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Antimagic Labeling for Product of Regular Graphs

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Abstract: An antimagic labeling of a graph $G = (V, E)$ is a bijection from the set of edges of G to $\{1, 2, \cdots, |E(G)|\}$ and such that any two vertices of G have distinct vertex sums where the vertex sum of a vertex v in $V(G)$ is nothing but the sum of all the incident edge labeling of G. In this paper, we discussed the antimagicness of rooted product and corona product of graphs. We proved that if we let G be a connected t-regular graph and H be a connected k-regular graph, then the rooted product of graph G and H admits antimagic labeling if $t \geq k$. Moreover, we proved that if we let G be a connected t-regular graph and H be a connected k-regular graph, then the corona product of graph G and H admits antimagic labeling for all $t, k \geq 2$.

Keywords: graph labeling; antimagic labeling; product graphs; rooted product; corona product; regular graph

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1. Introduction

Graphs that are considered in this paper are finite, undirected, connected, and simple. The concept of antimagic labeling of a graph was introduced by Hartsfield and Ringel [1].

An antimagic labeling of a graph G with m edges and n vertices is a one-to-one correspondence f between the edge set of G to the label set $\{1, 2, \cdots, m\}$ such that $\phi_f(u) \neq \phi_f(v)$, for any two distinct vertices of u, v in $V(G)$, where $\phi_f(v)$ is defined as the sum of the labels of the edges that are incident to a vertex v in G. A graph that has at least one antimagic labeling is called an antimagic graph.

It is clear that K_2 does not have any antimagic labeling. Except for K_2, it is believed that all other connected graphs admit at least some antimagic labeling. This is proposed as a conjecture by Hartsfield and Ringel [1] which states that "Every connected graph other than K_2 are antimagic". Hartsfield and Ringel [1] proved that stars, paths, cycles, wheels, complete graphs and complete bipartite graphs, $K_{2,m}, m \geq 3$ admit antimagic labeling.

In cryptography, Data Encryption Standard (DES) is a block cipher in which the data is encrypted in blocks. Compared to DES, the Advanced Encryption Standard (AES is a symmetric block cipher cryptographic algorithm used for carrying out the encryption) has more security. Antimagic labeling of a network (graph) is used in AES to perform the encryption of data in blocks of a size of 128 bits each. Likewise, antimagic labeling is used in many fields of engineering. By also using an antimagic labeling of a graph G, we can give a proper colouring to the graph G. For the study on antimagic labeling and its connection with the vertex colouring, refer to [2–4].

Alon et al. [5] confirmed that the conjecture stands true for some classes of graphs. That is, if G is a graph with n vertices and there exist a absolute constant c such that either $\delta(G) \geq c \log n$ or $\Delta(G) \geq n - 2$, then the graph G admits an antimagic labeling. Later, Yilma [6] proved that a graph with maximum degree greater than or equal to $n - 3$ is antimagic.

Researchers have adopted various new techniques to prove some interesting classes of graphs that have an antimagic labeling. For detailed survey, one can refer to [7–15].

Although researchers studied the antimagicness of several classes of graph, the conjecture of Hartsfield and Ringel still remains open, even for trees.

Regular graphs are well-studied networks in computer science and most of them are symmetric nature. So, studying the antimagicness of regular graphs is more attractive. Initially, the antimagicness of regular graphs were extensively studied by many researchers and finally, in 2016, it was shown that all regular graphs are antimagic. Cranston et al. [16] proved that all odd regular graphs are antimagic, while the antimagicness of the even regular graphs were verified by Chang et al. [17] in 2016. In 2015 , Bèrczi et al. [18] gave proof of the antimagicness of even regular graphs but they realized that the proof of the main theorem of the step uses an invalid assumption. Hence, 4 years later in 2019, they rectified the error and proved that all regular graphs admit antimagic labeling.

Theorem 1 (See [17])**.** *For $k \geq 2$, every k-regular graph is antimagic.*

Once regular graphs were proven to be antimagic, researchers focused on proving the antimagicness of graph products using the base as a regular graphs. Liang and Zhu [19] proved that if G is a k-regular graph and H is any arbitrary graph with $1 \leq |V(H)| - 1 \leq |E(H)|$, then the Cartesian product of graph G and H admits an antimagic labeling. Cheng [20] considers a regular graph G_1 and G_2 that has the degree bounded with some inequality, and in this case the Cartesian product of G_1 and G_2 again admits antimagic labeling. In addition, they investigated whether two or more regular graphs with positive degree (mandatorily not connected) admit an antimagic labeling. Wang and Hsiao [21] explored new classes of sparse antimagic graphs through Cartesian products. Additionally, Wang and Hsiao [21] considers G as an arbitrary graph and H as a $d-$regular graph with $d > 1$, and then they proved that the lexicographic product of graph G and H admits an antimagic labeling. Oudone Phanalasy et al. [22] proved that certain families of Cartesian products of regular graphs are antimagic. Daykin et al. [23] constructed two families of graphs known to be antimagic, namely sequential generalized corona graph and generalized snowflake graph. Wenhui et al. [24] investigated antimagicness for lexicographic product P_m and P_n where $m, n \geq 3$. Yingyu et al. [25] assumed G as a complete bipartite graph $K_{m,n}$ and H as a path graph P_k, and then they proved that the lexicographic product of graph G and H admits an antimagic labeling. Recently, Yingyu et al. [26] constructed oriented Eulerian circuit and used Siamese method to achieve an antimagic labeling for the composition of graph G and P_n. The antimagicness of joined graphs is considered by Wang et al. [27]. If G is a graph with minimum degree of at least r and H is a graph with maximum degree of at most $2r - 1$ then the join of G and H admits an antimagic labeling for $|V(H)| \geq |V(G)|$. Bača et al. [28] used the antimagic labeling of join graphs to prove the antimagicness of complete multi-partite graphs.

Inspired by the results on the antimagicness of product graphs, in this paper we discuss the antimagicness for rooted and corona products of regular graphs. More particularly, we proved that if we let G be a connected t-regular graph and H be a connected k-regular graph, then the rooted product of G and H admits antimagic labeling when $t \geq k$. We also proved that if we let G be a connected t-regular graph and H be a connected k-regular graph, then the corona product of G and H admits antimagic labeling for all $t, k \geq 2$.

A rooted graph H is a graph that has one vertex, named a root vertex, as its fixed vertex. Let G be a n vertex graph and $\mathcal{H}$ be a sequence of n rooted graphs $H_1, H_2, \ldots, H_n$ such that $H_i \cong H$ and v is the root vertex of H. The rooted product of the graphs G and H obtained from G such that $H_1, H_2, \cdots, H_n$ by identifying the root vertex of H_i to the ith vertex of G. The rooted product of graph G and H is denoted by $G \circ_v H$. The corona product of the graph G and H is the graph obtained by taking one copy of G and n copies of H, $H_i, 1 \leq i \leq n$ and joining the ith vertex of G to each vertex from the i^{th} copy of H and it is denoted by $G \odot H$.

2. Main Results

In this section, we prove our main results. Before proving the main result we prove some basic lemmas and observations which will be used in the main results.

Lemma 1. *If G is a k-regular graph with m edges, then for any vertex u in* $V(G)$, $\frac{k(k+1)}{2} \leq \phi_f(u) \leq km - \frac{k(k-1)}{2}$, *where f is an antimagic labeling of G.*

Proof. Let G be a k-regular graph with m edges. By Theorem 1 it admits an antimagic labeling. Let f be an antimagic labeling of G, then for any vertex u in $V(G)$, $\phi_f(u)$ takes minimum when their incident edges obtain labels from the set $\{1, 2, \cdots, k\}$ and $\phi_f(u)$ take the maximum value when their incident edges obtain labels from the set $\{m, m-1, \cdots, m-(k-1)\}$.

Hence, $\frac{k(k+1)}{2} \leq \phi_f(u) \leq km - \frac{k(k-1)}{2}$. □

Figure 1 shows the antimagic labeling of a 3-regular graph G with 9 edges. From Figure 1, we also observe that the range of the vertex sum of the vertices of G are from 6 to 22.

From Lemma 1, we have the following observation.

Observation 1. *If G be a k-regular graph with f as its antimagic labeling. Let u, v be any two vertices of G such that if* $\phi_f(u) \geq \phi_f(v)$ *then* $0 \leq \phi_f(u) - \phi_f(v) \leq km - k^2$.

An antimagic labeling of a 2-regular graph H with 3 edges is given in Figure 2. From Figure 2 we also observe that the range in the difference of the vertex sum of any two vertices of H are from 0 to 2.

Theorem 2. *Let G be a connected t-regular graph and let H be a connected k-regular graph,* $t \geq k$ *then the rooted product of G and H admits antimagic labeling.*

Proof. Let G be a t-regular graph with n vertices and m edges and let H be a k-regular graph with p vertices and q edges. By Theorem 1, the graphs G and H admit antimagic labeling. Let f and g be the antimagic labeling of G and H respectively. By definition of f, $f : E(G) \to \{1, 2, \cdots, m\}$ such that $\phi_f(u) \neq \phi_f(v)$ for any two distinct vertices u and v in G. By definition of g, $g : E(H) \to \{1, 2, \cdots, q\}$ such that $\phi_g(x) \neq \phi_g(y)$ for any two distinct vertices x and y in H.

Let us name the vertices of G as $v_1, v_2, \cdots, v_n$ such that,

$$\phi_f(v_1) < \phi_f(v_2) < \cdots < \phi_f(v_n) \tag{1}$$

and also name the vertices of H as $u_1, u_2, \cdots, u_p$ such that,

$$\phi_g(u_1) < \phi_g(u_2) < \cdots < \phi_g(u_p). \tag{2}$$

Construct the rooted product of G and H, $G \circ_v H$ by fixing the root vertex of H as u_p. Note that the number of edges in $G \circ_v H$ is $nq + m$. Let us name the vertices of $G \circ_v H$ as follows. The vertices of G are named as the same as the earlier, that is $v_1, v_2, \cdots, v_n$ and then name the vertices of H_i, for $i = 1, 2, \cdots, n$ (i^{th} isomorphic copy of H) as $u_1^i, u_2^i, \cdots, u_p^i$. That is the vertex u_l in H is now has the name u_l^i in H_i for $l = 1, 2, \cdots, p$. Note that $v_i = u_p^i$. That is, the set of vertices $\left\{u_p^1, u_p^2, \cdots, u_p^n\right\}$ induces the graph G. Before defining the antimagic labeling of $G \circ_v H$, we label the edges of H_i by using the edge labeling g of H as follows:

$$g_i : E(H_i) \to \{1, 2, \cdots, q\} \tag{3}$$

for an edge $e = (u_a^i, u_b^i)$ in H_i, $g_i(e) = g(e')$ where e' as an edge (u_a, u_b) in H. Then by definition of g and (2) for each i, $i = 1, 2, \cdots, n$.

$$\phi_{g_i}(u_1^i) < \phi_{g_i}(u_2^i) < \cdots < \phi_{g_i}(u_p^i). \tag{4}$$

where $u_1^i, u_2^i, \cdots, u_p^i$ are the vertices of H_i in $G \circ_v H$.

Now we define $h : E(G \circ_v H) \to \{1, 2, \cdots, nq + m\}$ by,

$$h(e) = \begin{cases} g_i(e) + (i-1)q, & \text{if } e \text{ is in } H_i \\ f(e) + nq, & \text{if } e \text{ is in } G \end{cases} \tag{5}$$

From the above labeling h, we observe that, for all $i, 1 \le i \le n$

$$\phi_h(v_i) = \phi_h(u_p^i) = \phi_f(v_i) + \phi_{g_i}(u_p^i) + kq(i-1) + tnq \tag{6}$$

and

$$\phi_h(u_l^i) = \phi_{g_i}(u_l^i) + kq(j-1) \text{ for every } l, l = 1, 2, \cdots, p-1. \tag{7}$$

In order to prove that h is an antimagic labeling of $G \circ_v H$, we need to prove that for any two distinct vertices x and y in $G \circ_v H$ such that $\phi_h(x) \ne \phi_h(y)$. We consider the following possible cases on the vertices of x and the vertices of y in $G \circ_v H$.

(i) x in H_i and y in H_j for $i, j = 1, 2, \cdots, n$ and $x \ne u_p^i$ & $y \ne u_p^j$.
(ii) x in G and y in H_j when $x \ne u_p^j$, $j = 1, 2, \cdots n$.
(iii) x and y are the vertices of G.

Case 1. For any two distinct vertices x and y in $G \circ_v H$, where x is in H_i and y in H_j, for $i, j, 1 \le i, j \le n$ and $x \ne u_p^i$ & $y \ne u_p^j$.

Case 1.1. When $i = j$, then both the vertex x and y are from H_i. By definition on the naming of the vertices of H_i, $x = u_r^i$ and $y = u_s^i$ for some $r, s, 1 \le r, s \le p-1, r \ne s$. Then by definition of h and by (7) we have,

$$\phi_h(x) = \phi_h(u_r^i) = \phi_{g_i}(u_r^i) + k(i-1)q$$
$$\phi_h(y) = \phi_h(u_s^i) = \phi_{g_i}(u_s^i) + k(i-1)q$$

Without loss of generality, we assume that $r < s$. By (4), we have $\phi_{g_i}(u_r^i) < \phi_{g_i}(u_s^i)$, therefore $\phi_h(x) < \phi_h(y)$. Hence $\phi_h(x) \ne \phi_h(y)$.

Case 1.2. When $i \ne j$, then the vertex x in H_i and the vertex y in H_j.

By definition on the naming of vertices of H_i and H_j, $x = u_r^i$ and $y = u_s^j$ for some r, s, $1 \le r, s \le q-1$.

Without loss of generality we assume $i < j$. Then by definition of h and by (7) we have,

$$\phi_h(x) = \phi_h(u_r^i) = \phi_{g_i}(u_r^i) + k(i-1)q$$
$$\phi_h(y) = \phi_h(u_s^j) = \phi_{g_j}(u_s^j) + k(j-1)q.$$

Case 1.2.1. If $r \le s$. Then by (2),

$$\phi_{g_i}(u_r^i) \le \phi_{g_j}(u_s^j). \tag{8}$$

Consider,

$$\phi_h(y) - \phi_h(x) = \phi_{g_j}(u_s^j) - \phi_{g_i}(u_r^i) + kq(j-i)$$
$$\phi_h(y) - \phi_h(x) > 0 \text{ since } j > i \text{ \& by (8)}$$
$$\Rightarrow \phi_h(y) \ne \phi_h(x)$$

Case 1.2.2. If $r > s$. Then by (2), $\phi_{g_i}(u_r^i) \geq \phi_{g_j}(u_s^j)$
Consider,

$$\phi_h(y) - \phi_h(x) = \phi_{g_j}(u_s^j) - \phi_{g_i}(u_r^i) + kq(j-i)$$

Since g_i and g_j are antimagic labeling of H_i and H_j respectively, by Observation 1, we have,

$$\phi_{g_j}(u_s^j) - \phi_{g_i}(u_r^i) \geq k^2 - kq$$
$$\therefore\ \phi_h(y) - \phi_h(x) \geq k^2 - kq + kq(j-i)$$
$$\phi_h(y) - \phi_h(x) \geq k^2 + kq(j-i-1).$$

$$\Rightarrow \phi_h(y) - \phi_h(x) > 0. \textit{ Since } k^2 > 0 \textit{ and } kq(j-i-1) \geq 0.$$
Hence, $\phi_h(y) \neq \phi_h(x)$

Case 2. For any two distinct vertices x and y in $G \circ_v H$ such that x in G and y in H_j for $j = 1, 2, \cdots, n$ such that $y \neq u_p^j$.

By definition on naming the vertices of G, $x = v_i = u_p^i$ and by definition on the naming of the vertices of H_j, $y = u_r^j$ for some r, $1 \leq r \leq p-1$. Then by definition of h and by (6) and (7). We have

$$\phi_h(x) = \phi_h(v_i) = \phi_h(u_p^i) = \phi_f(v_i) + \phi_{g_i}(u_p^i) + kq(i-1) + tnq.$$
$$\phi_h(y) = \phi_h(u_r^j) = \phi_{g_j}(u_r^j) + kq(j-1).$$

By (4), we have

$$\phi_{g_i}(u_p^i) > \phi_{g_j}(u_r^j) \tag{9}$$

Case 2.1. When $i \geq j$ for $1 \leq j \leq i \leq n$. Consider

$$\phi_h(x) - \phi_h(y) = \phi_f(v_i) + \phi_{g_i}(u_p^i) - \phi_{g_j}(u_r^j) + kq(i-j) + tnq$$

By (9),

$$\phi_h(x) - \phi_h(y) \geq \phi_f(v_i) + kq(i-j) + tnq$$

Since, $kq(i-j) \geq 0$, $tnq > 0$, $\phi_f(v_i) > 0 \Rightarrow$ $\phi_h(x) - \phi_h(y) > 0$. Hence, $\phi_h(x) \neq \phi_h(y)$.

Case 2.2. When $i < j$ for $1 \leq i < j \leq n$. Consider

$$\phi_h(x) - \phi_h(y) = \phi_f(v_i) + \phi_{g_i}(u_p^i) - \phi_{g_j}(u_r^j) + kq(i-j) + tnq$$

By (9),

$$\phi_h(x) - \phi_h(y) \geq \phi_f(v_i) + kq(i-j) + tnq$$

By definition of f, $\phi_f(v_i) \geq \dfrac{t(t+1)}{2}$.

$$\Rightarrow \phi_h(x) - \phi_h(y) \geq \frac{t(t+1)}{2} + kq(i-j) + tnq$$

Since $t \geq k$ & $-1 \leq i - j \leq 1 - n$

$$\Rightarrow \phi_h(x) - \phi_h(y) \geq \frac{k(k+1)}{2} + kq(1-n) + knq$$
$$\phi_h(x) - \phi_h(y) > 0 \because k > 0 \,\&\, kq > 0.$$
$$\Rightarrow \phi_h(x) \neq \phi_h(y)$$

Case 3. Both the vertices are from G. By definition on the naming of the vertices of G, $x = u_p^i$ and $y = u_p^j$ for some $i, j, 1 \leq i, j \leq n$ & $i \neq j$. By (6),

$$\phi_h(x) = \phi_h(u_p^i) = \phi_f(v_i) + \phi_{g_i}(u_p^i) + kq(i-1) + tnq$$
$$\phi_h(y) = \phi_h(u_p^j) = \phi_f(v_j) + \phi_{g_j}(u_p^j) + kq(j-1) + tnq$$

Without loss of generality, we consider $i < j$, therefore by (1) and $\phi_f(v_j) - \phi_f(v_i) > 0$ and by (4), $\phi_{g_i}(u_p^i) < \phi_{g_j}(u_p^j)$. Consider,

$$\phi_h(y) - \phi_h(x) = \phi_f(v_j) - \phi_f(v_i) + \phi_{g_j}(u_p^j) - \phi_{g_i}(u_p^i) + kq(j-i)$$

Since, $\phi_f(v_j) - \phi_f(v_i) > 0, \phi_{g_j}(u_p^j) - \phi_{g_i}(u_p^i) > 0$ & $kq(j-i) > 0. \Rightarrow \phi_h(y) - \phi_h(x) > 0$. Hence, $\phi_h(y) \neq \phi_h(x)$. $\square$

Figures 1–4 illustrate the proof Theorem 2. An antimagic labeling of a 3-regular graph G and the antimagic labeling of a 2-regular graph H are given in Figures 1 and 2, respectively. In Figure 3, the six copies of the graph H are considered with their labeling function $g_i, 1 \leq i \leq 6$. The rooted product of the graph G and H with their antimagic labeling is given in Figure 4. Here, the root vertex of the graph H is chosen as u_3.

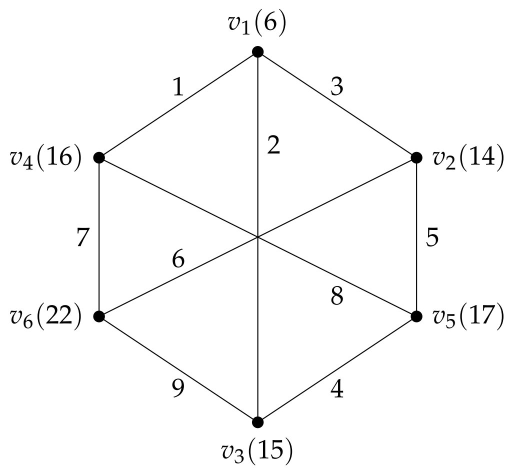

Figure 1. A graph G with $t = 3,\ n = 6,\ m = 9$.

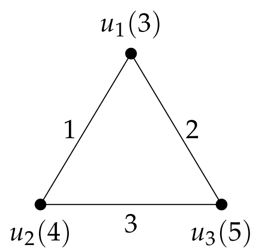

Figure 2. A graph H with $k = 2,\ p = 3,\ q = 3$.

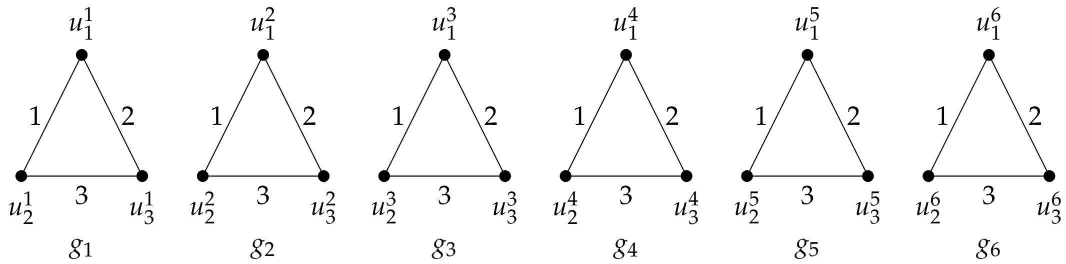

Figure 3. 6 copies of H.

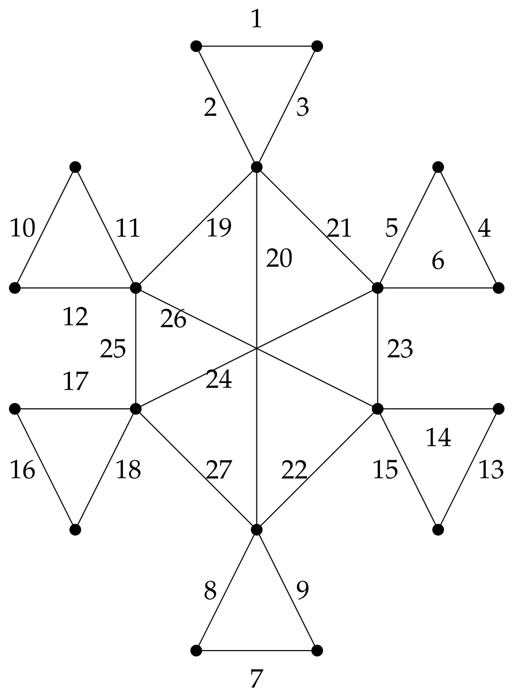

Figure 4. Antimagic labeling of $G \circ_v H$.

Theorem 3. *Let G be a connected t-regular graph and H be a connected k-regular graph then the corona product of G and H admits antimagic labeling, $\forall\ t, k \geq 2$.*

Proof. Let G be a t-regular graph with n vertices and m edges and let H be a k-regular graph with p vertices and q edges. As G and H are regular graphs, by Theorem 1, they admits an antimagic labeling. Let f and g be antimagic labeling of G and H respectively. By definition of f, $f : E(G) \to \{1, 2, \cdots, m\}$ such that $\phi_f(u) \neq \phi_f(v)$ for any two distinct vertices u and v in G. By definition of g, $g : E(H) \to \{1, 2, \cdots, q\}$ such that $\phi_g(x) \neq \phi_g(y)$ for any two distinct vertices x and y in H.

Let us name the vertices of G as $v_1, v_2, \cdots, v_n$ such that,

$$\phi_f(v_1) < \phi_f(v_2) < \cdots < \phi_f(v_n) \tag{10}$$

and also name the vertices of H as $u_1, u_2, \cdots, u_p$ such that,

$$\phi_g(u_1) < \phi_g(u_2) < \cdots < \phi_g(u_p). \tag{11}$$

Construct the corona product of G and H, $G \odot H$ by joining the edge with i^{th} vertex of G with each vertex from i^{th} copy of H. Note that the number of edges in $G \odot H$ is $n(p+q) + m$. Let us name the vertices of $G \odot H$ as follows. The vertices of G are named as the same as the earlier, that is $v_1, v_2, \cdots, v_n$ and then name the vertices of H_i, for $i = 1, 2, \cdots, n$ (i^{th} isomorphic copy of H) as $u_1^i, u_2^i, \cdots, u_p^i$. That is the vertex u_l in H is now has the name

u_l^i in H_i for $l = 1, 2, \cdots, p$. Before defining the antimagic labeling of $G \odot H$, we label the edges of H_i by using the edge labeling g of H as follows:

$$g_i : E(H_i) \to \{1, 2, \cdots, q\} \tag{12}$$

for an edge $e = (u_a^i, u_b^i)$ in H_i, $g_i(e) = g(e')$ where e' as an edge (u_a, u_b) in H that corresponding to the edge e in H_i. Then by definition of g_i and (11) for each i, $i = 1, 2, \cdots, n$.

$$\phi_{g_i}(u_1^i) < \phi_{g_i}(u_2^i) < \cdots < \phi_{g_i}(u_p^i). \tag{13}$$

where $u_1^i, u_2^i, \cdots, u_p^i$ are the vertices of H_i in $G \odot H$.

Now we define $h : E(G \odot H) \to \{1, 2, \cdots, n(p+q)+m\}$ by,

$$h(e) = \begin{cases} g_i(e) + (i-1)q, & \text{if } e \in E(H_i) \\ f(e) + n(p+q), & \text{if } e \in E(G) \\ l + nq + p(i-1), & \text{if } v_i u_l^i \in E(G \odot H) \text{ for } i = 1, 2, \cdots, n \;\&\; l = 1, 2, \cdots, p \end{cases} \tag{14}$$

From the above labeling h, we observe that, for all $i, 1 \le i \le n$

$$\phi_h(v_i) = \phi_f(v_i) + \frac{p(p+1)}{2} + npq + p^2(i-1) + nt(p+q) \tag{15}$$

and

$$\phi_h(u_l^i) = \phi_{g_i}(u_l^i) + nq + l + (kq+p)(i-1), \; \forall\, i = 1, 2, \cdots, n \;\&\; l = 1, 2, \cdots, p. \tag{16}$$

In order to prove that h is an antimagic labeling of $G \odot H$, we need to prove that for any two distinct vertices x and y in $G \odot H$ such that $\phi_h(x) \neq \phi_h(y)$. We consider the following possible cases on the vertices of x and the vertices of y in $G \odot H$.

(i) x in H_i and y in H_j for $i, j = 1, 2, \cdots, n$.
(ii) x in G and y in H_j for $j = 1, 2, \cdots, n$.
(iii) x and y are the vertices of G.

Case 1. For any two distinct vertices x and y in $G \odot H$, where x is in H_i and y in H_j, for each $i, j, 1 \le i, j \le n$.

Case 1.1. When $i = j$, then both the vertex x and y are from H_i. By definition on the naming of the vertices of H_i, $x = u_r^i$ and $y = u_s^i$ for some $r, s, 1 \le r, s \le p$. Then, by definition of h and by (16), we have

$$\phi_h(x) = \phi_h(u_r^i) = \phi_{g_i}(u_r^i) + nq + r + (kq+p)(i-1)$$
$$\phi_h(y) = \phi_h(u_s^i) = \phi_{g_i}(u_s^i) + nq + s + (kq+p)(i-1)$$

Without loss of generality, we assume that $r < s$. By (13), we have $\phi_{g_i}(u_r^i) < \phi_{g_i}(u_s^i)$, therefore $\phi_h(x) < \phi_h(y)$. Hence $\phi_h(x) \neq \phi_h(y)$.

Case 1.2. When $i \neq j$, then the vertex x in H_i and the vertex y in H_j. By definition on the naming of vertices of H_i and H_j, let $x = u_r^i$ and $y = u_s^j$ for some $r, s, 1 \le r, s \le p$.

Without loss of generality we assume $i < j$. Then by definition of h and by (16) we have

$$\phi_h(x) = \phi_h(u_r^i) = \phi_{g_i}(u_r^i) + nq + r + (kq+p)(i-1)$$
$$\phi_h(y) = \phi_h(u_s^j) = \phi_{g_j}(u_s^j) + nq + s + (kq+p)(j-1)$$

Case 1.2.1. If $r \le s$. Then by (13) we have, $\phi_{g_i}(u_r^i) \le \phi_{g_j}(u_s^j)$. Consider

$$\phi_h(y) - \phi_h(x) = \phi_{g_j}(u_s^j) - \phi_{g_i}(u_r^i) + (s-r) + (kq+p)(j-i)$$

Since $s-r\geq 0$ and $j-i>0$, $\phi_{g_j}(u_s^j)-\phi_{g_i}(u_r^i)\geq 0$

$$\text{Hence, } \phi_h(y)-\phi_h(x)>0 \Rightarrow \phi_h(y)\neq\phi_h(x)$$

Case 1.2.2. If $r>s$. Then by (13) we have, $\phi_{g_i}(u_r^i)>\phi_{g_j}(u_s^j)$ Consider $\phi_h(y)-\phi_h(x)=\phi_{g_j}(u_s^j)-\phi_{g_i}(u_r^i)+(s-r)+(kq+p)(j-i)$.

Since g_i and g_j are the antimagic labelings of H_i and H_j, respectively, and by Observation 1, $\phi_{g_j}(u_s^j)-\phi_{g_i}(u_r^i)\geq k^2-kq$ and also $1-p\leq s-r\leq -1$

$$\begin{aligned}\phi_h(y)-\phi_h(x)&\geq k^2-kq+1-p+kq(j-i)+p(j-i)\\ \phi_h(y)-\phi_h(x)&\geq k^2+1 \because 1\leq j-i\leq n-1\end{aligned}$$

Hence, $\phi_h(y)-\phi_h(x)>0$ (since $k^2>0$) $\Rightarrow \phi_h(y)\neq\phi_h(x)$.

Case 2. For any two distinct vertices x and y in $G\odot H$ such that x in G and y in H_j for $j=1,2,\cdots,n$.

By definition on naming the vertices of G, $x=v_i$ and by definition on the naming of the vertices of H_j, let $y=u_s^j$ for some s, $1\leq s\leq p$. Then, by definition of h and by (15) and (16), we have

$$\begin{aligned}\phi_h(x)&=\phi_h(v_i)=\phi_f(v_i)+\frac{p(p+1)}{2}+npq+p^2(i-1)+nt(p+q)\\ \phi_h(y)&=\phi_h(u_s^j)=\phi_{g_j}(u_s^j)+nq+s+(kq+p)(j-1).\end{aligned}$$

Consider

$$\begin{aligned}\phi_h(x)-\phi_h(y)=\phi_f(v_i)-\phi_{g_j}(u_s^j)+\frac{p(p+1)}{2}+npq+p^2(i-1)+ntp+ntq\\ -nq-s-(kq+p)(j-1)\end{aligned}$$

We apply maximum value for the negative terms of above equations i.e., $s\leq p,\ j\leq n$ and by Lemma 1, $\phi_{g_j}(u_s^j)\leq kq-\dfrac{k(k-1)}{2}$

$$\begin{aligned}\phi_h(x)-\phi_h(y)&=\phi_f(v_i)-kq+\frac{k(k-1)}{2}+\frac{p(p+1)}{2}+npq+p^2(i-1)+ntp\\ &\quad+ntq-nq-p-nkq-np+kq+p\\ &=\phi_f(v_i)+\frac{k(k-1)}{2}+\frac{p(p+1)}{2}+nq(p-k)+p^2(i-1)\\ &\quad+np(t-1)+nq(t-1)\end{aligned}$$

Since, $i-1\geq 0, t-1\geq 1, p\geq k$ and $k,p,\phi_f(v_i)>0$. Hence, $\phi_h(x)-\phi_h(y)>0 \Rightarrow \phi_h(x)\neq\phi_h(y)$.

Case 3. x and y are the vertices of G. By definition on the naming of the vertices of G, $x=v_i$ and $y=v_j$ for some $i,j, 1\leq i,j\leq n$ & $i\neq j$. By (15),

$$\begin{aligned}\phi_h(x)&=\phi_h(v_i)=\phi_f(v_i)+\frac{p(p+1)}{2}+npq+p^2(i-1)+nt(p+q)\\ \phi_h(y)&=\phi_h(v_j)=\phi_f(v_j)+\frac{p(p+1)}{2}+npq+p^2(j-1)+nt(p+q)\end{aligned}$$

Without loss of generality, we consider $i<j$, therefore by (10), $\phi_f(v_i)<\phi_f(v_j)$ Consider

$$\begin{aligned}\phi_h(y) - \phi_h(x) &= \phi_h(v_j) - \phi_h(v_i) \\ &= \phi_f(v_j) - \phi_f(v_i) + p^2(j-i) \text{ (since, } \phi_f(v_j) > \phi_f(v_i)) \\ \phi_h(y) - \phi_h(x) &> 0.\end{aligned}$$

Hence, $\phi_h(y) \neq \phi_h(x)$.

□

Figures 5–7 illustrate the proof Theorem 3. An antimagic labeling of a 2-regular graph G and the antimagic labeling of a 3-regular graph H are given in Figures 5 and 6 respectively. The corona product of the graph G and H with their antimagic labeling is given in Figure 7.

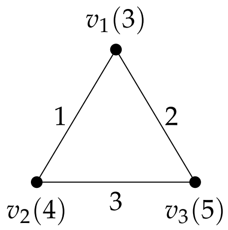

Figure 5. A graph G with $t = 2,\ n = 3,\ m = 3$.

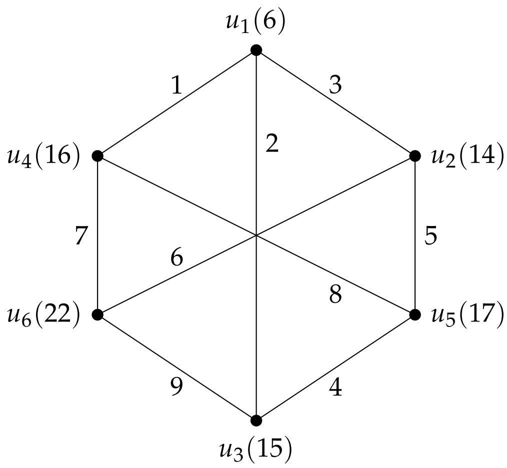

Figure 6. A graph H with $k = 3,\ p = 6,\ q = 9$.

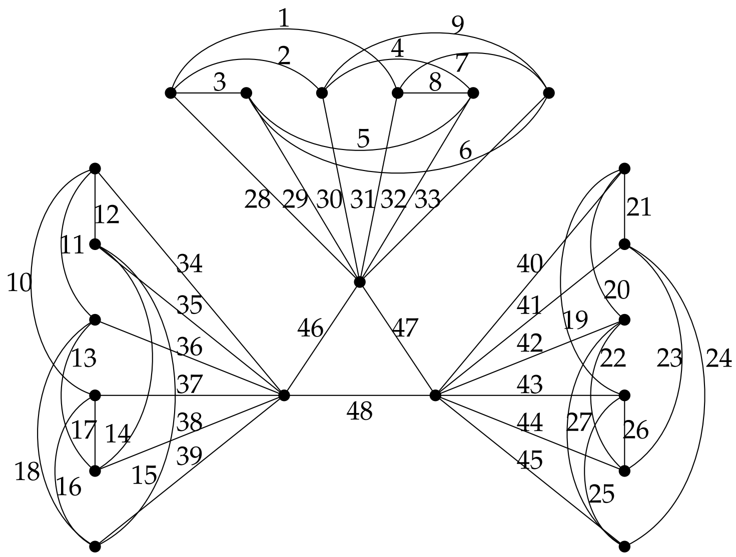

Figure 7. Antimagic labeling of $G \odot H$.

3. Conclusions

We have given an antimagic labeling for the rooted product of graphs G and H where G is a t-regular connected graph and H is a k-regular connected graph with the condition $t \geq k$. Moreover, we proved that there exists an antimagic labeling of the corona product of graph G and H where G is a t-regular connected graph and H is a k-regular connected graph for all $t, k \geq 2$.

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