The Existence and Uniqueness of Solution to Sequential Fractional Differential Equation with Affine Periodic Boundary Value Conditions

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Abstract: The solution to a sequential fractional differential equation with affine periodic boundary value conditions is investigated in this paper. The existence theorem of solution is established by means of the Leray–Schauder fixed point theorem and Krasnoselskii fixed point theorem. What is more, the uniqueness theorem of solution is demonstrated via Banach contraction mapping principle. In order to illustrate the main results, two examples are listed.

Keywords: sequential fractional derivative; affine periodic; boundary value problem; fixed point theorem; Banach contraction mapping principle

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1. Introduction

Recently, the investigation of fractional differential system has attracted extensive interest of researchers. Due to the past effects of the phenomenon under consideration, fractional differential system can build more accurate and precise models than integer differential systems; therefore, it is widely used in many domains, for instance, physics, biology, chemistry, astronomy, economics, control theory, and ecology. For relevant research on this results, we refer the interested readers to see [1–4].

Boundary value problem of fractional differential equation constitutes a very important and interesting class of problems, which arise in underground water flow, heat conduction, electromagnetic waves, membranes in nuclear reactors, etc. More and more scholars pay attention to this subject and achieve many excellent results. For instance, see [5–11] and the references therein. There are many types of boundary value problem, including the integral boundary value problem, multipoint boundary value problem, and periodic boundary value problem. In 2013, Li et al. [12] first proposed the affine-periodic system, which describes some physical phenomenon that is periodic in time and symmetric in space. Since then, scholars have done a lot of work on the affine periodic boundary value problem. In [13], Xu et al. have proved the existence of the affine-periodic solution to a Newton affine-periodic system by the lower and upper solutions method. For more details about affine periodic boundary value problem, we refer readers to see [14–17].

The research of sequential fractional differential equation has aroused widespread interests among scholars, since Miller and Ross first proposed the notion of sequential fractional derivative in [18] (p. 209). Many scholars have studied different types of fractional derivative, for instance, Riemann–Liouville fractional derivative, Caputo fractional derivative, and Hadamard fractional derivative. For Riemann–Liouville sequential fractional derivative, Bai studied the existence of solutions to a nonlinear impulsive fractional
differential equation supplemented with periodic boundary value condition in [19]. For Caputo sequential fractional derivative, Ahmad et al. [20] applied the fixed point theorem to research the existence of solution to a fractional differential equation with integral boundary conditions. In [21], Ahmad et al. considered a nonlinear fractional differential equation involving Hadamard sequential fractional derivative, under multi-point boundary conditions, they established the theory of existence and uniqueness of solution. For more research results on sequential fractional derivative, readers can be referred to the papers [22–26].

To our best knowledge, the boundary value problem of sequential fractional differential equation has been studied by many authors. However, there have not been any research results on the affine periodic boundary value problem of sequential fractional differential equation. According to the above analysis, we investigate the sequential fractional differential equation with affine periodic boundary value conditions:

\[
\begin{aligned}
(CD^\beta + \lambda CD^\alpha)z(t) &= g(t, z), t \in [0, T], \\
z(T) &= az(0), z'(T) = az'(0),
\end{aligned}
\]  

(1)

where \( CD^p \) expresses the Caputo fractional derivative, the order \( p \in \{ \alpha, \beta \} \) with \( 0 < \alpha < 1 < \beta < 2 \), and \( \beta = \alpha + 1 \). \( \lambda, a \in \mathbb{R} \), with \( a \neq 1, a \neq e^{-\lambda T} \), and \( g(t, z) : [0, T] \times C([0, T]; \mathbb{R}) \rightarrow \mathbb{R} \) is a continuous function. The contribution of this paper is the investigation of the issue of solution to the sequential fractional differential equation with affine periodic boundary value conditions. Firstly, we use two different methods to prove the existence theorem of the solution. On the basis of improving the condition, we prove the uniqueness of the solution to the equation. Most of the previous studies on affine periodic system are of integer-order derivative, this paper provides an idea for the study of fractional order affine periodic system.

The structure of this paper is as follows. Some definitions and lemmas are introduced in Section 2. The main results and the processes of proofs are presented in Section 3. In Section 4, we list two examples to illustrate our results.

2. Preliminaries

Let \( C([0, T]; \mathbb{R}) \) denotes a Banach space of continuous functions from \([0, T]\) into \( \mathbb{R} \) with the norm \( \|z\| = \max_{t \in [0, T]} |z| \). In the following, we will introduce a number of basic definitions and lemmas, which will be used thereafter. For more results, we refer the interested readers to see [27–31].

Definition 1. The Riemann–Liouville fractional integral of order \( p > 0 \) for a function \( w \) is defined as

\[
^{\mathcal{I}}_t^p w(t) = \frac{1}{\Gamma(p)} \int_0^t (t - \tau)^{p-1} w(\tau) d\tau, \quad t > 0,
\]

where \( \Gamma(\cdot) \) is the Gamma function.

Definition 2. The Caputo fractional derivative of order \( p > 0 \) for a function \( w \) can be written as

\[
^{\mathcal{C}}_t^p w(t) = \begin{cases} \\
\frac{1}{\Gamma(m-p)} \int_0^t (t - \tau)^{m-p-1} w^{(m)}(\tau) d\tau, & m - 1 < p < m (m \in \mathbb{N}^+), \\
\frac{1}{w^{(m)}(t)}, & p = m (m \in \mathbb{N})
\end{cases}
\]

for \( t > 0 \).

An important proposition of the Caputo fractional derivative needs to be provided, which will play a crucial role in our later proof:
Proposition 1 ([28]). For the given definitions, we have:

\[ \mathcal{T}^p(\mathcal{C} D^p w(t)) = w(t) - \sum_{i=1}^{m} a_{i-1} t^{i-1}, \quad t > 0 \]

where \( p \in (m - 1, m), \forall m \in \mathbb{N}^*, a_i (i = 1, 2, \cdots, m) \) are arbitrary constants.

Definition 3 ([18]). The sequential fractional derivative for a function \( w \) can be written as

\[ D^p w(t) = D^{p_1} D^{p_2} \cdots D^{p_m} w(t), \]

where \( p = (p_1, \cdots, p_m) \) is a multi-index.

Remark 1. The symbol \( D^p \) can denote the Grünwald-Letnikov, Riemann–Liouville, Caputo or any other kind of integro-differential operator. For more details, we refer readers to see [27] (p. 87).

Lemma 1 (Leray–Schauder fixed point theorem [29]). Let \( X \) be a Banach space, \( Y \subseteq X \) be nonempty, bounded and convex, \( Z \) be an open subset of \( Y \) with \( 0 \in Z \). Let map \( G : \overline{Z} \to Y \) be continuous and compact. Then, one of the following representations is true:

(i) there exist \( z \in \partial Z \) and \( \epsilon \in (0, 1) \) such that \( z = \epsilon G(z) \);
(ii) \( G \) has a fixed point \( z \in Z \).

Lemma 2 (Krasnoselskii fixed point theorem [30]). Let \( X \) be a Banach space, \( Y \subseteq X \) be nonempty, bounded, closed and convex. Let \( T_1, T_2 \) be two maps and satisfy:

(i) \( T_1 y_1 + T_2 y_2 \in Y, \forall y_1, y_2 \in Y \);
(ii) \( T_1 \) is continuous and compact;
(iii) \( T_2 \) is contractional.

Then, there exists \( w \in Y \) such that \( w = T_1 w + T_2 w \).

Lemma 3 (Arzela–Ascoli theorem [31]). Let \( \Omega \) be a compact space and \( K \) be a subset of \( C(\Omega, \mathbb{C}) \), \( K \) is relatively compact in \( (C(\Omega, \mathbb{C}), t\Omega) \) if and only if \( K \) is equicontinuous and \( \{ h(t) : h \in K, t \in \Omega \} \) is relatively compact in \( \mathbb{C} \), i.e.,

\[ \sup_{h \in K} |h(t)| < +\infty. \]

3. Main Results

Before presenting our main results, the linear variant is considered to describe the solution:

Lemma 4. For a given \( v(t) \in C[0, T] \), the unique solution of the \((T, a)\)-affine-periodic system

\[ \begin{cases} (C D^\beta + \lambda C D^\alpha) z(t) = v(t), t \in [0, T], \\ z(T) = a z(0), z'(T) = a z'(0), \end{cases} \tag{3} \]

is expressed as

\[ z(t) = \int_0^t \int_0^\tau e^{-\lambda (t-r)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau + \mu_1(t) \int_0^T \int_0^\tau e^{-\lambda (T-r)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau + \mu_2(t) \int_0^T (T-s)^{a-1} \frac{1}{\Gamma(a)} v(s) ds, \tag{4} \]

where \( \mu_1(t) = \frac{e^{-\lambda \tau}}{\tau} \) and \( \mu_2(t) = \frac{1}{\lambda (a-1)} - \frac{e^{-\lambda \tau}}{\lambda (a-1 - \tau)} \).
Proof. By Proposition 1, we take $I^a$ on (3) and gain that
\[ z'(t) + \lambda z(t) = \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} v(s) ds + c_1, \] (5)

using the method of constant variation, the solution to (3) can be expressed as
\[ z(t) = \left( \int_0^t \int_0^\tau e^{\lambda \tau} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau + c_2 + \frac{c_1}{\lambda} e^{\lambda t} \right) e^{-\lambda t} \]
\[ = \int_0^t \int_0^\tau e^{-\lambda(t-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau + c_2 e^{-\lambda t} + \frac{c_1}{\lambda}, \] (6)

where $c_1, c_2$ are arbitrary constants. Then
\[ z'(t) = -\lambda \int_0^t \int_0^\tau e^{-\lambda(t-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau + \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} v(s) ds - \lambda c_2 e^{-\lambda t}. \]

Following the boundary conditions for (3), we can find that
\[ c_1 = \frac{1}{a-1} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} v(s) ds, \]
\[ c_2 = \frac{1}{a-e^{-\lambda T}} \int_0^T \int_0^\tau e^{-\lambda(\tau-\tau')} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau - \frac{1}{\lambda(a-e^{-\lambda T})} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} v(s) ds. \]

Replacing the values of $c_1$ and $c_2$ into (6), the solution given by (4) is obtained. \qed

In order to simplify the following proofs, we present the estimate of the integral inequalities as follows. For $v(t) \in C[0, T]$, we get
\[ \left| \int_0^t \int_0^\tau e^{-\lambda(t-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau \right| \leq \left| \int_0^T \frac{\tau^{a-1}}{\Gamma(a+1)} v(s) ds \right| \leq \frac{T^a}{\Gamma(a+1)} \| v \|, \] (7)

Similarly, one has
\[ \left| \int_0^T \int_0^\tau e^{-\lambda(\tau-\tau')} \frac{(\tau-s)^{a-1}}{\Gamma(a)} v(s) ds d\tau' \right| \leq \frac{T^a}{\Gamma(a+1)} \left| \frac{1-e^{-\lambda T}}{\lambda} \right| \| v \|. \] (8)

For convenience, we let
\[ M = \frac{T^a}{\Gamma(a+1)} (M_\lambda + \bar{M}_1 M_\lambda + \bar{M}_2), \] (10)
\[ \bar{M} = \frac{T^a}{\Gamma(a+1)} (\bar{M}_1 M_\lambda + \bar{M}_2), \] (11)
where \( M_\lambda = \left| \frac{1-e^{-\lambda T}}{\lambda} \right| \), \( \mu_1(t) = \max_{t \in [0,T]} |\mu_1(t)| \), and \( \mu_2(t) = \max_{t \in [0,T]} |\mu_2(t)| \).

Now, we list the main results of this paper.

**Theorem 1.** Let \( g(t,z) : [0, T] \times C([0, T]; R) \rightarrow R \) be a continuous function, which satisfying the following hypotheses:

(H1) For all \( t \in [0, T] \) and \( z \in C([0, T]; R) \), there exist a positive continuous function \( h(t) \), and a nondecreasing continuous function \( \chi : [0, \infty) \rightarrow (0, \infty) \) such that

\[
|g(t, z)| \leq h(t) \chi(\|z\|).
\]

(H2) There exists a positive constant \( \rho \) such that

\[
\frac{\rho}{\|h\|\chi(\rho)} > M,
\]

where \( M \) is the constant given by (10). Then, the \((T, a)\)-affine-periodic system (1) admits at least one solution on \([0, T]\).

**Proof.** Let \( \Omega_\rho = \{z \in C([0, T]; R) : \|z\| < \rho\} \), where \( \rho \) is given in (H2). It is easy to know that \( \Omega_\rho \) is a bounded open subset of \( C([0, T]; R) \).

In light of Lemma 4, we introduce an operator \( \mathcal{G} : C([0, T]; R) \rightarrow C([0, T]; R) \), which is expressed by

\[
\mathcal{G}(z)(t) = \int_0^t \int_0^\tau e^{-\lambda(t-\tau)} \frac{(\tau-s)^a-1}{\Gamma(a)} g(s, z(s)) ds d\tau + \mu_1(t) \int_0^\tau \int_0^\tau e^{-\lambda(T-\tau)} \frac{(T-s)^a-1}{\Gamma(a)} g(s, z(s)) ds d\tau + \mu_2(t) \int_0^T \frac{(T-s)^a-1}{\Gamma(a)} g(s, z(s)) ds,
\]

where \( \mu_1(t), \mu_2(t) \) are given in (4). Then, we can transform the \((T, a)\)-affine-periodic system (1) into a fixed point problem, i.e., \( z = \mathcal{G}(z) \). What follows is to use Lemma 1 to solve the fixed point problem.

The proof is divided into four steps:

**Step 1.** The operator \( \mathcal{G} : C([0, T]; R) \rightarrow C([0, T]; R) \) is continuous.

Let \( \{z_m\} \) be a sequence such that \( z_m \rightarrow z \) in \( C([0, T]; R) \). Then, it holds that

\[
|G(z_m)(t) - G(z)(t)| \\
\leq \int_0^t \int_0^\tau e^{-\lambda(t-\tau)} \frac{(\tau-s)^a-1}{\Gamma(a)} |g(s, z_m(s)) - g(s, z(s))| ds d\tau + \mu_1(t) \int_0^\tau \int_0^\tau e^{-\lambda(T-\tau)} \frac{(T-s)^a-1}{\Gamma(a)} |g(s, z_m(s)) - g(s, z(s))| ds d\tau + \mu_2(t) \int_0^T \frac{(T-s)^a-1}{\Gamma(a)} |g(s, z_m(s)) - g(s, z(s))| ds,
\]

(13)
Noting the continuity of \( g(t, z) \), we get \( |g(t, z_m) - g(t, z)| \to 0 \) as \( m \to \infty \), which implies that
\[
\|G(z_m)(t) - G(z)(t)\| = \max_{t \in [0,T]} |G(z_m)(t) - G(z)(t)| \to 0 \text{ as } m \to \infty.
\]

**Step 2.** The operator \( \mathcal{G} : C([0, T]; R) \to C([0, T]; R) \) is equicontinuous.
Let \( z \in \overline{\Omega}_\rho \), for any \( 0 \leq t_1 < t_2 \leq T \), from (H1), we gain that
\[
|G(z)(t_2) - G(z)(t_1)| \leq \left| \int_0^{t_2} \int_0^T e^{-\lambda(t_2-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right|
- \left| \int_0^{t_1} \int_0^T e^{-\lambda(t_1-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right|
+ |\mu_1(t_2) - \mu_1(t_1)| \left| \int_0^T \int_0^T e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right|
+ |\mu_2(t_2) - \mu_2(t_1)| \left| \int_0^T \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s, z(s)) ds d\tau \right|
\]
\[
\rightarrow 0,
\]
for \( t_1 \to t_2 \), for any \( z \in \overline{\Omega}_\rho \). This means \( \mathcal{G} \) is equicontinuous.

**Step 3.** The operator \( \mathcal{G} : C([0, T]; R) \to C([0, T]; R) \) maps bounded sets into bounded sets.
For each \( z \in \overline{\Omega}_\rho \), \( t \in [0, T] \), owing to (H1) and (H2), one obtains
\[
|G(z)(t)| \leq \left| \int_0^t \int_0^T e^{-\lambda(t-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right|
+ |\mu_1(t)| \left| \int_0^T \int_0^T e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right|
+ |\mu_2(t)| \left| \int_0^T \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s, z(s)) ds d\tau \right|
\]
\[
\leq \left( \int_0^t \int_0^T e^{-\lambda(t-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} ds d\tau \right) \|h(t)\| \chi(|z|)
+ \left( \int_0^T e^{-\lambda(T-\tau)} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \|h(0)\| \chi(|z|)
+ \left( \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \|h(t)\| \chi(|z|)
\]
\[
\leq \left( \frac{T^\alpha}{\Gamma(\alpha + 1)} (M_1 + \chi_1 M_1 + \chi_2) \|h\| \chi(|z|)\right)
\leq M_1 \|h\| \chi(\rho)
< \rho,
\]
which yields \( \|g(z)(t)\| = \max_{t \in [0,T]} |g(z)(t)| < \rho \). That is, \( g(z) \in \overline{\Omega}_\rho \). Therefore, using Lemma 3, the operator \( G \) is compact.

**Step 4.** The operator \( G : C([0, T]; R) \rightarrow C([0, T]; R) \) has a fixed point.

Suppose \( z \in \partial \Omega_\rho \), there exists \( \varepsilon \in (0,1) \), such that \( z = \varepsilon G(z) \). It then follows from (15) that

\[
\rho = \|z\| = \varepsilon \|G(z)(t)\| < \varepsilon M \|h\| \chi(\rho) < \varepsilon \rho < \rho.
\]

Obviously, this leads to a contradiction. Invoking Leray–Schauder fixed point theorem (Lemma 1), the operator \( G \) has a fixed point, i.e., \( z = G(z), z \in \overline{\Omega}_\rho \), which means that the system (1) has at least one solution on \([0, T]\).  

Next, we use Lemma 2 to research the existence of solution to system (1), whose nonlinear function satisfies Lipschize condition. The following hypotheses on \( g(t,z) \) are required:

**Theorem 2.** Suppose \( g(t,z) : [0,T] \times C([0,T]; R) \rightarrow R \) be a continuous function, which satisfies the following conditions:

\( \text{(H3)} \) There exists a constant \( l > 0 \) such that

\[
|g(t,z_1) - g(t,z_2)| < l |z_1 - z_2|,
\]

\( \forall t \in [0,T], z_1, z_2 \in C([0,T]; R) \).

(\( \text{H4} \)) For all \( t \in [0,T] \), and each \( z \in C([0,T]; R) \), there exists a function \( \delta(t) \in C([0,T]; R^+) \) such that

\[
|g(t,z)| \leq \delta(t).
\]

Then, the \((T, a)\)-affine-periodic system (1) admits at least one solution on \([0, T]\) if

\[
\tilde{M} l < 1,
\]

where \( \tilde{M} \) is the constant given by (11).

**Proof.** We split the operator \( G : C([0, T]; R) \rightarrow C([0, T]; R) \) defined by Equation (12) as \( G = G_1 + G_2 \), where \( G_1 \) and \( G_2 \) are given by

\[
G_1(z)(t) = \int_0^t \int_0^T e^{-\lambda(t-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} g(s, z(s)) ds d\tau,
\]

and

\[
G_2(z)(t) = \mu_1(t) \int_0^T e^{-\lambda(T-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} g(s, z(s)) ds d\tau \\
+ \mu_2(t) \int_0^T (T-s)^{a-1} \frac{\chi(\rho) \rho}{\Gamma(a)} g(s, z(s)) ds,
\]

Let us set \( \|\delta\| = \max_{t \in [0,T]} |\delta(t)| \), and define a bounded set \( \Omega_\rho = \{ z \in C([0,T]; R) : \|z\| \leq \tilde{\rho} \} \), where \( \tilde{\rho} \geq M \|\delta\| \), \( M \) is the constant given by (10).

In what follows, we use three steps to complete the proof of the theorem.

**Step 1.** \( \forall z_1, z_2 \in \Omega_\rho, G_1(z_1)(t) + G_2(z_2)(t) \in \Omega_\rho \).
For each $z_1, z_2 \in \Omega_\rho$, $t \in [0, T]$, we have
\[
\begin{align*}
|G_1(z_1)(t) + G_2(z_2)(t)| & \leq \left| \int_0^t \int_0^T e^{-\lambda(t-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} g(s, z_1(s))dsd\tau \right| \\
& \quad + |\mu_1(t)| \left| \int_0^T \int_0^T e^{-\lambda(T-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} g(s, z_2(s))dsd\tau \right| \\
& \quad + |\mu_2(t)| \left| \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} g(s, z_2(s))ds \right| \\
& \leq \left( \int_0^t \int_0^T e^{-\lambda(t-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} dsd\tau \right) + |\mu_1| \left| \int_0^T \int_0^T e^{-\lambda(T-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} dsd\tau \right| \\
& \quad + |\mu_2| \left| \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} ds \right| \|\delta\| \\
& \leq \frac{T^a}{\Gamma(a+1)} (M_\lambda + |\mu_1| M_\lambda + |\mu_2|) \|\delta\| \\
& \leq \bar{\rho},
\end{align*}
\]  
that is, $\|G_1(z_1)(t) + G_2(z_2)(t)\| \leq \bar{\rho}$, which leads to $G_1(z_1)(t) + G_2(z_2)(t) \in \Omega_\rho$.

**Step 2.** $G_1 : C([0, T]; R) \rightarrow C([0, T]; R)$ is continuous and compact.

It is easy to get the continuity of $G_1 : C([0, T]; R) \rightarrow C([0, T]; R)$ from the continuity of $g$, what follows to consider the equicontinuity of $G_1$.

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, and $z \in \Omega_\rho$, one can find
\[
\begin{align*}
|G_1(z)(t_2) - G_1(z)(t_1)| & = \left| \int_0^{t_2} \int_0^T e^{-\lambda(t_2-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} g(s, z(s))dsd\tau \right| \\
& \quad - \left| \int_0^{t_1} \int_0^T e^{-\lambda(t_1-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} g(s, z(s))dsd\tau \right| \\
& \leq \left| \int_0^{t_1} \int_0^T e^{-\lambda(t_2-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} \delta(s)dsd\tau \right| \\
& \quad + \left| \int_0^{t_2} \int_0^T e^{-\lambda(t_2-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} \delta(s)dsd\tau \right| \\
& \quad + \left| \int_0^{t_1} \int_0^T e^{-\lambda(t_2-\tau)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} \delta(s)dsd\tau \right| \\
& \leq \left( \int_0^{t_1} \int_0^T e^{-\lambda t_2} \frac{(\tau-s)^{a-1}}{\Gamma(a)} dsd\tau \right) \|\delta\| \\
& \quad + \left( \int_0^{t_2} \int_0^T e^{-\lambda t_2} \frac{(\tau-s)^{a-1}}{\Gamma(a)} dsd\tau \right) \|\delta\| \\
& \quad + \left( \int_0^{t_1} \int_0^T e^{-\lambda t_2} \frac{(\tau-s)^{a-1}}{\Gamma(a)} dsd\tau \right) \|\delta\| \\
& \rightarrow 0
\end{align*}
\]  
as $t_1 \rightarrow t_2$, for any $z \in \Omega_\rho$, which shows that $G_1$ is equicontinuous. Moreover, $G_1$ is uniformly bounded on $\Omega_\rho$ as
\[
\|G_1(z)(t)\| = \max_{t \in [0, T]} |G_1(z)(t)| \leq \frac{T^a}{\Gamma(a+1)} M_\lambda \|\delta\| \leq \bar{\rho}.
\]

According to Lemma 3, the operator $G_1 : C([0, T]; R) \rightarrow C([0, T]; R)$ is compact.

**Step 3.** $G_2 : C([0, T]; R) \rightarrow C([0, T]; R)$ is contractional.
For \( t \in [0, T] \), \( z_1, z_2 \in \Omega_{\hat{\rho}} \), we can derive
\[
\left| G_2(z_1)(t) - G_2(z_2)(t) \right| \\
\leq |\mu_1(t)| \left| \int_{0}^{T} \int_{0}^{\tau} e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z_1(s)) - g(s, z_2(s)) ds d\tau \right| \\
+ |\mu_2(t)| \left| \int_{0}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| g(s, z_1(s)) - g(s, z_2(s)) \right| ds \right| \\
\leq \left( \frac{1}{M} \right) \left| \int_{0}^{T} \int_{0}^{\tau} e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} ds d\tau \right| + \left| \int_{0}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right| l \| z_1 - z_2 \| \\
= \tilde{M} l \| z_1 - z_2 \|.
\]

As \( 0 < \tilde{M} l < 1 \), the operator \( G_2 \) is contractional.

In summary, by Krasnoselskii fixed point theorem (Lemma 2), there exists \( z \in \Omega_{\hat{\rho}} \) such that \( z(t) = G_1(z)(t) + G_2(z)(t) = G(z)(t) \), i.e., \( z \) is the solution to the \( (T, a) \)-affine-periodic system (1).

Next, we prove the uniqueness of the solution to the system (1) by Banach contraction mapping principe.

**Theorem 3.** If the hypothesis (H3) holds, the \( (T, a) \)-affine-periodic system (1) admits a unique solution on \([0, T]\) if
\[
\tilde{M} l < 1,
\]
where \( M \) is the constant given by (10).

**Proof.** Firstly, we shall clear that \( G : C([0, T]; R) \rightarrow C([0, T]; R) \) maps bounded set into itself. For this purpose, let \( \sup_{t \in [0, T]} |g(t, 0)| = M_0 \), and choose \( \hat{\rho} > \frac{M M_0}{1 - \tilde{M} l} \) to show that \( G \Omega_{\hat{\rho}} \subset \Omega_{\hat{\rho}} \), where \( \Omega_{\hat{\rho}} = \{ z \in C([0, T]; R) : \| z \| \leq \hat{\rho} \} \). In light of (H3), one obtain that
\[
|g(t, z)| = |g(t, z) - g(t, 0) + g(t, 0)| \leq |g(t, z) - g(t, 0)| + |g(t, 0)| \\
\leq l \| z \| + M_0 \\
\leq l \hat{\rho} + M_0.
\]

Thus, for every \( z \in \Omega_{\hat{\rho}} \), apply (19) to get
\[
|G(z)(t)| \\
\leq \left| \int_{0}^{t} \int_{0}^{\tau} e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right| \\
+ |\mu_1(t)| \left| \int_{0}^{T} \int_{0}^{\tau} e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds d\tau \right| \\
+ |\mu_2(t)| \left| \int_{0}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, z(s)) ds \right| \\
\leq \left( \frac{T}{M} \right) \left| \int_{0}^{T} \int_{0}^{\tau} e^{-\lambda(T-\tau)} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} ds d\tau \right| + \left| \int_{0}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right| l \| z \| \\
= M l \| z \| + M_0 \\
= \tilde{M} l \| z \| + M_0 \\
\leq \hat{\rho},
\]

(20)
which indicates that \( \|G(z)(t)\| = \max_{t \in [0,T]} |G(z)(t)| \leq \bar{\rho}. \) That is, \( \mathcal{G} \Omega_{\bar{\rho}} \subset \Omega_{\bar{\rho}} \), which implies that \( \mathcal{G} \) maps \( \Omega_{\bar{\rho}} \) into itself.

Secondly, we claim that the operator \( \mathcal{G} : C([0,T]; R) \to C([0,T]; R) \) is contractional. For \( t \in [0,T], z_1, z_2 \in \Omega_{\bar{\rho}}, \) we can deduce

\[
\begin{align*}
|G(z_1)(t) - G(z_2)(t)| &\leq \left| \int_0^t \int_0^T e^{-\lambda(t-s)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} [g(s,z_1(s)) - g(s,z_2(s))] ds \right| d\tau
+ |\mu_1(t)\int_0^T \int_0^T e^{-\lambda(\tau-s)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} [g(s,z_1(s)) - g(s,z_2(s))] ds| d\tau \\
&\quad + |\mu_2(t)\int_0^T \frac{(\tau-s)^{a-1}}{\Gamma(a)} [g(s,z_1(s)) - g(s,z_2(s))] ds| d\tau
+ |\bar{\mu}_1(t)\int_0^T \frac{(\tau-s)^{a-1}}{\Gamma(a)} [g(s,z_1(s)) - g(s,z_2(s))] ds| d\tau
+ |\bar{\mu}_2(t)\int_0^T \frac{(\tau-s)^{a-1}}{\Gamma(a)} [g(s,z_1(s)) - g(s,z_2(s))] ds| d\tau
\end{align*}
\]

\[
\leq \left( \int_0^t \int_0^T e^{-\lambda(t-s)} \frac{(\tau-s)^{a-1}}{\Gamma(a)} ds \right)^{1/2} \int_0^T |g(s,z_1(s)) - g(s,z_2(s))| ds d\tau
\]

\[
\leq M_1\|z_1 - z_2\|.
\]

(21)

Due to \( 0 < M_1 < 1, \mathcal{G} : C([0,T]; R) \to C([0,T]; R) \) is contractional.

Thanks to Banach contraction mapping principle, the operator \( \mathcal{G} \) has a unique fixed point, which is the unique solution to \((\mathcal{T}, a)\)-affine-periodic system (1). The proof is completed. \( \square \)

4. Examples

Two examples are provided to verify Theorems 1 and 3 in this section.

**Example 1.** Let us consider the \((1,0)\)-affine-periodic problem:

\[
\begin{align*}
(CD_t^\alpha + 2C^{1/2}D_t^{1/2}) z(t) &= \frac{e^{-t}}{\sqrt{25+t^2}} (\sin z + \frac{z^2}{1+z^2}), \quad t \in [0,1], \\
z(1) &= \varepsilon z(0), \quad z'(1) = \varepsilon z'(0).
\end{align*}
\]

(22)

Here \( \alpha = \frac{1}{2}, \beta = \frac{3}{4}, \lambda = 2, a = c, T = 1, \) and \( g(t,z) = \frac{e^{-t}}{\sqrt{25+t^2}} (\sin z + \frac{z^2}{1+z^2}). \) Clearly, we have

\[
|g(t,z)| \leq \left| \frac{e^{-t}}{\sqrt{25+t^2}} \right| \cdot \left| \sin z + \frac{z^2}{1+z^2} \right| \leq h(t) \chi(\|z\|),
\]

where \( h(t) = \frac{e^{-t}}{\sqrt{25+t^2}}, \chi(\|z\|) = 1 + \frac{\|z\|^2}{2}. \)

With the above assumptions, we can obtain \( \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}, M_1 = \max_{t \in [0,1]} \left| \frac{e^{-t}}{\sqrt{25+t^2}} \right| \approx 0.2648, M_2 = \frac{\max_{t \in [0,1]} \left| \frac{e^{-t}}{\sqrt{25+t^2}} \right|}{\max_{\frac{z_{\text{int}}}{2\sqrt{1+z^2}}} \left( \frac{1}{\sin z + \frac{z^2}{1+z^2}} \right) \approx 0.9755, \|h\| = \frac{1}{2}. \) Then, using the condition (H2), we can find \( \rho > 0.2162, \) it follows by Theorem 1 that problem (22) has a solution.

**Example 2.** Let us consider the \((1,1)\)-affine-periodic problem:

\[
\begin{align*}
(CD_t^\frac{1}{2} - C^{1/2}D_t^{1/2}) z(t) &= \frac{z_{\text{int}}}{2\sqrt{1+z^2} \sqrt{100+t^2}}, \quad t \in [0,1], \\
z(1) &= \frac{1}{2} z(0), \quad z'(1) = \frac{1}{2} z'(0).
\end{align*}
\]

(23)

Here \( \alpha = \frac{1}{2}, \beta = \frac{1}{2}, \lambda = -1, a = c, T = 1, \) and \( g(t,z) = \frac{z_{\text{int}}}{2\sqrt{1+z^2} \sqrt{100+t^2}}. \) Obviously, one gets

\[
\|g(t,z_1) - g(t,z_2)\| \leq M_1\|z_1 - z_2\|,
\]
where \( l = \frac{1}{10} \).

With the given values, we can calculate \( \Gamma(\frac{1}{2}) \approx 0.8930, p_1 = \max_{t \in [0,1]} \left| \frac{e^{-t}}{t^\alpha} \right| \approx 1.1565, p_2 = \max_{t \in [0,1]} \left| \frac{1}{(t-1)\beta} - \frac{e^{-t}}{(1-\epsilon)} \right| \approx 1.1565, M_1 = \left| \frac{1-e^{-t}}{1-t} \right| \approx 1.7183, M \approx 5.4446, M_0 = 0 \), we choose \( \hat{\rho} \geq 0 \), using the condition (H3), we have \( 0 < Ml \leq 0.5445 < 1 \). By Theorem 3, the problem (23) has a unique solution.

5. Conclusions

In this paper, we investigate the existence and uniqueness of solution to a sequential fractional differential equation with affine periodic boundary value conditions. However, the two fractional derivatives in the text must meet the conditions: \( 0 < \alpha < 1 < \beta < 2 \), and \( \beta = \alpha + 1 \), which limit the application range of the differential equation. In the next study, we will research the existence of solution to a sequential fractional differential equation with the order \( 0 < \alpha < 1 < \beta < 2 \), which \( \alpha \) is independent of \( \beta \). What is more, the fractional differential equation with higher order \( n-1 < \alpha < n < \beta < n+1 \) will also be studied.

Due to the fact that the differential inclusion theory has a very wide range of applications in many fields, such as optimal control theory, dynamic system, and engineering technology. Therefore, the existence of solution to fractional differential inclusion problem deserves to be researched. We will further investigate the solution to a fractional differential inclusion with the affine periodic boundary value conditions in the future.

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