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Functional Differential Equations with Several Delays: Oscillatory Behavior

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Abstract: In this work, we study the asymptotic behavior of even-order delay functional differential equation. As an extension of the recent development in the study of oscillation, we obtain improved and simplified criteria that test the oscillation of solutions of the studied equation. We adopt an approach that improves the relationships between the solution with and without delay. The symmetry between the positive and negative solutions also plays a key role in simplifying the presentation of the main results. Finally, we attach an example to illustrate the results and compare them together with the previous results in the literature.

Keywords: delay differential equation; even order; sufficient conditions; noncanonical case

1. Introduction

Differential equations (DE) are essential for comprehending real-world issues and phenomena, or at the very least for understanding the properties of the equations that come from modeling these occurrences. However, DEs like the ones shown here that are used to solve real-world problems may not be explicitly solvable, that is, they may not have closed-form solutions. Only equations with basic forms accept explicit formulae for solutions. Different models of DEs have been produced in numerous domains in recent decades, which has sparked interest in qualitative theory of DEs study.

The highly rapid progress of research in the 20th century led to applications in biology, population, chemistry, medicine dynamics, social sciences, genetic engineering, economy, and other domains. All of these areas advanced, and new discoveries were made because of this type of mathematical modeling, see [1,2].

Oscillation theory is one of the branches of qualitative theory that studies the qualitative properties of solutions of differential equations, such as stability, symmetry, oscillation, and others, without finding solutions. The solutions of the studied equation are classified into three disjointed classes, which are positive eventually, negative eventually, and oscillatory solutions. The studied equation is characterized by the property of symmetry between the positive and negative solutions, which means that if \( x \) is a solution to the equation, then \( -x \) is also a solution. Therefore, the conditions that exclude positive solutions also exclude negative solutions, and thus they are conditions that guarantee the oscillation of the studied equation.

In this paper, we consider the delay differential equation (DDE)

\[
\left( a(s) \left( v(\tau_{i-1}(s)) \right) \right)' + \sum_{i=1}^{\ell} h_i(s) f(v(\sigma_i(s))) = 0, \quad s \geq s_0,
\]

(1)

Symmetry 2022, 14, 1570. https://doi.org/10.3390/sym14081570
https://www.mdpi.com/journal/symmetry
in the noncanonical case, that is,
\[ \delta_0(s_0) := \int_{s_0}^{\infty} \frac{1}{a^1/r(\theta)} \, d\theta < \infty. \] (2)

Throughout this study, we assume that \( r \) is a quotient of odd positive integers, \( \ell \) is a positive integers and \( \kappa \geq 4 \) is an even natural number, \( a, h_i \in C^1(I_0, \mathbb{R}^+) \), \( a'(s) \geq 0 \), \( \sigma_i \in C(I_0, \mathbb{R}^+) \), \( \sigma_i(s) \leq s \), \( \sigma_i'(s) > 0 \), \( \lim_{s \to \infty} \sigma_i(s) = \infty \), \( I_\beta := [s_\beta, \infty) \), \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( f(\upsilon) \geq \upsilon^\beta \) for \( \upsilon \neq 0 \).

By a solution of (1), we denote to a nontrivial real valued function \( \upsilon \) in \( C^{\kappa-1}([s_\upsilon, \infty)) \) for some \( s_\upsilon \geq s_0 \), which has the property \( a(\upsilon^{(\kappa-1)})' \in C^1([s_\upsilon, \infty)) \) and satisfies (1) on \([s_\upsilon, \infty)\). We take into account these solutions \( \upsilon \) of (1) such that \( \sup\{|\upsilon(u)|: \upsilon \geq s_\upsilon\} > 0 \) for every \( s_\upsilon \) in \([s_\upsilon, \infty)\). A solution \( \upsilon \) of (1) is said to be nonoscillatory if it is eventually positive or eventually negative; otherwise, it is said to be oscillatory.

There has recently been a surge of interest in developing sufficient conditions for oscillatory or non-oscillatory behavior of many types of functional differential equations. For the existence of solutions to DDEs, see \[3,4\].

The study of second-order differential equations and their properties have long been the subject of constant interest by researchers. We refer to \[5–9\] for more information, approaches, and methods on the oscillation of second-order neutral DDEs. Additionally, many techniques and methods have recently been developed to study the asymptotic behavior of higher-order DDEs, see for example \[10–14\].

For some related works, Zhang et al. \[15\] discussed the asymptotic properties of DDE
\[ \left(a(s)(\upsilon''(s))' + q(s)\upsilon^\beta(\sigma(s))\right) = 0, \] (3)
where \( r = \beta \) and obtained some oscillation criteria.

Baculikova et al. \[16\] proved that the oscillation of first-order DDE
\[ \upsilon'(s) + q(s)f \left( \frac{\delta s^{(\kappa-1)}(s)}{11(\kappa-1)a^{(\kappa-1)}(s)} \right) = 0 \]
guarantees the oscillation of even-order DDE
\[ \left[a(s)(\upsilon^{(\kappa-1)}(s))' + q(s)f(\upsilon(\sigma(s))) \right] = 0 \] (4)
in the case where
\[ \int_{s_0}^{\infty} \frac{1}{a^{1/r}(\theta)} \, d\theta = \infty, \] (5)
and obtained some comparison theorems for non-canonical case. Xing et al. \[17\] presented some theorems for oscillation of DDE of neutral type
\[ \left(a(s)(\upsilon(s) + p(t)\upsilon(\sigma(t)))^{(\kappa-1)})' + q(s)\upsilon^\beta(\sigma(s)) \right) = 0, \]
where \( r \leq 1 \) is the quotient of odd positive integers. Recently, in an interesting work, Moaaz and Muhib \[13\] extended and improved the results in \[15,16,18\] and obtained unusual conditions for testing the oscillation of the solutions of (4) where \( f(\upsilon) := \upsilon^\beta \) and \( \beta \) is a quotient of odd positive integers.

We begin in this paper with a simplification of the oscillation conditions of (1) as an extension of the approach used in [19]. We then improve the oscillation criteria by finding new relationships and improved inequalities. We suppose the Riccati transformations in the general form and we obtain new conditions for oscillation.
2. New Criteria for Oscillation

Let us define

\[ \delta_k(s) := \int_s^\infty \delta_{k-1}(\theta) \, d\theta, \quad \delta_0 \equiv \delta, \text{ for } k = 1, 2, \ldots, \kappa - 2 \]

and

\[ \sigma(s) := \min \{ \sigma_i(s) : i = 1, 2, \ldots, \kappa \} \].

**Lemma 1.** Let \( v \in C(I_0, (0, \infty)) \) be a solution of (1). Then \( \left( a(s) \left( v^{(k-1)}(s) \right)^r \right)' \leq 0 \), and we have the following cases, eventually:

1. \( v'(s) > 0, \ v^{(k-1)}(s) > 0, \ v^{(k)}(s) \leq 0 \);
2. \( v'(s) > 0, \ v^{(k-2)}(s) > 0, \ v^{(k-1)}(s) < 0 \);
3. \( v'(s) < 0, \ v^{(k-2)}(s) > 0, \ v^{(k-1)}(s) < 0 \).

**Proof.** From (1), we have

\[ \left( a(s) \left( v^{(k-1)}(s) \right)^r \right)' \leq -\sum_{i=1}^\ell h_i(s) v'(\sigma_i(s)) \leq 0. \]

From (1) and Lemma 2.2.1 in [20], there exist three possible cases (1)–(3) for \( s \geq s_1, s_1 \) that are large enough. The proof is complete. \( \square \)

**Theorem 1.** Let \( v \in C(I_0, (0, \infty)) \) be a solution of (1). If

\[ \limsup_{s \to \infty} \int_{s_1}^s \left( \frac{1}{a^{1/r}(\varphi)} \left( \int_{s_1}^\varphi \sum_{i=1}^\ell h_i(\theta) \delta_{k-2}(\sigma_i(\theta)) \, d\theta \right)^{1/r} \right) \, d\varphi = \infty, \tag{6} \]

then \( v \) satisfies Case (2).

**Proof.** Using Lemma 1, we have that one of the cases (1)–(3) holds.

First, we assume that (3) holds on \( I_1 \). Since \( \left( a(s) \left( v^{(k-1)}(s) \right)^r \right)' \leq 0 \), we have

\[ a(s) \left( v^{(k-1)}(s) \right)^r \leq a(s_1) \left( v^{(k-1)}(s_1) \right)^r := -L < 0, \tag{7} \]

which is

\[ a^{1/r}(s) v^{(k-1)}(s) \leq -L^{1/r}. \tag{8} \]

If we divide (8) by \( a^{1/r} \) and integrating the resulting inequality from \( s \) to \( \varphi \), we obtain

\[ v^{(k-2)}(\varphi) \leq v^{(k-2)}(s) - L^{1/r} \int_{s}^{\varphi} \frac{1}{a^{1/r}(\theta)} \, d\theta. \]

Letting \( \varphi \to \infty \), we obtain

\[ 0 \leq v^{(k-2)}(s) - L^{1/r} \delta_0(s). \tag{9} \]

Integrating (9) from \( s \) to \( \infty \), we obtain

\[ -v^{(k-3)}(s) \geq L^{1/r} \delta_1(s), \]

integrating the above inequality from \( s \) to \( \infty \) a total of \( (k-4) \) times, we find

\[ -v'(s) \geq L^{1/r} \delta_{k-3}(s). \tag{10} \]
Integrating (10) from $s$ to $\infty$, implies that

$$v(s) \geq L^{1/r} \delta_{k-2}(s).$$  \hfill (11)

From (1) and (11), we have

$$(a(s)(v^{(k-1)}(s))' \leq - \sum_{i=1}^{\ell} h_i(s) v'(\sigma_i(s))$$

$$\leq - L \sum_{i=1}^{\ell} h_i(s) \delta'_{k-2}(\sigma_i(s)).$$  \hfill (12)

Integrating (12) from $s_1$ to $s$, we obtain

$$a(s)(v^{(k-1)}(s))' \leq a(s_1)(v^{(k-1)}(s_1))' - L \int_{s_1}^{s} \sum_{i=1}^{\ell} h_i(\theta) \delta'_{k-2}(\sigma_i(\theta)) \, d\theta$$

$$\leq - L \int_{s_1}^{s} \sum_{i=1}^{\ell} h_i(\theta) \delta'_{k-2}(\sigma_i(\theta)) \, d\theta.$$  \hfill (13)

Integrating (13) from $s_1$ to $s$, we obtain

$$v^{(k-2)}(s) \leq v^{(k-2)}(s_1) - L^{1/r} \int_{s_1}^{s} \left( \frac{1}{a(\theta)} \int_{s_1}^{\theta} \sum_{i=1}^{\ell} h_i(\sigma_i(\theta)) \, d\theta \right)^{1/r} \, d\theta.$$

At $s \to \infty$, we obtain a contradiction with (6).

Now, let Case (1) holds on $I_1$. Now, from (2) and (6), we obtain that $\int_{s_1}^{s} \sum_{i=1}^{\ell} h_i(\theta) \delta'_{k-2}(\sigma_i(\theta)) \, d\theta$ must be unbounded. Further, since $\delta'_{k-2}(\theta) < 0$, it is easy to see that

$$\int_{s_1}^{s} \sum_{i=1}^{\ell} h_i(\theta) \, d\theta \to \infty \text{ as } s \to \infty.$$  \hfill (14)

Integrating (1) from $s_2$ to $s$, we find

$$a(s)(v^{(k-1)}(s))' \leq a(s_2)(v^{(k-1)}(s_2))' - \int_{s_2}^{s} \sum_{i=1}^{\ell} h_i(\theta) v'(\sigma_i(\theta)) \, d\theta$$

$$\leq a(s_2)(v^{(k-1)}(s_2))' - \int_{s_2}^{s} v'(\sigma(\theta)) \sum_{i=1}^{\ell} h_i(\theta) \, d\theta$$

$$\leq a(s_2)(v^{(k-1)}(s_2))' - v'(\sigma(s_2)) \int_{s_2}^{s} \sum_{i=1}^{\ell} h_i(\theta) \, d\theta,$$

which, from (14), contradicts to the fact that $v^{(k-1)} > 0$. The proof is complete. \hfill \(\square\)

**Theorem 2.** Let $v \in C(I_0, (0, \infty))$ be a solution of (1). If

$$\limsup_{s \to \infty} \delta_{k-2}(s) \int_{s_1}^{s} \sum_{i=1}^{\ell} h_i(\theta) \, d\theta > 1,$$

then $v$ satisfies Case (2).

**Proof.** Using Lemma 1, we have that one of the cases (1)–(3) holds. First, we suppose that (3) holds on $I_1$. Then,

$$v^{(k-2)}(s) \geq - \int_{s}^{\infty} a^{-1/r}(\theta) a^{1/r}(\theta) v^{(k-1)}(\theta) \, d\theta \geq - a^{1/r}(s) v^{(k-1)}(s) \delta_0(s).$$  \hfill (16)
Integrating (16) from \( s \) to \( \infty \), we obtain
\[
v^{(x-3)}(s) \leq a^{1/r}(s)v^{(x-1)}(s)\delta_1(s).
\]
Integrating the above inequality from \( s \) to \( \infty \) a total of \( (x - 4) \) times, we find
\[
v'(s) \leq a^{1/r}(s)v^{(x-1)}(s)\delta_{x-3}(s).
\]
Integrating (17) from \( s \) to \( \infty \), we arrive at
\[
v(s) \geq -a^{1/r}(s)v^{(x-1)}(s)\delta_{x-2}(s).
\]
Integrating (1) from \( s_1 \) to \( s \), we find
\[
a(s)(v^{(x-1)}(s))^r \leq a(s_1)(v^{(x-1)}(s_1))^r - \int_{s_1}^s \sum_{i=1}^\ell h_i(\theta)v'(\sigma_i(\theta))d\theta,
\]
since \( \sigma'(s) > 0 \), and \( \theta \leq s \), we obtain
\[
a(s)(v^{(x-1)}(s))^r \leq -v'(\sigma_1(s)) \int_{s_1}^s \sum_{i=1}^\ell h_i(\theta)d\theta.
\]
Since \( \sigma(s) \leq s \), we have
\[
a(s)(v^{(x-1)}(s))^r \leq -v'(s) \int_{s_1}^s \sum_{i=1}^\ell h_i(\theta)d\theta.
\]
From (18) and (20), we arrive at
\[
a(s)(v^{(x-1)}(s))^r \leq a(s)\left(v^{(x-1)}(s)\right)^r \delta_{x-2}(s) \int_{s_1}^s \sum_{i=1}^\ell h_i(\theta)d\theta.
\]
Dividing (21) by \( a(s)(v^{(x-1)}(s))^r \) and taking the limsup, we obtain
\[
\limsup_{s \to \infty} \delta_{x-2}(s) \int_{s_1}^s \sum_{i=1}^\ell h_i(\theta)d\theta \leq 1,
\]
which is a contradiction with (15).

Next, let Case (1) holds on \( l_1 \). It follows from (15) and \( \delta_{x-2}(s) < \infty \) that (14) holds. Now, we continue as in the proof of Theorem 1. The proof is complete. \( \square \)

**Lemma 2.** Assume that \( v \) satisfies Case (2). If
\[
\int_{s_0}^\infty \left( \frac{1}{a(v)} \int_{s_1}^s \sum_{i=1}^\ell h_i(\theta) \left( \frac{\lambda}{(\kappa - 2)^r} \sigma_i^{x-2}(\theta) \right)^r d\theta \right)^{1/r} dv = \infty,
\]
then \( \lim_{s \to \infty} v^{(x-2)}(s) = 0. \)

**Proof.** Let \( v \) be a positive solution of (1) and satisfies Case (2). It follows from the facts that \( v^{(x-2)}(s) > 0 \) and \( v^{(x-1)}(s) < 0 \) that \( \lim_{s \to \infty} v^{(x-2)}(s) = c \geq 0. \) Suppose that \( c > 0. \) Hence, there is a \( s_1 \geq s_0 \) such that
\[
v^{(x-2)}(\sigma_i(s)) \geq c \text{ for } s \geq s_1.
\]
From (1), we have
\[
\left(a(s)\left(v^{(s-1)}(s)\right)^r\right)' \leq -\sum_{i=1}^{\ell} h_i(s) v'\left(\sigma_i(s)\right), \tag{24}
\]
from [20] (Lemma 2.2.3), we see that
\[
v(s) \geq \frac{\lambda}{(k-2)!} s^{k-2} v(k-2)(s), \tag{25}
\]
using (24) and (25) becomes
\[
\left(a(s)\left(v^{(s-1)}(s)\right)^r\right)' \leq -\sum_{i=1}^{\ell} h_i(s) \left(\frac{\lambda}{(k-2)!} s^{k-2} \sigma_i(s)\right)^r \left(v(k-2)(\sigma_i(s))\right)' , \tag{26}
\]
from (23), we obtain
\[
\left(a(s)\left(v^{(s-1)}(s)\right)^r\right)' \leq -c^r \sum_{i=1}^{\ell} h_i(s) \left(\frac{\lambda}{(k-2)!} s^{k-2} \sigma_i(s)\right)^r , \tag{27}
\]
for \(s \geq s_1\). Integrating (27) twice from \(s_1\) to \(s\), we obtain
\[
v^{(s-1)}(s) \leq -c \left(\int_{s_1}^{s} \frac{1}{a(s)} \int_{s_1}^{s} h_i(\theta) \left(\frac{\lambda}{(k-2)!} s^{k-2} \sigma_i(\theta)\right)^r d\theta\right)^{1/r}
\]
and
\[
v^{(k-2)}(s) \leq v^{(k-2)}(s_1) - c \int_{s_1}^{s} \left(\frac{1}{a(v)} \int_{s_1}^{v} h_i(\theta) \left(\frac{\lambda}{(k-2)!} s^{k-2} \sigma_i(\theta)\right)^r d\theta\right)^{1/r} dv.
\]
Letting \(s \to \infty\) and using (22), we get that \(\lim_{s \to \infty} v^{(k-2)}(s) = -\infty\), which contradicts \(v^{(k-2)}(s) > 0\). Thus, the proof is complete. \(\square\)

**Lemma 3.** Assume that (22) holds, \(v(s) \in C([0, 0, \infty))\) is a solution of (1) and satisfies Case (2). If there exists a constant \(\mu \geq 0\) such that
\[
\frac{\delta(s)}{a^{1/3}(s)\delta^2(s)} \left(\int_{s_1}^{s} \sum_{i=1}^{\ell} G_i(\theta)d\theta\right)^{1/r} \geq \mu, \tag{28}
\]
for some \(\lambda \in (0, 1)\), then
\[
\left(\frac{v^{(k-2)}(s)}{\delta^2(s)}\right)' \leq 0, \tag{29}
\]
where
\[
G_i(\theta) = h_i(\theta) \left(\frac{\lambda s^{k-2}(\theta)}{(k-2)!}\right)^r.
\]
**Proof.** Proceeding as in the proof of Lemma 2, we obtain that (26) holds. Integrating (26) from \(s_1\) to \(s\), we find
\[
a(s)\left(v^{(s-1)}(s)\right)' - a(s_1)\left(v^{(s-1)}(s_1)\right)' \leq -\int_{s_1}^{s} \sum_{i=1}^{\ell} G_i(\theta) \left(v^{(k-2)}(\sigma_i(\theta))\right)' d\theta,
\]
since \( \sigma'(s) > 0 \), and \( \theta \leq s \), we obtain

\[
a(s) \left( v^{(k-1)}(s) \right)^r - a(s_1) \left( v^{(k-1)}(s_1) \right)^r \leq - \left( v^{(k-2)}(\sigma_i(s)) \right)^r \int_{s_1}^s \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta,
\]

and so

\[
a(s) \left( v^{(k-1)}(s) \right)^r \leq a(s_1) \left( v^{(k-1)}(s_1) \right)^r - \left( v^{(k-2)}(\sigma_i(s)) \right)^r \int_{s_0}^s \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta
\]

\[
+ \left( v^{(k-2)}(\sigma_i(s)) \right)^r \int_{s_0}^{s_1} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta.
\]

Using Lemma 2, we obtain that \( \lim_{s \to \infty} v^{(k-2)}(s) = 0 \). Thus, there is a \( s_2 \geq s_1 \) such that

\[
a(s_1) \left( v^{(k-1)}(s_1) \right)^r + \left( v^{(k-2)}(\sigma_i(s)) \right)^r \int_{s_0}^{s_2} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta < 0, \text{ for every } s \geq s_2,
\]

thus (30) becomes

\[
a(s) \left( v^{(k-1)}(s) \right)^r \leq - \left( v^{(k-2)}(\sigma_i(s)) \right)^r \int_{s_0}^{s_2} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta
\]

\[
\leq - \left( v^{(k-2)}(s) \right)^r \int_{s_0}^{s_2} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta,
\]

and so

\[
v^{(k-1)}(s) \leq - \frac{v^{(k-2)}(s)}{\delta_i^0(s)} \left( \int_{s_0}^{s_2} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta \right)^{1/r}.
\]

Next, we have that

\[
\left( \frac{v^{(k-2)}(s)}{\delta_i^0(s)} \right)^r = \frac{\delta_i^{k-2}(s) v^{(k-1)}(s) + \mu \delta_i^{k-3}(s) v^{(k-2)}(s)}{\delta_i^{k-2}(s)}.
\]

This implies

\[
\delta_i^{k-2}(s) v^{(k-1)}(s) + \mu \delta_i^{k-3}(s) v^{(k-2)}(s)
\]

\[
\leq - \delta_i^{k-2}(s) \frac{v^{(k-2)}(s)}{\delta_i^0(s)} \left( \int_{s_0}^{s_2} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta \right)^{1/r} + \mu \delta_i^{k-3}(s) v^{(k-2)}(s)
\]

\[
\leq \left( \frac{\delta_i^{k-2}(s)}{\delta_i^0(s)} \right) \left( \int_{s_0}^{s_2} \frac{\ell}{\delta_i^0(s)} G_i(\theta) d\theta \right)^{1/r} + \mu \delta_i^{k-3}(s) v^{(k-2)}(s).
\]

It follows from (28) that \( \delta_i^{k-3}(s) v^{(k-1)}(s) + \mu \delta_i^{k-2}(s) v^{(k-2)}(s) \leq 0 \), which, with (32), implies the function \( v^{(k-2)}(s) / \delta_i^{k-2}(s) \) is nonincreasing. This completes the proof. \( \square \)

**Theorem 3.** Assume that (22) and (28) hold, \( v(s) \in C(I_0, (0, \infty)) \) is a solution of (1) and \( r \geq 1 \). If there exists a positive function \( \rho(s) \in C^1([s_0, \infty)) \) such that

\[
\limsup_{s \to \infty} \int_{s_0}^s W(\theta) - \frac{a(\theta) \rho(\theta)}{(r + 1)(r+1)} \left( \frac{\rho'(\theta)}{\rho(\theta)} + \frac{1 + r}{a^{1/r}(\theta) \delta(\theta)} \right)^{r+1} d\theta = \infty,
\]

(33)
where
\[
W(s) := \rho(s) \sum_{i=1}^\ell h_i(s) \left( \frac{\lambda}{(k-2)!} \sigma_{k-2}^r(s) \right) \delta_{k-2}^\mu (\tau_i(s)) + (1 - r) \frac{\rho(s)}{a^{1/r}(s)} \delta^{r+1}(s),
\]
for some \( \lambda \in (0, 1) \), then Case 2 does not satisfied.

**Proof.** Assume the contrary that (1) has a positive solution \( v(s) \) and satisfies Case (2). Noting that \( a(s)(v^{(k-1)}(s))^r \) is non-increasing, we have
\[
v^{(k-2)}(s) - v^{(k-2)}(s) = \int_s^v \frac{1}{a^{1/r}(\theta)} \left( a(\theta) \left( v^{(k-1)}(\theta) \right) \right)^r d\theta 
\leq a^{1/r}(s)v^{(k-1)}(s) \int_s^v \frac{1}{a^{1/r}(\theta)} d\theta.
\]
Letting \( v \to \infty \), we obtain
\[
-v^{(k-2)}(s) \leq a^{1/r}(s)v^{(k-1)}(s)\delta(s). 
\tag{34}
\]
Define the function \( \omega(s) \) by
\[
\omega(s) := \rho(s) \left( a(s)v^{(k-1)}(s)^r \right) \left( v^{(k-2)}(s)^r \right) \delta(s).
\tag{35}
\]
From (34), we have \( \omega(s) > 0 \) for \( s \geq s_1 \). Differentiating (35), we obtain
\[
\omega'(s) = \frac{\rho'(s)}{\rho(s)} \omega(s) + \rho(s) \left( \frac{a(s)v^{(k-1)}(s)^r}{v^{(k-2)}(s)^r} - ra(s)v^{(k-1)}(s)^{r+1} \right) \left( \omega(s) - \frac{1}{\delta(s)} \right)^{(r+1)/r}
\]
which follows from (1) and (35) that
\[
\omega'(s) \leq \frac{\rho'(s)}{\rho(s)} \omega(s) - \frac{\rho(s)}{v^{(k-2)}(s)^r} \sum_{i=1}^\ell h_i(s) v^r(\sigma_i(s)) \left( \omega(s) - \frac{1}{\delta(s)} \right)^{(r+1)/r}
+ \frac{r\rho(s)}{a^{1/r}(s)} \delta^{r+1}(s).
\tag{36}
\]
From (25) and (36), we have
\[
\omega'(s) \leq \frac{\rho'(s)}{\rho(s)} \omega(s) - \frac{\rho(s)}{v^{(k-2)}(s)^r} \sum_{i=1}^\ell h_i(s) \left( \frac{\lambda}{(k-2)!} \sigma_{k-2}^r(s) \right) \left( v^{(k-2)}(\tau_i(s)) \right)^r
\]
\[
- \frac{r\rho(s)}{a^{1/r}(s)} \left( \omega(s) - \frac{1}{\delta(s)} \right)^{(r+1)/r} + \frac{r\rho(s)}{a^{1/r}(s)} \delta^{r+1}(s),
\]
using (29), we obtain
\[
\omega'(s) \leq \frac{\rho'(s)}{\rho(s)} \omega(s) - \frac{\rho(s)}{v^{(k-2)}(s)^r} \sum_{i=1}^\ell h_i(s) \left( \frac{\lambda}{(k-2)!} \sigma_{k-2}^r(s) \right) \left( v^{(k-2)}(\tau_i(s)) \right)^r \delta_{k-2}^\mu (\sigma_i(s))
\]
\[
- \frac{r\rho(s)}{a^{1/r}(s)} \left( \omega(s) - \frac{1}{\delta(s)} \right)^{(r+1)/r} + \frac{r\rho(s)}{a^{1/r}(s)} \delta^{r+1}(s).
that is
\[
\omega'(s) \leq \frac{\rho'(s)}{\rho(s)} \omega(s) - \rho(s) \sum_{i=1}^{\ell} h_i(s) \left( \frac{\lambda}{(k-2)!} \sigma_i^{k-2}(s) \right) \frac{\delta_{k-2}^{\mu}(\sigma_i(s))}{\delta_{k-2}^{\mu}(s)} \\
- \frac{\rho(s)}{a^{1/r}(s)} \left( \frac{\omega(s)}{\rho(s)} - \frac{1}{\delta'(s)} \right)^{(r+1)/r} + \frac{\rho(s)}{a^{1/r}(s) \delta(r+1)}. \tag{37}
\]

Using the inequality
\[
\Omega^{(r+1)/r}_1 - (\Omega_1 - \Omega_2)^{(r+1)/r} \leq \frac{\Omega^{1/r}_1}{r} [(1 + r) \Omega_1 - \Omega_2], \quad \Omega_1 \Omega_2 \geq 0,
\]
with \( \Omega_1 = \omega/\rho, \Omega_2 = 1/\delta' \), we obtain
\[
\omega'(s) \leq \left( \frac{\rho'(s)}{\rho(s)} + \frac{1 + r}{a^{1/r}(s) \delta(s)} \right) \omega(s) - \rho(s) \sum_{i=1}^{\ell} h_i(s) \left( \frac{\lambda}{(k-2)!} \sigma_i^{k-2}(s) \right) \frac{\delta_{k-2}^{\mu}(\sigma_i(s))}{\delta_{k-2}^{\mu}(s)} \\
- \frac{r}{a^{1/r}(s)^{1/r}(s) \rho^{1/r}(s) \omega^{(r+1)/r}(s)} - \frac{\rho(s)}{a^{1/r}(s) \delta(r+1)} + \frac{\rho(s)}{a^{1/r}(s) \delta(r+1)}.
\]

By using the inequality
\[
v_y - V_y^{(r+1)/r} \leq \frac{r^r}{(r+1)^{(r+1)/r}} V^{r+1}, \quad V > 0,
\]
with \( v = \rho'/\rho + (1 + r) / \left( a^{1/r} \delta' \right), \quad V = r / \left( a^{1/r} \rho^{1/r} \right) \) and \( y = \omega \), we find
\[
\omega'(s) \leq -\rho(s) \sum_{i=1}^{\ell} h_i(s) \left( \frac{\lambda}{(k-2)!} \sigma_i^{k-2}(s) \right) \frac{\delta_{k-2}^{\mu}(\sigma_i(s))}{\delta_{k-2}^{\mu}(s)} + (r - 1) \frac{\rho(s)}{a^{1/r}(s) \delta(r+1)} \\
+ \frac{a(s) \rho(s)}{(r+1)^{(r+1)}} \left( \frac{\rho'(s)}{\rho(s)} + \frac{1 + r}{a^{1/r}(\theta) \delta(\theta)} \right)^{(r+1)}.
\]

Integrating this inequality from \( s_1 \) to \( s \), we find
\[
\int_{s_1}^{s} \left( \int_{s_1}^{\theta} W(\theta) - \frac{a(\theta) \rho(\theta)}{(r+1)^{(r+1)}} \left( \frac{\rho'(\theta)}{\rho(\theta)} + \frac{1 + r}{a^{1/r}(\theta) \delta(\theta)} \right)^{(r+1)} \right) d\theta \leq \omega(s_1),
\]
which contradicts (33). This completes the proof. \( \square \)

**Theorem 4.** Suppose that \( r \geq 1, (6), (22) \) and (28) hold. If there is a function \( \rho \in C^1([s_0, \infty), (0, \infty)) \) such that (33) holds, then (1) is oscillatory.

**Proof.** We conclude from assuming the contrary that there exists a \( s_1 \in [s_0, \infty) \) such that \( v(s) > 0 \) and \( v(\sigma_i(s)) > 0 \) for \( s \geq s_1 \). From Lemma 1, we have three possible cases (1)–(3). It follows from Theorem 1 that the (6) ensure that \( v \) satisfies Case (2). However, from Theorem 3, we have that (33) contrasts with Case (2). Then, the proof is complete. \( \square \)
Theorem 5. Suppose that \( r \geq 1, (15), (22) \) and (28) hold. If there is a function \( \rho \in C^1([s_0, \infty), (0, \infty)) \) such that (33) holds, then (1) is oscillatory.

Proof. The proof is quite similar to the proof of Theorem 4, so it has been omitted. \( \Box \)

Example 1. Consider the DDE

\[
(s^6v'''(s))' + h_1 s^2 v\left(\frac{s}{2}\right) + h_2 s^2 v\left(\frac{s}{3}\right) = 0,
\]

where \( h_1 \) and \( h_2 > 0 \). We note that \( r = 1, a(s) = s^6, \varphi_1(s) = s/2, \) and \( \varphi_2(s) = s/3 \). Hence, it is easy to see that

\[
\delta(\theta) = \frac{1}{50\theta^5}, \quad \delta_1(\theta) = \frac{1}{20\theta^4} \quad \text{and} \quad \delta_2(\theta) = \frac{1}{60\theta^3}.
\]

If we choose \( \rho(\theta) = 1/\theta^5 \) and

\[
\mu = \lambda_1 \left( \frac{h_1}{120} + \frac{h_2}{270} \right),
\]

then (6), (22) and (28) are satisfied, and

\[
W(\theta) = \frac{1}{\theta^5} \left( h_1 \theta^2 \left( \frac{\lambda_2 \theta^2}{8} \right) \left( 2^3 \right) \lambda_1 \left( \frac{h_1}{120} + \frac{h_2}{270} \right) + h_2 \theta^2 \left( \frac{\lambda_2 \theta^2}{18} \right) \left( 3^3 \right) \lambda_1 \left( \frac{h_1}{120} + \frac{h_2}{270} \right) \right).
\]

Now, condition (33) is satisfied if

\[
h_1 \left( \frac{\lambda_2}{2} \right) \left( 2^3 \right) \lambda_1 \left( \frac{h_1}{120} + \frac{h_2}{270} \right) + h_2 \left( \frac{\lambda_2}{18} \right) \left( 3^3 \right) \lambda_1 \left( \frac{h_1}{120} + \frac{h_2}{270} \right) > \frac{25}{4},
\]

for \( \lambda_i \in (0, 1) \). Therefore, from Theorem 4, Equation (38) is oscillatory if (39) holds.

Example 2. Consider the DDE

\[
(s^4v'''(s))' + \sum_{i=1}^{\ell} h_i v(\beta_is) = 0,
\]

where \( h_i > 0 \) and \( \beta_i \in (0, 1) \). We note that \( r = 1, a(s) = s^4, h_i(s) = h_i \), and \( \varphi_i(s) = \beta_i s, \) where \( i = 1, 2, \ldots, \ell \). Hence, it is easy to see that

\[
\delta(\theta) = \frac{1}{3\theta^3}, \quad \delta_1(\theta) = \frac{1}{6\theta^2} \quad \text{and} \quad \delta_2(\theta) = \frac{1}{6\theta}.
\]

If we choose \( \rho(\theta) = 1/\theta^3 \) and

\[
\mu = \frac{1}{3} \sum_{i=1}^{\ell} h_i \frac{\lambda}{21} \beta_i^2,
\]

then, condition (15) and (22) become

\[
\int_{s_0}^{\infty} \left( \frac{1}{\theta^4} \int_{s_0}^{\theta} \sum_{i=1}^{\ell} h_i \frac{\lambda}{21} \beta_i^2 \theta^2 d\theta \right) d\theta = \infty,
\]

and

\[
\limsup_{s \to \infty} \frac{1}{6s} \int_{s_0}^{s} \sum_{i=1}^{\ell} h_i d\theta > 1,
\]
respectively. Thus, condition (41) holds if

\[ \sum_{i=1}^{\ell} h_i > 6. \]  

(42)

Now, we have

\[ W(\theta) = \frac{1}{\theta^3} \sum_{i=1}^{\ell} h_i \left( \frac{\lambda}{2! \beta_i^2} \theta^2 \right) \left( \frac{1}{\beta_i^3} \right)^{\frac{1}{2} \sum_{j=1}^{\ell} h_j \frac{1}{2} \beta_i^2}. \]

Therefore, condition (33) is satisfied if

\[ \sum_{i=1}^{\ell} h_i \left( \frac{\lambda}{2! \beta_i^2} \right) \left( \frac{1}{\beta_i^3} \right)^{\frac{1}{2} \sum_{j=1}^{\ell} h_j \frac{1}{2} \beta_i^2} > \left( \frac{3}{4} \right)^2, \]  

(43)

for some \( \lambda \in (0, 1) \). Hence, using Theorem 5, Equation (40) is oscillatory if (42) and (43) hold.

**Remark 1.** Consider the special case of (40) when \( \ell = 1 \), namely (of Euler type)

\[ (s^4 v''''(s))' + h_1 v(s/2) = 0, \]  

(44)

Note that conditions (42) and (43) reduce to \( h_1 > 6 \) and \( h_1 > 12.533 \), respectively. Then, Equation (44) is oscillatory if \( h_1 > 12.533 \).

On the other hand, using Corollary 2.1 in [18] or Corollary 2.4 in [13], Equation (44) is oscillatory if \( h_1 > 18 \). Therefore, our results provide a better criterion for oscillation, as it guarantees the oscillation of Equation (44) if \( h_1 \in (12.533, 18] \), while the relevant results fail.

3. Conclusions

In this paper, we aim to establish new oscillation criteria for DDE (1). We start by excluding two possible cases of positive solutions using one condition, as in Theorems 1 and 2. The development of the study of the oscillatory behavior of solutions of DDEs depends on the development of the inequalities and relationships used in the study. Proceeding from this, we used the condition (28) to obtain a new monotonical property of \( v(\kappa - 2) \), which in turn contributes to finding a better estimate of the ratio \( (v(\kappa - 2) \circ \sigma_1) / v(\kappa - 2) \). This estimate plays a role in improving the criterion that ensures that no solutions meet case (2) in Lemma 1. By combining the conditions for excluding all possible cases of positive solutions, we obtain a new oscillation criterion.

Author Contributions: Conceptualization, B.A. and A.M.; formal analysis, O.M. and M.A.; Investigation, C.C. and B.A.; methodology, C.C. and A.M.; writing—original draft preparation, O.M.; writing—review and editing, B.A., A.M. and M.A. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number PNNURSP2022R216, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This research was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project number PNNURSP2022R216, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.
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