A New Extension to the Intuitionistic Fuzzy Metric-like Spaces

Fahim Uddin 1, Umar Ishtiaq 2, Khalil Javed 3, Suhad Subhi Aiadi 4, Muhammad Arshad 3, Nizar Souayah 5,6,* and Nabil Mlaiki 4

1 Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan; fahim@sms.edu.pk
2 Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan; umar.ishtiaq000@gmail.com
3 Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan; khaliljaved15@gmail.com (K.J.); marshad@iiu.edu.pk (M.A.)
4 Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia; ssubhi@psu.edu.sa (S.S.A.); nmlaiki@psu.edu.sa (N.M.)
5 Department of Natural Sciences, Community College Al-Riyadh, King Saud University, Riyadh 11451, Saudi Arabia
6 Ecole Supérieure des Sciences Economiques et Commerciales de Tunis, Université de Tunis, Tunis 1002, Tunisia
* Correspondence: nsouayah@ksu.edu.sa

Abstract: In this manuscript, we introduce the concept of intuitionistic fuzzy controlled metric-like spaces via continuous t-norms and continuous t-conorms. This new metric space is an extension to intuitionistic fuzzy controlled metric-like spaces, controlled metric-like spaces and controlled fuzzy metric spaces, and intuitionistic fuzzy metric spaces. We prove some fixed-point theorems and we present non-trivial examples to illustrate our results. We used different techniques based on the properties of the considered spaces notably the symmetry of the metric. Moreover, we present an application to non-linear fractional differential equations.

Keywords: symmetric metric spaces; fuzzy metric spaces; fixed point theorems; non-linear fractional differential equations

1. Introduction

In today’s multifaceted environment, uncertainty and fuzziness are widespread in many applications. Zadeh [1] pioneered the concept of fuzzy sets (FSs) to capture the ambiguity and fuzziness of information. Since its origin, many extensions of FSs have been proposed to better represent sophisticated information, including intuitionistic fuzzy sets (IFSs), picture FSs, q-rung orthopair FSs, and neutrosophic sets.

Recently, Harandi [2] initiated the concept of metric-like spaces, which generalized the notion of metric spaces in a nice way. Alghamdi et al. [3] used the concept metric-like spaces to introduce the notion of b-metric-like spaces. Mlaiki [4] introduced the concept of controlled metric type spaces and proved various results. Mlaiki et al. [5] proposed the notion of controlled metric-like spaces (CMLSs) and proved several results for contractive mappings.

generalized the concept of controlled type metric spaces and introduced the concept of Controlled fuzzy metric spaces (CFMS). In this sequel, Shukla and Abbas [15] generalized the concept of metric-like spaces and introduced fuzzy metric-like spaces (FMLSs). Javed et al. [16] proposed fuzzy b-metric-like spaces. Shukla et al. [17] proposed an amazing notion of 1-M complete FMSs and proved various theorems.

The approach of intuitionistic fuzzy metric spaces (IFMSs) via continuous t-norms and continuous t-conorms was presented by Park in [18]. Rafi and Noorani [19] proved several fixed-point results in the context of IFMSs. Sintunavarat and Kumam [20] proved fixed point theorems for a generalized intuitionistic fuzzy contraction in IFMSs. Later, Konwar [21] presented intuitionistic fuzzy b-metric space (IFBMS). Alaca et al. [22] and Mohamad [23] proved several fixed-point results. Saadati and Park [24] did amazing work on intuitionistic fuzzy topological spaces. In addition, Sezen in [14] introduced the concept of metric-like spaces and introduced fuzzy metric-like spaces (FMLSs). Javed generalized the concept of controlled type metric spaces and introduced the concept of intuitionistic fuzzy controlled metric-like spaces (IFCMLSs) by using the approach in [5], also to extend various fixed point (FP) results on intuitionistic fuzzy topological spaces. In addition, Sezen in [14], introduced the concept of 1-M complete FMSs and proved various theorems.

The goal of this manuscript is to introduce intuitionistic fuzzy controlled metric-like spaces (IFCMLSs) by using the approach in [5], also to extend various fixed point (FP) results for contraction mappings, which is an improvement of the present literature’s methodology using different techniques based on the properties of contractions and the considered metric such as the triangle inequality and the symmetry. In closing, and inspired by work carried out in [25–29], we present an application of our results to fractional differential equations.

2. Preliminaries

Now, we start this section by listing various helpful definitions for readers and $I = [0, 1]$ be used in this study.

Definition 1 ([11]). A binary operation $\ast : I \times I \rightarrow I$ is called a continuous t-norm (CTN) if:

(a1) $\nu \ast \omega = \omega \ast \nu, \forall \nu, \omega \in I$;
(b1) $\ast$ is continuous.
(c1) $\nu \ast 1 = \nu, \forall \nu \in I$;
(d1) $(\nu \ast \omega) \ast \kappa = \nu \ast (\omega \ast \kappa), \forall \nu, \omega, \kappa \in I$;
(e1) If $\nu \leq \kappa$ and $\omega \leq d$, with $\nu, \omega, \kappa, d \in I$, then $\nu \ast \omega \leq \kappa \ast d$.

Definition 2 ([11]). A binary operation $\circ : I \times I \rightarrow I$ is called a continuous t-conorm (CTCN) if:

(i) $\nu \circ \omega = \omega \circ \nu$, for all $\nu, \omega \in I$;
(ii) $\circ$ is continuous.
(iii) $\nu \circ 0 = 0$, for all $\nu \in I$;
(iv) $(\nu \circ \omega) \circ \kappa = \nu \circ (\omega \circ \kappa)$, for all $\nu, \omega, \kappa \in I$;
(v) If $\nu \leq \kappa$ and $\omega \leq d$, with $\nu, \omega, \kappa, d \in I$, then $\nu \circ \omega \leq \kappa \circ d$.

Definition 3 ([23]). Let $K \neq \emptyset$. A mapping $\mathcal{L} : K \times K \rightarrow [1, \infty)$, fulfilling the following assertions:

a. $\mathcal{L}(\bar{a}, \bar{a}) = 0$ implies $\bar{a} = \bar{a}$;

b. $\mathcal{L}(\bar{b}, \bar{a}) = \mathcal{L}(\bar{a}, \bar{b})$;

c. $\mathcal{L}(\bar{b}, \bar{a}) \leq \mathcal{L}(\bar{b}, \bar{d}) + \mathcal{L}(\bar{d}, \bar{b})$;

for all $\bar{a}, \bar{b}, \bar{d} \in K$. Then $(K, \mathcal{L})$ is called a metric-like space.

Definition 4 ([24]). Let $K \neq \emptyset$. A function $\psi : K \times K \rightarrow [1, \infty)$ and a mapping $\mathcal{L} : K \times K \rightarrow \mathbb{R}^+$, fulfilling the following assertions:

I. $\mathcal{L}(\bar{b}, \bar{a}) = 0$ implies $\bar{b} = \bar{a}$;

II. $\mathcal{L}(\bar{b}, \bar{a}) = \mathcal{L}(\bar{a}, \bar{b})$;

III. $\mathcal{L}(\bar{b}, \bar{a}) \leq \psi(\bar{b}, \bar{d}) \mathcal{L}(\bar{b}, \bar{d}) + \psi(\bar{d}, \bar{b}) \mathcal{L}(\bar{d}, \bar{b})$;

for all $\bar{b}, \bar{a}, \bar{d} \in K$. Then $(K, \mathcal{L})$ is named a CMLS.

Definition 5 ([21]). Let $K \neq \emptyset$. Suppose $\ast$ be a CTN and $\mathcal{N}_b$ be a FS on $K \times K \times (0, \infty)$. A three tuple $(K, \mathcal{N}_b, \ast)$ is called FMLS, if it is fulfilling the following assertions, for all $\bar{b}, \bar{a} \in K$ and $\bar{a}, \omega > 0$:
(F1). \( \mathcal{N}_a(p, a, \omega) > 0 \);
(F2). \( \mathcal{N}_a(p, a, \omega) = 1 \) implies \( p = a \);
(F3). \( \mathcal{N}_a(p, a, \omega) = \mathcal{N}_b(\bar{a}, p, \omega) \);
(F4). \( \mathcal{N}_a(p, \delta, (\omega + \theta)) \geq \mathcal{N}_b(p, a, \omega) + \mathcal{N}_a(a, \delta, \theta) \);
(F5). \( \mathcal{N}_b(p, a) : (0, \infty) \to [0, 1] \) is continuous.

Definition 6 ([4]). Let \( K \neq \emptyset \). Suppose \(*\) be a CTN, \( \circ \) be a CTCN, \( b \geq 1 \) and \( \mathcal{N}_a, \mathcal{R}_b \) be FSs on \( K \times K \times (0, \infty) \). If \( (K, \mathcal{N}_a, \mathcal{R}_b, *, \circ) \) verifies the following for all \( p, a \in K \) and \( \theta, \omega > 0 \):

(CL1). \( \mathcal{N}_a(p, a, \omega) + \mathcal{R}_b(p, a, \omega) \leq 1 \);
(CL2). \( \mathcal{N}_a(p, a, \omega) > 0 \);
(CL3). \( \mathcal{N}_a(p, a, \omega) = 1 \) ⇔ \( p = a \);
(CL4). \( \mathcal{N}_a(p, a, \omega) = \mathcal{R}_b(\bar{a}, p, \omega) \);
(CL5). \( \mathcal{N}_a(p, \delta, b(\omega + \theta)) \geq \mathcal{N}_b(p, a, \omega) + \mathcal{N}_a(a, \delta, \theta) \);
(CL6). \( \mathcal{N}_a(p, a) \) is a non decreasing (ND) function of \( \mathbb{R}^+ \) and \( \lim_{\omega \to \infty} \mathcal{N}_b(p, a, \omega) = 1 \);
(CL7). \( \mathcal{R}_b(p, a, \omega) > 0 \);
(CL8). \( \mathcal{R}_b(p, a, \omega) = 0 \) ⇔ \( p = a \);
(CL9). \( \mathcal{R}_b(p, a, \omega) = \mathcal{R}_b(\bar{a}, p, \omega) \);
(CL10). \( \mathcal{R}_b(p, \delta, b(\omega + \theta)) \leq \mathcal{R}_b(p, a, \omega) \circ \mathcal{R}_a(\bar{a}, \delta, \theta) \);
(CL11). \( \mathcal{R}_b(p, a) \) is a non increasing (NI) function of \( \mathbb{R}^+ \) and \( \lim_{\omega \to \infty} \mathcal{R}_b(p, a, \omega) = 0 \).

Then \( (K, \mathcal{N}_a, \mathcal{R}_b, *, \circ) \) is an IFBMS.

3. Main Results

In this section, we present the concept of an IFCMLS and prove several FP results.

Definition 7. Suppose \( K \neq \emptyset \), assume a five tuple \( (K, \mathcal{N}_a, \mathcal{R}_b, *, \circ) \) where \(*\) is a CTN, \( \circ \) is a CTCN, \( \phi : K \times K \to [1, \infty) \) and \( \mathcal{N}_a, \mathcal{R}_b \) are FSs on \( K \times K \times (0, \infty) \). If \( (K, \mathcal{N}_a, \mathcal{R}_b, *, \circ) \) meet the below circumstances for all \( p, a \in K \) and \( \theta, \omega > 0 \):

(CL1). \( \mathcal{N}_a(p, a, \omega) + \mathcal{R}_b(p, a, \omega) \leq 1 \);
(CL2). \( \mathcal{N}_a(p, a, \omega) > 0 \);
(CL3). \( \mathcal{N}_a(p, a, \omega) = 1 \) implies \( p = a \);
(CL4). \( \mathcal{N}_a(p, a, \omega) = \mathcal{R}_b(\bar{a}, p, \omega) \);
(CL5). \( \mathcal{N}_a(p, \delta, (\omega + \theta)) \geq \mathcal{N}_a(\bar{a}, p, \omega) \); \( \mathcal{R}_b(p, a, \omega) \) + \( \mathcal{N}_a(\bar{a}, \delta, \theta) \);
(CL6). \( \mathcal{N}_a(p, a) \) is ND function of \( \mathbb{R}^+ \) and \( \lim_{\omega \to \infty} \mathcal{N}_a(p, a, \omega) = 1 \);
(CL7). \( \mathcal{R}_b(p, a, \omega) > 0 \);
(CL8). \( \mathcal{R}_b(p, a, \omega) = 0 \) implies \( p = a \);
(CL9). \( \mathcal{R}_b(p, a, \omega) = \mathcal{R}_b(\bar{a}, p, \omega) \);
(CL10). \( \mathcal{R}_b(p, \delta, (\omega + \theta)) \leq \mathcal{R}_b(p, a, \omega) \); \( \mathcal{R}_b(\bar{a}, \delta, \theta) \);
(CL11). \( \mathcal{R}_b(p, a) \) is NI function of \( \mathbb{R}^+ \) and \( \lim_{\omega \to \infty} \mathcal{R}_b(p, a, \omega) = 0 \).

Then \( (K, \mathcal{N}_a, \mathcal{R}_b, *, \circ) \) is an IFCMLS.

Example 1. Let \( K = \{1, 2, 3\} \) and \( a : K \times K \to [1, \infty) \) be a function given by \( a(p, a) = p + a + 1 \). Define \( \mathcal{N}_a, \mathcal{R}_b : K \times K \times (0, \infty) \to [0, 1] \) as,

\[
\mathcal{N}_a(p, a, \omega) = \frac{\omega}{\omega + \max\{p, a\}}
\]

In addition,

\[
\mathcal{R}_b(p, a, \omega) = \frac{\max\{p, a\}}{\omega + \max\{p, a\}}.
\]
Then \((K, \mathcal{N}_\phi, \mathcal{R}_\phi, *, \circ)\) is an IFCMLS with CTN \(a * b = ab\) and CTCN \(a \circ b = \max\{a, b\}\).

**Proof.** (CL1)–(CL4), (CL6)–(CL9) and (CL11) are obvious, here we prove (CL5) and (CL10).

Let \(p = 1, \bar{a} = 2\) and \(\delta = 3\). Then,

\[
\mathcal{N}_\phi(1, 3, \omega + \theta) = \frac{\omega + \theta}{\omega + \theta + \max\{1, 3\}} = \frac{\omega + \theta}{\omega + \theta + 3}.
\]

On the other hand,

\[
\mathcal{N}_\phi\left(1, 2, \frac{\omega}{a(1, 2)}\right) = \frac{\omega}{\omega + \max\{1, 2\}} = \frac{\omega}{\omega + 2} = \frac{\omega}{\omega + 8}.
\]

In addition,

\[
\mathcal{N}_\phi\left(2, 3, \frac{\theta}{a(2, 3)}\right) = \frac{\theta}{\theta + \max\{2, 3\}} = \frac{\theta}{\theta + 3} = \frac{\theta}{\theta + 18}.
\]

That is,

\[
\frac{\omega + \theta}{\omega + \theta + 3} \geq \frac{\omega}{\omega + 8} \cdot \frac{\theta}{\theta + 18}.
\]

Then it satisfied, for all \(\omega, \theta > 0\). Hence,

\[
\mathcal{N}_\phi(p, \delta, \omega + \theta) \geq \mathcal{N}_\phi\left(p, \bar{a}, \frac{\omega}{a(p, \bar{a})}\right) \circ \mathcal{N}_\phi\left(\bar{a}, \delta, \frac{\theta}{a(\bar{a}, \delta)}\right).
\]

Now,

\[
\mathcal{R}_\phi(1, 3, \omega + \theta) = \frac{\max\{1, 3\}}{\omega + \theta + \max\{1, 3\}} = \frac{3}{\omega + \theta + 3}.
\]

On the other hand,

\[
\mathcal{R}_\phi\left(1, 2, \frac{\omega}{a(1, 2)}\right) = \frac{\max\{1, 2\}}{\omega + \max\{1, 2\}} = \frac{2}{\omega + 2} = \frac{8}{\omega + 8}.
\]

In addition,

\[
\mathcal{R}_\phi\left(2, 3, \frac{\theta}{a(2, 3)}\right) = \frac{\max\{2, 3\}}{\theta + \max\{2, 3\}} = \frac{3}{\theta + 3} = \frac{18}{\theta + 18}.
\]

That is,

\[
\frac{3}{\omega + \theta + 3} \leq \max\left\{\frac{8}{\omega + 8}, \frac{18}{\theta + 18}\right\}.
\]

Then it satisfied, for all \(\omega, \theta > 0\). Hence,

\[
\mathcal{R}_\phi(p, \delta, \omega + \theta) \leq \mathcal{R}_\phi\left(p, \bar{a}, \frac{\omega}{a(p, \bar{a})}\right) \circ \mathcal{R}_\phi\left(\bar{a}, \delta, \frac{\theta}{a(\bar{a}, \delta)}\right).
\]
Similarly, all other cases can be investigated. Hence, \((K, \mathcal{R}_\phi, \mathcal{R}_\psi, \ast, \circ)\) is an IFCMLS. □

**Remark 1.** The preceding example satisfied as well for CTN \(a \ast b = \min\{a, b\}\) and CTCN \(a \circ b = \max\{a, b\}\).

**Example 2.** Let \(K = \{1, 2, 3\}\) and \(\alpha : K \times K \to [1, \infty)\) be a function given by \(\alpha(p, a) = p + a + 1\). Define \(\mathcal{R}_\phi, \mathcal{R}_\psi : K \times K \to (0, \infty) \to [0, 1]\) as,

\[
\mathcal{R}_\phi(p, a, \omega) = \frac{\omega + \min\{p, a\}}{\omega + \max\{p, a\}}
\]

Then \((K, \mathcal{R}_\phi, \mathcal{R}_\psi, \ast, \circ)\) is an intuitionistic fuzzy controlled metric-like space with CTN \(a \ast b = ab\) and CTCN \(a \circ b = \max\{a, b\}\).

**Proof.** It is not difficult to check. □

**Proposition 1.** Let \(K = [0, 1]\) and \(\alpha : K \times K \to [0, 1]\) be a function given by \(\alpha(p, a) = 2(p + a + 1)\). Define \(\mathcal{R}_\phi, \mathcal{R}_\psi\) as,

\[
\mathcal{R}_\phi(p, a, \omega) = e^{-\frac{\max\{p, a\}}{\omega}}, \quad \mathcal{R}_\psi(p, a, \omega) = 1 - e^{-\frac{\max\{p, a\}}{\omega}} \text{ for all } n \in \mathbb{N}, \ p, a \in K, \omega > 0.
\]

Then \((K, \mathcal{R}_\phi, \mathcal{R}_\psi, \ast, \circ)\) is an intuitionistic fuzzy controlled metric-like space with CTN \(a \ast b = ab\) and CTCN \(a \circ b = \max\{a, b\}\).

**Remark 2.** The above proposition also satisfied for CTN \(a \ast b = \min\{a, b\}\) and CTCN \(a \circ b = \max\{a, b\}\).

**Proposition 2.** Let \(K = [0, 1]\) and \(\alpha : K \times K \to [0, 1]\) be a function given by \(\alpha(p, a) = 2(p + a + 1)\). Define \(\mathcal{R}_\phi, \mathcal{R}_\psi\) as,

\[
\mathcal{R}_\phi(p, a, \omega) = \left[e^{-\frac{\max\{p, a\}}{\omega}}\right]^{-1}, \quad \mathcal{R}_\psi(p, a, \omega) = 1 - \left[e^{-\frac{\max\{p, a\}}{\omega}}\right]^{-1} \text{ for all } n \in \mathbb{N}, \ p, a \in K, \omega > 0.
\]

Then \((K, \mathcal{R}_\phi, \mathcal{R}_\psi, \ast, \circ)\) is an intuitionistic fuzzy controlled metric-like space with CTN \(a \ast b = ab\) and CTCN \(a \circ b = \max\{a, b\}\).

**Example 3.** Let \(K = (0, \infty)\), define \(\mathcal{R}_\phi, \mathcal{R}_\psi : K \times K \times (0, \infty) \to [0, 1]\) by,

\[
\mathcal{R}_\phi(p, a, \omega) = \frac{\omega}{\omega + \max\{p, a\}}, \quad \mathcal{R}_\psi(p, a, \omega) = \frac{\max\{p, a\}}{\omega + \max\{p, a\}}
\]

for all \(p, a \in K\) and \(\omega > 0\), define CTN “\(\ast\)” by \(v \ast \omega = v \cdot \omega\) and CTCN “\(\circ\)” by \(v \circ \omega = \max\{v, \omega\}\) and define “\(\phi\)” by,

\[
\phi(p, \bar{a}) = \begin{cases} 1 & \text{if } \bar{a} = \bar{a}, \\ \frac{1 + \max\{p, \bar{a}\}}{\min\{p, \bar{a}\}} & \text{if } \bar{a} \neq \bar{a} \end{cases}
\]

Then \((K, \mathcal{R}_\phi, \mathcal{R}_\psi, \ast, \circ)\) be an IFCMLS.
Proof. (CL1)–(CL4), (CL6)–(CL9) and (CL11) are obvious, here we prove (CL5) and (CL10)

\[ \max\{\mathcal{P}, \delta\} \leq \phi(\mathcal{P}, \bar{a}) \max\{\mathcal{P}, \bar{a}\} + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\} \]

This implies,

\[ \omega \theta \max\{\mathcal{P}, \delta\} \leq \phi(\mathcal{P}, \bar{a}) \left( \omega \theta + \theta^2 \right) \max\{\mathcal{P}, \bar{a}\} + \phi(\bar{a}, \delta) \left( \omega \theta + \omega^2 \right) \max\{\bar{a}, \delta\} \]

Then,

\[ \omega \theta \max\{\mathcal{P}, \delta\} \leq \phi(\mathcal{P}, \bar{a})(\omega + \theta) \max\{\mathcal{P}, \bar{a}\} + \phi(\bar{a}, \delta)(\omega + \theta) \omega \max\{\bar{a}, \delta\} \]

Therefore,

\[ \omega \theta(\omega + \theta) + \omega \theta \max\{\mathcal{P}, \delta\} \leq \omega \theta(\omega + \theta) + \phi(\mathcal{P}, \bar{a})(\omega + \theta) \max\{\mathcal{P}, \bar{a}\} + \phi(\bar{a}, \delta)(\omega + \theta) \omega \max\{\bar{a}, \delta\} \]

This implies,

\[ \omega \theta[\omega + \max\{\mathcal{P}, \delta\}] \leq (\omega + \theta)[\omega + \phi(\mathcal{P}, \bar{a}) \max\{\mathcal{P}, \bar{a}\} + \phi(\bar{a}, \delta) \omega \max\{\bar{a}, \delta\}] \]

Then,

\[ \omega \theta[\omega + \max\{\mathcal{P}, \delta\}] \leq (\omega + \theta)[\omega + \phi(\mathcal{P}, \bar{a}) \max\{\mathcal{P}, \bar{a}\}][\theta + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\}] \]

This implies,

\[ \frac{(\omega + \theta)}{(\omega + \theta) + \max\{\mathcal{P}, \delta\}} \geq \frac{\omega \theta}{\omega + \phi(\mathcal{P}, \bar{a}) \max\{\mathcal{P}, \bar{a}\} + \phi(\bar{a}, \delta) \omega \max\{\bar{a}, \delta\}} \]

This implies,

\[ \frac{(\omega + \theta)}{(\omega + \theta) + \max\{\mathcal{P}, \delta\}} \geq \frac{\omega}{\omega + \phi(\mathcal{P}, \bar{a}) \max\{\mathcal{P}, \bar{a}\}} \cdot \frac{\theta}{\theta + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\}} \]

Then,

\[ \frac{(\omega + \theta)}{(\omega + \theta) + \max\{\mathcal{P}, \delta\}} \geq \frac{\omega}{\frac{\bar{a}}{\phi(\mathcal{P}, \bar{a})}} + \max\{\bar{a}, \delta\} \cdot \frac{\theta}{\frac{\bar{a}}{\phi(\bar{a}, \delta)}} + \max\{\bar{a}, \delta\} \]

Hence,

\[ \kappa_{\phi}(\mathcal{P}, \delta, (\omega + \theta)) \geq \kappa_{\phi} \left( \mathcal{P}, \bar{a}, \frac{\omega}{\phi(\mathcal{P}, \bar{a})} \right) \cdot \kappa_{\phi} \left( \bar{a}, \delta, \frac{\theta}{\phi(\bar{a}, \delta)} \right) \]

(CL5) is satisfied.

\[ \max\{\mathcal{P}, \delta\} = \max\{\mathcal{P}, \delta\} \max\{1, 1\} \]

\[ \max\{\mathcal{P}, \delta\} = \max\{\mathcal{P}, \delta\} \max \left\{ \max\{\mathcal{P}, \bar{a}\} \max\{\bar{a}, \delta\} \right\} \]

\[ \max\{\mathcal{P}, \delta\} \leq \left[ (\omega + \theta) + \max\{\mathcal{P}, \bar{a}\} \right] \max \left\{ \max\{\mathcal{P}, \bar{a}\} \max\{\bar{a}, \delta\} \right\} \]
Therefore,
\[
\max\{\beta, \delta\} \leq [(\omega + \theta) + \max\{\beta, \delta\}]^{\max\{\phi(\beta, \delta)\delta\max\{\beta, \delta\}, \phi(\beta, \delta)\delta\max\{\beta, \delta\}\}}
\]

Then,
\[
\max\{\beta, \delta\} \leq \max\{\phi(\beta, \delta)\delta\max\{\beta, \delta\}, \phi(\beta, \delta)\delta\max\{\beta, \delta\}\}
\]

This implies,
\[
\max\{\beta, \delta\} \leq \max\{\frac{\max\{\beta, \delta\}}{\phi(\beta, \delta)}, \frac{\max\{\beta, \delta\}}{\phi(\beta, \delta)}\}
\]

Hence,
\[
\mathcal{R}_\phi(\beta, \delta, (\omega + \theta)) \leq \mathcal{R}_\phi\left(\beta, \delta, \frac{\omega}{\phi(\beta, \delta)}\right) \ast \mathcal{R}_\phi\left(\delta, \delta, \frac{\theta}{\phi(\beta, \delta)}\right)
\]

(CL10) is satisfied. □

**Definition 8.** Let \((\mathcal{K}, \mathcal{N}_\phi, \mathcal{R}_\phi, *, \omega)\) be a IFCMLS. Then,

(i). \(A\) sequence \(\{\beta_n\}\) in \(\mathcal{K}\) is said to be \(G\)-Cauchy sequence (GCS) if and only if for all \(\omega > 0\), \(\lim_{n \to \infty} \mathcal{N}_\phi(\beta_n, \beta_{n+q}, \omega)\) and \(\lim_{n \to \infty} \mathcal{R}_\phi(\beta_n, \beta_{n+q}, \omega)\), exists and is finite.

(ii). \(A\) sequence \(\{\beta_n\}\) in \(\mathcal{K}\) is named to be \(G\)-convergent (GC) to \(\beta\) in \(\mathcal{K}\), if and only if for all \(\omega > 0\),
\[
\lim_{n \to \infty} \mathcal{N}_\phi(\beta_n, \beta, \omega) = \mathcal{N}_\phi(\beta, \beta, \omega) \text{ and } \lim_{n \to \infty} \mathcal{R}_\phi(\beta_n, \beta, \omega) = \mathcal{R}_\phi(\beta, \beta, \omega).
\]

(iii). A IFCMLS is named to be complete iff each GCS is convergent, i.e.,
\[
\lim_{n \to \infty} \mathcal{N}_\phi(\beta_n, \beta_{n+q}, \omega) = \mathcal{N}_\phi(\beta, \beta, \omega) \text{ and } \lim_{n \to \infty} \mathcal{R}_\phi(\beta_n, \beta_{n+q}, \omega) = \mathcal{R}_\phi(\beta, \beta, \omega).
\]

**Theorem 1.** Suppose \((\mathcal{K}, \mathcal{N}_\phi, \mathcal{R}_\phi, *, \omega)\) be a \(G\)-complete IFCMLS in the company of \(\phi : \mathcal{K} \times \mathcal{K} \to [1, \infty)\) and suppose that,
\[
\lim_{\omega \to \infty} \mathcal{N}_\phi(\beta, \delta, \omega) = 1 \text{ and } \lim_{\omega \to \infty} \mathcal{R}_\phi(\beta, \delta, \omega) = 0
\]

for all \(\beta, \delta \in \mathcal{K}\) and \(\omega > 0\). Let \(\xi : \mathcal{K} \to \mathcal{K}\) be a mapping satisfying,
\[
\mathcal{N}_\phi(\xi(\beta), \xi(\delta), \delta, \omega) \geq \mathcal{N}_\phi(\beta, \delta, \omega) \text{ and } \mathcal{R}_\phi(\xi(\beta), \xi(\delta), \delta, \omega) \leq \mathcal{R}_\phi(\beta, \delta, \omega)
\]

for all \(\beta, \delta \in \mathcal{K}\), and \(\omega > 0\), where \(0 < \epsilon < 1\). Furthermore, assume that for every \(\alpha \in \mathcal{K}\),
\[
\lim_{n \to \infty} \phi(\beta_n, \alpha) \& \lim_{n \to \infty} \phi(\alpha, \beta_n)
\]

In addition,
\[
\lim_{n, m \to \infty} \phi(\beta_n, \beta_m) \& \lim_{n, m \to \infty} \phi(\beta_m, \beta_n)
\]

exists and are finite, where \(\beta_n = \xi^n \beta_0 = \xi \beta_{n-1}\), for all \(n \in \mathbb{N}\) and \(\beta_0\) be arbitrary point of \(\mathcal{K}\). Then \(\xi\) has a unique FP.
Proof. Let $p_0$ be an arbitrary point of $K$ and define a sequence $p_n$ by $p_n = \xi^n p_0 = \xi p_{n-1}, n \in \mathbb{N}$. Using (2) for all $\omega > 0$, we examine,

$$N_f(p_n, p_{n+1}, \varepsilon \omega) = N_f(\xi p_{n-1}, \xi p_n, \varepsilon \omega) \geq N_f(p_{n-1}, p_n, \omega) \geq N_f(p_{n-2}, p_n, \frac{\omega}{\epsilon})$$

$$\geq N_f(p_{n-3}, p_{n-2}, \frac{\omega}{\epsilon^2}) \geq \cdots \geq N_f(p_0, p_1, \frac{\omega}{\epsilon^{n-1}})$$

In addition,

$$R_f(p_n, p_{n+1}, \varepsilon \omega) = R_f(\xi p_{n-1}, \xi p_n, \varepsilon \omega) \leq R_f(p_{n-1}, p_n, \omega) \leq R_f(p_{n-2}, p_{n-1}, \frac{\omega}{\epsilon})$$

$$\leq R_f(p_{n-3}, p_{n-2}, \frac{\omega}{\epsilon^2}) \leq \cdots \leq R_f(p_0, p_1, \frac{\omega}{\epsilon^{n-1}})$$

We obtain,

$$N_f(p_n, p_{n+1}, \varepsilon \omega) \geq N_f(p_0, p_1, \frac{\omega}{\epsilon^{n-1}})$$

and

$$R_f(p_n, p_{n+1}, \varepsilon \omega) \leq R_f(p_0, p_1, \frac{\omega}{\epsilon^{n-1}})$$

for any $q \in \mathbb{N}$, using (CL5) and (CL10), we deduce,

$$N_f(p_n, p_{n+q}, \omega) \geq N_f(p_n, p_{n+1}, \frac{\omega}{\epsilon (\phi(p_n, p_{n+1}))}) \ast N_f(p_{n+1}, p_{n+q}, \frac{\omega}{\epsilon (\phi(p_{n+1}, p_{n+q}))})$$

$$\geq N_f(p_n, p_{n+1}, \frac{\omega}{\epsilon (\phi(p_n, p_{n+1}))}) \ast N_f(p_{n+1}, p_{n+2}, \frac{\omega}{(2)\epsilon^2 (\phi(p_{n+1}, p_{n+2}))(\phi(p_{n+1}, p_{n+3}))})$$

$$\geq N_f(p_n, p_{n+1}, \frac{\omega}{(2)\epsilon^2 (\phi(p_{n+1}, p_{n+2}))(\phi(p_{n+1}, p_{n+3}))})$$

$$\geq N_f(p_{n+1}, p_{n+2}, \frac{\omega}{(2)\epsilon^3 (\phi(p_{n+1}, p_{n+2}))(\phi(p_{n+1}, p_{n+3}))(\phi(p_{n+1}, p_{n+4}))})$$

$$\geq N_f(p_{n+1}, p_{n+2}, \frac{\omega}{(2)\epsilon^4 (\phi(p_{n+1}, p_{n+2}))(\phi(p_{n+1}, p_{n+3}))(\phi(p_{n+1}, p_{n+4}))(\phi(p_{n+1}, p_{n+5}))})$$

$$\ast \ast \ast$$

In addition,

$$R_f(p_n, p_{n+q}, \omega) \leq R_f(p_n, p_{n+1}, \frac{\omega}{\epsilon (\phi(p_n, p_{n+1}))}) \ast R_f(p_{n+1}, p_{n+q}, \frac{\omega}{\epsilon (\phi(p_{n+1}, p_{n+q}))})$$

$$\leq R_f(p_n, p_{n+1}, \frac{\omega}{\epsilon (\phi(p_n, p_{n+1}))}) \ast R_f(p_{n+1}, p_{n+2}, \frac{\omega}{(2)\epsilon^2 (\phi(p_{n+1}, p_{n+2}))(\phi(p_{n+1}, p_{n+3}))})$$

$$\leq R_f(p_{n+1}, p_{n+2}, \frac{\omega}{(2)\epsilon^2 (\phi(p_{n+1}, p_{n+2}))(\phi(p_{n+1}, p_{n+3}))})$$

$$\leq \cdots$$
\[ \circ \mathcal{R}_\phi \left( P_{n+2}, P_{n+3}, \phi(P_{n+1}, P_{n+4}) \right) \]
\[ \circ \mathcal{R}_\phi \left( P_{n+3}, P_{n+4}, \phi(P_{n+1}, P_{n+5}) \right) \]
\[ \leq \mathcal{R}_\phi \left( P_{n+1}, P_{n+2}, \phi(P_{n+1}, P_{n+4}) \right) \]
\[ \circ \mathcal{R}_\phi \left( P_{n+2}, P_{n+3}, \phi(P_{n+1}, P_{n+5}) \right) \]
\[ \circ \mathcal{R}_\phi \left( P_{n+3}, P_{n+4}, \phi(P_{n+1}, P_{n+6}) \right) \]

Using (4) in the above inequalities, we deduce,
\[ \mathcal{K}_\phi \left( P_n, P_{n+q}, \omega \right) \leq \mathcal{K}_\phi \left( P_0, P_{1}, \frac{\phi(P_{n+1}, P_{n+4})}{2^{(e)^n-1}} \right) \]
\[ \circ \mathcal{R}_\phi \left( P_0, P_{1}, \frac{\phi(P_{n+1}, P_{n+4})}{2^{(e)^n-1}} \right) \]
\[ \circ \mathcal{R}_\phi \left( P_0, P_{1}, \frac{\phi(P_{n+1}, P_{n+4})}{2^{(e)^n-1}} \right) \]
\[ \circ \mathcal{R}_\phi \left( P_0, P_{1}, \frac{\phi(P_{n+1}, P_{n+4})}{2^{(e)^n-1}} \right) \]

In addition,
\[ \mathcal{R}_\phi \left( P_n, P_{n+q}, \omega \right) \leq \mathcal{R}_\phi \left( P_0, P_{1}, \frac{\text{Lim}}{2^{(e)^n-1}} \right) \]
\[ \circ \mathcal{R}_\phi \left( P_0, P_{1}, \frac{\text{Lim}}{2^{(e)^n-1}} \right) \]
\[ \circ \mathcal{R}_\phi \left( P_0, P_{1}, \frac{\text{Lim}}{2^{(e)^n-1}} \right) \]

Using (1), for \( n \to \infty \), we deduce,
\[ \lim_{n \to \infty} \mathcal{K}_\phi \left( P_n, P_{n+q}, \omega \right) = 1 * 1 * \cdots * 1 = 1 \]

In addition,
\[ \lim_{n \to \infty} \mathcal{R}_\phi \left( P_n, P_{n+q}, \omega \right) = 0 \circ 0 \cdots \circ 0 = 0 \]

i.e., \( \{P_n\} \) is a GCS. Therefore, \( (K, \mathcal{K}_\phi, \mathcal{R}_\phi, \circ) \) is a G-complete IFCMLS, there exists,
\[ \lim_{n \to \infty} P_n = \mathcal{P} \]
Now examine that \( b \) is a FP of \( \xi \), using (CL5), (CL10) and (1), we deduce,

\[
\mathcal{R}_\phi(p, \xi p, \omega) \geq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \ast \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[
\mathcal{R}_\phi(p, \xi p, \omega) \geq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \ast \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[
\mathcal{R}_\phi(p, \xi p, \omega) \geq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \ast \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[
\mathcal{R}_\phi(p, \xi p, \omega) \geq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \ast \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[\to 1 = 1 \]

as \( n \to \infty \), and,

\[
\mathcal{R}_\phi(p, \xi p, \omega) \leq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \circ \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[
\mathcal{R}_\phi(p, \xi p, \omega) \leq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \circ \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[
\mathcal{R}_\phi(p, \xi p, \omega) \leq \mathcal{R}_\phi(p, p_{n+1}, \frac{c}{\xi(p, p_{n+1})}) \circ \mathcal{R}_\phi(p_{n+1}, \xi p, \frac{c}{\xi(p, p_{n+1})})
\]

\[\to 0 = 0 \]

as \( n \to \infty \). Hence, \( \xi p = p \), a FP. To examine uniqueness, assume that \( \xi c = c \) for some \( c \in K \), then,

\[
1 \geq \mathcal{R}_\phi(c, p, \omega) = \mathcal{R}_\phi(\xi c, \xi p, \omega) \geq \mathcal{R}_\phi(c, p, \frac{c}{\xi}) = \mathcal{R}_\phi(\xi c, \xi p, \frac{c}{\xi})
\]

\[
\geq \mathcal{R}_\phi(c, p, \frac{c}{\xi}) \geq \cdots \geq \mathcal{R}_\phi(c, p, \frac{c}{\xi}) \to 1 \text{ as } n \to \infty,
\]

In addition,

\[
0 \leq \mathcal{R}_\phi(c, p, \omega) = \mathcal{R}_\phi(\xi c, \xi p, \omega) \leq \mathcal{R}_\phi(c, p, \frac{c}{\xi}) = \mathcal{R}_\phi(\xi c, \xi p, \frac{c}{\xi})
\]

\[
\leq \mathcal{R}_\phi(c, p, \frac{c}{\xi}) \leq \cdots \leq \mathcal{R}_\phi(c, p, \frac{c}{\xi}) \to 0 \text{ as } n \to \infty,
\]

by using (CL3) and (CL8), \( p = c. \square \)

**Definition 9.** Let \( (K, \mathcal{R}_\phi, \mathcal{R}_\phi, \ast, \circ) \) be a IFCMLS. A map \( \xi : K \to K \) is intuitionistic fuzzy controlled (IFC) contraction if there exists \( 0 < c < 1 \), such that,

\[
\frac{1}{\mathcal{R}_\phi(\xi p, \xi a, \omega)} - 1 \leq c \left[ \frac{1}{\mathcal{R}_\phi(p, a, \omega)} - 1 \right]
\]

(5)

In addition,

\[
\mathcal{R}_\phi(\xi p, \xi a, \omega) \leq c \mathcal{R}_\phi(p, a, \omega),
\]

(6)

for all \( p, a \in K \) and \( \omega > 0 \).

Now, we prove the theorem for IFCMLS.

**Theorem 2.** Let \( (K, \mathcal{R}_\phi, \mathcal{R}_\phi, \ast, \circ) \) be a G-complete IFCMLS with \( \phi : K \times K \to [1, \infty) \) and suppose that,

\[
\lim_{\omega \to \infty} \mathcal{R}_\phi(p, a, \omega) = 1 \text{ and } \lim_{\omega \to \infty} \mathcal{R}_\phi(p, a, \omega) = 0
\]

(7)

for all \( p, a \in K \) and \( \omega > 0 \). Let \( \xi : K \to K \) be an IFC contraction. Further, suppose that for every \( a \in K \),

\[
\lim_{n \to \infty} \phi(p_n, a) \text{ & } \lim_{n \to \infty} \phi(\xi p_n, a)
\]

In addition,

\[
\lim_{n,m \to \infty} \phi(p_n, p_m) \text{ & } \lim_{n,m \to \infty} \phi(p_m, p_n)
\]

exist and are finite, where \( p_n = \xi^n p_0 = \xi p_{n-1} \), for all \( n \in \mathbb{N} \) and \( p_0 \) be arbitrary point of \( K \).

Then \( \xi \) has a unique FP.
Proof. Let \( p_0 \) be an arbitrary point of \( K \) and define a sequence \( p_n \) by \( p_n = \xi p_{n-1} \), \( n \in \mathbb{N} \). Using Equations (5) and (6) for all \( \omega > 0, n > q \), we acquire,

\[
\frac{1}{n} \sum_{k=1}^{n} 1 = \frac{1}{n} \sum_{k=1}^{n} \xi^{k-1} p_0 = \xi^q p_{n-q-1} - 1
\]

Continuing in this way, we acquire,

\[
\frac{1}{n} \sum_{k=1}^{n} 1 = \frac{1}{n} \sum_{k=1}^{n} \xi^{k-1} (1 - \varepsilon) + \xi^{n-2} (1 - \varepsilon) + \cdots + (1 - \varepsilon) \left( 1 - \frac{1}{n} \right)
\]

We obtain,

\[
\frac{1}{n} \sum_{k=1}^{n} 1 \leq \epsilon_n (p_n, p_{n+1}, \omega) \tag{8}
\]

In addition,

\[
\mathcal{R}_q (p_n, p_{n+1}, \omega) = \mathcal{R}_q (p_{n-1}, p_n, \omega) \leq \epsilon \mathcal{R}_q (p_n, p_{n+1}, \omega) = \epsilon \mathcal{R}_q (p_{n-2}, p_{n-1}, \omega) \tag{9}
\]

For any \( q \in \mathbb{N} \), using (CL5) and (CL10), we deduce,
\[\Re \phi \left( P_n, P_{n+q}, \omega \right) \leq n+1 \Re \phi \left( P_{n+1}, P_{n+q+2}, (2^\phi (P_{n+1}, P_{n+q+2})) \right) \]

In addition,
Therefore,
\[
\lim_{n \to \infty} \gamma_{\phi}(P_{n}, P_{n+q}, \omega) = 1 * 1 * \cdots * 1 = 1
\]
In addition,
\[
\lim_{n \to \infty} \varphi_{\phi}(P_{n}, P_{n+q}, \omega) = 0 \circ 0 \circ \cdots \circ 0 = 0
\]
i.e., \(\{P_{n}\}\) is a GCS. Since \((K, \varphi_{\phi}, \gamma_{\phi}, \ast, \circ)\) be a G-complete IFCMLS, there exists,
\[
limit_{n \to \infty} P_{n} = \mathcal{P}.
\]
Now examine that \(\mathcal{P}\) is a FP of \(\xi\), using (CL5) and (CL10), we deduce,
\[
\frac{1}{\gamma_{\phi}(\xi P_{n}, \mathcal{P}, \omega)} - 1 \leq \frac{1}{\gamma_{\phi}(P_{n}, \mathcal{P}, \omega)} - 1
\]
\[
\Rightarrow \frac{1}{\gamma_{\phi}(\xi P_{n}, \mathcal{P}, \omega) + (1-\epsilon)} \leq \gamma_{\phi}(P_{n}, \xi P_{n}, \omega)
\]
Using above inequality, we obtain,
\[
\gamma_{\phi}(P_{n}, \xi P_{n}, \omega) \geq \gamma_{\phi}(P_{n+1}, \xi P_{n+1}, \omega) \ast \gamma_{\phi}(P_{n+1}, \xi P_{n+1}, \omega)
\]
\[
\geq \gamma_{\phi}(P_{n}, \xi P_{n}, \omega) \ast \gamma_{\phi}(P_{n}, \xi P_{n}, \omega)
\]
\[
\geq \gamma_{\phi}(P_{n}, \xi P_{n}, \omega) \ast \gamma_{\phi}(P_{n}, \xi P_{n}, \omega)
\]
as \(n \to \infty\), and,
\[
\varphi_{\phi}(P_{n}, \xi P_{n}, \omega) \leq \varphi_{\phi}(P_{n+1}, \xi P_{n+1}, \omega) \circ \varphi_{\phi}(P_{n+1}, \xi P_{n+1}, \omega)
\]
\[
\leq \varphi_{\phi}(P_{n}, \xi P_{n}, \omega) \circ \varphi_{\phi}(P_{n}, \xi P_{n}, \omega)
\]
\[
\leq \varphi_{\phi}(P_{n}, \xi P_{n}, \omega) \circ \varphi_{\phi}(P_{n}, \xi P_{n}, \omega)
\]
This shows that \(\xi \mathcal{P} = \mathcal{P}\), a FP. To examine the uniqueness, assume that \(\xi c = c\) for some \(c \in K\), then,
\[
\frac{1}{\gamma_{\phi}(P_{n}, c, \omega)} - 1 = \frac{1}{\gamma_{\phi}(\xi P_{n}, \xi c, \omega)} - 1 \leq \frac{1}{\gamma_{\phi}(P_{n}, c, \omega)} - 1
\]
a contradiction, and,
\[
\varphi_{\phi}(P_{n}, c, \omega) = \varphi_{\phi}(\xi P_{n}, \xi c, \omega) \leq \varphi_{\phi}(P_{n}, c, \omega) \leq \varphi_{\phi}(P_{n}, c, \omega)
\]
a contradiction. Therefore, we must have \(\gamma_{\phi}(P_{n}, c, \omega) = 1\) and \(\varphi_{\phi}(P_{n}, c, \omega) = 0\), hence \(\mathcal{P} = c\). \(\square\)

**Example 4.** Let \(K = [0, 1]\). Define \(\phi\) by,
\[
\phi(P, \bar{a}) = \begin{cases} 
1 & \text{if } P = \bar{a}, \\
\frac{1 + \max\{P, \bar{a}\}}{\min\{P, \bar{a}\}} & \text{if } P \neq \bar{a} \neq 0
\end{cases}
\]
Furthermore, take,
\[
\gamma_{\phi}(P, \bar{a}, \omega) = e^{-\frac{\max\{P, \bar{a}\}}{\mu}} \text{ and } \varphi_{\phi}(P, \bar{a}, \omega) = 1 - e^{-\frac{\max\{P, \bar{a}\}}{\mu}}
\]
with \( v \ast \omega = v . \omega \) and \( \nu \circ \omega = \max \{ \nu, \omega \} \). Then \( (K, \mathcal{R}_\gamma, \mathcal{R}_{\phi}, \ast, \circ) \) is a \( G \)-complete IFCMLS. Observe that \( \lim_{\omega \to \infty} \mathcal{R}_\phi(P, \tilde{a}, \omega) \) and \( \lim_{\omega \to \infty} \mathcal{R}_\phi(P, \tilde{a}, \omega) \) exists and finite. Define \( \xi : K \to K \) by,

\[
\xi(P) = \begin{cases} 
0 & \text{if } P \in [0, \frac{1}{2}], \\
\frac{p}{4} & \text{if } P \in \left(\frac{1}{2}, 1\right]
\end{cases}
\]

Then we have for cases:

I. If \( P, \tilde{a} \in [0, \frac{1}{2}] \), then \( \xi P = \xi \tilde{a} = 0 \);

II. If \( P \in [0, \frac{1}{2}] \) and \( \tilde{a} \in \left(\frac{1}{2}, 1\right] \), then \( \xi P = 0 \) and \( \xi \tilde{a} = \frac{\tilde{a}}{2} \);

III. If \( \tilde{a} \in [0, \frac{1}{2}] \) and \( P \in \left(\frac{1}{2}, 1\right] \), then \( \xi \tilde{a} = 0 \) and \( \xi P = \frac{P}{2} \);

IV. If \( P, \tilde{a} \in \left(\frac{1}{2}, 1\right] \), then \( \xi P = \frac{1}{4} \) and \( \xi \tilde{a} = \frac{\tilde{a}}{4} \);

In all (I)–(IV) cases,

\[
\mathcal{N}_\phi(\xi P, \xi \tilde{a}, \epsilon \omega) \geq \mathcal{N}_\phi(\tilde{a}, P, \omega) \quad \text{and} \quad \mathcal{R}_\phi(\xi P, \xi \tilde{a}, \epsilon \omega) \leq \mathcal{R}_\phi(P, \tilde{a}, \omega)
\]

are satisfied for \( \epsilon \in \left[\frac{1}{2}, 1\right) \), and also,

\[
\frac{1}{\mathcal{R}_\phi(\xi P, \xi \tilde{a}, \omega)} - 1 \leq \epsilon \left[ \frac{1}{\mathcal{R}_\phi(\tilde{a}, P, \omega)} - 1 \right] \quad \text{and} \quad \mathcal{R}_\phi(\xi P, \xi \tilde{a}, \omega) \leq \epsilon \mathcal{R}_\phi(P, \tilde{a}, \omega)
\]

satisfied for \( \epsilon \in \left[\frac{1}{2}, 1\right) \).

Observe that \( \lim_{n \to \infty} \phi(P_n, \tilde{a}) \) and \( \lim_{n \to \infty} \phi(\tilde{a}, P_n) \) exists and finite. Furthermore, observe that all circumstances of Theorems 1 and 2 are fulfilled, and 0 is a unique FP of \( \xi \).

**Open Problem 1.** It is related to dealing with the Kannan contraction, Chatterjee contraction and Suzuki contraction in the sense of IFCMLS.

### 4. Application to Nonlinear Fractional Differential Equations

In present section, we aim to apply Theorem 3 to obtain the existence and uniqueness of a solution to a nonlinear fractional differential equation (NFDE),

\[
D^p_{0+} P(t) = g(t, P(t)), \quad 0 < t < 1
\]

with the boundary conditions,

\[
P(0) + P'(0) = 0, \quad P(1) + P'(1) = 0,
\]

where \( 1 < p \leq 2 \) is a number, \( D^p_{0+} \) is the Caputo fractional derivative and \( g : [0, 1] \times [0, \infty) \to [0, \infty) \) is a continuous function. Let \( K = C([0, 1], \mathbb{R}) \) denote the space of all continuous functions defined on \( [0, 1] \) equipped with the CTN \( c \ast d = c . d \) and CTCN \( c \circ d = \max\{c, d\} \) for all \( c, d \in [0, 1] \) and define an IFCMLS on \( K \) as follows:

\[
\mathcal{N}_\phi(P, \delta, \omega) = \frac{\alpha \omega}{\alpha \omega + \gamma \max\left\{ \sup_{t \in [0, 1]} P(t), \sup_{t \in [0, 1]} \delta(t) \right\} \delta'}
\]

\[
\mathcal{R}_\phi(P, \delta, \omega) = \frac{\gamma \max\left\{ \sup_{t \in [0, 1]} P(t), \sup_{t \in [0, 1]} \delta(t) \right\} \delta'}{\alpha \omega + \gamma \max\left\{ \sup_{t \in [0, 1]} P(t), \sup_{t \in [0, 1]} \delta(t) \right\} \delta'}
\]
for all $\omega > 0$ and $\mathcal{P}, \delta \in K$ with the control function,

$$\Phi(\mathcal{P}, \delta) = 1 + \max\left\{ \sup_{t \in (0,1]} B(t), \sup_{t \in (0,1]} \delta(t) \right\}^6.$$ 

Observe that if $\mathcal{P} \in K$ solves (10) whenever $\mathcal{P} \in K$ solves the below integral equation:

$$B(t) = \frac{1}{\Gamma(p)} \int_{t}^{1} (1-s)^{p-1}(1-t)g(s, B(s))ds + \frac{1}{\Gamma(p-1)} \int_{0}^{1} (1-s)^{p-2}(1-t)g(s, B(s))ds$$

$$+ \frac{1}{\Gamma(p)} \int_{0}^{t} (t-s)^{p-1}g(s, B(s))ds.$$ 

**Theorem 3.** The integral operator $\varphi : K \to K$ is given by,

$$\varphi B(t) = \frac{1}{\Gamma(p)} \int_{t}^{1} (1-s)^{p-1}(1-t)g(s, B(s))ds + \frac{1}{\Gamma(p-1)} \int_{0}^{1} (1-s)^{p-2}(1-t)g(s, B(s))ds$$

$$+ \frac{1}{\Gamma(p)} \int_{0}^{t} (t-s)^{p-1}g(s, B(s))ds,$$

where $g : [0,1] \times [0,\infty) \to [0,\infty)$ fulfilling the following criteria:

$$\max\left\{ \sup_{s \in [0,1]} g(s, B(s)), \sup_{s \in [0,1]} g(s, \delta(s)) \right\} \leq \frac{1}{4} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\},$$

$$\sup_{t \in (0,1)} \frac{1}{4096} \left[ \frac{1-t}{\Gamma(p+1)} + \frac{1-t}{\Gamma(p)} + \frac{\mu_{\mathcal{P}}}{\Gamma(p+1)} \right] \leq \tilde{a} < 1.$$ 

Then NFDE has a unique solution in $X$.

**Proof.**

$$\max\left\{ \varphi B(t), \varphi \delta(t) \right\}^6 = \left( \frac{1-t}{\Gamma(p)} \int_{t}^{1} (1-s)^{p-1} \max\left\{ \sup_{s \in [0,1]} g(s, B(s)), \sup_{s \in [0,1]} g(s, \delta(s)) \right\} ds \right.$$

$$+ \frac{1-t}{\Gamma(p-1)} \int_{0}^{1} (1-s)^{p-2} \max\left\{ \sup_{s \in [0,1]} g(s, B(s)), \sup_{s \in [0,1]} g(s, \delta(s)) \right\} ds$$

$$+ \frac{1}{\Gamma(p)} \int_{0}^{t} (t-s)^{p-1} \max\left\{ \sup_{s \in [0,1]} g(s, B(s)), \sup_{s \in [0,1]} g(s, \delta(s)) \right\}^6$$

$$\leq \left( \frac{1-t}{\Gamma(p)} \int_{0}^{1} (1-s)^{p-1} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\} \frac{1}{4} ds \right.$$

$$+ \frac{1-t}{\Gamma(p-1)} \int_{0}^{1} (1-s)^{p-2} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\} \frac{1}{4} ds$$

$$+ \frac{1}{\Gamma(p)} \int_{0}^{t} (t-s)^{p-1} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\} \frac{1}{4} ds \right)^6$$

$$\leq \frac{1}{4} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\}^6 \left( \frac{1-t}{\Gamma(p)} \int_{0}^{1} (1-s)^{p-1} ds \right.$$ 

$$+ \frac{1-t}{\Gamma(p-1)} \int_{0}^{1} (1-s)^{p-2} ds + \frac{1}{\Gamma(p)} \int_{0}^{t} (t-s)^{p-1} ds \right)^6$$

$$\leq \frac{1}{4} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\}^6 \sup_{t \in (0,1)} \left[ \frac{1-t}{\Gamma(p+1)} + \frac{1-t}{\Gamma(p)} + \frac{\mu_{\mathcal{P}}}{\Gamma(p+1)} \right]^6$$

$$= \tilde{a} \max\left\{ \sup_{s \in [0,1]} B(s), \sup_{s \in [0,1]} \delta(s) \right\}^6,$$

where,

$$\tilde{a} = \sup_{t \in (0,1)} \frac{1}{4096} \left[ \frac{1-t}{\Gamma(p+1)} + \frac{1-t}{\Gamma(p)} + \frac{\mu_{\mathcal{P}}}{\Gamma(p+1)} \right]^6.$$
Therefore, the above inequality,

\[
\max \left\{ \sup_{t \in [0,1]} \zeta b(t), \sup_{t \in [0,1]} \zeta \delta(t) \right\} \leq \alpha \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\} \\
\Rightarrow \alpha \omega + \frac{1}{6} \max \left\{ \sup_{t \in [0,1]} \zeta b(t), \sup_{t \in [0,1]} \zeta \delta(t) \right\} \leq \alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\} \\
\Rightarrow \frac{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}}{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}} \leq \frac{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}}{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}} \\
\Rightarrow R_\phi (\zeta b, \zeta \delta, \alpha \omega) \geq R_\phi (b, \delta, \omega),
\]

In addition,

\[
\Rightarrow \gamma \sup_{t \in [0,1]} \max \left\{ \sup_{t \in [0,1]} \zeta b(t), \sup_{t \in [0,1]} \zeta \delta(t) \right\} \leq \frac{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}}{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}} \\
\Rightarrow \gamma \sup_{t \in [0,1]} \max \left\{ \sup_{t \in [0,1]} \zeta b(t), \sup_{t \in [0,1]} \zeta \delta(t) \right\} \leq \frac{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}}{\alpha \omega + \gamma \max \left\{ \sup_{t \in [0,1]} b(t), \sup_{t \in [0,1]} \delta(t) \right\}} \\
\Rightarrow R_\phi (\zeta b, \zeta \delta, \alpha \omega) \leq R_\phi (b, \delta, \omega),
\]

for some \(a, \gamma > 0\). Observe that the conditions of the Theorem 1 are fulfilled. Resultantly, \(\zeta\) has a unique fixed point; accordingly, the specified NFDE has a unique solution. □

5. Conclusions

We present intuitionistic fuzzy controlled metric-like spaces in this paper and established several new types of fixed-point theorems in this new context. Moreover, we provided non-trivial examples and an application to non-linear fractional differential equations is given to demonstrate the viability of the proposed method. Our findings and concepts expand and generalize the existing literature. The structures of intuitionistic fuzzy double controlled metric-like spaces, intuitionistic pentagonal fuzzy controlled metric-like spaces, and neutrosophic controlled metric-like spaces etc. can all be extended using this study.

Author Contributions: F.U.: writing—original draft, methodology; U.I.: conceptualization, supervision, writing—original draft; K.J.: conceptualization, writing—original draft; S.S.A.: methodology, writing—original draft; M.A.: conceptualization, supervision, writing—original draft; N.S.: conceptualization, supervision, writing—original draft, N.M.: investigation, writing—original draft. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors S. Subhi and N. Mlaiki want to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of Interest: The authors declare no conflict of interest.

References