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Characterizations of Well-Posedness for Generalized Hemivariational Inequalities Systems with Derived Inclusion Problems Systems in Banach Spaces

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Abstract: In real Banach spaces, the concept of α-well-posedness is extended to the class of generalized hemivariational inequalities systems consisting of two parts which are of symmetric structure mutually. First, certain concepts of α-well-posedness for generalized hemivariational inequalities systems are put forward. Second, certain metric characterizations of α-well-posedness for generalized hemivariational inequalities systems are presented. Lastly, certain equivalence results between strong α-well-posedness of both the system of generalized hemivariational inequalities and its system of derived inclusion problems are established.

Keywords: generalized hemivariational inequalities systems; well-posedness; characterizations; Clarke's generalized subdifferential; derived inclusion problems systems

MSC: 49J40; 47J22; 49J52

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1. Introduction

Tykhonov's well-posedness put forward in [1] has been playing an important role in the study of optimization problems and their related problems such as variational inequalities, inclusion problems, Nash equilibrium problems, etc. For more than the last 50 years, a large number of results regarding well-posedness for optimization problems have been established in the literature; these can be seen, e.g., in [2–11] and the references therein. In particular, Lucchetti and Patrone [12] extended the concept of well-posedness for optimization problems to the variational inequalities in 1981. Using Ekeland's variational principle, they presented the characterization of Tykhonov's well-posedness for minimization problems involving convex and lower semicontinuous (l.s.c.) functions on nonempty, convex and closed sets.

In 1995, Goeleven and Mentagui [13] first put forward the notion of well posedness for hemivariational inequalities (HVIs) and established certain elementary results for well-posed HVIs. Very recently, Wang et al. [14] built the equivalence between the well-posedness of both the hemivariational inequalities system (SHVI) and its derived inclusion problems system (SDIP), i.e., an inclusion problems system which is equivalent to the SHVI. Meanwhile, Ceng, Liou and Wen [15] extended the concept of α-well-posedness to the class of generalized hemivariational inequalities (GHVIs), gave certain metric characterizations of α-well-posedness for GHVIs, and established the equivalence between α-well-posedness of both the GHVI and its derived inclusion problem (DIP), i.e., an inclusion problem which is equivalent to the GHVI. Additionally, Ceng and Lin [16] introduced and considered the α-well-posedness for systems of mixed quasivariational-like inequalities (SMQVLIs) in Banach spaces, and furnished certain metric characterizations of α-well-posedness for SMQVLIs.

Suppose that V_k is a real Banach space with its dual V_k^* for $k = 1, 2$. For $k = 1, 2$, we denote by $\langle \cdot, \cdot \rangle_{V_k^* \times V_k}$ the duality pairing between V_k and V_k^* and by $|| \cdot ||_{V_k}$ and $|| \cdot ||_{V_k^*}$ the norms of spaces V_k and V_k^*, respectively. It is well known that the product space $\mathcal{V} = V_1 \times V_2$ is still a real Banach space endowed with the norm below:

$$||\mathbf{u}||_{\mathcal{V}} = ||u_1||_{V_1} + ||u_2||_{V_2} \quad \forall \mathbf{u} = (u_1, u_2) \in \mathcal{V}.$$

For $k = 1, 2$, let $A_k : V_1 \times V_2 \to 2^{V_k^*}$ be a nonempty set-valued mapping, $J : V_1 \times V_2 \to \mathbf{R}$ be a locally Lipschitz functional on $\mathcal{V}$ and f_k be a given point in V_k^*.

In this paper, we consider the system of generalized hemivariational inequalities (SGHVI), which consists of finding $\mathbf{u} = (u_1, u_2) \in \mathcal{V}$ s.t. for certain $(\omega_1, \omega_2) \in A_1(u_1, u_2) \times A_2(u_1, u_2)$,

$$(\mathbf{SGHVI}) \quad \begin{cases} \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq 0 & \forall v_1 \in V_1, \\ \langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0 & \forall v_2 \in V_2, \end{cases}$$

where, for $k \neq j = 1, 2$, $J_k^\circ(u_k, u_j; v_k - u_k)$ indicates Clarke's generalized directional derivative of functional $J(\cdot, u_j)$ at u_k in the direction $v_k - u_k$, with $J(\cdot, u_j)$ being a functional on V_k for any fixed $u_j \in V_j$, that is,

$$J_k^\circ(u_k, u_j; v_k - u_k) = \limsup_{w \to u_k, \lambda \downarrow 0} \frac{J(w + \lambda(v_k - u_k), u_j) - J(w, u_j)}{\lambda}.$$

It is worth pointing out that the above SGHVI consists of two parts, which are of symmetric structure mutually.

In particular, if A_k is a single-valued mapping for $k = 1, 2$, then the above SGHVI reduces to the following system of hemivariational inequalities (SHVI) investigated in [14]:

Find $\mathbf{u} = (u_1, u_2) \in \mathcal{V}$ s.t.

$$(\mathbf{SHVI}) \quad \begin{cases} \langle A_1(u_1, u_2) - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq 0 & \forall v_1 \in V_1, \\ \langle A_2(u_1, u_2) - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0 & \forall v_2 \in V_2. \end{cases}$$

Inspired by the above research works on well posedness, we shall extend the concept of α-well-posedness to the class of SGHVIs in Banach spaces, present certain metric characterizations of α-well-posedness for SGHVIs, and establish the equivalence between the α-well-posedness of both the SGHVI and its SDIP. The architecture of this article is organized below: in Section 2, we present some concepts and basic tools for further use. In Section 3, we define certain notions of α-well-posedness for SGHVIs and, under two assumptions imposed on the operators involved, provide certain metric characterizations of α-well-posedness for SGHVIs. In Section 4, we establish two equivalence results between the α-well-posedness of both the SGHVI and its SDIP.

2. Preliminaries

First of all, we recall certain vital concepts and helpful results on nonlinear analysis, optimization theory and nonsmooth analysis, which can be found in [17–21] . Let E be a real Banach space with its dual E^*. Let v and $\{v_n\}$ be a point and a sequence in E, and let v^* and $\{v_n^*\}$ be a point and a sequence in E^*, respectively. We use the notations $v_n \to v$, $v_n \rightharpoonup v$ and $v_n^* \overset{*}{\rightharpoonup} v^*$ to represent the strong convergence of $\{v_n\}$ to v, the weak convergence of $\{v_n\}$ to v and the weak* convergence of $\{v_n^*\}$ to v^*, respectively. Recall that, if E is not reflexive, then the weak* topology of E^* is weaker than its weak topology and that if E is reflexive, then the weak* topology of V^* coincides with its weak topology. It is readily known that if $\{v_n\} \subset E$, $\{v_n^*\} \subset E^*$, $v_n \to v$ in E and $v_n^* \overset{*}{\rightharpoonup} v^*$ in E^*, then $\langle v_n^*, v_n \rangle_{E^* \times E} \to \langle v^*, v \rangle_{E^* \times E}$ as $n \to \infty$.

Definition 1. *Let $\varphi : E \to \mathbf{R}$ be a functional on E. φ is referred to as being*

(i) Lipschitz continuous on E iff $\exists L > 0$ s.t.

$$|\varphi(v_1) - \varphi(v_2)| \leq L||v_1 - v_2||_E \quad \forall v_1, v_2 \in E;$$

(ii) Locally Lipschitz continuous on E iff $\forall v \in E$, $\exists$ (neighborhood) $N(v)$ and $\exists L_v > 0$ s.t.

$$|\varphi(v_1) - \varphi(v_2)| \leq L_v||v_1 - v_2||_E \quad \forall v_1, v_2 \in N(v).$$

Definition 2. *Let V_1, V_2 be two real Banach spaces and $J : V_1 \times V_2 \to \mathbf{R}$ be a functional on $V_1 \times V_2$. The functional J is referred to as being:*

(i) Lipschitz continuous in the first variable iff the functional $J(\cdot, v_2) : V_1 \to \mathbf{R}$ is Lipschitz continuous on V_1 for any fixed $v_2 \in V_2$;

(ii) Locally Lipschitz continuous in the first variable, iff the functional $J(\cdot, u_2) : V_1 \to \mathbf{R}$ is locally Lipschitz continuous on V_1 for any fixed $v_2 \in V_2$.

In a similar way, the Lipschitz continuity and locally Lipschitz continuity of the functional $J : V_1 \times V_2 \to \mathbf{R}$ in the second variable can be formulated, respectively.

Suppose that $\varphi : E \to \mathbf{R}$ be a locally Lipschitz functional on E, u is a given point and v is a directional vector in E. The Clarke's generalized directional derivative (CGDD) of φ at the point u in the direction v, denoted by $\varphi^\circ(u; v)$, is formulated below

$$\varphi^\circ(u; v) = \limsup_{w \to u, \lambda \downarrow 0} \frac{\varphi(w + \lambda v) - \varphi(w)}{\lambda}.$$

According to the CGDD, Clarke's generalized subdifferential (CGS) of φ at u, denoted by $\partial\varphi(u)$, is the set in the dual space E^*, formulated below

$$\partial\varphi(u) = \{\xi \in E^* : \varphi^\circ(u; v) \geq \langle \xi, v \rangle_{E^* \times E}, \forall v \in E\}.$$

The following proposition provides some basic properties for the CGDD and the CGS; as can be seen in, e.g., [18,20,22–24]and the references therein.

Proposition 1. *Let $\varphi : E \to \mathbf{R}$ be a locally Lipschitz functional on E and let $u, v \in E$ be two given elements. Then:*

(i) The function $v \mapsto \varphi^\circ(u; v)$ is finite, positively homogeneous, subadditive and thus convex on E;

(ii) $\varphi^\circ(u; v)$ is upper semicontinuous (u.s.c.) on $E \times E$ as a function of (u, v), as a function of v alone, is Lipschitz continuous on E;

(iii) $\varphi^\circ(u; -v) = (-\varphi)^\circ(u; v)$;

(iv) For all $u \in E$, $\partial\varphi(u)$ is a nonempty, convex, bounded and weak-compact set in E^*;*

(v) For all $v \in E$, one has

$$\varphi^\circ(u; v) = \max\{\langle \xi, v \rangle_{E^* \times E} : \xi \in \partial\varphi(u)\};$$

(vi) The graph of the Clarke's generalized subdifferential $\partial\varphi(u)$ is closed in $E \times (w^* - E^*)$ topology, with $(w^* - E^*)$ being the space E^* endowed with the weak topology, i.e., if $\{u_n\} \subset E$ and $\{u_n^*\} \subset E^*$ are sequences s.t. $u_n^* \in \partial\varphi(u_n)$, $u_n \to u$ in E and $u_n^* \to u^*$ weakly* in E^*, then $u^* \in \partial\varphi(u)$.*

Definition 3. *(i) A single-valued operator $T : E \to E^*$ is referred to as being monotone, iff*

$$\langle Tu - Tv, u - v \rangle_{E^* \times E} \geq 0 \quad \forall u, v \in E;$$

(ii) A set-valued operator $F : E \to 2^{E^*}$ is referred to as being monotone, iff

$$\langle u^* - v^*, u - v \rangle \geq 0 \quad \forall u, v \in E, u^* \in F(u), v^* \in F(v).$$

Definition 4 (see [19])**.** *Let S be a nonempty set in E. The measure of noncompactness (MNC) μ of the set S is formulated below*

$$\mu(S) = \inf\{\epsilon > 0 : S \subset \bigcup_{k=1}^{n} S_k \text{ and } \operatorname{diam}(S_k) < \epsilon \ \forall k \in \{1, 2, ..., n\}\},$$

where $\operatorname{diam}(S_k)$ *indicates the diameter of set S_k.*

Let A_1, A_2 be the nonempty subsets of E. The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between A_1 and A_2 is formulated by

$$\mathcal{H}(A_1, A_2) = \max\{e(A_1, A_2), e(A_2, A_1)\},$$

where $e(A_1, A_2) = \sup_{a \in A_1} d(a, A_2)$ with $d(a, A_2) = \inf_{b \in A_2} ||a - b||_E$. It is worth pointing out that certain additional properties of the Hausdorff metric between two sets can be found in [19]. In addition, we note that [25], if A_1 and A_2 are compact subsets in E, we know that $\forall a \in A_1$, $\exists b \in A_2$ s.t.

$$||a - b||_E \leq \mathcal{H}(A_1, A_2).$$

Definition 5 (see [26])**.** *Let $\mathcal{H}(\cdot, \cdot)$ be the Hausdorff metric on the collection $CB(E^*)$ of all nonempty, closed and bounded subsets of E^*, formulated below*

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

for A and B in $CB(E^*)$. A set-valued operator $F : E \to CB(E^*)$ is referred to as being

(i) $\mathcal{H}$-hemicontinuous, if for any $u, v \in E$, the function $t \mapsto \mathcal{H}(F(u + t(v - u)), F(u))$ from $[0, 1]$ into $[0, +\infty)$ is continuous at 0^+;

(ii) $\mathcal{H}$-continuous, if $\forall \epsilon > 0$ and $\forall$ (fixed) $u_0 \in E$, $\exists \delta > 0$ s.t. $\forall v \in E$ with $||v - u_0||_E < \delta$, one has $\mathcal{H}(F(v), F(u_0)) < \epsilon$.

It is remarkable that the $\mathcal{H}$-continuity ensures the $\mathcal{H}$-hemicontinuity, but the converse is generally not true. In the end, we recall a theorem in [27], which is very vital for deducing our main results.

Theorem 1 (see [27])**.** *Suppose that C is nonempty, closed and convex in E and C^* is nonempty, closed, convex and bounded in E^*. Let $\varphi : E \to \mathbf{R}$ be a proper convex l.s.c. functional and $v \in C$ be arbitrary. Assume that $\forall u \in C$, $\exists u^*(u) \in C^*$ s.t.*

$$\langle u^*(u), u - v \rangle_{E^* \times E} \geq \varphi(v) - \varphi(u).$$

Then, $\exists v^* \in C^*$ s.t.

$$\langle v^*, u - v \rangle_{E^* \times E} \geq \varphi(v) - \varphi(u) \quad \forall u \in C.$$

3. Metric Characterizations of Well-Posedness for SGHVIs

In this section, we introduce certain notions of α-well-posedness for SGHVIs and establish certain metric characterizations of α-well-posedness for SGHVIs under certain appropriate conditions.

On the basis of certain notions of well-posedness in [2,15,16,26,28–34], we first introduce certain definitions of α-well-posedness for SGHVIs. For $k = 1, 2$, let $\alpha_k : V_k \to [0, +\infty)$

be convex, continuous, and positively homogeneous, i.e., $\alpha_k(sv_k) = s\alpha_k(v_k)$ for all $v_k \in V_k$ and $s \geq 0$.

Definition 6. *A sequence $\{\mathbf{u}^n\} \subset V_1 \times V_2$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is referred to as being an α-approximating sequence with $\alpha = (\alpha_1, \alpha_2)$ for the SGHVI iff $\exists(\omega_1^n, \omega_2^n) \in A_1(u_1^n, u_2^n) \times A_2(u_1^n, u_2^n), n \in \mathbf{N}$ and $\exists\{\epsilon_n\} \subset [0, +\infty)$ with $\epsilon_n \to 0$ $(n \to \infty)$ s.t.*

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n \alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1, \\ \langle \omega_2^n - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n \alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2. \end{cases}$$

In particular, if for $k = 1, 2$, A_k is single-valued and $\alpha_k(x_k - y_k) = ||x_k - y_k||_{V_k}$ $\forall x_k, y_k \in V_k$, then $\{\mathbf{u}^n\}$ is referred to as being an approximating sequence for SHVI (see [14]).

Definition 7. *The SGHVI is referred to as being strongly (and weakly, respectively) α-well-posed with $\alpha = (\alpha_1, \alpha_2)$ iff it has a unique solution and every α-approximating sequence for the SGHVI converges strongly (and weakly, respectively) to the unique solution. In particular, if for $k = 1, 2$, A_k is single-valued and $\alpha_k(x_k - y_k) = ||x_k - y_k||_{V_k}$ $\forall x_k, y_k \in V_k$, then the SHVI is referred to as being strongly (and weakly, respectively) well-posed (see [14]).*

It is evident that the strong α-well-posedness of the SGHVI ensures the weak α-well-posedness of the SGHVI, but the converse is generally not valid.

Definition 8. *The SGHVI is referred to as being strongly (and weakly, respectively) α-well-posed in the generalized sense if the solution set of the SGHVI is nonempty and, for every α-approximating sequence, there always exists a subsequence converging strongly (and weakly, respectively) to some point of the solution set. In particular, if for $k = 1, 2$, A_k is single-valued and $\alpha_k(x_k - y_k) = ||x_k - y_k||_{V_k}$ $\forall x_k, y_k \in V_k$, then the SHVI is referred to as being strongly (and weakly, respectively) well-posed in the generalized sense (see [14]).*

In a similar way, the strong α-well-posedness in the generalized sense for the SGHVI ensures the weak α-well-posedness in the generalized sense for the SGHVI, but the converse is not valid in general. Obviously, the notions of strong and weak α-well-posedness of the SGHVI put forward in this paper are quite different from those of Definitions 3.1–3.2 and 3.4 in Wang et al. [14]. In order to establish the metric characterizations of α-well-posedness for SGHVI, for any $\epsilon > 0$, we first formulate two sets in $\mathcal{V} = V_1 \times V_2$ below:

$$\begin{aligned} \Omega_\alpha(\epsilon) \; = \{ & (u_1, u_2) \in V_1 \times V_2 : \text{ for some } (\omega_1, \omega_2) \in A_1(u_1, u_2) \times A_2(u_1, u_2), \\ & \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq -\epsilon \alpha_1(v_1 - u_1) \;\; \forall v_1 \in V_1, \\ & \langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq -\epsilon \alpha_2(v_2 - u_2) \;\; \forall v_2 \in V_2 \}, \end{aligned}$$

and

$$\begin{aligned} \Delta_\alpha(\epsilon) \; = \{ & (u_1, u_2) \in V_1 \times V_2 : \text{ for all } (v_1, v_2) \in V_1 \times V_2, \\ & \langle \nu_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq -\epsilon \alpha_1(v_1 - u_1) \;\; \forall \nu_1 \in A_1(v_1, u_2), \\ & \langle \nu_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq -\epsilon \alpha_2(v_2 - u_2) \;\; \forall \nu_2 \in A_2(u_1, v_2) \}. \end{aligned}$$

In order to show certain properties of sets $\Omega_\alpha(\epsilon)$ and $\Delta_\alpha(\epsilon)$, we first impose certain hypotheses on the operators A_1, A_2 and J in the SGHVI.

(HA): (a) $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ is monotone in the first variable, i.e., $\forall u_1, v_1 \in V_1$ and $u_2 \in V_2$,

$$\langle v_1^* - u_1^*, v_1 - u_1 \rangle_{V_1^* \times V_1} \geq 0 \quad \forall v_1^* \in A_1(v_1, u_2), u_1^* \in A_1(u_1, u_2);$$

(b) $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ is monotone in the second variable, i.e., $\forall u_1 \in V_1$ and $u_2, v_2 \in V_2$,

$$\langle v_2^* - u_2^*, v_2 - u_2 \rangle_{V_2^* \times V_2} \geq 0 \quad \forall v_2^* \in A_2(u_1, v_2), u_2^* \in A_2(u_1, u_2);$$

(c) $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ is a nonempty compact-valued mapping which is $\mathcal{H}$-hemicontinuous;
(d) $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ is a nonempty compact-valued mapping which is $\mathcal{H}$-hemicontinuous;
(e) $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ is a nonempty compact-valued mapping which is $\mathcal{H}$-continuous;
(f) $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ is a nonempty compact-valued mapping which is $\mathcal{H}$-continuous.

(HJ): (a) $J : V_1 \times V_2 \to \mathbf{R}$ is locally Lipschitz with respect to the first variable and second variable on $V_1 \times V_2$;
(b) $J(u_1, u_2) + J(v_1, v_2) = J(u_1, v_2) + J(v_1, u_2)$ $\forall \mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathcal{V} = V_1 \times V_2$.

Lemma 1 (see ([14], Lemma 3.6))**.** *Suppose that the functional $J : V_1 \times V_2 \to \mathbf{R}$ satisfies the hypotheses (a), (b) in* **(HJ)***. Then, for any sequence $\mathbf{u}^n = (u_1^n, u_2^n) \in \mathcal{V}$ strongly converging towards $\mathbf{u} = (u_1, u_2) \in \mathcal{V}$ and $v_k^n \in V_k$ strongly converging towards $v_k \in V_k$, one has*

$$\limsup_{n \to \infty} J_k^\circ(u_1^n, u_2^n; v_k^n) \leq J_k^\circ(u_1, u_2; v_k), \tag{1}$$

where $k = 1, 2$.

Proposition 2. *Suppose that $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ and $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ satisfy the hypotheses (a), (b), (c), (d) in* **(HA)** *and $J : V_1 \times V_2 \to \mathbf{R}$ satisfies the hypothesis* **(HJ)***. Then, $\Omega_\alpha(\epsilon) = \Delta_\alpha(\epsilon)$ $\forall \epsilon > 0$.*

Proof. From the monotonicity of operators A_1 in the first variable and A_2 in the second variable, it follows that $\langle \nu_1 - \omega_1, v_1 - u_1 \rangle_{V_1^* \times V_1} \geq 0$ $\forall \nu_1 \in A_1(v_1, u_2), \omega_1 \in A_1(u_1, u_2)$, and $\langle \nu_2 - \omega_2, v_2 - u_2 \rangle_{V_2^* \times V_2} \geq 0$ $\forall \nu_2 \in A_2(u_1, v_2), \omega_2 \in A_2(u_1, u_2)$. Hence, it is easy to see that $\Omega_\alpha(\epsilon) \subset \Delta_\alpha(\epsilon)$ for any $\epsilon > 0$. Thus, it is sufficient to show that $\Delta_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon)$. In fact, arbitrarily pick a fixed $\mathbf{u} = (u_1, u_2) \in \Delta_\alpha(\epsilon)$. Then, $\forall (v_1, v_2) \in V_1 \times V_2$, one has

$$\begin{cases} \langle \nu_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq -\epsilon \alpha_1(v_1 - u_1) & \forall \nu_1 \in A_1(v_1, u_2), \\ \langle \nu_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq -\epsilon \alpha_2(v_2 - u_2) & \forall \nu_2 \in A_2(u_1, v_2). \end{cases} \tag{2}$$

For any $\mathbf{w} = (w_1, w_2) \in V_1 \times V_2$ and $t \in (0, 1)$, letting $v_1 := w_{1,t} = u_1 + t(w_1 - u_1)$ and $v_2 := w_{2,t} = u_2 + t(w_2 - u_2)$ in (2), we deduce from the positive homogeneousness of α_1 and α_2 that

$$\begin{cases} \langle \omega_{1,t} - f_1, t(w_1 - u_1) \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; t(w_1 - u_1)) \geq -\epsilon t \alpha_1(w_1 - u_1) & \forall \omega_{1,t} \in A_1(w_{1,t}, u_2), \\ \langle \omega_{2,t} - f_2, t(w_2 - u_2) \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; t(w_2 - u_2)) \geq -\epsilon t \alpha_2(w_2 - u_2) & \forall \omega_{2,t} \in A_2(u_1, w_{2,t}). \end{cases}$$

Using Proposition 1 (i), we know that the CGDD is of positive homogeneousness with respect to its direction. So it follows that

$$\begin{cases} \langle \omega_{1,t} - f_1, w_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; w_1 - u_1) \geq -\epsilon \alpha_1(w_1 - u_1), \ \forall \omega_{1,t} \in A_1(w_{1,t}, u_2), \\ \langle \omega_{2,t} - f_2, w_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; w_2 - u_2) \geq -\epsilon \alpha_2(w_2 - u_2), \ \forall \omega_{2,t} \in A_2(u_1, w_{2,t}). \end{cases} \tag{3}$$

Since $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ and $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ are nonempty compact-valued mappings, $A_1(w_{1,t}, u_2), A_1(u_1, u_2), A_2(u_1, w_{2,t})$ and $A_2(u_1, u_2)$ are nonempty compact sets.

Hence, by Nadler's result [25], we deduce that $\forall t \in (0,1)$, $\omega_{1,t} \in A_1(w_{1,t}, u_2)$ and $\omega_{2,t} \in A_2(u_1, w_{2,t})$, $\exists \nu_{1,t} \in A_1(u_1, u_2)$ and $\nu_{2,t} \in A_2(u_1, u_2)$ s.t.

$$\begin{cases} ||\omega_{1,t} - \nu_{1,t}||_{V_1^*} \leq \mathcal{H}(A_1(w_{1,t}, u_2), A_1(u_1, u_2)), \\ ||\omega_{2,t} - \nu_{2,t}||_{V_2^*} \leq \mathcal{H}(A_2(u_1, w_{2,t}), A_2(u_1, u_2)). \end{cases}$$

Since for $k = 1,2$, $A_k(u_1, u_2)$ is compact, without loss of generality, we may assume that $\nu_{k,t} \to \omega_k \in A_k(u_1, u_2)$ as $t \to 0^+$. It is obvious that $(w_{1,t}, u_2) = (u_1, u_2) + t[(w_1, u_2) - (u_1, u_2)]$ and $(u_1, w_{2,t}) = (u_1, u_2) + t[(u_1, w_2) - (u_1, u_2)]$. Since A_k is $\mathcal{H}$-hemicontinuous for $k = 1,2$, we obtain that

$$\begin{cases} ||\omega_{1,t} - \nu_{1,t}||_{V_1^*} \leq \mathcal{H}(A_1(w_{1,t}, u_2), A_1(u_1, u_2)) \to 0 \text{ as } t \to 0^+, \\ ||\omega_{2,t} - \nu_{2,t}||_{V_2^*} \leq \mathcal{H}(A_2(u_1, w_{2,t}), A_2(u_1, u_2)) \to 0 \text{ as } t \to 0^+, \end{cases}$$

which immediately implies that for $k = 1,2$,

$$||\omega_{k,t} - \omega_k||_{V_k^*} \leq ||\omega_{k,t} - \nu_{k,t}||_{V_k^*} + ||\nu_{k,t} - \omega_k||_{V_k^*} \to 0 \text{ as } t \to 0^+. \tag{4}$$

Thus, taking the limit as $t \to 0^+$ at both sides of the inequalities in (3), we infer from (4) that

$$\begin{cases} \langle \omega_1 - f_1, w_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; w_1 - u_1) \geq -\epsilon\alpha_1(w_1 - u_1), \\ \langle \omega_2 - f_2, w_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; w_2 - u_2) \geq -\epsilon\alpha_2(w_2 - u_2), \end{cases}$$

which, together with the arbitrariness of $\mathbf{w} = (w_1, w_2) \in V_1 \times V_2$, implies that $\Delta_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon)$. This completes the proof. □

Lemma 2. *Suppose that $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ and $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ satisfy the hypotheses (a), (b), (e), (f) in* **(HA)**, *and $J : V_1 \times V_2 \to \mathbf{R}$ satisfies the hypothesis* **(HJ)**. *Then, for any $\epsilon > 0$, $\Omega_\alpha(\epsilon) = \Delta_\alpha(\epsilon)$ is closed in $\mathcal{V} = V_1 \times V_2$.*

Proof. Since the $\mathcal{H}$-continuity guarantees the $\mathcal{H}$-hemicontinuity, using Proposition 2, one has $\Omega_\alpha(\epsilon) = \Delta_\alpha(\epsilon)$ $\forall \epsilon > 0$. Let $\mathbf{u}^n = (u_1^n, u_2^n) \in \Delta_\alpha(\epsilon)$ be a sequence strongly converging towards $\mathbf{u} = (u_1, u_2)$ in $\mathcal{V} = V_1 \times V_2$. Then, $\forall n \geq 1$, $\exists (\omega_1^n, \omega_2^n) \in A_1(u_1^n, u_2^n) \times A_2(u_1^n, u_2^n)$ s.t.

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon\alpha_1(v_1 - u_1^n), \ \forall v_1 \in V_1, \\ \langle \omega_2^n - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon\alpha_2(v_2 - u_2^n), \ \forall v_2 \in V_2. \end{cases} \tag{5}$$

Since $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ and $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ are nonempty compact-valued mappings, $A_1(u_1^n, u_2^n), A_1(u_1, u_2), A_2(u_1^n, u_2^n)$ and $A_2(u_1, u_2)$ are nonempty compact sets. Hence, by Nadler's result [25], one knows that for $\omega_1^n \in A_1(u_1^n, u_2^n)$ and $\omega_2^n \in A_2(u_1^n, u_2^n)$, $\exists \nu_1^n \in A_1(u_1, u_2)$ and $\exists \nu_2^n \in A_2(u_1, u_2)$ s.t.

$$\begin{cases} ||\omega_1^n - \nu_1^n||_{V_1^*} \leq \mathcal{H}(A_1(u_1^n, u_2^n), A_1(u_1, u_2)), \\ ||\omega_2^n - \nu_2^n||_{V_2^*} \leq \mathcal{H}(A_2(u_1^n, u_2^n), A_2(u_1, u_2)). \end{cases}$$

Furthermore, since for $k = 1,2$, $A_k(u_1, u_2)$ is compact, without loss of generality, we may assume that $\nu_k^n \to \omega_k \in A_k(u_1, u_2)$ as $n \to \infty$. For $k = 1,2$, we note that A_k is $\mathcal{H}$-continuous. Thus, we obtain that

$$\begin{cases} ||\omega_1^n - \nu_1^n||_{V_1^*} \leq \mathcal{H}(A_1(u_1^n, u_2^n), A_1(u_1, u_2)) \to 0 \text{ as } n \to \infty, \\ ||\omega_2^n - \nu_2^n||_{V_2^*} \leq \mathcal{H}(A_2(u_1^n, u_2^n), A_2(u_1, u_2)) \to 0 \text{ as } n \to \infty, \end{cases}$$

which immediately implies that, for $k = 1, 2$,

$$||\omega_k^n - \omega_k||_{V_k^*} \leq ||\omega_k^n - \nu_k^n||_{V_k^*} + ||\nu_k^n - \omega_k||_{V_k^*} \to 0 \text{ as } n \to \infty. \tag{6}$$

It therefore follows from (6) that

$$\begin{cases} \lim\limits_{n\to\infty} \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} = \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1}, \\ \lim\limits_{n\to\infty} \langle \omega_2^n - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} = \langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2}. \end{cases} \tag{7}$$

Moreover, by the hypothesis **(HJ)** on the functional J, Lemma 1 ensures that

$$\begin{cases} \limsup\limits_{n\to\infty} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \leq J_1^\circ(u_1, u_2; v_1 - u_1), \\ \limsup\limits_{n\to\infty} J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \leq J_2^\circ(u_1, u_2; v_2 - u_2). \end{cases} \tag{8}$$

Furthermore, using the continuity of α_1 and α_2, we obtain that, for $k = 1, 2$,

$$\lim_{n\to\infty} \alpha_k(v_k - u_k^n) = \alpha_k(v_k - u_k). \tag{9}$$

Therefore, taking the limsup as $n \to \infty$ at both sides of the inequalities in (5), we conclude from (7)–(9) that

$$\begin{cases} \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq -\epsilon\alpha_1(v_1 - u_1) \quad \forall v_1 \in V_1, \\ \langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq -\epsilon\alpha_2(v_2 - u_2) \quad \forall v_2 \in V_2, \end{cases}$$

which implies that $\mathbf{u} = (u_1, u_2) \in \Omega_\alpha(\epsilon) = \Delta_\alpha(\epsilon)$. Thus, $\Omega_\alpha(\epsilon) = \Delta_\alpha(\epsilon)$ is closed in $\mathcal{V} = V_1 \times V_2$. This completes the proof. □

Theorem 2. *Suppose that* $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ *satisfy the hypothesis (d) in* **(HA)**, $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ *satisfy the hypothesis (e) in* **(HA)**, *and* $J : V_1 \times V_2 \to \mathbf{R}$ *satisfy the hypothesis* **(HJ)**. *Then, the SGHVI is strongly* α*-well-posed if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset \; \forall \epsilon > 0 \quad \text{and} \quad \text{diam}(\Omega_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0.$$

Proof. Necessity. Assume that the SGHVI is strongly α-well-posed. Then, the SGHVI admits a unique solution $\mathbf{u} = (u_1, u_2) \in V_1 \times V_2$, i.e., for certain $(\omega_1, \omega_2) \in A_1(u_1, u_2) \times A_2(u_1, u_2)$,

$$\textbf{SGHVI} \quad \begin{cases} \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq 0 \quad \forall v_1 \in V_1, \\ \langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0 \quad \forall v_2 \in V_2. \end{cases}$$

This ensures that $\mathbf{u} \in \Omega_\alpha(\epsilon) \; \forall \epsilon > 0$, i.e., $\Omega_\alpha(\epsilon) \neq \emptyset \; \forall \epsilon > 0$. If $\text{diam}(\Omega_\alpha(\epsilon)) \not\to 0$ as $\epsilon \to 0$, then there exists $\mathbf{u}^n = (u_1^n, u_2^n)$, $\mathbf{p}^n = (p_1^n, p_2^n) \in \Omega_\alpha(\epsilon_n)$, $d > 0$ and $0 < \epsilon_n \to 0$ such that

$$||\mathbf{u}^n - \mathbf{p}^n||_{V_1 \times V_2} = ||u_1^n - p_1^n||_{V_1} + ||u_2^n - p_2^n||_{V_2} > d. \tag{10}$$

By the definition of the α-approximating sequence for the SGHVI, $\{\mathbf{u}^n\}$ and $\{\mathbf{p}^n\}$ are two α-approximating sequences. Thus, it follows from the strong α-well-posedness of SGHVI that $\{\mathbf{u}^n\}$ and $\{\mathbf{p}^n\}$ both strongly converge towards the unique solution $\mathbf{u}$, which contradicts (10).

Sufficiency. Suppose that $\Omega_\alpha(\epsilon) \neq \emptyset$ and $\text{diam}(\Omega_\alpha(\epsilon)) \to 0$ as $\epsilon \to 0$. We claim that the SGHVI is strongly α-well-posed. In fact, let $\{\mathbf{u}^n\}$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ be an

α-approximating sequence for the SGHVI. Then, there exist $(\omega_1^n, \omega_2^n) \in A_1(u_1^n, u_2^n) \times A_2(u_1^n, u_2^n), n \in \mathbf{N}$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ $(n \to \infty)$ such that

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n \alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1, \\ \langle \omega_2^n - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n \alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2, \end{cases}$$

which implies $\mathbf{u}^n \in \Omega_\alpha(\epsilon_n)$ $\forall n \geq 1$. Since $\operatorname{diam}(\Omega_\alpha(\epsilon_n)) \to 0$ as $n \to \infty$, $\{\mathbf{u}^n\}$ is a Cauchy sequence in $\mathcal{V} = V_1 \times V_2$. Without loss of generality, we may assume that $\{\mathbf{u}^n\}$ strongly converges towards $\mathbf{u} = (u_1, u_2)$ in $\mathcal{V} = V_1 \times V_2$.

Now, we claim that $\mathbf{u}$ is a unique solution to the SGHVI. Indeed, since operators A_1 and A_2 are $\mathcal{H}$-continuous on $\mathcal{V} = V_1 \times V_2$, the functional J satisfies the hypothesis **(HJ)**, and α_1 and α_2 are continuous, so we can obtain by similar arguments to those in (7)–(9) that

$$\begin{aligned} & \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \\ & \geq \lim_{n \to \infty} \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + \limsup_{n \to \infty} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \\ & = \limsup_{n \to \infty} \{ \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \} \\ & \geq \lim_{n \to \infty} -\epsilon_n \alpha_1(v_1 - u_1^n) = \lim_{n \to \infty} -\alpha_1(\epsilon_n(v_1 - u_1^n)) \\ & = 0. \end{aligned}$$

By a similar way, one has

$$\langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0.$$

Therefore, $\mathbf{u}$ is a solution to the SGHVI.

Finally, we claim the uniqueness of solutions of the SGHVI. Suppose that $\mathbf{u}'$ is another solution to the SGHVI. Since, for any $\epsilon > 0$, $\mathbf{u}, \mathbf{u}' \in \Omega_\alpha(\epsilon)$, $||\mathbf{u} - \mathbf{u}'||_{V_1 \times V_2} \leq \operatorname{diam}(\Omega_\alpha(\epsilon))$, which together with the condition $\operatorname{diam}(\Omega_\alpha(\epsilon)) \to 0$ as $\epsilon \to 0$, guarantees that $\mathbf{u} = \mathbf{u}'$. This completes the proof. □

Theorem 3. *Suppose that $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ and $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ satisfy the hypotheses (a), (b), (e) and (f) in* **(HA)** *and $J : V_1 \times V_2 \to \mathbf{R}$ satisfy the hypothesis* **(HJ)**. *Then, the SGHVI is strongly α-well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset \; \forall \epsilon > 0 \quad \text{and} \quad \mu(\Omega_\alpha(\epsilon)) \to 0 \; (\epsilon \to 0).$$

Proof. Necessity. Suppose that the SGHVI is strongly α-well-posed in the generalized sense. Then, the solution set S of the SGHVI is nonempty, i.e., $S \neq \emptyset$. This ensures that $\Omega_\alpha(\epsilon) \neq \emptyset \; \forall \epsilon > 0$ because $S \subset \Omega_\alpha(\epsilon)$. Moreover, we claim here that the solution set S of the SGHVI is compact. In fact, for any sequence $\{\mathbf{u}^n\} \subset S$ with $\mathbf{u}^n = (u_1^n, u_2^n)$, $\{\mathbf{u}^n\}$ is an α-approximating sequence for the SGHVI and thus there exists a subsequence of $\{\mathbf{u}^n\}$ strongly converging towards a certain element of S, which implies that S is compact. To complete the proof of the necessity, we claim that $\mu(\Omega_\alpha(\epsilon)) \to 0$ as $\epsilon \to 0$. From $S \subset \Omega_\alpha(\epsilon)$, it follows that

$$\mathcal{H}(\Omega_\alpha(\epsilon), S) = \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} = e(\Omega_\alpha(\epsilon), S).$$

Since the solution set S is compact, one has

$$\mu(\Omega_\alpha(\epsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\epsilon), S) = 2e(\Omega_\alpha(\epsilon), S).$$

Now, to prove $\mu(\Omega_\alpha(\epsilon)) \to 0$ as $\epsilon \to 0$, it is sufficient to show that $e(\Omega_\alpha(\epsilon), S) \to 0$ as $\epsilon \to 0$. On the contrary, assume that $e(\Omega_\alpha(\epsilon), S) \not\to 0$ as $\epsilon \to 0$. Then, there exists a constant $l > 0$, a sequence $\{\epsilon_n\} \subset [0, \infty)$ with $\epsilon_n \to 0$ and $\mathbf{u}^n \in \Omega_\alpha(\epsilon_n)$ such that

$$\mathbf{u}^n \notin S + B(0, l), \tag{11}$$

where $B(0,l)$ is the closed ball centered at 0 with radius l. Since $\mathbf{u}^n \in \Omega_\alpha(\epsilon_n)$ with $\epsilon_n \to 0$, $\{\mathbf{u}^n\}$ is an α-approximating sequence for SGHVI. Thus, there exists a subsequence converging strongly towards a certain element $\mathbf{u} \in S$ due to the strong α-well-posedness in the generalized sense for SGHVI. This contradicts (11). Consequently, $\mu(\Omega_\alpha(\epsilon)) \to 0$ as $\epsilon \to 0$.

Sufficiency. Assume that $\Omega_\alpha(\epsilon) \neq \emptyset\ \forall \epsilon > 0$ and $\mu(\Omega_\alpha(\epsilon)) \to 0\ (\epsilon \to 0)$. We claim that the SGHVI is strongly α-well-posed in the generalized sense. In fact, we observe that

$$S = \bigcap_{\epsilon>0} \Omega_\alpha(\epsilon).$$

Furthermore, since $\mu(\Omega_\alpha(\epsilon)) \to 0\ (\epsilon \to 0)$ and $\Omega_\alpha(\epsilon)$ is nonempty and closed for any $\epsilon > 0$ (due to Lemma 2), it follows from the theorem in ([19], p. 412) that S is nonempty compact and

$$e(\Omega_\alpha(\epsilon), S) = \mathcal{H}(\Omega_\alpha(\epsilon), S) \to 0 \quad \epsilon \to 0. \tag{12}$$

Now, to show the strong α-well-posedness in the generalized sense for the SGHVI, let $\{\mathbf{u}^n\} \subset V_1 \times V_2$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ be an α-approximating sequence for the SGHVI. Then, there exists $(\omega_1^n, \omega_2^n) \in A_1(u_1^n, u_2^n) \times A_2(u_1^n, u_2^n), n \in \mathbf{N}$ and $\{\epsilon_n\} \subset [0, +\infty)$ with $\epsilon_n \to 0\ (n \to \infty)$ such that

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n \alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1, \\ \langle \omega_2^n - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n \alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2, \end{cases}$$

which yields $\mathbf{u}^n \in \Omega_\alpha(\epsilon_n)$. This, together with (12), leads to

$$d(\mathbf{u}^n, S) \leq e(\Omega_\alpha(\epsilon_n), S) \to 0.$$

Since S is compact, there exists $\bar{\mathbf{u}}^n \in S$ such that

$$||\mathbf{u}^n - \bar{\mathbf{u}}^n||_{V_1 \times V_2} = d(\mathbf{u}^n, S) \to 0.$$

Again from the compactness of the solution set S, one knows that $\{\bar{\mathbf{u}}^n\}$ has a subsequence $\{\bar{\mathbf{u}}^{n_k}\}$ strongly converging towards a certain element $\bar{\mathbf{u}} \in S$. Thus, it follows that

$$||\mathbf{u}^{n_k} - \bar{\mathbf{u}}||_{V_1 \times V_2} \leq ||\mathbf{u}^{n_k} - \bar{\mathbf{u}}^{n_k}||_{V_1 \times V_2} + ||\bar{\mathbf{u}}^{n_k} - \bar{\mathbf{u}}||_{V_1 \times V_2} \to 0,$$

which immediately implies that the subsequence $\{\mathbf{u}^{n_k}\}$ of $\{\mathbf{u}^n\}$ strongly converges towards $\bar{\mathbf{u}}$. Therefore, the SGHVI is strongly α-well-posed in the generalized sense. This completes the proof. □

It is remarkable that Proposition 2, Lemma 2 and Theorems 2–3 improve, extend and develop Lemmas 3.7–3.8 and Theorems 3.10–3.11 in [14] to a great extent because the SGHVI is more general than the SHVI considered in Lemmas 3.7–3.8 and Theorems 3.10–3.11 of [14].

4. Equivalence for Well-Posedness of the SGHVI and SDIP

In this section, we first introduce the systems of inclusion problems (SIPs) in the product space $\mathcal{V} = V_1 \times V_2$ and then define the concept of α-well-posedness for SIPs. Moreover, we show the equivalence results between the α-well-posedness of the SGHVI and α-well-posedness of its SDIP.

Let V_1 and V_2 be two real Banach spaces with V_1^* and V_2^* being their dual spaces, respectively. Suppose that, for $k = 1, 2$, Γ_k is a nonempty set-valued mapping from $V_1 \times V_2$ to V_k^*. A system of inclusion problems (SIP) associated with mappings Γ_1 and Γ_2 is formulated below:

Find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$(\mathbf{SIP}) \quad \begin{cases} 0_1 \in \Gamma_1(u_1, u_2), \\ 0_2 \in \Gamma_2(u_1, u_2), \end{cases} \tag{13}$$

where for $k = 1, 2$, $0_k \in V_k^*$ represents the zero element in V_k^*. For simplicity, we use the symbols below:

$$\mathbf{u} = (u_1, u_2) \in V_1 \times V_2, \ \mathbf{0} = (0_1, 0_2) \in V_1^* \times V_2^* \ \text{ and } \ \mathbf{\Gamma}(\mathbf{u}) = (\Gamma_1(\mathbf{u}), \Gamma_2(\mathbf{u})) \in V_1^* \times V_2^*.$$

This allows us to simplify the SIP as follows:
Find $\mathbf{u} \in \mathcal{V} = V_1 \times V_2$ such that

$$\mathbf{0} \in \mathbf{\Gamma}(\mathbf{u}).$$

Definition 9. *A sequence $\{\mathbf{u}^n\} \subset V_1 \times V_2$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is called an α-approximating sequence for the SIP if $\exists \mathbf{p}^n = (p_1^n, p_2^n) \in \mathbf{\Gamma}(\mathbf{u}^n)$, $n \in \mathbf{N}$ and $\exists\{\epsilon_n\} \subset [0, +\infty)$ with $||\mathbf{p}^n||_{V_1^* \times V_2^*} + \epsilon_n \to 0$ as $n \to \infty$, s.t.*

$$\begin{cases} \langle p_1^n, v_1 - u_1^n \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(v_1 - u_1^n) \quad \forall v_1 \in V_1, \ n \in \mathbf{N}, \\ \langle p_2^n, v_2 - u_2^n \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(v_2 - u_2^n) \quad \forall v_2 \in V_2, \ n \in \mathbf{N}. \end{cases}$$

Definition 10. *The SIP is referred to as being strongly (and weakly, respectively) α-well-posed if it has a unique solution and every α-approximating sequence converges strongly (and weakly, respectively) to the unique solution of the SIP.*

Definition 11. *The SIP is referred to as being strongly (and weakly, respectively) α-well-posed in the generalized sense if the solution set S of the SIP is nonempty and every α-approximating sequence has a subsequence strongly converging (and weakly, respectively) towards a certain element of the solution set S.*

In order to show that the α-well-posedness for the SGHVI is equivalent to the α-well-posedness for its SDIP, we first furnish a lemma which establishes the equivalence between the SGHVI and SDIP.

Lemma 3. $\mathbf{u} = (u_1, u_2) \in V_1 \times V_2$ *is a solution to the SGHVI if and only if it solves the following SDIP:*
Find $\mathbf{u} = (u_1, u_2) \in V_1 \times V_2$ such that

$$(\mathbf{SDIP}) \quad \begin{cases} f_1 \in A_1(u_1, u_2) + \partial_1 J(u_1, u_2), \\ f_2 \in A_2(u_1, u_2) + \partial_2 J(u_1, u_2), \end{cases}$$

where, for $k \neq j = 1, 2$, $\partial_k J(u_1, u_2)$ denotes the CGS of $J(\cdot, u_j)$ at u_k.

Proof. First of all, we claim the necessity. In fact, assume that $\mathbf{u} = (u_1, u_2) \in \mathcal{V} = V_1 \times V_2$ is a solution of the SGHVI, i.e., for certain $(\omega_1, \omega_2) \in A_1(u_1, u_2) \times A_2(u_1, u_2)$,

$$\begin{cases} \langle \omega_1 - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq 0 \quad \forall v_1 \in V_1, \\ \langle \omega_2 - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0 \quad \forall v_2 \in V_2. \end{cases} \tag{14}$$

For any $\mathbf{w} = (w_1, w_2) \in V_1 \times V_2$, letting $v_1 = u_1 + w_1 \in V_1$ and $v_2 = u_2 + w_2 \in V_2$ in (14), we obtain that

$$\begin{cases} J_1^\circ(u_1, u_2; w_1) \geq \langle f_1 - \omega_1, w_1 \rangle_{V_1^* \times V_1}, \\ J_2^\circ(u_1, u_2; w_2) \geq \langle f_2 - \omega_2, w_2 \rangle_{V_2^* \times V_2}. \end{cases}$$

It follows from the definition of the CGS and the arbitrariness of $w_k \in V_k, k = 1, 2$ that

$$\begin{cases} f_1 \in \omega_1 + \partial_1 J(u_1, u_2) \subseteq A_1(u_1, u_2) + \partial_1 J(u_1, u_2), \\ f_2 \in \omega_2 + \partial_2 J(u_1, u_2) \subseteq A_2(u_1, u_2) + \partial_2 J(u_1, u_2), \end{cases}$$

which implies that $\mathbf{u} = (u_1, u_2) \in V_1 \times V_2$ is a solution to the SDIP.

Sufficiency. Suppose that $\mathbf{u} = (u_1, u_2) \in \mathcal{V} = V_1 \times V_2$ is a solution to the SDIP, i.e.,

$$\begin{cases} f_1 \in A_1(u_1, u_2) + \partial_1 J(u_1, u_2), \\ f_2 \in A_2(u_1, u_2) + \partial_2 J(u_1, u_2). \end{cases}$$

It follows that, for $k = 1, 2$, there exist $\omega_k \in A_k(u_1, u_2)$ and $\eta_k \in \partial_k J(u_1, u_2)$ such that

$$\begin{cases} f_1 = \omega_1 + \eta_1, \\ f_2 = \omega_2 + \eta_2. \end{cases} \tag{15}$$

For any $\mathbf{v} = (v_1, v_2) \in \mathcal{V} = V_1 \times V_2$, by multiplying both sides of the equalities in (15) with $v_1 - u_1 \in V_1$ and $v_2 - u_2 \in V_2$, respectively, we deduce, by the definition of the CGS, that

$$\begin{aligned} \langle f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} &= \langle \omega_1 + \eta_1, v_1 - u_1 \rangle_{V_1^* \times V_1} \\ &= \langle \omega_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + \langle \eta_1, v_1 - u_1 \rangle_{V_1^* \times V_1} \\ &\le \langle \omega_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1), \end{aligned}$$

and

$$\begin{aligned} \langle f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} &= \langle \omega_2 + \eta_2, v_2 - u_2 \rangle_{V_2^* \times V_2} \\ &= \langle \omega_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + \langle \eta_2, v_2 - u_2 \rangle_{V_2^* \times V_2} \\ &\le \langle \omega_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2). \end{aligned}$$

Therefore, $\mathbf{u}$ is a solution of the SGHVI. This completes the proof. □

Let E be a real reflexive Banach space with its dual E^*. We denote by $\mathcal{J}$ the normalized duality mapping from E^* to its dual $E^{**}(= E)$ formulated by

$$\mathcal{J}(\nu) = \{x \in E : \langle \nu, x \rangle_{E^* \times E} = ||\nu||_{E^*}^2 = ||x||_E^2\} \quad \forall \nu \in E^*.$$

Theorem 4. *Let V_1 and V_2 be real reflexive Banach spaces. Then, the SGHVI is strongly α-well-posed if and only if its SDIP is strongly α-well-posed.*

Proof. Necessity. Suppose that the SGHVI is strongly α-well-posed. Then there exists a unique $\mathbf{u} = (u_1, u_2) \in \mathcal{V} = V_1 \times V_2$ settling the SGHVI. It follows from Lemma 3 that $\mathbf{u}$ is the unique solution of the SDIP. To show the strong α-well-posedness for the SDIP, we let $\mathbf{u}^n = (u_1^n, u_2^n)$ be an α-approximating sequence for the SDIP. We claim that $\mathbf{u}^n \to \mathbf{u}$ as $n \to \infty$. In fact, one knows that there exists a sequence $\mathbf{p}^n = (p_1^n, p_2^n) \in V_1^* \times V_2^*, n \in \mathbf{N}$ and a sequence $\{\epsilon_n\} \subset [0, +\infty)$, such that for each $k = 1, 2,\quad p_k^n \in A_k(u_1^n, u_2^n) - f_k + \partial_k J(u_1^n, u_2^n)$, $||\mathbf{p}^n||_{V_1^* \times V_2^*} + \epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} \langle p_1^n, v_1 - u_1^n \rangle_{V_1^* \times V_1} \ge -\epsilon_n \alpha_1(v_1 - u_1^n) \quad \forall v_1 \in V_1, n \in \mathbf{N}, \\ \langle p_2^n, v_2 - u_2^n \rangle_{V_2^* \times V_2} \ge -\epsilon_n \alpha_2(v_2 - u_2^n) \quad \forall v_2 \in V_2, n \in \mathbf{N}. \end{cases} \tag{16}$$

It is obvious that for $k = 1, 2$, there exists $\omega_k^n \in A_k(u_1^n, u_2^n)$ and $\eta_k^n \in \partial_k J(u_1^n, u_2^n)$, such that

$$\begin{cases} p_1^n = \omega_1^n - f_1 + \eta_1^n, \\ p_2^n = \omega_2^n - f_2 + \eta_2^n. \end{cases} \tag{17}$$

For $k \neq j = 1,2$, using the definition of the CGS $\partial_k J(u_1^n, u_2^n)$ of $J(\cdot, u_j^n)$ at u_k^n and multiplying both sides of the equalities in (17) with $v_k - u_k^n \in V_k$, we obtain from (16) that

$$\begin{aligned}&\langle \omega_1^n - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n)\\ &\geq \langle \omega_1^n - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + \langle \eta_1^n, v_1 - u_1^n\rangle_{V_1^*\times V_1}\\ &= \langle p_1^n, v_1 - u_1^n\rangle_{V_1^*\times V_1} \geq -\epsilon_n\alpha_1(v_1 - u_1^n) \quad \forall v_1 \in V_1,\end{aligned}$$

and

$$\begin{aligned}&\langle \omega_2^n - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n)\\ &\geq \langle \omega_2^n - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + \langle \eta_2^n, v_2 - u_2^n\rangle_{V_2^*\times V_2}\\ &= \langle p_2^n, v_2 - u_2^n\rangle_{V_2^*\times V_2} \geq -\epsilon_n\alpha_2(v_2 - u_2^n) \quad \forall v_2 \in V_2,\end{aligned}$$

Therefore, we deduce that $\{\mathbf{u}^n\}$ is an α-approximating sequence for the SGHVI. Thus, it follows from the strong α-well-posedness for the SGHVI that $\{\mathbf{u}^n\}$ strongly converges towards the unique solution $\mathbf{u}$. This ensures that the SDIP is strongly α-well-posed.

Sufficiency. Suppose that the SDIP is strongly α-well-posed. Then, there exists a unique solution $\mathbf{u}$ of the SDIP, which, together with Lemma 3, implies that $\mathbf{u}$ is also the unique solution of the SGHVI. Let $\{\mathbf{u}^n\}$ be an α-approximating sequence for the SGHVI. Then, there exist $(\omega_1^n, \omega_2^n) \in A_1(u_1^n, u_2^n) \times A_2(u_1^n, u_2^n), n \in \mathbf{N}$ and $\{\epsilon_n\} \subset [0, +\infty)$ with $\epsilon_n \to 0$ $(n \to \infty)$ such that

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n\alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1,\\ \langle \omega_2^n - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n\alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2.\end{cases} \tag{18}$$

Using Proposition 1 (v), one observes that

$$\begin{cases} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) = \max\{\langle h_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} : h_1 \in \partial_1 J(u_1^n, u_2^n)\},\\ J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) = \max\{\langle h_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} : h_2 \in \partial_2 J(u_1^n, u_2^n)\}.\end{cases}$$

Thus, for any $(v_1, v_2) \in V_1 \times V_2$, there exist $h_1(u_1^n, u_2^n, v_1) \in \partial_1 J(u_1^n, u_2^n)$ and $h_2(u_1^n, u_2^n, v_2) \in \partial_2 J(u_1^n, u_2^n)$ such that

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + \langle h_1(u_1^n, u_2^n, v_1), v_1 - u_1^n\rangle_{V_1^*\times V_1} \geq -\epsilon_n\alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1,\\ \langle \omega_2^n - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + \langle h_2(u_1^n, u_2^n, v_2), v_2 - u_2^n\rangle_{V_2^*\times V_2} \geq -\epsilon_n\alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2.\end{cases} \tag{19}$$

By Proposition 1 (iv), we know that $\partial_1 J(u_1^n, u_2^n)$ and $\partial_2 J(u_1^n, u_2^n)$ are nonempty, convex, bounded and closed subsets in V_1^* and V_2^*, respectively, which imply that, for each $k = 1,2$, the set $\{\omega_k^n + h_k - f_k : h_k \in \partial_k J(u_1^n, u_2^n)\}$ is also nonempty, convex, bounded and closed in V_k^*. Therefore, for each $k = 1,2$, it follows from (19) and Theorem 1 with $\varphi_k(\cdot) = \epsilon_n\alpha_k(\cdot - u_k^n)$, which is proper, convex and continuous, that there exists a $h_k^n \in \partial_k J(u_1^n, u_2^n)$, which is independent on v_k, such that

$$\begin{cases} \langle \omega_1^n - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + \langle h_1^n, v_1 - u_1^n\rangle_{V_1^*\times V_1} \geq -\epsilon_n\alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1,\\ \langle \omega_2^n - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + \langle h_2^n, v_2 - u_2^n\rangle_{V_2^*\times V_2} \geq -\epsilon_n\alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2.\end{cases} \tag{20}$$

Therefore, it follows that

$$\begin{cases} \langle p_1^n, v_1 - u_1^n\rangle_{V_1^*\times V_1} \geq -\epsilon_n\alpha_1(v_1 - u_1^n) & \forall v_1 \in V_1,\\ \langle p_2^n, v_2 - u_2^n\rangle_{V_2^*\times V_2} \geq -\epsilon_n\alpha_2(v_2 - u_2^n) & \forall v_2 \in V_2,\end{cases} \tag{21}$$

where $p_k^n = \omega_k^n - f_k + h_k^n$ for $k = 1,2$. It is readily known that for $k = 1,2$,

$$p_k^n = \omega_k^n - f_k + h_k^n \in A_k(u_1^n, u_2^n) - f_k + \partial_k J(u_1^n, u_2^n). \tag{22}$$

Then, to show that $||\mathbf{p}^n||_{V_1^* \times V_2^*} \to 0$ as $n \to \infty$, it is sufficient to show that $||p_k^n||_{V_k^*} \to 0$ as $n \to \infty$ for $k = 1, 2$, that is, for any $\varepsilon > 0$, there exists an integer $N \geq 1$ such that $||p_k^n||_{V_k^*} < \varepsilon$ for all $n \geq N$. In fact, note that V_k is reflexive, i.e., $V_k = V_k^{**}$. According to the normalized duality mapping $\mathcal{J}_k$ from V_k^* to its dual $V_k^{**}(= V_k)$ formulated below

$$\mathcal{J}_k(\nu_k) = \{x_k \in V_k : \langle \nu_k, x_k \rangle_{V_k^* \times V_k} = ||\nu_k||^2_{V_k^*} = ||x_k||^2_{V_k}\} \quad \forall \nu_k \in V_k^*,$$

we know that for each $n \in \mathbf{N}$, there exists $\tilde{j}_k(p_k^n) \in \mathcal{J}_k(p_k^n)$ such that

$$\langle p_k^n, \tilde{j}_k(p_k^n) \rangle_{V_k^* \times V_k} = ||p_k^n||^2_{V_k^*} = ||\tilde{j}_k(p_k^n)||^2_{V_k}.$$

For $k = 1, 2$, putting $v_k = u_k^n - \tilde{j}_k(p_k^n)$ in (21), we obtain

$$\langle p_k^n, -\tilde{j}_k(p_k^n) \rangle_{V_k^* \times V_k} \geq -\epsilon_n \alpha_k(-\tilde{j}_k(p_k^n)),$$

that is,

$$||p_k^n||^2_{V_k^*} \leq \epsilon_n \alpha_k(-\tilde{j}_k(p_k^n)). \tag{23}$$

If $||p_k^n||_{V_k^*} \not\to 0$ as $n \to \infty$, then there exists $\varepsilon_k > 0$ and for each $j \geq 1$, there exists $p_k^{n_j}$ such that

$$||p_k^{n_j}||_{V_k^*} \geq \varepsilon_k.$$

Taking into account $||\epsilon_{n_j} \frac{\tilde{j}_k(p_k^{n_j})}{||p_k^{n_j}||_{V_k^*}}||_{V_k} \to 0$ as $j \to \infty$, and using the positive homogeneousness and continuity of α_k, we conclude from (23) that

$$\varepsilon_k \leq ||p_k^{n_j}||_{V_k^*} \quad \leq \frac{\epsilon_{n_j}}{||p_k^{n_j}||_{V_k^*}} \alpha_k(-\tilde{j}_k(p_k^{n_j})) = \alpha_k(-\epsilon_{n_j} \frac{\tilde{j}_k(p_k^{n_j})}{||p_k^{n_j}||_{V_k^*}}) \to 0 \text{ as } j \to \infty,$$

which reaches a contradiction. This means that $||p_k^n||_{V_k^*} \to 0$ as $n \to \infty$ for $k = 1, 2$. Hence, the sequence $\{\mathbf{u}^n\}$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is an α-approximating sequence for SDIP. Thus, it follows from the strong α-well-posedness for the SDIP that $\{\mathbf{u}^n\}$ strongly converges towards the unique solution $\mathbf{u}$ in $V_1 \times V_2$. Therefore, the SGHVI is strongly α-well-posed. This completes the proof. □

Using arguments similar to those in the proof of Theorem 4, one can easily prove the following equivalence between the strong α-well-posedness in the generalized sense for the SGHVI and the strong α-well-posedness in the generalized sense for the SDIP. In fact, we first denote by $\mho$ the solution set of the SGHVI. Note that the SGHVI is strongly α-well-posed $\Leftrightarrow \mho = \{\mathbf{u}\}$ and $\forall$ (α-approximating sequence) $\{\mathbf{u}_n\}$ for the SGHVI it holds $\mathbf{u}_n \to \mathbf{u}$, and that the SGHVI is strongly α-well-posed in the generalized sense $\Leftrightarrow \mho \neq \emptyset$ and $\forall$ (α-approximating sequence) $\{\mathbf{u}_n\}$, $\exists \{\mathbf{u}_{n_j}\} \subset \{\mathbf{u}_n\}$ s.t. $\mathbf{u}_{n_j} \to \mathbf{u}$ for some $\mathbf{u} \in \mho$. After substituting the strong α-well-posedness in the generalized sense for the SGHVI (and SDIP, respectively) into the strong α-well-posedness for the SGHVI (and SDIP, respectively) in the proof of Theorem 4, we can deduce the conclusion of the following Theorem 5.

Theorem 5. *Let V_1 and V_2 be real reflexive Banach spaces. Then, the SGHVI is strongly α-well-posed in the generalized sense if and only if its SDIP is strongly α-well-posed in the generalized sense.*

It is remarkable that, not only in [14] (Theorem 4.5), Wang et al. proved that the SHVI is strongly well-posed if and only if its SDIP is strongly well-posed, but also in [14] (Theorem 4.6), they proved that the SHVI is strongly well-posed in the generalized sense if and only if its SDIP is strongly well-posed in the generalized sense. Compared with

Theorems 4.5 and 4.6 of [14], our Theorems 4 and 5 improve and extend them in the following aspects:

(i) The strong well-posedness for the SHVI and its SDIP in [14] (Theorem 4.5) is extended to develop the strong α-well-posedness for the SGHVI and its SDIP in our Theorem 4.

(ii) The strong well-posedness in the generalized sense for the SHVI and its SDIP in [14] (Theorem 4.6) is extended to develop the strong α-well-posedness in the generalized sense for the SGHVI and its SDIP in our Theorem 5.

5. Conclusions

In this article, we extended the concept of α-well-posedness to the class of generalized hemivariational inequalities systems (SGHVIs) consisting of the two parts which are of symmetric structure mutually. In real Banach spaces, we first put forward certain concepts of α-well-posedness for SGHVIs, and then provide certain metric characterizations of α-well-posedness for SGHVIs. Additionally, we establish certain equivalence results of strong α-well-posedness for both the SGHVI and its system of derived inclusion problems (SDIP). In particular, these equivalence results of strong α-well-posedness (i.e., Theorems 4 and 5) improve and extend Theorems 4.5 and 4.6 of [14] in the following aspects:

(i) The strong well-posedness for the SHVI and its SDIP in [14] (Theorem 4.5) is extended to develop the strong α-well-posedness for the SGHVI and its SDIP in our Theorem 4.

(ii) The strong well-posedness in the generalized sense for the SHVI and its SDIP in [14] (Theorem 4.6) is extended to develop the strong α-well-posedness in the generalized sense for the SGHVI and its SDIP in our Theorem 5.

On the other hand, for $k = 1, 2$, let $G_k : V_k \to \mathbf{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous functional, and $\bar{g}_k : V_k \to V_k$ be a continuous mapping. Denote by $\mathrm{dom}G_k$ the efficient domain of functional G_k, that is, $\mathrm{dom}G_k := \{u_k \in V_k : G_k(u_k) < +\infty\}$. Consider the system of generalized strongly variational–hemivariational inequalities (SGSVHVI), which consists of finding $\mathbf{u} = (u_1, u_2) \in \mathcal{V} = V_1 \times V_2$ such that for some $(\omega_1, \omega_2) \in A_1(\bar{g}_1(u_1), u_2) \times A_2(u_1, \bar{g}_2(u_2))$,

$$\begin{cases} \langle \omega_1 - f_1, v_1 - \bar{g}_1(u_1)\rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - \bar{g}_1(u_1)) + G_1(v_1) - G_1(\bar{g}_1(u_1)) \geq 0 \quad \forall v_1 \in V_1, \\ \langle \omega_2 - f_2, v_2 - \bar{g}_2(u_2)\rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - \bar{g}_2(u_2)) + G_2(v_2) - G_2(\bar{g}_2(u_2)) \geq 0 \quad \forall v_2 \in V_2. \end{cases}$$

It is worth mentioning that the above SGSVHVI also consists of two parts which are of symmetric structure mutually.

In particular, if $G_k(v_k) = 0\ \forall v_k \in V_k$ and $\bar{g}_k$ is the identity mapping on V_k, then the above SGSVHVI reduces to the SGHVI considered in this article. Additionally, if A_k is a single-valued mapping for $k = 1, 2$, then the above SGSVHVI reduces to the SHVI considered in [14].

Finally, it is worth mentioning that part of our future research is aiming to generalize and extend the well-posedness results for SGHVIs in this article to the above class of SGSVHVIs in real Banach spaces.

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