \times n}$, then we have $\dim(X) = \begin{cases}
\frac{m^2 + m}{2}, & n = 2m; \\
\frac{(m + 1)^2}{2}, & n = 2m + 1.
\end{cases}$ According to Definition 6, we select a standard basis as

$$\{S_{11}, S_{22}, \ldots, S_{m1}, \ldots, S_{nn}, S_{m+1,1}, \ldots, S_{m+1,n-1-1}, \ldots, S_{n-1,1}, S_{n-1,2}, S_{n1}\},$$

where $S_{ij} = (s_{pq})_{n \times n}$, $s_{ij} = s_{ji} = s_{n+1-i,n+1-j} = 1$, and the other elements are zero. For any $A = (a_{ij})_{n \times n} \in X$, there exist $V_n(A) = (a_1, a_2)^T$ in the form of (5), such that

$\begin{align*}
V_r(A) &= \sum_{i=1}^{m} \sum_{j=1}^{m} V_r(S_{ij})a_{ij} + \sum_{i=m+1}^{n} \sum_{j=1}^{n-m} V_r(S_{ij})a_{ij} \\
&= [V_r(S_{11}), \ldots, V_r(S_{m1}), \ldots, V_r(S_{nn}), \ldots, V_r(S_{n-1,1}), V_r(S_{n-1,2}), V_r(S_{n1})] V_n(A) \\
&= \mathcal{H}_1 V_n(A),
\end{align*}$

in which $\mathcal{H}_1 \in \mathbb{R}^{n^2 \times (m^2 + m)} / \mathbb{R}^{n^2 \times (m+1)^2}$, thus we have (7). \qed

Lemma 6. Let $A = (a_{ij}) \in \mathbb{S}^{B}^{n \times n}$, then

$$V_r(A) = \mathcal{H}_2 V_n(A),$$

(8)

when $n = 2m$, $\mathcal{H}_2 \in \mathbb{R}^{n^2 \times (m^2 - m)}$, and

$$\mathcal{H}_2 = \begin{bmatrix}
\delta_{ij} & 0 & 0 & \cdots & 0 \\
0 & \delta_{ij} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \delta_{ij} & 0 \\
0 & 0 & \cdots & 0 & \delta_{ij}
\end{bmatrix},$$

when $n = 2m + 1$, $\mathcal{H}_2 \in \mathbb{R}^{n^2 \times (m+1)^2}$, and

$$\mathcal{H}_2 = \begin{bmatrix}
\delta_{ij} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \delta_{ij} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \delta_{ij} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \delta_{ij} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \delta_{ij} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \delta_{ij} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \delta_{ij} & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \delta_{ij}
\end{bmatrix}.$$
where $n = 2m + 1$, $\mathcal{H}_2 \in \mathbb{R}^{n \times n}$, and
\[
\mathcal{H}_2 = \begin{bmatrix}
-S_2 & 0 & \cdots & 0 & -S_m & \cdots & 0 \\
-S_m & 0 & \cdots & 0 & -S_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -S_2 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -S_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -S_m & \cdots & -S_2
\end{bmatrix}.
\]

We select the standard basis as
\[
\{S_{21}, S_{31}, \ldots, S_{m1}, \ldots, S_{mn}, S_{m+1,1}, S_{m+1,n-(m+1)}, \ldots, S_{n-2,1}, S_{n-2,2}, S_{n-1,1}\},
\]
where $S_{ij} = (S_{pq})_{n \times n}$, $S_{ij} = -S_{i+1-j,n+1-j} = 1$, and the other elements are zero. Since the method is similar to that of Lemma 6, we omit the detailed proof.

Next, we study the $\mathcal{H}$ representation of split quaternion bisymmetric and skew bisymmetric matrices.

**Theorem 2.** Let $X = X_1 + X_2i + X_3j + X_4k \in \mathbb{BQ}_0^{n \times n}$, then
\[
V_r(\overrightarrow{X}) = \mathcal{H}V_A(\overrightarrow{X}),
\]
where $\mathcal{H} = \text{diag}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_2)$, $V_A(\overrightarrow{X}) = \begin{pmatrix} V_4(X_1) \\ V_4(X_2) \\ V_4(X_3) \\ V_4(X_4) \end{pmatrix}$.

**Proof of Theorem 2.** $X = X_1 + X_2i + X_3j + X_4k \in \mathbb{BQ}_0^{n \times n}$, then we have $X_1 \in \mathbb{BR}^{n \times n}$, $X_j \in \mathbb{SBR}^{n \times n}$ ($j = 2, 3, 4$). According to Lemmas 5 and 6,
\[
V_r(\overrightarrow{X}) = \begin{pmatrix} V_r(X_1) \\ V_r(X_2) \\ V_r(X_3) \\ V_r(X_4) \end{pmatrix} = \begin{pmatrix} H_1V_4(X_1) \\ H_2V_4(X_2) \\ H_2V_4(X_3) \\ H_2V_4(X_4) \end{pmatrix} = \begin{pmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & H_2 \end{pmatrix} \begin{pmatrix} V_4(X_1) \\ V_4(X_2) \\ V_4(X_3) \\ V_4(X_4) \end{pmatrix}.
\]

Obviously, the formula (9) can be obtained. \(\square\)

**Theorem 3.** Let $X = X_1 + X_2i + X_3j + X_4k \in \mathbb{SBQ}_0^{n \times n}$, then
\[
V_r(\overrightarrow{X}) = \mathcal{H}V_B(\overrightarrow{X}),
\]
in which $\mathcal{H} = \text{diag}(\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_1)$, $V_B(\overrightarrow{X}) = \begin{pmatrix} V_4(X_1) \\ V_4(X_2) \\ V_4(X_3) \\ V_4(X_4) \end{pmatrix}$.

4. The Solutions of Problems 1, 2

In this section, according to the real representation and $\mathcal{H}$ representation method of split quaternion matrices, we convert the problems 1, 2 of the split quaternion matrix Equation (1) into corresponding problems of the real matrix equation. When $f(X) \in \{X, X^H, X^{iH}, X^{iH}X^{kH}\}$, the solution of different types of equations are given as follows.
4.1. Equation $X - Af(X)B = C$ with $f(X) = X^H$ or $f(X) = X^{\eta H}$

In this subsection, we consider Equation (1) with $f(X) = X^H$, or $f(X) = X^{\eta H}$, $(\eta = i, j, k)$ namely,

$$ X - AX^H B = C, \quad (11) $$

$$ X - AX^{\eta H} B = C. \quad (12) $$

Theorem 4. Suppose $A, B, C \in \mathbb{Q}_s^{n \times n}$, denote $\tilde{A} = [I_{4n^2} - (A^R \otimes (\tilde{B})^T)GM]H \in \mathbb{R}^{4n^2 \times p}$, where $p = \begin{cases} n^2 - n, & n = 2m; \\ n^2 - n + 1, & n = 2m + 1. \end{cases} \mathcal{M} = \begin{cases} \mathcal{M}_1, & f(X) = X^H; \\ \mathcal{M}_\eta, & f(X) = X^{\eta H}. \end{cases}$ then the split quaternion matrix Equation (11)/(12) has a solution $X \in \mathbb{BQ}_s^{n \times n}$ if and only if

$$ (\tilde{A}^T - I_{4n^2}) V_r(\bar{C}) = 0. \quad (13) $$

If (13) holds, the solution set of (11)/(12) over $\mathbb{BQ}_s^{n \times n}$ can be expressed as

$$ Q_B = \{ X \in \mathbb{BQ}_s^{n \times n} | V_A(\bar{X}) = \tilde{A}^T V_r(\bar{C}) + (I_p - \tilde{A}^T \tilde{A})y, \ \forall y \in \mathbb{R}^p \}. \quad (14) $$

When $\text{rank}(\tilde{A}) = p$, the matrix Equation (11)/(12) has a unique bisymmetric solution $X_B$, which satisfies

$$ V_A(\bar{X}_B) = \tilde{A}^T V_r(\bar{C}). \quad (15) $$

Proof of Theorem 4. According to Propositions 1 and 2 we have

$$ X - Af(X)B = C \iff X^R - (Af(X)B)^R = C^R \iff \bar{X} - \overline{Af(X)B} = \overline{C}. $$

Therefore, the split quaternion matrix Equation (11)/(12) has a bisymmetric solution if and only if for any $X \in Q_B$ satisfies

$$ \| \bar{X} - \overline{Af(X)B} - \overline{C} \| = 0. $$

According to Proposition 2, Lemmas 1–3 and Theorem 2, we have

$$ \| \bar{X} - \overline{Af(X)B} - \overline{C} \| = \| V_r(\bar{X}) - V_r(A^R f(X)^R \bar{B}) - V_r(\bar{C}) \| = \| V_r(\bar{X}) - (A^R \otimes (\tilde{B})^T) V_r(f(X)^R) - V_r(\bar{C}) \| = \| V_r(\bar{X}) - (A^R \otimes (\tilde{B})^T) \mathcal{M} V_r(f(X)^R) - V_r(\bar{C}) \| = \| [I_{4n^2} - (A^R \otimes (\tilde{B})^T) \mathcal{M}] V_r(\bar{X}) - V_r(\bar{C}) \| = \| \tilde{A} V_A(\bar{X}) - V_r(\bar{C}) \|. $$

Thus, for $X \in Q_B$, $\| \bar{X} - \overline{Af(X)B} - \overline{C} \| = 0$ if and only if $\| \tilde{A} V_A(\bar{X}) - V_r(\bar{C}) \| = 0.$

For real matrix equation

$$ \tilde{A} V_A(\bar{X}) = V_r(\bar{C}). \quad (16) $$
By the proposition of the Moore–Penrose generalized inverse, we get
\[ \left\| \tilde{A} V_{\tilde{A}} \left( \overrightarrow{X} \right) - V_{\tilde{A}} \left( \overrightarrow{C} \right) \right\| = \left\| \tilde{A} \tilde{A}^\dagger V_{\tilde{A}} \left( \overrightarrow{X} \right) - V_{\tilde{A}} \left( \overrightarrow{C} \right) \right\| = \left\| \tilde{A} \tilde{A}^\dagger V_{\tilde{A}} \left( \overrightarrow{C} \right) - V_{\tilde{A}} \left( \overrightarrow{C} \right) \right\| = \left\| \left( \tilde{A} \tilde{A}^\dagger - I_{4n^2} \right) V_{\tilde{A}} \left( \overrightarrow{C} \right) \right\| . \]

For \( X \in Q_B \), we obtain
\[ X - Af(X)B = C \iff \left\| \left( \tilde{A} \tilde{A}^\dagger - I_{4n^2} \right) V_{\tilde{A}} \left( \overrightarrow{C} \right) \right\| = 0 \iff \left( \tilde{A} \tilde{A}^\dagger - I_{4n^2} \right) V_{\tilde{A}} \left( \overrightarrow{C} \right) = 0. \]

So, (13) holds. By Lemma 4, if (13) holds, the solution set of (11)/(12) over \( \mathbb{B} Q_{4\times n}^{n \times n} \) which satisfies (16) can be expressed as
\[ V_{\tilde{A}} \left( \overrightarrow{X} \right) = \tilde{A}^\dagger V_{\tilde{A}} \left( \overrightarrow{C} \right) + \left( I_p - \tilde{A}^\dagger \tilde{A} \right) y, \quad \forall y \in \mathbb{R}^p. \]

When \( \text{rank}(\tilde{A}) = p \), the matrix Equation (11)/(12) has a unique bisymmetric solution \( X_B \), which satisfies
\[ V_{\tilde{A}} \left( \overrightarrow{X_B} \right) = \tilde{A}^\dagger V_{\tilde{A}} \left( \overrightarrow{C} \right). \]

\[ \Box \]

Now, we study the skew bisymmetric solution of the Equation (11)/(12) in Problem 2. Since the method is similar to that of Problem 1, we only describe the results and omit the detailed proof.

**Theorem 5.** Suppose \( A, B, C \in \mathbb{Q}^{n \times n} \), denote \( \tilde{A} = [I_{4n^2} - (A^R \otimes \left( \overrightarrow{B} \right)^T) G] M \tilde{H} \in \mathbb{R}^{4n^2 \times q} \), where \( q = \begin{cases} n^2 + n, & n = 2m; \\ n^2 + n + 1, & n = 2m + 1. \end{cases} \), \( M = \begin{cases} M_1, & f(X) = X^H; \\ M_H, & f(X) = X^H. \end{cases} \), then the split quaternion matrix Equation (11)/(12) has a solution \( X \in \mathbb{B} Q_{4\times n}^{n \times n} \) if and only if
\[ (\tilde{A} \tilde{A}^\dagger - I_{4n^2}) V_{\tilde{A}} \left( \overrightarrow{C} \right) = 0. \]

If (17) holds, the solution set of (11)/(12) over \( \mathbb{B} Q_{4\times n}^{n \times n} \) can be expressed as
\[ Q_{SB} = \left\{ X \in \mathbb{B} Q_{4\times n}^{n \times n} | V_{\tilde{A}} \left( \overrightarrow{X} \right) = \tilde{A}^\dagger V_{\tilde{A}} \left( \overrightarrow{C} \right) + \left( I_q - \tilde{A}^\dagger \tilde{A} \right) y, \quad \forall y \in \mathbb{R}^q \right\}. \]

When \( \text{rank}(\tilde{A}) = q \), the matrix Equation (11)/(12) has a unique skew bisymmetric solution \( X_A \), which satisfies
\[ V_{\tilde{A}} \left( \overrightarrow{X_A} \right) = \tilde{A}^\dagger V_{\tilde{A}} \left( \overrightarrow{C} \right). \]

4.2. Equation \( X - AXB = C \)

In this subsection, we consider Equation (1) with \( f(X) = X \), namely,
\[ X - AXB = C. \]

**Theorem 6.** Suppose \( A, B, C \in \mathbb{Q}^{n \times n} \), denote \( \tilde{P} = [I_{4n^2} - (A^R \otimes \left( \overrightarrow{B} \right)^T) G] \tilde{H} \in \mathbb{R}^{4n^2 \times p} \), where \( p = \begin{cases} n^2 - n, & n = 2m; \\ n^2 - n + 1, & n = 2m + 1. \end{cases} \), then the split quaternion matrix Equation (20) has a solution \( X \in \mathbb{B} Q_{4\times n}^{n \times n} \) if and only if
\[ (\tilde{P} \tilde{P}^\dagger - I_{4n^2}) V_{\tilde{A}} \left( \overrightarrow{C} \right) = 0. \]
If (21) holds, the solution set of (20) over \( \mathbb{BQ}^{n \times n}_q \) can be expressed as
\[
Q_B = \left\{ X \in \mathbb{BQ}^{n \times n}_q | V_A(\overline{X}) = \overline{\beta^t V_r(\overline{C})} + (I_q - \overline{\beta^t \beta})y, \quad \forall y \in \mathbb{R}^q \right\}. \tag{22}
\]

When \( \text{rank}(\overline{P}) = p \), the matrix Equation (20) has a unique bisymmetric solution \( X_B \), which satisfies
\[
V_A(\overline{X}_B^*) = \overline{\beta^t V_r(\overline{C})}. \tag{23}
\]

**Theorem 7.** Suppose \( A, B, C \in \mathbb{Q}^{n \times n}_q \), denote \( \overline{P} = \left[ I_{4n^2} - (A^R \otimes (\overline{B}^T)^T) \right] \mathcal{G} \), \( \overline{\mathcal{H}} \in \mathbb{R}^{4n^2 \times 4n^2} \), where
\[
q = \begin{cases} 
2n^2 + n, & n = 2m; \\
2n^2 + n + 1, & n = 2m + 1.
\end{cases}
\]
then the split quaternion matrix Equation (20) has a solution \( X \in \mathbb{SBQ}^{n \times n}_q \) if and only if
\[
(\overline{P} \overline{P}^t - I_{4n^2}) V_r(\overline{C}) = 0. \tag{24}
\]
If (24) holds, the solution set of (20) over \( \mathbb{SBQ}^{n \times n}_q \) can be expressed as
\[
Q_{SB} = \left\{ X \in \mathbb{SBQ}^{n \times n}_q | V_A(\overline{X}) = \overline{\beta^t V_r(\overline{C})} + (I_q - \overline{\beta^t \beta})y, \quad \forall y \in \mathbb{R}^q \right\}. \tag{25}
\]

When \( \text{rank}(\overline{A}) = q \), the matrix Equation (20) has a unique skew bisymmetric solution \( X_A \), which satisfies
\[
V_A(\overline{X}_A^*) = \overline{\beta^t V_r(\overline{C})}. \tag{26}
\]

5. Numerical Algorithm and Experiments

In this section, by using of the results in Section 4, we propose the numerical algorithms.

**Example 2.** Consider the split quaternion matrix equation
\[
X - AX^iI_B = C, \tag{27}
\]
in which \( A, B, C \in \mathbb{Q}^{n \times n}_q \). Let \( n = 3k \), \( K = 1: 14 \), \( A \) and \( B \) can be generated randomly in Matlab with
\[
A = \text{rand}(n) + \text{rand}(n)i + \text{rand}(n)j + \text{rand}(n)k, \\
B = \text{rand}(n) + \text{rand}(n)i + \text{rand}(n)j + \text{rand}(n)k.
\]
According to the characteristics of bisymmetric matrix, randomly generated \( X_B = X_1 + X_2i + X_3j + X_4k \in \mathbb{BQ}^{n \times n}_q \). Compute \( C = X_B - A(X_B)^{iH}B \). Obviously, the split quaternion matrix Equation (27) has the unique solution \( X_B \). According to Algorithm 1.

**Algorithm 1:** (Problem 1)

1. Input \( A_i, B_i, C_i \in \mathbb{R}^{n \times n} \), \( i = 1, 2, 3, 4 \), output \( A^R, \overline{B}, V_r(\overline{C}) \);
2. Input \( \mathcal{G}, M, \mathcal{H} \), output the matrix \( \overline{A}, \overline{P} \);
3. According to the formula (15)/(23), output the unique bisymmetric solution \( X_B \) of (1).

We compute the numerical solution \( X_B^* = X_1^* + X_2^*i + X_3^*j + X_4^*k \), denote \( \varepsilon_1 = \log_{10}\|X_B^* - X_B\|_F \). By calculation, we obtain the relation between \( n \) and the error \( \varepsilon_1 \), shown in Figure 1.
Example 3. Consider the split quaternion matrix equation

\[ X - AXB = C, \]  

(28)

in which \( A, B, C \in \mathbb{Q}_{s}^{n \times n} \). Similar to Example 2, let \( n = 3K, K = 1: 14 \), Randomly generate \( A \) and \( B \) in Matlab. According to the characteristics of skew bisymmetric matrix, randomly generated \( X_A = X_1 + X_2i + X_3j + X_4k \in \mathbb{SBQ}_{s}^{n \times n} \). Compute \( C = X_A - AX_AB \). The split quaternion matrix Equation (28) has the unique solution \( X_A \). According to Algorithm 2.

Algorithm 2: (Problem 2)

1. Input \( A_i, B_i, C_i \in \mathbb{R}^{n \times n}, (i = 1, 2, 3, 4) \), output \( A_R, B, V_r(\overline{C}) \);
2. Input \( \mathcal{G}, \mathcal{M}, \overline{\mathcal{H}} \), output the matrix \( \overline{A}, \overline{P} \);
3. According to the formula (19)/(26), output the unique skew bisymmetric solution \( X_A \) of (1).

We compute the numerical solution \( X_A^* \), denote \( \varepsilon_2 = \log_{10} \| X_A^* - X_A \|_F \). By calculation, we obtain the relation between \( n \) and the error \( \varepsilon_2 \), shown in Figure 2.

6. Conclusions

In this paper, based on the real representation of the split quaternion matrix and \( \mathcal{H} \) representation method, we investigate the solutions of split quaternion matrix Equation (1)
over $\mathbb{BQ}_0^{n\times n}$ and $\mathbb{BQ}_S^{n\times n}$. Then, we established the equivalent solvability conditions and general expressions of the (skew) bisymmetric solution for the split quaternion matrix Equation (1). The algorithms only involve real operations and the solution expressions we proposed only involve real matrices. Therefore, this method is very convenient, and the final numerical examples also illustrate its efficiency and superiority.

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**References**


