Multiple Solutions for a Class of BVPs of Second-Order Discontinuous Differential Equations with Impulse Effects

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Abstract: This paper deals with a class of boundary value problems of second-order differential equations with impulses and discontinuity. The existence of single or multiple positive solutions to discontinuous differential equations with impulse effects is established by using the nonlinear alternative of Krasnoselskii’s fixed point theorem for discontinuous operators on cones. Finally, an example is given to illustrate the main results.

Keywords: multiple solutions; discontinuous differential equations; boundary value problems; impulsive differential equations; fixed point theory

1. Introduction

Sophus Lie proposed the symmetry theory of differential equations in the late 19th century. Lie symmetry theory unifies and extends existing techniques for constructing explicit solutions to differential equations. The basic idea of Lie groups is to find symmetry groups of given equations. It is an effective method to analyze differential equations. It is usually used to reduce the order of ordinary differential equations (ODEs). We can obtain its explicit solution of the first order ODE.

As an important branch of ODEs, impulsive differential equations (IDEs) are regarded as a vital mathematical tool in some areas of research. IDEs are related to mechanical systems, theoretical physics, chemistry, control theory and so on. Hence, they are regarded as an effective mathematical tool to solve some real-world problems in the applied sciences (see, for instance, [1–4]). Many authors are devoted to the existence of solutions to IDEs.

Recently, boundary value problems (BVPs) in IDEs have been extensively studied. Correspondingly, some basic results for BVPs of second-order IDEs have been obtained by many authors. For example, W. Ding and M. Han studied the existence of solutions to the periodic BVPs for second-order IDEs in [5]. Ref. [6] was concerned with multiple non-negative solutions to periodic BVPs for second-order IDEs. Additionally, refs. [7–10] researched some BVPs for various types of differential equations, such as fourth-order differential systems with parameters, nonlinear fourth-order differential equations, φ-Caputo fractional differential equations and fractional (p, q)-difference equations, respectively.

As we know, continuity is a clear limitation to its applicability in degree theory. The existence of BVP solutions is usually established by transformation into fixed-point problems of integral operators. However, there many differential equations are discontinuous in many fields, such as mechanical engineering, mechanics, automatic control, neural network, biology and so on. As the corresponding operators are not continuous, most discontinuous differential equations, see [11,12], fall outside that scope. To overcome this problem, R. Figueroa et al. define a new topological degree for discontinuous operators and derived some fixed-point theorems for these operators in [13], such as discontinuous operators’ Schauder and Krasnoselskii fixed point theorem.
Moreover, R. Figueroa and G. Infante in [14] considered a class of BVPs of second-order discontinuous differential equation using Schauder fixed point theorem. In [15], J. R. Lopez studied a class of discontinuous one-dimensional beam equations involving BVP. By using Krasnoselskii’s fixed point theorem on cones, the existence of non-negative solutions is obtained. However, there are few studies on multiple solutions for integral BVPs of second-order discontinuous differential equations with impulse effects. The purpose of the present paper is to fill this gap.

This paper is strongly motivated by the above discussions. It studies multiple solutions for the following BVP:

\[
\begin{aligned}
&x''(\tau) = g(\tau)f(\tau,x(\tau)), \text{ a.e. } \tau \in J' ; \\
&\Delta x|_{\tau=\tau_k} = I_k(x(\tau_k)), \; k = 1, 2, \ldots, m ; \\
&\Delta x'|_{\tau=\tau_k} = 0, \; k = 1, 2, \ldots, m ; \\
&\theta x(0) - \phi x(1) = \int_{0}^{\tau_1} v_1(s)x(s)ds ; \\
&\pi x'(0) - \omega x'(1) = \int_{0}^{\tau_1} v_2(s)x(s)ds ,
\end{aligned}
\]

where \( \theta > \theta > 0, \; \pi > \omega > 0, \; g, \; v_1, \; v_2 \geq 0 \text{ a.e. on } J = [0,1], \; g, \; v_1, \; v_2 \in L^1(0,1) \text{ and } f : J \times \mathbb{R}^+ \rightarrow \mathbb{R}^+, \; \mathbb{R}^+ = [0, +\infty) . \; I_k \in C(\mathbb{R}^+, \mathbb{R}^+) \text{ is an impulsive function, } k = 1, 2, \ldots, m , \; 0 < \tau_1 < \tau_2 < \cdots < \tau_m < 1 , \; J' = J \setminus \{ \tau_1, \cdots, \tau_m \} . \; \text{And } \Delta x|_{\tau=\tau_k} , \; \Delta x'|_{\tau=\tau_k} \text{ denotes the jump of } x(\tau) \text{ and } x'(\tau) \text{ at } \tau = \tau_k , \text{ respectively. This paper has the following features. Firstly, compared with [16,17], BVP (1) is a second-order discontinuous differential equation with impulse effects. The nonlinear term } f \text{ here is discontinuous over countable curve families. Secondly, the integral boundary value conditions we consider are more extensive. Thirdly, the method adopted in this paper has some advantages compared to some of the reference methods listed above. The tool used here is multivalued analysis. A special cone is constructed by studying the properties of Green’s function. Therefore, the existence of positive solutions can be established by Krasnoselskii’s fixed point theorem for discontinuous operators.}

The remainder of this paper is organized as follows. Section 2 gives some basic definitions and some preliminary results. The existence results are given and proved in Section 3. Finally, Section 4 indicate an example to illustrate the main results .

2. Preliminaries

In this section, we first present some preliminary facts.

Let

\[
PC(J) = \{ x : [0,1] \rightarrow \mathbb{R}, \; x \in C(J'), \text{ and } x(\tau_k^+) , \; x(\tau_k^-) \text{ exists,} \\
\text{and } x(\tau_k^-) = x(\tau_k) , \; 1 \leq k \leq m \}
\]

and

\[
PC^1(J) = \{ x : [0,1] \rightarrow \mathbb{R}, \; x \in C^1(J'), \text{ and } x'(\tau_k^+) , \; x'(\tau_k^-) \text{ exists,} \\
\text{and } x'(\tau_k^-) = x'(\tau_k) , \; 1 \leq k \leq m \} .
\]

They are Banach spaces with the norm

\[
\| x \|_0 = \sup_{0 \leq t \leq 1} | x(\tau) | = \| x \|_\infty
\]

and

\[
\| x \|_1 = \max\{ \| x \|_\infty , \| x' \|_\infty \} ,
\]

respectively.
For convenience, we denote $A_j = \int_0^1 v_j(s)ds$, $P_j = \int_0^1 \frac{(\theta - \tau)s + \theta}{(\theta - \phi)(\pi - \phi)} v_j(s)ds$, $Q_j = \frac{1}{\theta - \phi} A_j$, $(j = 1, 2)$, $\Gamma = (1 - P_2)(1 - Q_1) - P_1 Q_2$ and $Q_M = \max\{Q_2, 1 - Q_1\}$.

**Lemma 1.** Suppose that $(1 - P_2)(1 - Q_1) \neq P_1 Q_2$. Then, for $v \in L(J, \mathbb{R}^+)$, the following boundary value problem

$$
\begin{align*}
\begin{cases}
\ddot{x}(\tau) = v(\tau), \text{ a.e. } \tau \in [0, 1]; \\
\Delta x|_{\tau = \tau_k} = I_k(x(\tau_k)), k = 1, 2, \ldots, m; \\
\Delta x|_{\tau = 0} = \Delta x|_{\tau = 1} = 0, k = 1, 2, \ldots, m; \\
\theta x(0) - \theta x(1) = \int_0^1 v_1(s)x(s)ds; \\
\pi x'(0) - \omega x'(1) = \int_0^1 v_2(s)x(s)ds,
\end{cases}
\end{align*}
$$

(2)

has a positive solution

$$
x(\tau) = \int_0^1 \mathbb{H}_1(\tau, s)v(s)ds + \sum_{i=1}^{m} \mathbb{H}_2(\tau, \tau_i)I_i(x(\tau_i)),
$$

where

$$
\mathbb{H}_1(\tau, s) = G(\tau, s) + \sum_{n=1}^{2} \varphi_n(\tau) \int_0^1 G(s, \zeta)v_n(\zeta)d\zeta,
$$

$$
\mathbb{H}_2(\tau, \tau_i) = \begin{cases}
\frac{\theta}{\theta - \phi} + \frac{\phi}{\theta - \phi} \sum_{n=1}^{2} A_n \varphi_n(\tau), & 0 \leq \tau \leq \tau_i \leq 1; \\
\frac{\theta}{\theta - \phi} + \frac{\phi}{\theta - \phi} \sum_{n=1}^{2} A_n \varphi_n(\tau), & 0 \leq \tau_i < \tau \leq 1,
\end{cases}
$$

$$
\varphi_1(\tau) = \frac{(\pi - \omega)(1 - P_2) + [\theta + (\theta - \phi)\tau]Q_2}{(\theta - \phi)(\pi - \omega)\Gamma},
$$

$$
\varphi_2(\tau) = \frac{(\pi - \omega)P_1 + [\theta + (\theta - \phi)\tau](1 - Q_1)}{(\theta - \phi)(\pi - \omega)\Gamma}.
$$

and

$$
G(\tau, s) = \begin{cases}
\frac{\theta\phi}{(\theta - \phi)(\pi - \omega)} + \frac{\theta(1 - s)}{\theta - \phi} + \frac{\omega\tau}{\pi - \omega} + (\tau - s), & 0 \leq s \leq \tau \leq 1; \\
\frac{\theta\phi}{(\theta - \phi)(\pi - \omega)} + \frac{\theta(1 - s)}{\theta - \phi} + \frac{\omega\tau}{\pi - \omega}, & 0 \leq \tau \leq s \leq 1.
\end{cases}
$$

**Proof.** First, by integrating both sides of Equation (2), we have

$$
x(\tau) = \int_0^\tau (\tau - s)v(s)ds - c_k - d\tau, \text{ for } \tau \in (\tau_k, \tau_{k+1}],
$$

(3)

where $\tau_0 = 0$, $\tau_{m+1} = 1$. Then,

$$
x'(\tau) = \int_0^\tau v(s)ds - d, \tau \in (\tau_k, \tau_{k+1}].
$$

From Equation (2), we know

$$
-\theta c_0 - \theta \int_0^1 (1 - s)v(s)ds - c_m - d = \int_0^1 v_1(s)x(s)ds,
$$

(4)

$$
-\pi d - \omega \int_0^1 v(s)ds - d = \int_0^1 v_2(s)x(s)ds.
$$

(5)

$$
d_{k+1} - d_k = 0,
$$
and
\[ c_{k-1} - c_k = I_k(x(\tau_k)). \]  \hspace{1cm} (6)

In (5), it is clear that
\[ d := d_0 = \cdots = d_m = \frac{-\int_0^1 v_2(s)x(s)ds + \omega \int_0^1 v(s)ds}{\pi - \omega}. \]  \hspace{1cm} (7)

From (4), (6) and (7), one can easily see that
\[ c_0 = \frac{-1}{\theta - \theta} \left[ \int_0^1 v_1(s)x(s)ds + \theta \int_0^1 (1 - s)v(s)ds + \theta \sum_{i=1}^m I_i(x(\tau_i)) \right] \]
\[ + \frac{\theta}{\pi - \omega} \left[ \int_0^1 v_2(s)x(s)ds + \omega \int_0^1 v(s)ds \right], \]  \hspace{1cm} (8)

and
\[ c_k = c_0 - \frac{k}{\theta - \theta} \left[ \int_0^1 v_1(s)x(s)ds + \theta \int_0^1 (1 - s)v(s)ds + \theta \sum_{i=1}^m I_i(x(\tau_i)) \right] \]
\[ + \frac{\theta}{\pi - \omega} \left[ \int_0^1 v_2(s)x(s)ds + \omega \int_0^1 v(s)ds \right] - \sum_{i=1}^k I_i(x(\tau_i)). \]  \hspace{1cm} (9)

Therefore, for \( k = 0, 1, 2, \ldots, m \), \( m \) (8) and (9) imply that
\[ c_k + d\tau = \frac{-1}{\theta - \theta} \left[ \int_0^1 v_1(s)x(s)ds + \theta \int_0^1 (1 - s)v(s)ds + \theta \sum_{i=1}^m I_i(x(\tau_i)) \right] \]
\[ + \frac{\theta}{\pi - \omega} \left[ \int_0^1 v_2(s)x(s)ds + \omega \int_0^1 v(s)ds \right] - \sum_{i=1}^k I_i(x(\tau_i)) \]
\[ + \frac{\theta}{\pi - \omega} \left[ \int_0^1 v_1(s)x(s)ds + \theta \int_0^1 (1 - s)v(s)ds + \theta \sum_{i=1}^m I_i(x(\tau_i)) \right] \]
\[ - \frac{\theta}{\theta - \theta} \sum_{i=1}^m I_i(x(\tau_i)) - \frac{k}{\theta - \theta} \sum_{i=1}^k I_i(x(\tau_i)). \]
\[ \left. \right. \]  \hspace{1cm} (10)

Now, by (10) and (3), for \( \tau \in [0, \tau_1] \), we obtain
\[
x(\tau) = \int_0^T (\tau - s) v(s) ds + \int_0^1 h_1 x(s) ds + \frac{1}{(\theta - \tau)(\tau - \Theta)} \int_0^1 v_2(s) u(s) ds \\
+ \frac{\theta}{\theta - \tau} \int_0^T (1 - s) v(s) ds + \frac{\omega[(\theta - \tau) + \Theta]}{(\theta - \tau)(\tau - \Theta)} \int_0^1 v(s) ds \\
+ \frac{\theta}{\theta - \tau} \sum_{i=1}^{m} I_i(x(\tau_i)) + \sum_{i=1}^{m} I_i(x(\tau_i)) \\
= \int_0^T (\tau - s) v(s) ds + \frac{\theta}{(\theta - \tau)(\tau - \Theta)} \int_0^T v(s) ds \\
+ \frac{\theta}{\theta - \tau} \int_0^T (1 - s) v(s) ds + \pi \pi \int_0^T v(s) ds + \frac{\theta}{\theta - \tau}(\tau - \Theta) \int_0^1 v(s) ds \\
+ \frac{1}{\theta - \tau} \int_0^T v_1(s) x(s) ds + \frac{\theta}{(\theta - \tau)} \int_0^1 v_2(s) x(s) ds \\
+ \frac{\theta}{\theta - \tau} \sum_{i=1}^{m} I_i(x(\tau_i)) \\
= \int_0^1 G(s, x) v(s) ds + \frac{1}{\theta - \tau} \int_0^T v_1(s) x(s) ds + \frac{(\theta - \tau) + \Theta}{(\theta - \tau)(\tau - \Theta)} \int_0^1 v_2(s) x(s) ds \\
+ \frac{\theta}{\theta - \tau} \sum_{i=1}^{m} I_i(x(\tau_i)).
\]

From (6),

\[
\int_0^1 v_1(s) \int_0^1 G(s, \varphi) v(\varphi) d\varphi ds = (1 - Q_1) \int_0^1 v_1(s) x(s) ds - P_1 \int_0^1 v_2(s) x(s) ds \\
- A_1 \sum_{i=1}^{m} I_i(x(\tau_i)),
\]

\[
\int_0^1 v_2(s) \int_0^1 G(s, \varphi) v(\varphi) d\varphi ds = -Q_2 \int_0^1 v_1(s) x(s) ds + (1 - P_2) \int_0^1 v_2(s) x(s) ds \\
- A_2 \sum_{i=1}^{m} I_i(x(\tau_i)),
\]

Hence,

\[
\int_0^1 v_1(s) x(s) ds = \frac{1}{P_1} [(1 - P_2) \int_0^1 v_1(s) \int_0^1 G(s, \varphi) v(\varphi) d\varphi ds + A_1 \sum_{i=1}^{m} I_i(x(\tau_i))] \\
+ P_1 \int_0^1 v_2(s) \int_0^1 G(s, \varphi) v(\varphi) d\varphi ds + A_2 \sum_{i=1}^{m} I_i(x(\tau_i))],
\]

and

\[
\int_0^1 v_2(s) x(s) ds = \frac{1}{Q_2} [Q_2 \int_0^1 v_1(s) \int_0^1 G(s, \varphi) v(\varphi) d\varphi ds + A_1 \sum_{i=1}^{m} I_i(x(\tau_i))] \\
+ (1 - Q_1) \int_0^1 v_2(s) \int_0^1 G(s, \varphi) v(\varphi) d\varphi ds + A_2 \sum_{i=1}^{m} I_i(x(\tau_i))],
\]

which together with (6) implies that
\[ x(\tau) = \int_0^1 G(\tau, s) \nu(s) ds + \frac{1}{\theta - \bar{\theta}} \int_0^1 \nu_1(s) x(s) ds + \frac{\theta - \bar{\theta}}{(\theta - \bar{\theta})(\bar{\theta} - \omega)} \int_0^1 v_2(s) x(s) ds + \frac{\theta}{\theta - \bar{\theta}} \sum_{i=1}^m I_i(x(\tau_i)) \]

\[ = \int_0^1 G(\tau, s) \nu(s) ds + \varphi_1(\tau) \left[ \int_0^1 \nu_1(s) \int_0^1 G(s, \varrho) \nu(\varrho) d\varrho ds + A_1 \frac{\theta}{\theta - \bar{\theta}} \sum_{i=1}^m I_i(x(\tau_i)) \right] \]

\[ + \varphi_2(\tau) \left[ \int_0^1 v_2(s) \int_0^1 G(s, \varrho) \nu(\varrho) d\varrho ds + A_2 \frac{\theta}{\theta - \bar{\theta}} \sum_{i=1}^m I_i(x(\tau_i)) \right] + \frac{\theta}{\theta - \bar{\theta}} \sum_{i=1}^m I_i(x(\tau_i)) \]

\[ = \int_0^1 G(\tau, s) \nu(s) ds + \sum_{n=1}^2 \varphi_n(\tau) \left[ \int_0^1 \nu_n(s) \int_0^1 G(s, \varrho) \nu(\varrho) d\varrho ds + A_n \frac{\theta}{\theta - \bar{\theta}} \sum_{i=1}^m I_i(x(\tau_i)) \right] \]

Similar to the above process, by substituting (10) into (3), for \( \tau \in J_k = (\tau_k, \tau_{k+1}] \), one can obtain that
\[ x(\tau) = \int_0^1 G(\tau, s)v(s)ds + \frac{1}{\theta - \vartheta} \int_0^1 v_1(s)x(s)ds + \frac{(\theta - \vartheta)\tau + \theta}{(\theta - \vartheta)(\pi - \varrho)} \int_0^1 v_2(s)x(s)ds \]

\[ + \frac{\vartheta}{\theta - \vartheta} \sum_{i=1}^m I_i(x(\tau_i)) + \sum_{i=1}^k I_i(x(\tau_i)) \]

\[ = \int_0^1 G(\tau, s)v(s)ds + \varphi_1(\tau) \left[ \int_0^1 v_1(s) \int_0^1 G(s, \varphi)v(\varphi)d\varphi ds \right] \]

\[ + A_1 \left( \frac{\vartheta}{\theta - \vartheta} \sum_{i=1}^m I_i(x(\tau_i)) + \sum_{i=1}^k I_i(x(\tau_i)) \right) \]

\[ + A_2 \left( \frac{\vartheta}{\theta - \vartheta} \sum_{i=1}^m I_i(x(\tau_i)) + \sum_{i=1}^k I_i(x(\tau_i)) \right) \]

\[ = \int_0^1 G(\tau, s)v(s)ds + \varphi_1(\tau) \left[ \int_0^1 v_1(s) \int_0^1 G(s, \varphi)v(\varphi)d\varphi ds \right] \]

\[ + A_1 \left( \frac{\vartheta}{\theta - \vartheta} \sum_{i=1}^m I_i(x(\tau_i)) + \sum_{i=1}^k I_i(x(\tau_i)) \right) \]

\[ + A_2 \left( \frac{\vartheta}{\theta - \vartheta} \sum_{i=1}^m I_i(x(\tau_i)) + \sum_{i=1}^k I_i(x(\tau_i)) \right) \]

Assume that the following condition is satisfied throughout this paper: Hypothesis 1 (H1). \( Q_1 < 1, P_2 < 1, (1 - Q_1)(1 - P_2) > P_1Q_2. \)

**Lemma 2.** The functions \( H_1 \) and \( H_2 \) have the following properties:

1. \( H_1(\tau, s), H_2(\tau, \tau_i) > 0 \) for all \( \tau, s \in [0, 1], i = 1, 2, \ldots, m; \)
(2) \[ \sigma M(s) \leq m(s) \leq H_1(\tau, s) \leq M(s) \text{ for all } \tau, s \in [0, 1]; \]

(3) \[ \sigma H_2(1, 0) \leq H_2(\tau, \tau_i) \leq H_2(1, 0) \text{ for all } \tau \in [0, 1], i = 1, 2, \ldots, m; \]

where

\[
M(s) = G(1, 0) + \sum_{n=1}^{2} \varphi_n(1) \int_{0}^{1} G(s, \varrho) v_n(\varrho) d\varrho,
\]

\[
m(s) = G(0, 1) + \sum_{n=1}^{2} \varphi_n(0) \int_{0}^{1} G(s, \varrho) v_n(\varrho) d\varrho,
\]

and \( \sigma = \min \left\{ \frac{\partial \omega}{\partial \varpi}, \frac{\vartheta}{\Pi} \right\}, \Pi = \frac{1 + \sum_{n=1}^{2} A_n \varphi_n(0)}{1 + \sum_{n=1}^{2} A_n \varphi_n(1)} \).

**Proof.** First, it is obvious that \( H_1(\tau, s), H_2(\tau, \tau_i) > 0 \) for all \( \tau, s \in [0, 1], i = 1, 2, \ldots, m. \)

For a given \( s \in [0, 1], H_1(\tau, s) \) is increasing with \( \tau \) for \( \tau \in I \). Then,

\[
M(s) = G(1, 0) + \sum_{n=1}^{2} \varphi_n(1) \int_{0}^{1} G(s, \varrho) v_n(\varrho) d\varrho
\]

\[
= \frac{\partial \omega}{(\vartheta - \vartheta)(\pi - \omega)} + \frac{\vartheta}{\vartheta - \vartheta} + \frac{\omega}{\pi - \omega} + 1
\]

\[
+ \int_{0}^{1} G(s, \varrho) v_1(\varrho) d\varrho \left[ \frac{(\pi - \omega)(1 - P_2) + \vartheta Q_2}{(\vartheta - \vartheta)(\pi - \omega)\Gamma} \right]
\]

\[
+ \int_{0}^{1} G(s, \varrho) v_2(\varrho) d\varrho \left[ \frac{(\pi - \omega)P_1 + \vartheta(1 - Q_1)}{(\vartheta - \vartheta)(\pi - \omega)\Gamma} \right],
\]

\[
m(s) = G(0, 1) + \sum_{n=1}^{2} \varphi_n(0) \int_{0}^{1} G(s, \varrho) v_n(\varrho) d\varrho
\]

\[
= \frac{\partial \omega}{(\vartheta - \vartheta)(\pi - \omega)}
\]

\[
+ \int_{0}^{1} G(s, \varrho) v_1(\varrho) d\varrho \left[ \frac{(\pi - \omega)(1 - P_2) + \vartheta Q_2}{(\vartheta - \vartheta)(\pi - \omega)\Gamma} \right]
\]

\[
+ \int_{0}^{1} G(s, \varrho) v_2(\varrho) d\varrho \left[ \frac{(\pi - \omega)P_1 + \vartheta(1 - Q_1)}{(\vartheta - \vartheta)(\pi - \omega)\Gamma} \right],
\]

\[
\geq \frac{\partial \omega}{\vartheta\Gamma} M(s)
\]

\[
\geq \sigma M(s).
\]
Thus,
$$\sigma M(s) \leq m(s) \leq H_1(\tau, s) \leq M(s) \text{ for all } \tau, s \in [0, 1].$$

Now, we are in a position to prove (3).

Firstly, for $0 \leq \tau \leq \tau_i \leq 1 (i = 1, 2, \ldots, m)$, one can easily obtain that
$$H_2(\tau, \tau_i) = \frac{\theta}{\theta - \theta} + \frac{\theta}{\theta - \theta} \sum_{n=1}^{2} A_n \varphi_n(\tau)$$
$$\geq \frac{\theta}{\theta} \left[ 1 + \sum_{n=1}^{2} A_n \varphi_n(0) \right]$$
$$\geq \frac{\theta}{\theta} \Pi$$
$$\geq \sigma.$$

Secondly, for $0 \leq \tau_i < \tau \leq 1 (i = 1, 2, \ldots, m)$, we get
$$H_2(\tau, \tau_i) = \frac{\theta}{\theta - \theta} + \frac{\theta}{\theta - \theta} \sum_{n=1}^{2} A_n \varphi_n(\tau)$$
$$\geq \frac{\theta}{\theta} \left[ 1 + \sum_{n=1}^{2} A_n \varphi_n(1) \right]$$
$$\geq \frac{\theta}{\theta} \Pi$$
$$\geq \sigma.$$

Therefore, $\sigma H_2(1, 0) \leq H_2(\tau, \tau_i) \leq H_2(1, 0)$ for all $\tau \in [0, 1], i = 1, 2, \ldots, m$. \(\square\)

**Lemma 3** ([18]). The set $F \subset PC([0, 1], R^n)$ is relatively compact if, and only if,

1. $F$ is bounded, that is, $\|x\| \leq C$ for each $x \in F$ and some $C > 0$;
2. $F$ is quasi-equicontinuous in $[0, 1]$. That is to say, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $x \in F, k \in N; \tau_1, \tau_2 \in (t_{k-1}, t_k)$ and $|\tau_1 - \tau_2| < \delta$, we have $|x(\tau_1) - x(\tau_2)| < \varepsilon$.

In the following, let $(X, \| \cdot \|)$ be a Banach space. $\Omega$ is a nonempty open subset of $(X, \| \cdot \|)$. $T: \overline{\Omega} \rightarrow X$ is an operator which is not necessarily continuous.

**Definition 1** ([19]). The closed-convex Krasovskij envelope (cc-envelope) of an operator $T: \overline{\Omega} \rightarrow X$ is the multivalued mapping $T: \overline{\Omega} \rightarrow 2^X$ given by

$$Tx = \bigcap_{\varepsilon > 0} \sigma T(B_\varepsilon(x) \cap \overline{\Omega})$$

for every $x \in \overline{\Omega},$

where $B_\varepsilon(x)$ denotes the closed ball centered at $x$ and radius $\varepsilon$, and $\sigma$ means closed convex hull.
Lemma 4 ([13,19]). \( y \in \mathbb{T}_x \) if, for every \( \varepsilon > 0 \) and every \( \rho > 0 \), there exist \( m \in \mathbb{N} \) and a finite family of vectors \( x_i \in \mathbb{B}_x(x) \cap \mathbb{T} \) and coefficients \( \lambda_i \in [0,1] (i = 1,2,\ldots,m) \), such that
\[
\|y - \sum_{i=1}^{m} \lambda_i x_i\| < \rho.
\]

Next, we introduce Krasnosel’skii’s fixed-point theorems for discontinuous operators on cones. Denote \( P_R = \{ x \in P : \|x\| < R \} \) for \( R > 0 \).

Lemma 5 ([13]). Let \( R > 0, 0 \in \Omega_i \subset P_R \) be relatively open subsets of \( P (i = 1,2) \). \( T : \mathbb{P}_R \to P \) is a mapping such that \( T \mathbb{P}_R \) is relatively compact and it fulfills condition
\[
x \cap \mathbb{T}_x \subset \{ Tx \}
\]
in \( \mathbb{P}_R \).

(a) If \( \lambda x \not\in \mathbb{T}_x \) for all \( x \in P \) with \( x \in \partial \Omega_1 \) and all \( \lambda \geq 1 \), then \( i(T, \Omega_1, P) = 1 \).

(b) If there exists \( \omega \in P \) with \( \|\omega\| \neq 0 \) such that \( x \not\in \mathbb{T}_x + \lambda \omega \) for every \( \lambda \geq 0 \) and all \( x \in P \) with \( x \in \partial \Omega_2 \), then \( i(T, \Omega_2, P) = 0 \).

Lemma 6 ([13]). Condition (a) in Lemma 5 is satisfied if one of the following two conditions holds:

(i) \( y \neq u \) for all \( y \in \mathbb{T}_u \) with \( x \in P \) and \( \|x\| = r_1 \).

(ii) \( \|y\| < \|x\| \) for all \( y \in \mathbb{T}_x \) and all \( x \in P \) with \( \|x\| = r_1 \).

Analogously, assumption (b) in Lemma 5 holds if one of the following two conditions holds:

(i) \( y \neq u \) for all \( y \in \mathbb{T}_u \) with \( x \in P \) and \( \|x\| = r_1 \).

(ii) If \( \|y\| > \|x\| \) for all \( y \in \mathbb{T}_x \) and all \( x \in P \) with \( \|x\| = r_2 \).

Now, we define the admissible discontinuity curves where we allow the nonlinearities \( f \) to be discontinuous.

Definition 2. We say that \( \gamma : J \to \mathbb{R}^+, \gamma \in \text{PC}^1(J) \cap W^{2,1}(J') \) is an admissible discontinuity curve for the differential system (1) if \( \gamma \) satisfies the boundary value conditions of (1), \( \Delta \gamma|_{\tau=\tau_k} = 0 \), \( k = 1, \ldots, m \), and one of the following conditions holds:

(i)
\[
\left\{
\begin{array}{l}
\gamma''(\tau) = g(\tau)f(\tau, \gamma(\tau)), \text{ a.e. } \tau \in J'; \\
\Delta \gamma|_{\tau=\tau_k} = \Delta_k(\gamma(\tau_k)), k = 1, \ldots, m;
\end{array}
\right.
\]

(ii) there exist \( \phi, \Phi \in L^1(J), \phi(\tau), \Phi(\tau) > 0 \) a.e. for \( \tau \in [0,1], S, Q \subset J, m(S \cap Q) = 0, m(S \cup Q) > 0 \), and \( \varepsilon > 0 \) such that
\[
\left\{
\begin{array}{l}
\gamma''(\tau) + \Phi(\tau) < g(\tau)f(\tau, x), \text{ a.e. } \tau \in Q, x \in [\gamma(\tau) - \varepsilon, \gamma(\tau) + \varepsilon]; \\
\gamma''(\tau) - \Phi(\tau) > g(\tau)f(\tau, x), \text{ a.e. } \tau \in S, x \in [\gamma(\tau) - \varepsilon, \gamma(\tau) + \varepsilon]; \\
\gamma''(\tau) = g(\tau)f(\tau, \gamma(\tau)), \text{ a.e. } \tau \in J' \setminus (S \cup Q); \\
\Delta \gamma|_{\tau=\tau_k} = \Delta_k(\gamma(\tau_k)), k = 1, \ldots, m;
\end{array}
\right.
\]

(iii) there exist \( k \in \{1, 2, \ldots, m\} \) such that
\[
\left\{
\begin{array}{l}
\gamma''(\tau) = g(\tau)f(\tau, \gamma(\tau)), \text{ a.e. } \tau \in J'; \\
\Delta \gamma|_{\tau=\tau_k} \neq \Delta_k(\gamma(\tau_k));
\end{array}
\right.
\]

(iv) there exist \( \phi, \Phi \in L^1(J), \phi(\tau), \Phi(\tau) > 0 \) a.e. for \( \tau \in [0,1], S, Q \subset J, m(S \cap Q) = 0, m(S \cup Q) > 0 \), and \( \varepsilon > 0 \), \( k \in \{1, 2, \ldots, m\} \), such that
\[
\begin{cases}
\gamma''(\tau) + \phi(\tau) < g(\tau)f(\tau, x), & \text{a.e. } \tau \in Q, \ x \in [\gamma(\tau) - \varepsilon, \gamma(\tau) + \varepsilon]; \\
\gamma''(\tau) - \phi(\tau) > g(\tau)f(V, x), & \text{a.e. } \tau \in S, \ x \in [\gamma(\tau) - \varepsilon, \gamma(\tau) + \varepsilon]; \\
\gamma''(\tau) = g(\tau)f(\tau, \gamma(\tau)), & \text{a.e. } \tau \in J' \setminus (S \cup Q); \\
\triangle \gamma|_{\tau=\kappa} \neq I_k(\gamma(\kappa)).
\end{cases}
\] (16)

If (i) holds, then we say that \( \gamma \) is viable for the BVP (1); if one of (ii)-(iv) holds, we say that \( \gamma \) is inviable.

3. Existence Results

Let \( E = PC^1[0, 1], \ P := \{x \in E : x(\tau) \geq \sigma \|x\|_1, \ \forall \tau \in [0, 1]\} \), and \( P_r := \{x \in P : \|x\|_1 \leq r\} \).

Then, \( (E, \|\cdot\|_1) \) is a real Banach space and \( P \) is a cone on \( E \). We can recall that \( u \in PC^1(f) \cap W^{2,1}(f) \) is a BVP solution (1) if (and only if) \( u \) is a solution of the following integral equation:

\[
x(\tau) = \int_0^1 \mathbb{H}_1(\tau, s)g(s)f(s, x(s))ds + \sum_{i=1}^{m} \mathbb{H}_2(\tau, \tau_i)I_i(x(\tau_i)).
\] (17)

Define operator \( T : P \to E \) as follows:

\[
Tx(\tau) := \int_0^1 \mathbb{H}_1(\tau, s)g(s)f(s, u(s))ds + \sum_{i=1}^{m} \mathbb{H}_2(\tau, \tau_i)I_i(x(\tau_i)), \ x \in P.
\] (18)

For any \( x \in P \), by \( g \in L(0, 1) \), the continuity of \( \mathbb{H}_1 \) and the assumption of \( f, T \) is well-defined. Next, we will find the positive fixed point of \( T \) in the following work.

Set

\[
N_1 = \left( \int_0^1 M(s)g(s)ds \right)^{-1}, \ N_2 = \left( \int_0^1 m(s)g(s)ds \right)^{-1},
\]

\[
N_3 = (sup_{\tau \in [0, 1]} \int_0^1 (\mathbb{H}_1)'(\tau, s)g(s)ds)^{-1}, \ N_4 = (inf_{\tau \in [0, 1]} \int_0^1 (\mathbb{H}_1)'(\tau, s)g(s)ds)^{-1},
\]

\[
N_5 = sup_{\tau \in [0, 1], i \in \{1, \ldots, m\}} \mathbb{H}_2(\tau, \tau_i), \ N_6 = inf_{\tau \in [0, 1], i \in \{1, \ldots, m\}} \mathbb{H}_2(\tau, \tau_i),
\]

\[
N_7 = sup_{\tau \in [0, 1], i \in \{1, \ldots, m\}} \mathbb{H}_2(\tau, \tau_i), \ N_8 = inf_{\tau \in [0, 1], i \in \{1, \ldots, m\}} \mathbb{H}_2(\tau, \tau_i).
\]

Now, let us list the following assumptions.

Hypothesis 2 (H2). \( f : I \times \mathbb{R}^+ \to \mathbb{R}^+ \) is such that:

(a) For any \( x \in \mathbb{R}^+ \), the mapping \( \tau \in I \mapsto f(\cdot, x) \) is measurable;

(b) For each \( r > 0 \) there exists \( R > 0 \), such that \( f(\tau, x) \leq R \) for a.e. \( \tau \in I \) and all \( x \in [0, r] \).

Hypothesis 3 (H3). \( g \) is measurable and \( g(\tau) \geq 0 \) a.e. for \( \tau \in [0, 1] \).

Hypothesis 4 (H4). There are admissible discontinuity curves \( \gamma_n : I \to \mathbb{R}^+, \ n \in \mathbb{N} \), such that the function \( u \mapsto f(\tau, x) \) is continuous in \([0, \infty) \setminus \bigcup_{n \in \mathbb{N}} \{\gamma_n(\tau)\} \) for a.e. \( \tau \in I \).

Hypothesis 5 (H5).

\[
\lim_{x \to 0^+} \inf_{\tau \in [0, 1]} \frac{f(\tau, x)}{x} > \frac{4}{3\sigma} \frac{1}{N_2} + mN_6|^{-1}; \quad \lim_{x \to 0^+} \frac{T_k(x)}{x} > \frac{4}{3\sigma} \frac{1}{N_2} + mN_6|^{-1}.
\]

Hypothesis 6 (H6).

\[
\lim_{x \to 0^+} \inf_{\tau \in [0, 1]} \frac{f(\tau, x)}{x} < \frac{4}{5} \frac{1}{N_1} + mN_5|^{-1}; \quad \lim_{x \to 0^+} \frac{T_k(x)}{x} < \frac{4}{5} \frac{1}{N_1} + mN_5|^{-1}.
\]

Hypothesis 7 (H7).

\[
\lim_{x \to 0^+} \inf_{\tau \in [0, 1]} \frac{f(\tau, x)}{x} < \frac{4}{5} \frac{1}{N_1} + mN_5|^{-1}; \quad \lim_{x \to 0^+} \frac{T_k(x)}{x} \frac{4}{5} \frac{1}{N_1} + mN_5|^{-1}.
\]
Hypothesis 8 (H8).
\[ \lim_{x \to 0^+} \inf_{r \in [0,1]} \frac{f(\tau,x)}{x} > \frac{4}{3\sigma} \cdot \frac{1}{N_2} + mN_6 \cdot \lim_{x \to 0^+} \frac{T_k(x)}{x} > \frac{4}{3\sigma} \cdot \frac{1}{N_2} + mN_6. \]

Lemma 7. The operator \( T : P \to P \) is well-defined and maps bounded sets into relatively compact sets.

Proof of Lemma 7. First, we shall show that \( T : P \to P \) is well-defined. It is obvious that \( T x(\tau) \geq 0 \), for \( x \in [0,1] \). From the expression of \( T \), we have
\[
(Tx)'(\tau) = \int_{0}^{1} (\mathbb{H}_1)'(\tau,s)g(s)f(s,x(s))ds + \sum_{i=1}^{m} (\mathbb{H}_2)'(\tau,\tau_i)I_i(x(\tau_i)), \ x \in P,
\]
\[
(\mathbb{H}_1)'(\tau,s) = \begin{cases} \frac{\pi}{\pi - \omega} + \sum_{n=1}^{2} \phi_n'(\tau) \int_{0}^{1} G(s,\theta)\nu_n(\theta)d\theta, & 0 \leq s \leq \tau \leq 1; \\ \frac{\omega}{\pi - \omega} + \sum_{n=1}^{2} \phi_n'(\tau) \int_{0}^{1} G(s,\theta)\nu_n(\theta)d\theta, & 0 \leq \tau \leq s \leq 1,
\end{cases}
\]
and
\[
(\mathbb{H}_2)'(\tau,\tau_i) = \begin{cases} \frac{\theta}{\theta - \theta} \sum_{n=1}^{2} A_n\psi_n'(\tau), & 0 \leq \tau \leq \tau_i \leq 1; \\ \frac{\theta}{\theta - \theta} \sum_{n=1}^{2} A_n\psi_n'(\tau), & 0 \leq \tau_i < \tau \leq 1.
\end{cases}
\]

Hence,
\[ \|Tx\|_0 \geq \|(Tx)'\|_0. \]

Then,
\[
\|T\|_1 = \sigma \|Tx\|_0 \leq \sigma \left[ \int_{0}^{1} M(s)g(s)f(s,x(s))ds + \sum_{i=1}^{m} \mathbb{H}_2(1,0)I_i(x(\tau_i)) \right] 
\leq \int_{0}^{1} m(s)g(s)f(s,x(s))ds + \sum_{i=1}^{m} \mathbb{H}_2(0,1)I_i(x(\tau_i)) 
\leq \min_{\tau \in [0,1]} \|Tx(\tau)\|.
\]

Next, we are in the position to prove that \( T \) maps bounded sets into relatively compact sets by Lemma 3. Indeed, it is enough to show that, from each \( P_r = \{ x \in P : \|x\|_1 \leq r \} \), \( T(P_r) \) is a relatively compact set, where \( r > 0 \).

In fact, from \( \mathcal{I}_k \subset \mathbb{C}(\mathbb{R}^+,\mathbb{R}^+) \), there exists \( \mathcal{M}_k > 0 \) such that \( \mathcal{I}_k(x) \leq \mathcal{M}_k \) for \( u \in [0,r](k = 1, 2, \ldots, m) \). By (H2), \( T(P_r) \) is bounded. Moreover, for a.e. \( \tau \in \mathcal{I}_k \) and any \( x \in P_r \), we have
\[
(Tx)''(\tau) = g(\tau)f(\tau,x(\tau)) \leq Rg(\tau),
\]
which implies that
\[
|(Tx)'(\tau_2) - (Tx)'(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |(Tx)''(\tau)|d\tau \leq \int_{\tau_1}^{\tau_2} Rg(\tau)d\tau,
\]
where \( \tau_1, \tau_2 \in \mathcal{I}_k \) with \( \tau_1 < \tau_2 \). Hence, \( T \) is quasi-equicontinuous. \( \square \)

Lemma 8. Assume that (H4) is valid and let \( \mathcal{T} \) be the cc-envelope of an operator \( T : P_R \to P \). Then,
\[ x \cap Tx \subset \{ Tx \}, \ for \ all \ x \in P_R. \]
Proof of Lemma 8. Fix \( x \in P_R \) and let \( T_n = \{ \tau \in I : x(\tau) = \gamma_n(\tau) \} (n \in \mathbb{N}) \).

Case 1: \( m(T_n) = 0 \) for all \( n \in \mathbb{N} \).

Using (H4), we obtain \( f(\tau, x_3(\tau)) \rightarrow f(\tau, x(\tau)) \) for a.e. \( \tau \in I \) if \( x_3 \rightarrow x \) in \( P_R \). From (H2) and (H3), it is easy to see that \( TX \) converges to \( TX \) in \( P_R \). Hence, \( T \) is continuous at \( x \).

By Proposition 2.2 in [19], we have \( TX = TX \).

Case 2: There exists \( n \in \mathbb{N} \) such that \( \gamma_n \) is inviable and \( m(T_n) > 0 \). Let \( B = \{ n : m(T_n) > 0, \gamma_n \text{ is inviable} \} \). Case 2 will be divided into three subcases to prove.

Case 2.1: The above \( \gamma_n \) satisfies (ii) in Definition 2. Then, there exist \( \phi, \overline{\phi} \in L^1(\mathcal{F}), \phi(\tau), \overline{\phi}(\tau) > 0 \) for a.e. \( \tau \in [0,1] \), \( S_n, Q_n \subset I, m(S_n \cap Q_n) = 0, m(S_n \cup Q_n) > 0, \) and \( \epsilon > 0 \) such that

\[
\begin{align*}
&\gamma''_n(\tau) + \overline{\phi}(\tau) < g(\tau)f(\tau, \gamma_n(\tau)), \text{ a.e. } \tau \in Q_n, x \in [\gamma_n(t) - \epsilon, \gamma_n(\tau) + \epsilon]; \\
&\gamma''_n(\tau) - \phi(t) > g(\tau)f(\tau, \gamma_n(\tau)), \text{ a.e. } \tau \in S_n, x \in [\gamma_n(t) - \epsilon, \gamma_n(\tau) + \epsilon]; \\
&\gamma''_n(\tau) = g(\tau)f(\tau, \gamma_n(\tau)), \text{ a.e. } \tau \in J \setminus (S_n \cup Q_n);
\end{align*}
\]

(19)

(I) \( m(\{ \tau \in S_n \cup Q_n | x(\tau) = \gamma_n(\tau) \}) \approx 0 \) for all \( n \in B \).

We will prove that if \( x \in TX \), then \( x = TX \). By \( m(\{ \tau \in S_n \cup Q_n | x(\tau) = \gamma_n(\tau) \}) \approx 0 \), for a.e. \( \tau \in T_n \), we have \( \gamma''_n(\tau) = g(\tau)f(\tau, \gamma_n(\tau)) \), which implies \( x''(\tau) = g(\tau)f(\tau, x) \). Therefore, \( x''(\tau) = g(\tau)f(\tau, x) \) a.e. \( \tau \in \bigcup_{n \in B} T_n \).

Next, we will prove that \( u''(\tau) = g(\tau)f(\tau, x) \) a.e. in \( J \setminus \bigcup_{n \in B} T_n \), which shows \( x = TX \) if \( x \in TX \).

Since \( x \in TX \), for each \( k \in \mathbb{N} \), by Lemma 4 with \( \epsilon = \rho = \frac{1}{p} \), we can find functions \( x_{p,i} \in B_1(x) \cap P_R \) and coefficients \( \lambda_{p,i} \in [0, 1] (i = 1, 2, \ldots, m(p)) \) such that \( \sum_{i=1}^{m(p)} \lambda_{p,i} = 1 \) and

\[
||x - \sum_{i=1}^{m(p)} \lambda_{p,i} T x_{p,i}||_1 < \frac{1}{p}.
\]

Let \( v_p = \sum_{i=1}^{m(p)} \lambda_{p,i} T x_{p,i} \). We have that \( v_p \rightarrow x' \) uniformly when \( p \rightarrow \infty \) in \( J \) and

\[
||x_{p,i} - x'||_1 \leq \frac{1}{p},
\]

for all \( p \in \mathbb{N}, i \in \{ 1, 2, \ldots, m(p) \} \).

Moreover, \( g(\tau)f(\tau, \cdot) \) is continuous at \( x(\tau) \) for a.e. \( \tau \in J \setminus \bigcup_{n \in B} T_n \). Hence, there is some \( p_0 = p(\tau) \in \mathbb{N} \) such that

\[
|g(\tau)f(\tau, u_{p,i}(\tau)) - g(\tau)f(\tau, x(\tau))| < \epsilon \text{ for all } i \in \{ 1, 2, \ldots, m(p) \},
\]

for any \( \epsilon > 0 \), where \( p \in \mathbb{N}, p \geq p_0 \). Therefore,

\[
|v''_p(\tau) - g(\tau)f(\tau, x(\tau))| \leq \sum_{i=1}^{m(p)} \lambda_{p,i} |g(\tau)f(\tau, x_{p,i}(\tau)) - g(\tau)f(\tau, x(\tau))| < \epsilon.
\]

Hence, \( v''_p(\tau) \rightarrow g(\tau)f(\tau, x(\tau)) \) for a.e. \( \tau \in J \setminus \bigcup_{n \in B} T_n \). Then, by Corollary 3.10 in [19], we have \( x''(\tau) = g(\tau)f(\tau, x) \) for a.e. \( \tau \in J \setminus \bigcup_{n \in B} T_n \).

(II) There exists \( n \in B \) such that \( m(\{ \tau \in S_n \cup Q_n | x(\tau) = \gamma_n(\tau) \}) > 0 \). Without loss of generality, suppose \( m(\{ \tau \in S_n | u(\tau) = \gamma_n(\tau) \}) > 0 \). In this case, we will show that \( x \not\in TX \).
First, by (H2), for a.e. $\tau \in J$, there exists $H_0 > 0$ such that $f(\tau, x(\tau)) < H_0$. Set $F(\tau) = g(\tau)H_0$ and $A = \{\tau \in S_n \mid x(\tau) = \gamma_n(\tau)\} (n \in \mathbb{N})$. Then there exists at least one interval $f_0(k_0 \in \{1, \ldots, m\})$ such that $m(f_0 \cap A) > 0$. Let $A = f_0 \cap A$. Noticing that $F \in L(f)$, by Lemma 3.8 in [19], there is a measurable set $A_0 \subset A$ with $m(A_0) = m(A) > 0$ such that, for all $\tau_0 \in A_0$, we have

$$\lim_{\tau \to \tau_0^+} \frac{2}{4} \int_{[\tau_0, \tau]} F(s) ds = 0 = \lim_{\tau \to \tau_0^-} \frac{2}{4} \int_{[\tau_0, \tau]} F(s) ds.$$  \hspace{3cm} \text{(20)}$$

Moreover, by Corollary 3.9 in [19], there exists $A_1 \subset A_0$ with $m(A_0 \setminus A_1) = 0$ such that, for all $\tau_0 \in A_1$, we have

$$\lim_{\tau \to \tau_0^+} \frac{1}{4} \int_{[\tau_0, \tau]} F(s) ds = 1 = \lim_{\tau \to \tau_0^-} \frac{1}{4} \int_{[\tau_0, \tau]} F(s) ds.$$ \hspace{3cm} \text{(21)}$$

Fix a point $\tau_0 \in A_1$. From (20) and (21); the following inequalities are satisfied:

$$2 \int_{[\tau_0, \tau^+]} F(s) ds < \frac{1}{4} \int_{\tau_0^+} \phi(s) ds,$$

$$\int_{[\tau_0, \tau]} \phi(s) ds \geq \int_{[\tau_0, \tau^+]} \phi(s) ds > \frac{1}{2} \int_{\tau_0^+} \phi(s) ds,$$

$$2 \int_{[\tau_0^-, \tau]} F(s) ds < \frac{1}{4} \int_{\tau_0^-} \phi(s) ds,$$

$$\int_{[\tau_0^-, \tau]} \phi(s) ds > \frac{1}{2} \int_{\tau_0^-} \phi(s) ds,$$

where $\tau_0 < \tau_0$ and $\tau_0 > \tau_0$ and $\tau_0$, $\tau_0$ sufficiently close to $\tau_0$.

Now we prove that $x \not\in T_x$.

Claim: There exists $\rho > 0$, for every finite family $x_i \in B_\rho(x) \cap \overline{B}_{\mathbb{R}}$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \ldots, m_1$) with $\sum_{i=1}^{m_1} \lambda_i = 1$ such that

$$\|x - \sum_{i=1}^{m_1} \lambda_i T_x_i\| \geq \rho.$$
\( v'(q_0) - v'(\tau_-) = \int_{\tau_-}^{q_0} v''(s)\,ds = \int_{[\tau_-, q_0] \cap A} v''(s)\,ds + \int_{[\tau_- , q_0] \setminus A} v''(s)\,ds \)

\[
\begin{align*}
&< \int_{[\tau_- , q_0] \cap A} x''(s)\,ds - \int_{[\tau_- , q_0] \cap A} \phi(s)\,ds + \int_{[\tau_- , q_0] \setminus A} F(s)\,ds \\
&= x'(q_0) - x'(\tau_-) - \int_{[\tau_- , q_0] \setminus A} x''(s)\,ds - \int_{[\tau_- , q_0] \setminus A} \phi(s)\,ds + \int_{[\tau_- , q_0] \setminus A} F(s)\,ds \\
&\leq x'(q_0) - x'(\tau_-) - \int_{[\tau_- , q_0] \setminus A} \phi(s)\,ds + 2 \int_{[\tau_- , q_0] \setminus A} F(s)\,ds \\
&< x'(q_0) - x'(\tau_-) - \frac{1}{4} \int_{\tau_-}^{q_0} \phi(s)\,ds.
\end{align*}
\]

Choosing
\[
\rho = \min\left\{ \frac{1}{4} \int_{\tau_-}^{q_0} \phi(s)\,ds, \frac{1}{4} \int_{\tau_-}^{q_0} \phi(s)\,ds \right\}.
\]

Hence, \( \|x - v\|_1 \geq v'(t_-) - x'(t_-) \geq \rho \), if \( v'(q_0) \geq x'(q_0) \).

By replacing \( \tau_- \) with \( \tau_+ \) and a similar calculation, one can obtain that \( \|x - v\|_1 \geq \rho \) if \( v'(q_0) \leq x'(q_0) \).

Case 2.2: The above \( \gamma_n \) satisfies (iii) in Definition 2. Then, there exists \( k \in \{1, 2, \ldots, m\} \) such that
\[
\gamma_n' (\tau) = g(\tau)f(\tau, \gamma_n(\tau)), \text{ a.e. } \tau \in I',
\]
\[
\Delta \gamma_n|_{\tau=k} \neq \mathcal{I}_k(\gamma_n(\tau_k)), \ k = 1, \ldots, m.
\]

From the continuity of \( \mathcal{I}_k \), we suppose that there exist \( \Lambda, \epsilon > 0 \) such that \( \Delta \gamma_n|_{\tau=k} + \Lambda < \mathcal{I}_k(z), z \in [\gamma_{n-1}(\tau_k) - \epsilon, \gamma_n(\tau_k) + \epsilon] \).

(I) \( x(\tau_k) \neq \gamma_n(\tau_k) \) or \( x(\tau_k^+) \neq \gamma_n(\tau_k^+) \).

From (29), we have \( x'(\tau_k) = g(\tau) f(\tau, x(\tau_k)) \) for a.e. \( \tau \in \bigcup_{n \in B} T_n \). Without loss of generality, similar to the proof process of (I) in Case 2.1, one can obtain that \( x \notin T_k \) or \( x = T \) for all \( x \in P_R \).

Claim: Let \( \epsilon > 0 \) and \( \rho = \frac{\Lambda}{2} \), for every finite family \( x_i \in B_{\epsilon}(x) \cap P_R \) and \( \lambda_i \in [0, 1] \) \((i = 1, 2, \ldots, m_1)\) with \( \sum_{i=1}^{m_1} \lambda_i = 1 \), we have
\[
\|x - \sum_{i=1}^{m_1} \lambda_i T x_i\|_1 \geq \rho.
\]

Let \( x_i \) and \( \lambda_i \) be as in the claim and denote \( v = \sum_{i=1}^{m_1} \lambda_i T x_i \). In view of \( |x_i(\tau_k) - x(\tau_k)| = |x_i(\tau_k) - \gamma_n(\tau_k)| < \epsilon_1 \), one can see that
\[
\begin{align*}
\Delta v|_{\tau=k} &= \sum_{i=1}^{m_1} \lambda_i (\Delta T x_i|_{\tau-k}) = \sum_{i=1}^{m_1} \lambda_i (\mathcal{I}_k(x_i(\tau_k))) \\
&> \sum_{i=1}^{m_1} \lambda_i (\Delta \gamma_n|_{\tau=k} + \Lambda) \\
&= \Delta \gamma_n|_{\tau=k} + \Lambda \\
&= \Delta x|_{\tau=k} + \Lambda,
\end{align*}
\]

which implies that
\[
v(\tau_k^+) - x(\tau_k^+) > v(\tau_k) - x(\tau_k) + \Lambda \geq -|v(\tau_k) - x(\tau_k)| + \Lambda.
\]
Hence,
\[ \|x - v\|_1 \geq \frac{\Lambda}{2}. \]

The claim is proven.

Case 2.3: The above \( \gamma_n \) satisfies (iv) in Definition 2. By a similar process to that proving Case 2.1 and Case 2.2, one can also obtain that
\[ x \cap Tx \subset \{Tx\}, \text{ for all } x \in P_R. \]

Case 3: \( m(T_n) > 0 \) for \( n \in \mathbb{N} \) such that \( \gamma_n \) is viable.

Let \( B_1 = \{n : m(T_n) > 0 \text{ and } \gamma_n \text{ is viable}\} \). We will show that, in this case, \( x \in Tx \)
implies \( x = Tx \).

For each \( n \in B_1 \) and a.e. \( \tau \in T_n \),
\[ x''(\tau) = \gamma''_n(\tau) = g(\tau)f(\tau, \gamma_n(\tau)) = g(\tau)f(\tau, x(\tau)). \]

Hence, \( x''(\tau) = g(\tau)f(\tau, x(\tau)) \) a.e. in \( B = \bigcup_{n \in B_1} T_n \). Now, by process of proving (I) in Case 2.1, one can obtain that \( x''(\tau) = g(\tau)f(\tau, x(\tau)) \) a.e. in \( J \setminus B \) if \( x \in Tx \). Hence, \( x = Tx \). \( \square \)

**Theorem 1.** Assume that (H1)–(H6) hold. Then, BVP (1) admits at least one positive solution.

**Proof of Theorem 1.** We need only to prove \( T \) has at least one positive fixed point in \( P \cap (B_{R_1} \setminus \partial B_{r_1}) \).

Claim 1: There exists \( r_1 > 0 \), such that \( y \not\approx x \) for all \( y \in Tx \) and all \( x \in P \) with \( \|x\|_1 = r_1 \).

(H5) implies that \( \varepsilon_0 \) and \( r_1 > 0 \), such that
\[ f(\tau, x) > (\lambda + \varepsilon_0)x_1, \quad T_0(x) > (\lambda + \varepsilon_0)x, \quad \tau \in [0, 1], \quad x \in \left[0, \rho_1^1\right]. \]  (30)

Supposing that \( x \in P \) with \( \|x\|_1 = r_1 \), for every finite family \( x_i \in B_\varepsilon(x) \cap P \) and \( \lambda_i \in [0, 1] \)
\( i = 1, 2, \ldots, m_2 \), with \( \sum \lambda_i = 1 \), and \( \varepsilon \in (0, \rho_1^1] \), we have
\[ v(\tau) = \sum_{i=1}^{m_2} \lambda_i T(x_i(\tau)) \]
\[ = \sum_{i=1}^{m_2} \lambda_i \left[ \int_0^1 H_1(\tau, s) g(s) f(s, u_i(s)) ds + \sum_{i=1}^{m_2} H_2(\tau, \tau_i) I_i(x_i(\tau_i)) \right] \]
\[ > \sum_{i=1}^{m_2} \lambda_i (\lambda + \varepsilon_0) \left[ \int_0^1 H_1(\tau, s) g(s) x_i(s) ds + \sum_{i=1}^{m_2} H_2(\tau, \tau_i)x_i(\tau_i) \right] \]
\[ \geq \sum_{i=1}^{m_2} \lambda_i (\lambda + \varepsilon_0) \left[ \frac{\sigma\|x_i\|_1}{N_2} + mN_6\sigma\|x_i\|_1 \right] \]
\[ \geq \sigma(\|x\|_1 - \varepsilon)(\lambda + \varepsilon_0) \left[ \frac{1}{N_2} + mN_6 \right] \]
\[ \geq r_1 = \|x\|_1. \]

Hence, \( y \not\approx x \) for all \( y \in Tx \) with \( x \in P \) and \( \|x\|_1 = r_1 \). And by Lemma 6, one can obtain that
\[ i(T, P \cap \partial B_{r_1}, P) = 0. \]  (31)

Claim 2: There exists \( R_1 > r_1 > 0 \) such that \( \|y\|_1 < \|x\|_1 \) for all \( y \in Tx \) and all \( x \in P \)
with \( \|x\|_1 = R_1 \).
Hypothesis 9 (H9): There exist $R > 0$ such that

$$f(t, u) < (\bar{\lambda} - \epsilon_1)u, \quad f_t(u) < (\bar{\lambda} - \epsilon_1)u, \quad t \in [0, 1], \quad u \geq \frac{3}{4}R.$$ 

Choosing $R_1 > \max\{r_1, \frac{3R_1}{4r_1}\}$, for $x \in \partial P_{R_1}$, one can see that

$$x(\tau) \geq \sigma\|x\|_1 = \sigma R_1 > \frac{3}{4}R.$$ 

Supposing that $x \in P$ with $\|x\|_1 = R_1$, then, for every finite family $x_i \in B_x(x) \cap P$ and $\lambda_i \in [0, 1] (i = 1, 2, \ldots, m_3)$, with $\sum \lambda_i = 1$, and $e \in (0, r_1/4]$ we have

$$v(\tau) = \sum_{i=1}^{m_3} \lambda_i T(x_i(\tau))$$

$$= \sum_{i=1}^{m_3} \lambda_i \int_0^1 H_1(\tau, s)g(s)f(s, u_i(s))ds + \sum_{i=1}^{m_3} \sum_{k=1}^{m} H_2(\tau, \tau_i)\mathcal{I}_i(x_i(\tau_i))$$

$$\leq \sum_{i=1}^{m_3} \lambda_i \int_0^1 H_1(\tau, s)g(s)(\bar{\lambda} - \epsilon_1)x_i(s)ds + \sum_{i=1}^{m_3} \sum_{k=1}^{m} H_2(\tau, \tau_i)(\bar{\lambda} - \epsilon_1)x_i(\tau_i)$$

$$\leq \sum_{i=1}^{m_3} \lambda_i (\|x_i\|_1(\bar{\lambda} - \epsilon_1) + N\|\mathcal{I}_i\|_1 + mN_5)$$

$$\leq (R_1 + e)(\bar{\lambda} - \epsilon_1)\left[\frac{1}{N_1} + mN_5\right]$$

$$< R_1 = \|x\|_1.$$ 

Therefore, if $y \in T x$, then this is the limit of a sequence of functions $v$ as above, so $\|y\|_1 < \|x\|_1$, for all $y \in T x$ and all $x \in P$ with $\|x\|_1 = R_1$. Using Lemma 6, we get

$$i(T, P \cap \partial B_{R_1}, P) = 1.$$ 

Together with (31), we have

$$i(T, P \cap (B_{R_1} \setminus \overline{B_{r_1}}), P) = 1 - 0 = 1.$$ 

In sum, BVP (1) admits at least one positive solution. 

**Theorem 2.** Under the assumptions (H1)–(H4), (H7) and (H8), suppose

Hypothesis 9 (H9): There exist $R > 0$ such that $f^R < \frac{N_1 R}{2}$ and $\sum_{k=1}^{m} T_k^R < \frac{R}{2N_5}$, where

$$f^R := \sup_{\tau \in [0, 1], 0 \leq |x| \leq \frac{5R}{4}} f(\tau, x), \quad T_k^R := \sup_{0 \leq |x| \leq \frac{5R}{4}} T_k(x).$$
Then, BVP (1) admits at least two positive solutions.

**Proof of Theorem 2.** We only need to prove that $T$ has at least two positive fixed points in $P \cap (B_{r_1} \setminus B_{r_2})$ and $P \cap (B_{r_3} \setminus B_{r_2})$, respectively.

First, by (H7), there exist $r_2$ and $\varepsilon_2 \in (0, r)$ such that

$$f(\tau, x) < (v - \varepsilon_2)x, \quad I_k(x) < (v - \varepsilon_2)x, \quad \tau \in [0, 1], \quad x \in [0, \frac{5}{4}r_2].$$

We claim that for $\mu \geq 1$,

$$\mu x \notin \mathbb{T}x, \quad \forall x \in P \cap \partial B_{r_2}. \quad (34)$$

In fact, suppose that there exist $x \in P \cap \partial B_{r_2}, \mu \geq 1$ such that $\mu x(\tau) = T\nu(\tau)$ for some $\nu \in \overline{B}_{\varepsilon}(x) \cap P$, i.e.,

$$\mu x(\tau) = \int_0^1 H_1(\tau, s)g(s)f(s, \nu(s))ds + \sum_{i=1}^m I_i(\nu(t_i)) < (v - \varepsilon_2)(\|x\|_1 + \varepsilon)[\frac{1}{N_1} + mN_g] < r_2.$$  

Then,

$$\mu x'(\tau) = \int_0^1 H_1'(\tau, s)g(s)f(s, \nu(s))ds + \sum_{i=1}^m I_i'(\nu(t_i)) < (v - \varepsilon_2)(\|x\|_1 + \varepsilon)[\frac{1}{N_3} + mN_g] \leq (v - \varepsilon_2)(\|x\|_1 + \varepsilon)[\frac{1}{N_1} + mN_g] < r_2.$$  

Taking the supremum for $t \in [0, 1]$,

$$\mu \|x\|_1 = \mu r_2 < r_2, \quad (35)$$

which is a contradiction.

Now, given $p \in \mathbb{N}$, we can similarly prove that $\mu x \neq \sum_{i=1}^p \lambda_i T v_i$ for any $v_i \in B_k(x) \cap P$ and $\lambda_i \in [0, 1]$ with $\sum_{i=1}^p \lambda_i = 1$. Hence, $\mu x \notin \text{co}(T(B_k(x) \cap P))$.

Second, the assumption (H8) implies that there exist $\varepsilon_3 > 0, \mathcal{R} > r_2$ such that

$$f(\tau, x) > (\bar{v} + \varepsilon_3)x, \quad I_k(x) > (\bar{v} + \varepsilon_3), \quad \tau \in [0, 1], \quad x \geq \frac{3}{4} \mathcal{R}.$$  

Choosing $R_2 > \max\{r_1, \frac{3\mathcal{R}}{4\sigma}\}$, for any $x \in \partial P_{R_2}$, one can obtain that

$$x(\tau) \geq \sigma \|x\|_1 = \sigma R_2 > \frac{3}{4} \mathcal{R}.$$  

We claim that, for all $x \in P \cap \partial B_{R_2}$ and $\mu \geq 0$,

$$x \notin \mathbb{T}x + \mu e,$$

where $e(\tau) \equiv 1$.  

In fact, that there exist \( x \in P \cap \partial B_{R_2}, \mu \geq 0 \) such that \( x = T v + \mu \epsilon \) for some \( v \in \overline{B}_{\epsilon}(x) \cap P \),

\[
x(\tau) = \int_0^1 H_1(\tau,s)g(s)f(s,v(s))ds + \sum_{i=1}^m H_2(\tau,\tau_i)I_i(v(\tau_i)) + \mu
\]

\[
\geq (R_2 - \epsilon)\sigma(\bar{v} + \epsilon)\left[\frac{1}{N_2} + mN_0\right] + \mu
\]

\[
> R_2 + \mu.
\]

This, together with the definition of \( \| \cdot \|_1 \), guarantees that

\[ R_2 = \| x \|_1 \geq \max_{\tau \in [0,1]} x(\tau) > R_2 + \mu, \quad (36) \]

which is a contradiction because \( \mu \geq 0 \).

Given \( p \in \mathbb{N} \), we prove similarly that \( x \neq \sum_{i=1}^p \lambda_i T v_i + \mu \epsilon \) for any \( v_i \in B_{\epsilon}(x) \cap P \) and \( \lambda_i \in [0,1] (i = 1, \ldots, p) \) with \( \sum_{i=1}^p \lambda_i = 1 \). Therefore, \( x \notin \text{co}(T(B_{\epsilon}(x) \cap P)) + \mu \epsilon \).

Now, we can see that \( x \notin T x + \mu \epsilon \). By Lemma 5, one can obtain that \( i(T, P \cap \partial B_{R_2}, P) = 1 \) and \( i(T, P \cap \partial B_{R_2}, P) = 0 \). Hence,

\[ i(T, P \cap (B_{R_2} \setminus \overline{B}_{R_2}), P) = 0 - 1 = -1. \quad (37) \]

Third, by (H9), there exist \( R_3 > R_2 \) and \( \epsilon \in (0, \frac{R_2}{4}] \) such that \( f^{R_3} < \frac{N_1 R_3}{2} \) and \( \sum_{k=1}^m T_k^{R_3} < \frac{R_3}{2N_3} \).

Similar to the process above, we have

\[ i(T, P \cap \partial B_{R_3}, P) = 1. \]

Then,

\[ i(T, P \cap (B_{R_3} \setminus \overline{B}_{R_2}), P) = 1 - 0 = 1. \]

Together with (37), BVP (1) admits at least two positive solutions in \( P \cap (B_{R_3} \setminus \overline{B}_{R_2}) \) and \( P \cap (B_{R_3} \setminus \overline{B}_{R_2}) \), respectively. \( \square \)

4. Example

Example 1. Consider the following BVP

\[
\begin{cases}
    u''(t) = f(t, u), \text{ a.e. } t \in [0,1]; \\
    \Delta u|_{t=t_1} = I_1(u(t_1)); \\
    \Delta u'|_{t=t_1} = 0; \\
    3u(0) - u(1) = \int_0^1 \frac{1}{2} u(s)ds; \\
    3u'(0) - u'(1) = \int_0^1 u(s)ds,
\end{cases}
\]

where \( 0 < t_1 < 1, I_1(u) = \frac{u^2}{10^3} \) and

\[
f(t,u) = \begin{cases}
    \frac{u^2}{10^3}(\cos^2(\frac{1}{t^2-u}) + 1), & u \neq t^2, 0 \leq t \leq 1; \\
    \frac{t^4}{500}, & u = t^2, 0 \leq t \leq 1.
\end{cases}
\]

Conclusion: BVP (38) has at least two positive solutions.
Proof of Example 1. First, it is easy to see show that $f$ satisfies (H2). For a.e. $t \in J$, the function $u \to f(t, u)$ is continuous on
\[ \mathbb{R}^+ \setminus \bigcup_{n \in I} \{ \gamma_n(t) \}, \]
where for each $n \in \mathbb{Z} \setminus \{0\}$ the curves $\gamma_n(t) = t^2 - n^{-1}$ and $\gamma_0(t) = t^2$ are admissible discontinuity curves with
\[ 1 > \gamma_n''(t) - 1 > f(t, z), \quad t \in [0, 1], \quad z \in [\gamma_n(t) - 1, \gamma_n(t) + 1]. \]

By calculation, one can obtain that
\[ A_1 = \frac{1}{2}, \quad A_2 = 1, \quad P_1 = Q_1 = \frac{1}{4}, \quad P_2 = Q_2 = \frac{1}{2}, \]
\[ \Gamma = \frac{1}{8} > 0, \quad \varphi_1(t) = 2t + 2, \quad \varphi_2(t) = 3t + \frac{5}{2}, \]
\[ G(t, s) = \begin{cases} 
( t - s ) + \frac{1}{4} + \frac{1 - s}{2} + \frac{t}{2}, & 0 \leq t \leq 1; \\
\frac{1}{4} + \frac{1 - s}{2} + \frac{t}{2}, & 0 \leq t \leq s \leq 1. 
\end{cases} \]
\[ H_2(t, l_i) = \begin{cases} 
\frac{1}{2} + \frac{1}{2} (4t + \frac{7}{2}), & 0 \leq t \leq l_i \leq 1; \\
\frac{3}{2} + \frac{3}{2} (4t + \frac{7}{2}), & 0 \leq t_i < t < 1. 
\end{cases} \]

Thus, by calculation, we can obtain that $(N_1)^{-1} \approx 10.458$, $(N_2)^{-1} \approx 4.375$, $(N_3)^{-1} \approx 5.333$, $(N_4)^{-1} \approx 4.333$, $N_5 = \frac{51}{4}$, $N_6 = \frac{9}{4}$, $N_7 = 6$, $N_8 = 2$. Choosing $v = 0.03$ and $\bar{v} = 2$, which satisfies $5v(\frac{1}{N_1} + mN_5) \leq 4$ and $3\sigma \bar{v}(\frac{1}{N_2} + mN_6) \geq 4$.

Therefore,
\[ \lim_{u \to 0^+} \sup_{t \in [0, 1]} \frac{f(t, u)}{u} = 0 < v; \quad \lim_{u \to 0^+} \frac{I_k(u)}{u} = 0 < v. \]
\[ \lim_{u \to +\infty} \inf_{t \in [0, 1]} \frac{f(t, u)}{u} = +\infty > \bar{v}; \quad \lim_{u \to +\infty} \frac{I_k(u)}{u} = +\infty > \bar{v}. \]

In addition, notice that $(N_1)^{-1} \approx 10.458$, $N_5 = \frac{51}{4}$ and choose $R_3 = 10$ such that (H9) is satisfied.

Consequently, all conditions in Theorem 2 hold. □

5. Conclusions

In this paper, we studied the existence of positive solutions for a class of impulsive BVPs for second-order discontinuous differential equations. The main results are established by means of multivalued analysis and Krasnoselskii’s fixed point theorem for discontinuous operators.

Author Contributions: Conceptualization and visualization, Y.L. (Yansheng Liu); formal analysis and investigation, Y.W.; writing original draft and investigation, Y.L. (Yating Li). All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by NNSF of P.R. China (62073202), Natural Science Foundation of Shandong Province (ZR2020MA007), and Doctoral Research Funds of Shandong Management University (SDMUD202010), QiHang Research Project Funds of Shandong Management University (QH2020202).

Institutional Review Board Statement: Not applicable.
Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the editor and anonymous referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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