Article

Dynamics and Exact Traveling Wave Solutions of the Sharma–Tasso–Olver–Burgers Equation

Yan Zhou † and Jinsen Zhuang *,†

School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China; zy4233@hqu.edu.cn
* Correspondence: zzjinsen@hqu.edu.cn or zzjinsen@ustc.edu.cn
† These authors contributed equally to this work.

Abstract: In this paper, to study the Sharma–Tasso–Olver–Burgers equation, we focus on the geometric properties and the exact traveling wave solutions. The corresponding traveling system is a cubic oscillator with damping, and it has time-dependent and time-independent first integral. For all bounded orbits of the traveling system, we give the exact explicit kink wave solutions.

Keywords: Sharma–Tasso–Olver–Burgers equation; integral system; exact traveling wave solution; phase portrait; kink wave solution

1. Introduction

Recently, Yan and Lou (2020) [1] considered new types of soliton molecules in the Sharma–Tasso–Olver–Burgers (STOB) equation:

\[ u_t + \alpha (3u^2 + 3u^2 u_x + 3uu_{xx} + u_{xxx}) + \beta (2uu_x + u_{xx}) = 0. \]  \( (1) \)

Clearly, it is the Burgers equation when \( \alpha = 0 \). It reduces to the Sharma–Tasso–Olver equation when \( \beta = 0 \). Thus, the STOB equation is an integrable nonlinear evolution equation that is a combination of the well-known Burgers equation and the STO equation. From the point of view of either mathematics or physics (see Olver (1977) [2], Lian and Lou (2005) [3], He et al. (2013) [4], Gomez and Hernandez (2017) [5], El-Rashidy (2020) [6] and Li (2019) [7]), the Burgers system and the Sharma–Tasso–Olver equation have been widely investigated using various effective methods, including the inverse scattering method, Lie group method, Hirota’s bilinear method, etc. Lian and Lou (2005) [3] used the simple symmetry reduction procedure to obtain infinitely many symmetries and exact solutions with new soliton fission and fusion phenomena for the STO equation. He et al. (2013) [4] proposed an improved \( G'/G \)-expansion method to study the solitons and periodic solutions for the STO equation. El-Rashidy (2020) [6] used the extension exponential rational function method to deduce the new and general traveling wave solutions for the STO equation and \((2 + 1)\)-dimensional STO equation.

Therefore, it is of important significance to study the structure and properties of these integrable systems (PDEs). In particular, for the STOB equation, Yan and Lou (2020) [1] investigated soliton molecules and their fission and fusion phenomena by introducing a velocity resonance mechanism. Gomez and Hernandez (2017) [5] used the improved tanh-coth method, as well as the Exp-function method, to investigate the traveling wave solutions. However, to date, little is known about the geometric structure and dynamic behaviors of the traveling wave solutions of the STOB equation. It is very important to study these properties, which can help us understand the physical meaning and practical applications of the STOB equation.

The aim of this work is twofold. To study the Sharma–Tasso–Olver–Burgers (STOB) equation, we first discuss the geometric properties and exact traveling wave solutions. Second, we implement different first integrals, such as the time-dependent first integral...
and time-independent first integral, to finish the study of the properties and exact traveling wave solutions.

To study the traveling wave solutions of Equation (1), we let \( u = \phi(\xi) \), \( \xi = x - ct \) and substitute them into Equation (1). Now, integrating once and letting the integration constant be 0, we have that

\[
\phi'' + \left(3\phi + \frac{\beta}{\alpha}\right)\phi' + \phi\left(\phi^2 + \frac{\beta}{\alpha}\phi - \frac{c}{\alpha}\right) = 0,
\]

where \( \phi' = \frac{d\phi}{d\xi} \).

Note that Equation (2) is a cubic nonlinear oscillator with damping. It is well known that a cubic nonlinear oscillator with damping can be equivalently changed into a planar dynamical system:

\[
\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\left(3\phi + \frac{\beta}{\alpha}\right)y - \phi\left(\phi^2 + \frac{\beta}{\alpha}\phi - \frac{c}{\alpha}\right).
\]

From the perspective of the theory of dynamical systems, it is interesting to study the connection between orbits in phase portraits and exact solutions. Equivalently, it is necessary to study the geometric features of all known exact solutions in greater depth (for examples, see Li (2014) [8], Li and Zhu (2016) [9], and Li and Feng (2016) [10]). Under some parametric conditions, we study the phase portraits of system (3) in this paper and give corresponding parametric representations for all bounded orbits.

We have organized the paper as follows. In Section 2, system (3) has a time-dependent first integral. We discuss the integrable cubic nonlinear oscillator with damping and present the exact solutions. In Section 3, system (3) has a time-independent first integral. We discuss the integrable cubic nonlinear oscillator with damping and present the exact solutions.

2. Exact Solutions and Dynamics of System (3) with a Time-Dependent First Integral

In the current section, we study the exact solutions and dynamics of system (3) with a time-dependent first integral. We always assume that \( \alpha > 0, \beta > 0, \) and \( c < 0. \) In other cases, the study of system (3) is similar.

Clearly, when \( \Delta = \beta^2 + 4\alpha c > 0, \) we obtain three equilibrium points of system (3) at \( O(0,0), E_1(\phi_1,0) \) and \( E_2(\phi_2,0), \) where \( \phi_1 = -\frac{1}{2\alpha}(\beta + \sqrt{\Delta}), \phi_2 = -\frac{1}{2\alpha}(\beta - \sqrt{\Delta}) \) with \( \phi_1 < \phi_2 < 0. \) When \( \Delta = 0, \) there exists only one simple equilibrium point \( O(0,0) \) and a double equilibrium point \( E_{12}\left(-\frac{\beta}{2\alpha},0\right). \) When \( \Delta < 0, \) there exists only one simple equilibrium point \( O(0,0). \)

At the equilibrium point \( (\phi,0), \) setting \( M(\phi,0) \) as the coefficient matrix of the linearized system of (3), we find that

\[
J(0,0) = \det M(0,0) = -\frac{c}{\alpha} < 0, \quad J(\phi,0) = \det M(\phi,0) = \phi\left(2\phi + \frac{\beta}{\alpha}\right) = \pm \frac{\phi\sqrt{\Delta}}{\alpha}.
\]

Thus, \( J(\phi_1,0) > 0, J(\phi_2,0) < 0. \) In addition,

\[
(\text{Trace}M(0,0))^2 - 4J(0,0) = \frac{\Delta}{\alpha^2} > 0, (\text{Trace}M(\phi_1,0))^2 - 4J(\phi_1,0) = \phi_2^2 > 0.
\]

It is well known from the theory of planar dynamical systems that an equilibrium point of a planar integrable system is (1) a saddle point if \( J < 0, \) (2) a center point (a node point) if \( J > 0 \) and \( (\text{Trace}M)^2 - 4J < 0 \) (\( > 0, \)) or (3) a cusp if \( J = 0 \) and the Poincaré index of the equilibrium point is 0.

Hence, when \( c < 0, \) we have that (1) the equilibrium point \( O(0,0) \) is a stable node point, (2) the equilibrium point \( E_1(\phi_1,0) \) is an unstable node point, and (3) the equilibrium point \( E_2(\phi_2,0) \) is a saddle point.
Using such a qualitative analysis, we obtain the phase portraits of system (3), which are presented in Figure 1a–c. Note that we choose the following parameters: (a) $\alpha = 2, \beta = 10.8, c = -12.28$; (b) $\alpha = 1.3, c = -3.3, \beta = \sqrt{-4}\alpha = 4.142463035$; (c) $\alpha = 2, \beta = 2.7, c = -16.27$.

Figure 1. Different phase portraits of system (3). (a) $\Delta > 0$, (b) $\Delta = 0$, and (c) $\Delta < 0$.

In Figure 1, we see that system (3) has infinitely many heteroclinic orbits when $\Delta \geq 0$. They connect the two equilibrium points $E_1(\phi_1, 0)$ and $O(0, 0)$. These orbits lead to a family of kink wave solutions of Equation (1). In addition, when $\Delta > 0$, there exists a heteroclinic orbit that connects the equilibrium points $E_1(\phi_1, 0)$ and $E_2(\phi_2, 0)$. There exists a heteroclinic orbit that connects the equilibrium points $E_2(\phi_2, 0)$ and $O(0, 0)$.

By using the results given by Chandrasekar et al. (2006) [11], we know that the equation

$$\phi'' + (k_1\phi + k_2)\phi' + \frac{k_1^2}{3}\phi^3 + \frac{k_1k_2}{3}\phi^2 + \lambda\phi = 0$$

is an integrable system. Obviously, Equation (2) is the special case of (4) with $k_1 = 3, k_2 = \frac{\beta}{\alpha}, \lambda = -\frac{\beta}{\alpha}$. Therefore, when $\Delta \neq 0$, system (3) possesses the first integral depending on $\xi$ as follows:

$$H_1(\phi, y, \xi) = \left(3y - \frac{3}{\phi}(-\frac{\beta}{\alpha} + \omega)\phi + 3\phi^2\right)e^{\frac{-\omega\xi}{\alpha}},$$

where $\omega = \sqrt{\frac{\beta}{\alpha}}$. When $\Delta = 0$, system (3) has the first integral

$$H_2(\phi, y, \xi) = -\xi + \frac{\phi}{\left(\frac{\beta}{\alpha}\phi + \phi^2 + y\right)}.$$

Equations (5) and (6) suggest the following result.

Theorem 1. (i) When $\omega > 0$, the exact parametric representation for the orbits of system (3) is

$$\phi(\xi) = \phi_+(\xi) = \frac{6c(C_1e^{\omega\xi} - 1)}{3\omega_c(1 + C_1e^{\omega\xi}) - (\beta + \omega C_2e^{\frac{\beta + \omega C_2}{\omega_c}x}) - 3\beta(1 - C_1e^{\omega\xi})},$$

where $C_1$ and $C_2$ are two arbitrary integral constants.

(ii) When $\omega = 0$, the exact parametric representation for the orbits of system (3) is

$$\phi(\xi) = \frac{3(C_1 + \xi)}{3C_2e^{\frac{\beta}{\alpha}x} - \frac{\alpha^2}{\beta}(2 + \frac{\beta}{\alpha}(C_1 + \xi))}.$$
(iii) When $\omega < 0$, the exact parametric representation for the orbits of system (3) is

$$
\phi(\xi) = \frac{C_1 \cos(\omega_0 \xi + C_2)}{e^{\left(\frac{\xi}{\alpha}\right)^2} + \frac{2\xi^2 C_1}{\beta^2 + 4\alpha^2 \omega_0^2} \left[2\omega_0 \sin(\omega_0 \xi + C_2) - \frac{\beta}{\pi} \cos(\omega_0 \xi + C_2)\right]},
$$

where $\omega_0 = \frac{1}{2\alpha} \sqrt{-\Delta}$.

**Remark 1.** The exact solution (7b) is the correct result, which changes the incorrect Formula (58) in Chandrasekar et al. (2006) [11].

**Remark 2.** (1) When $\Delta > 0$ and $C_1 = 0$, Equation (7) becomes

$$
\phi(\xi) = \phi_a(\xi) = \frac{6c}{3(\beta - \alpha \omega) + (\beta + \alpha \omega) C_2 e^{\left(\frac{\xi}{\alpha}\right)^2} \xi},
$$

and

$$
\phi(\xi) = \phi_b(\xi) = \frac{6c}{(\beta - \alpha \omega) \left[3 + C_2 e^{\left(\frac{\xi}{\alpha}\right)^2} \xi\right]}.
$$

When $\beta - \alpha \omega > 0$, $C_2 > 0$, Equations (10) and (11) lead to two families of monotonic kink wave solutions of system (1) (see Figure 2a).

(2) When $\Delta > 0$ and $C_2 = 0$, Equation (7) becomes

$$
\phi(\xi) = \phi_c(\xi) = \frac{2c(C_1 e^{\omega_0 \xi} - 1)}{(\beta + \alpha \omega) C_2 e^{\omega_0 \xi} - (\beta - \alpha \omega)},
$$

When $\beta - \alpha \omega > 0$, $C_1 < 0$, Equation (12) leads to a family of monotonic kink wave solutions of system (1) (see Figure 2a).

(3) When $\Delta > 0$ and $C_1 \neq 0$, $C_2 \neq 0$, Equation (7) can give a family of non-monotonic kink wave solutions (see Figure 2b). The parameters are given as $\alpha = 2$, $\beta = 10.8$, $c = -12.28$, $\Delta = 4.6$, and $\omega = 2.144761059$.

![Figure 2. Three-dimensional profiles of kink waves of Equation (1).](image-url)

(a) (b)

3. Exact Solutions and Dynamics of System (3) with a Time-Independent First Integral

In this section, we take $c = -\frac{2\beta^2}{9\alpha^2}$ in system (3), i.e., we consider the system

$$
\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -(3\phi + \frac{\beta}{\alpha}) y - \phi \left(\phi^2 + \frac{\beta}{\alpha} \phi + \frac{2\beta^2}{9\alpha^2}\right).
$$


From the results in Chandrasekar et al. (2006) [11], Mak and Harko (2005) [12], and Li and Zhu (2016) [9], we know that system (13) has the first integral

\[ H_3(\phi, y) = \left( y + \phi^2 + \frac{2\beta}{3\alpha} \phi \right)^{-3} \left[ y \left( y + \frac{3}{2} \phi^2 + \frac{\beta}{\alpha} \phi \right) + \frac{2}{9} \left( \frac{3}{2} \phi^2 + \frac{\beta}{\alpha} \phi \right)^2 \right] = h, \]  

(14)

Clearly, system (13) has three equilibrium points: \( O(0, 0) \), \( E_1\left(-\frac{2\beta}{3\alpha}, 0\right) \), and \( E_2\left(-\frac{\beta}{\alpha}, 0\right) \). We write that

\[ h_2 = H_3\left(-\frac{\beta}{\alpha}, 0\right) = -\frac{6\alpha}{27}. \]

From the analysis in Section 2, for system (13), we have \( \Delta = \frac{1}{9} \beta^2 > 0 \). Hence, under the parameters \( \alpha = 2, \beta = 5, \) and \( \Delta = \frac{25}{9} \), Figure 3 gives the phase portrait of system (13).

![Figure 3](image)

**Figure 3.** The phase portrait of system (13), when \( \alpha = 2, \beta = 5, \) and \( \Delta = \frac{25}{9} \).

Figure 4 gives some figures of the level curves of \( H_3(\phi, y) = h \).

![Figure 4](image)

**Figure 4.** The level curves of \( H_3(\phi, y) = h \). (a) \( h = h_2 \); (b) \( h = 0 \); (c) \( 0 < h < h_m \); (d) \( h_m < h < \infty \).

We first discuss the level curves defined by \( H_3(\phi, y) = h_2 \). Now, the function

\[ y \left( y + \frac{3}{2} \phi^2 + \frac{\beta}{\alpha} \phi \right) + \frac{2}{9} \left( \frac{3}{2} \phi^2 + \frac{\beta}{\alpha} \phi \right)^2 - h_2 \left( y + \phi^2 + \frac{2\beta}{3\alpha} \phi \right)^3 = 0 \]

(15)

can be written as

\[ \left( y + \phi \left( \phi + \frac{\beta}{3\alpha} \right) \right) \left( y + \phi \left( \phi + \frac{2\beta}{3\alpha} \right) \right) \left( y + \left( \phi + \frac{\beta}{3\alpha} \right) \left( \phi + \frac{2\beta}{3\alpha} \right) \right) = 0. \]

(16)

Thus, corresponding to the two heteroclinic orbits shown in Figure 4a, we have two kink wave solutions of Equation (1):

\[ \phi(\xi) = -\frac{\beta}{3\alpha(1 + e^{\alpha/\xi})}. \]

(17)
and

$$\phi(\xi) = -\frac{\beta(2 + e^{\omega_0 \xi})}{3\alpha(1 + e^{\omega_0 \xi})},$$

(18)

where \(\omega_0 = \frac{\beta}{3\alpha}\).

Considering the level curves of \(H_3(\phi, y) = 0\) together with (14), we see that \(H_3(\phi, y) = 0\) can be written as follows when \(h = 0\):

$$y = -\phi\left(\frac{1}{2}\phi + \frac{\beta}{3\alpha}\right), \text{ and } y = -\phi\left(\phi + \frac{2\beta}{3\alpha}\right),$$

(19)

Clearly, the two curves in (19) are heteroclinic orbits, and they connect the equilibrium points \(E_1\left(-\frac{2\beta}{3\alpha}, 0\right)\) and \(O(0, 0)\) (see Figure 4b).

From the first equation of system (13), the following exact parametric representations are obtained:

$$\phi(\xi) = \frac{-2\beta}{3\alpha(1 + e^{\omega_0 \xi})},$$

(20)

where \(j = 1, 2, \omega_1 = \frac{2\beta}{3\alpha}, \omega_2 = \frac{4\beta}{3\alpha}\). Now, two monotonic kink wave solutions of system (13) are given by Equation (20).

Now, from Figure 4c, it can be seen that, for each \(h \in (0, h_m)\), the level curve of \(H_3(\phi, y) = h\) leads to two heteroclinic orbits. They connect the equilibrium points \(E_1\left(-\frac{2\beta}{3\alpha}, 0\right)\) and \(O(0, 0)\). Moreover, one determines a monotonic kink wave solution, and the other determines a non-monotonic kink wave solution of system (13). Therefore, corresponding to all \(h \in (0, h_m)\), the level curves \(H_3(\phi, y) = h\) lead to two families of monotonic and non-monotonic kink wave solutions of system (13), and the number of solutions is uncountably infinite.

With the transformation \(\varphi = \phi + \frac{\beta}{3\alpha}\), we can represent system (13) as the following symmetric form:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -3\varphi y + \left(\frac{\beta^2}{9\alpha^2}\varphi - \varphi^3\right)$$

(21)

with the first integral

$$H_3(\varphi, y) = \left(y + \varphi^2 - \frac{\beta^2}{9\alpha^2}\right)^{-3} \left[y\left(y + \frac{3}{2}\varphi^2 - \frac{\beta^2}{6\alpha^2}\right) + \frac{3}{2} \left(\frac{3}{2}\varphi^2 - \frac{\beta^2}{6\alpha^2}\right)^2\right] = h.$$ 

(22)

For the heteroclinic orbits (see Figure 4c) defined by \(H_3(\varphi, y) = h, h \in (0, h_m)\), we have

$$y\left(y + \frac{3}{2}\varphi^2 - \frac{\beta^2}{6\alpha^2}\right) + \frac{2}{3} \left(\frac{3}{2}\varphi^2 - \frac{\beta^2}{6\alpha^2}\right)^2 - h\left(y + \varphi^2 - \frac{\beta^2}{9\alpha^2}\right)^3 = 0.$$ 

(23)

This implies that

$$y = y_1(\varphi) = \frac{1}{18ha^2} \left[A - B\varphi^2 + 3\alpha\sqrt{A - B\varphi^2}\right],$$

$$y = y_2(\varphi) = \frac{1}{18ha^2} \left[A - B\varphi^2 - 3\alpha\sqrt{A - B\varphi^2}\right],$$

(24)

where \(A = A(h) = 9\alpha^2 + 2h\beta^2, B = B(h) = 18ha^2\). Now, we consider the two heteroclinic orbits when \(0 < h < h_m\). Clearly, the upper arc \(y = y_1(x)\) intersects with the \(\varphi\)-axis at two points, \((\pm\varphi_m, 0)\), where \(\varphi_m = \sqrt{\frac{A}{B}}\), while the lower arc \(y = y_2(x)\) intersects with the \(\varphi\)-axis at four points: \((\pm\varphi_m, 0)\) and \(E_\pi\left(\pm\frac{\beta}{3\alpha}, 0\right)\).
Now, from the first equations of systems (21) and (24), the uncountably infinitely many non-monotonic kink wave solutions of system (13) have the following parametric representations (see Figure 5b):

\[
\phi(\xi) = -\frac{\beta}{3\alpha} + \varphi(\xi) = -\frac{\beta}{3\alpha} \mp \frac{1}{\sqrt{h(h) \cosh(\omega \xi) - 3\alpha}} \left[ A(h) - \left( \frac{2h\beta^2}{\sqrt{A(h) \cosh(\omega \xi) - 3\alpha}} \right)^2 \right]^{1/2},
\]
for \( \xi \in (-\infty, 0) \) and \( \xi \in [0, \infty) \), respectively.

where \( \omega = \beta \sqrt{2h} \).

Regarding the lower arc of the level curve \( y = y_2(x) \), the uncountably infinitely many monotonic kink wave solutions of system (13) have the following parametric representations (see Figure 5a):

\[
\phi(\xi) = -\frac{\beta}{3\alpha} + \varphi(\xi) = -\frac{\beta}{3\alpha} \mp \frac{1}{\sqrt{h(h) \cosh(\omega \xi) + 3\alpha}} \left[ A(h) - \left( \frac{2h\beta^2}{\sqrt{A(h) \cosh(\omega \xi) + 3\alpha}} \right)^2 \right]^{1/2},
\]
for \( \xi \in (-\infty, 0) \) and \( \xi \in [0, \infty) \), respectively.

![Figure 5](image)

**Figure 5.** Profiles of the kink waves of Equation (13). (a) Monotonic kink wave solution; (b) Non-monotonic kink wave solution.

We summarize the findings in the following result.

**Theorem 2.** Assume that \( c = -\frac{2\beta}{3\alpha} \).

(i) In the case that the two heteroclinic orbits \( H_3(\phi, y) = h_2 \) of system (13) connect the equilibrium points \( E_1 \left( -\frac{2\beta}{3\alpha}, 0 \right) \), \( E_2 \left( -\frac{\beta}{3\alpha}, 0 \right) \), and \( O(0, 0) \), Equation (1) has two monotonic kink wave solutions in (17) and (18).

(ii) In the case that the two heteroclinic orbits \( H_3(\phi, y) = 0 \) of system (13) connect the equilibrium points \( E_1 \left( -\frac{2\beta}{3\alpha}, 0 \right) \) and \( O(0, 0) \), Equation (1) has two monotonic kink wave solutions in (20).

(iii) In the case that the uncountably infinitely many heteroclinic orbits \( H_3(\phi, y) = h \), \( h \in (h_1, 0) \), and \( h \in (h_m, \infty) \) of system (13) connect two node points, \( E_1 \left( -\frac{2\beta}{3\alpha}, 0 \right) \) and \( O(0, 0) \), Equation (1) has monotonic kink wave solutions in (26). The number of solutions is uncountably infinite.

(iv) In the case that the heteroclinic orbits \( H_3(\phi, y) = h \), \( h \in (0, h_m) \) connect two node points, \( E_1 \left( -\frac{2\beta}{3\alpha}, 0 \right) \) and \( O(0, 0) \), Equation (1) has non-monotonic kink wave solutions and monotonic kink wave solutions in (25) and (26). The number of solutions is uncountably infinite.
Author Contributions: Conceptualization, Y.Z. and J.Z.; Funding acquisition, Y.Z. and J.Z.; Methodology, Y.Z. and J.Z.; Software, Y.Z. and J.Z.; Writing – original draft, Y.Z. and J.Z. Both authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by the National Natural Science Foundation of China (11871231, 12071162, 11701191) and the Natural Science Foundation of Fujian Province (2021J01303, 2022J01303).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to express our deepest gratitude to Jibin Li for his valuable instructions, suggestions, and extremely careful reading of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References
8. Li, J. On the exact traveling wave solutions of (2+1)-dimensional higher order Broer–Kaup equation. *Int. J. Bifurc. Chaos* 2014, 24, 1450007. [CrossRef]
10. Li, J.; Feng, Z. Quadratic and cubic nonlinear oscillators with damping and their applications. *Int. J. Bifurc. Chaos* 2016, 26, 1650050. [CrossRef]