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Revisiting Additive Consistency of Hesitant Fuzzy Linguistic Preference Relations

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Abstract: Consistency has always been a hot topic in the study of decision-making based on preference relations. This paper focuses on the consistency of hesitant fuzzy linguistic preference relations (HFLPRs). Firstly, a new definition of the additive consistency of HFLPRs is given. Secondly, to examine whether an HFLPR is additively consistent, two equivalent programming models are constructed. Thirdly, for inconsistent HFLPRs, the corresponding consistency improvement model is further proposed, where only upper triangular elements in the HFLPRs are considered in view of the symmetry of HFLPRs. Using the consistency improvement model, an inconsistent HFLPR can be adjusted to the consistent one, which retains the original information as much as possible. Fourthly, a hesitant fuzzy linguistic weight vector is introduced and a programming model is constructed to derive the weight vector. Finally, the feasibility and effectiveness of the proposed method are illustrated by numerical examples and comparative analysis. This result demonstrates that the consistency model proposed considers each element of HFLPRs such that the consistent HFLPRs derived fully retain the original information. Moreover, only some preference values in the HFLPR are adjusted, and no preference value is out of range of the predefined HFLTSs.

Keywords: hesitant fuzzy linguistic preference relation; additive consistency; hesitant fuzzy linguistic weight vector; hesitant fuzzy linguistic term set

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1. Introduction

Torra [1] proposed the concept of hesitant fuzzy sets (HFSs), which allows several values between [0, 1] to indicate its membership. In reality, decision-makers (DMs) may show hesitant preference for alternatives on account of various factors in the decision-making process. Hence, HFS has been widely used as a tool to express DMs' hesitation. Due to the complexity of real decision-making situations, DMs may have difficulty in assigning appropriate numerical values to express their preferences. In this case, DMs tend to use linguistic terms rather than numerical values to express preferences. Due to the above considerations, Rodriguez et al. [2] proposed the hesitant fuzzy linguistic term sets (HFLTSs), which allow DMs to express their preferences by using several possible linguistic variables. Therefore, with the advantages of both HFSs and fuzzy linguistic sets, HFLTSs enable DMs to express their preferences more flexibly and conveniently in decision-making.

Based on HFLTSs, Rodriguez et al. [2] developed the concept of hesitant fuzzy linguistic preference relations (HFLPRs) as a tool to deal with DMs' hesitant degree of preference for several possible linguistic terms over the paired of alternatives. Recently, more and more research [3–14] on HFLPRs has been performed. Dong et al. [3] proposed a new distance formula to measure the distance between two HFLTSs and developed the consensus level measure of HFLPRs. Song et al. [4] proposed a definition of multiplicative consistency of HFLPRs. Tang and Meng [5] introduced some definitions of multiplicative hesitant fuzzy linguistic preference relations (MHFLPRs) and corresponding consistency definitions. Wu et al. [6] proposed a new formula to measure the similarity between HFLTSs and a consensus improvement process based on the local modification mechanism.

Tang et al. [7] proposed a new definition of interval linguistic hesitant fuzzy preference relations (ILHFPRs) and a concept of additive consistency, and then, Tang et al. [8] proposed a new definition of multiplicative interval linguistic hesitant fuzzy preference relations (MILHFPRs) and discussed the consistency issue. Chen et al. [9] considered the worst consistency (WCI) of HFLPRs and constructed two models to improve the WCI index and consensus level. Zhang and Chen [10] proposed a group decision-making (GDM) method based on the acceptable multiplicative consistency and consensus of HFLPRs. Ren et al. [11] proposed a kernel-based algorithm and a consensus measure for HFLPRs. Zheng et al. [12] proposed hesitant degree and fuzzy degree function of HTLTSs and established a model to normalize different lengths. Xu et al. [13] proposed a new AHP method and some models to improve the consistency and consensus levels. Li et al. [14] proposed a model to obtain the incomplete elements for an IHFLPR and an iterative algorithm to reach consensus.

Consistency is regarded as an important indicator for DMs to avoid illogic when comparing alternatives. Based on cardinal consistency of preference relations, including addition consistency [15] and multiplication consistency [16], researchers have proposed different methods to define the consistency of HFLPRs. Both additive consistency and multiplicative consistency of HFLPRs have been widely discussed [17–26]. Ren et al. [17] proposed a GDM method based on consistency and consensus measurement of HFLPRs, and proposed a hesitant fuzzy linguistic geometric consistency index (HFLGCI) and a worst consensus index of HFLPRs. Xu et al. [18] proposed two additive consistency definitions for HFLPRs: completely additive consistency (CAC) and weakly additive consistency (WAC). Zhang and Wu [19] developed the multiplicative consistency of HFLPRs and defined a consistency indicator to measure the degree of deviation between the original and the consistent HFLPRs. Zhu and Xu [20] introduced an additive consistency concept of HFLPRs and developed some consistency and acceptable consistency measures for HFLPRs. Xu and Wang [21] proposed the additive consistency of hesitant 2-tuple fuzzy linguistic preference relations (H2TFLPRs) and proposed revised definition of H2TFLPRs based on Zhu and Xu [20]. Feng et al. [22] proposed an additively consistent definition of HFLPRs and developed goal programming models to measure consistency. Li et al. [23] proposed an interval consistency index of HFLPRs, which consists of the worst consistency index and the best consistency index of HFLPRs. Zhang and Chen [24] proposed a method to solve weight vector of HFLPRs, based on which the consistency index and acceptable definition of multiplicative consistency of HFLPR are defined. Liu et al. [25] established a new model to make HFLPRs achieve the maximum consistency degree. In addition, Liu et al. [26] calculated the missing elements of incomplete HFLPRs according to the best and worst consistency.

In the existing studies, there are still many issues deserving further discussion and improvement. The consistency definitions [22,26] only take part of the original HFLPR information into account and tend to be too loose. In the consistency optimization models [19–21], the original preference relations need to be normalized, where the HFLTSs of HFLPR are further artificially processed to have the same length by adding or deleting some specified linguistic variables. Such process distorts the original preference information, and some preference values in the adjusted preference relations may not belong to the original linguistic term set. In consistency optimization models [19–21,24], almost all the preference values in the original preference relation are adjusted, which may not be accepted by corresponding DMs, and cannot keep the original preference information as much as possible. In addition, in the additive consistency model [20,21], the preference values obtained by additive consistency may be out of range of the predefined HFLTSs.

In order to better deal with the above-mentioned issues, this paper proposes a new consistency concept and corresponding consistency improvement models for HFLPRs. The main novelties of this paper are listed as follows:

(1) A new additive consistency definition of HFLPRs is introduced. This consistency definition takes each linguistic term of HFLPR into account. As a result, the consistency definition can fully reflect consistent information of original HFLPRs.

(2) To judge if an HFLPR is additively consistent, some programming models are developed to measure the consistency of the HFLPR. The consistency test method does not add or remove any value of the HFLTS, which avoids distorting the original preference information of the HFLPR.
(3) For an HFLPR not satisfying additive consistency, a consistency improvement model is developed to derive the corresponding consistent HFLPR, where only some preference values in the HFLPR are adjusted and no preference value is out of range of the predefined HFLTS. Moreover, the deviation between the corresponding consistent HFLPR and the original one is minimal, which ensures the new constructed additively consistent HFLPR keeps the original information as much as possible.
(4) To determine the ranking order of the alternatives, hesitant fuzzy linguistic weight vector (HFLWV) is defined and the corresponding model is established to derive the HFLWV.

The remainder of this paper is organized as follows. Section 2 introduces some fundamental definitions. In Section 3, new concept of additive consistency of HFLPR is introduced. Then, two programming models are developed to measure the consistency of an HFLPR. For inconsistent HFLPRs, a consistency improvement model is constructed to adjust the inconsistent HFLPRs. In addition, a programming model is constructed to derive the priority weights for additively consistent HFLPRs. In Section 4, some numerical examples and comparisons with the existing methods are presented to illustrate the effectiveness of the proposed method. Conclusions are given in Section 5.

2. Preliminaries

In this section, the relevant basic definitions and some consistency definitions for HFLPRs are reviewed.

For decision-making problems under linguistic environment, a linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$ is always utilized, where g+*1* is named the cardinality of S.

2.1. Basic Definitions

Definition 1 [27]. *Let $\beta \in [0, g]$ be a value derived from the result of a symbolic aggregation operation in $S = \{s_0, s_1, \ldots, s_g\}$. The equivalent information for β in the 2-tuple is obtained by the function as follows:*

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5) \tag{1}$$

$$\Delta(\beta) = \begin{cases} \mathrm{s}_i & i = round(\beta) \\ \alpha = \beta - i\alpha \in [-0.5, 0.5) \end{cases} \tag{2}$$

where round denotes the rounding operation.

Definition 2 [27]. *Suppose $S = \{s_0, s_1, \ldots, s_g\}$ is a linguistic term set and (s_i, α) is a 2-tuple. There always exists the following function Δ^{-1} which can transform a 2-tuple into its equivalent numerical value $\gamma \in [0, g]$. The function is defined as follows:*

$$\Delta^{-1} : S \times [-0.5, 0.5) \rightarrow [0, g] \tag{3}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha \tag{4}$$

In addition, let (s_i, α) and (s_j, γ) be 2-tuples, then:

(1) If $i < j$, then (s_i, α) is smaller than (s_j, γ).
(2) If $i = j$, then:
    (a) If $\alpha = \gamma$, then (s_i, α) and (s_j, γ) represent the same information.
    (b) If $\alpha < \gamma$, then (s_i, α) is smaller than (s_j, γ).

Definition 3 [28]. *Suppose* $S = \{s_0, s_1, \ldots, s_g\}$ *is a linguistic term set, where* $g + 1$ *is odd. An HFLTS,* $H_s = \left\{ s_{\sigma(l)} \middle| s_{\sigma(l)} \in S, l = 1, 2, \ldots, \#H_s \right\}$, *is an ordered finite subset of consecutive linguistic terms of* S, *where* $s_{\sigma(l)}$ *is the lth linguistic term of* H_s *and* $\#H_s$ *is the number of linguistic terms of* H_s.

Definition 4 [2]. *For HFLTS* H_s, *there are the following operations:*

(1) *Lower bound:* $H_s^- = \min(s_i), \forall s_i \in H_s$.
(2) *Upper bound:* $H_s^+ = \max(s_i), \forall s_i \in H_s$.

To rank HFLTSs, Liu and Jiang [29] defined the following score function:

$$Score(H_s) = \frac{1}{\#H_s} \sum_{l=1}^{\#H_s} \Delta^{-1}\left(s_{\sigma(l)}\right) \tag{5}$$

When two HFLTSs have the same score function value, their degree of accuracy can be compared by the following precision function.

$$H(H_s) = \Delta^{-1}(H_s^-) - \Delta^{-1}(H_s^+) + g \tag{6}$$

The larger the value of $H(H_s)$, the better performance it has.

Definition 5 [3]. *Suppose* $S = \{s_0, s_1, \ldots, s_g\}$ *is a linguistic term set. For two HFLTSs* H_s^1 *and* H_s^2, *the distance between* H_s^1 *and* H_s^2 *can be measured by*

$$D\left(H_s^1, H_s^2\right) = \left|\Delta^{-1}\left(H_s^{1-}\right) - \Delta^{-1}\left(H_s^{2-}\right)\right| + \left|\Delta^{-1}\left(H_s^{1+}\right) - \Delta^{-1}\left(H_s^{2+}\right)\right| \tag{7}$$

where $H_s^{1-}\left(H_s^{2-}\right)$ *and* $H_s^{1+}\left(H_s^{2+}\right)$ *denote the lower and upper bounds of* $H_s^1\left(H_s^2\right)$, *respectively.*

Definition 6 [30]. *A fuzzy linguistic preference relation (FLPR) is defined as* $A = (a_{ij})_{n \times n}$, *where* $a_{ij} \in S$ *denotes the preference degree of alternative* x_i *to* x_j, *which satisfies*

$$\Delta^{-1}(a_{ij}) + \Delta^{-1}(a_{ji}) = g, i, j = 1, 2, \ldots, n \tag{8}$$

$$a_{ii} = s_{g/2}, i = 1, 2, \ldots, n \tag{9}$$

Definition 7 [2]. *Suppose* $S = \{s_0, s_1, \ldots, s_g\}$ *is a linguistic term set. An HFLPR is expressed as* $H = (h_{ij})_{n \times n}$, *where* $h_{ij} \in S$ *represents the preference degree of alternative* x_i *to* x_j, *if the following conditions are satisfied:*

$$\Delta\left(\Delta^{-1}\left(h_{ij}^{\sigma(r)}\right) + \Delta^{-1}\left(h_{ji}^{\sigma(r)}\right)\right) = s_g, i, j = 1, 2, \ldots, n \tag{10}$$

$$h_{ii} = s_{g/2}, i = 1, 2, \ldots, n \tag{11}$$

$$\#h_{ij} = \#h_{ji}, i, j = 1, 2, \ldots, n \tag{12}$$

$$h_{ij}^{\sigma(r)} < h_{ij}^{\sigma(r+1)}, h_{ji}^{\sigma(r)} < h_{ji}^{\sigma(r+1)}, i, j = 1, 2, \ldots, n \tag{13}$$

where $h_{ij} = \left\{h_{ij}^r \middle| r = 1, 2, \ldots, \#h_{ij}\right\}$ *(*$\#h_{ij}$ *is the number of linguistic terms in* h_{ij}*),* $h_{ij}^{\sigma(r)}$ *is the rth linguistic term in* h_{ij}.

2.2. Some Consistency Definitions

Definition 8 [30]. *An FLPR* $A = (a_{ij})_{n \times n}$ *is additively consistent if*

$$\Delta^{-1}(a_{ij}) = \Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) - \frac{g}{2}, i, j, k = 1, \ldots, n \tag{14}$$

According to the additive consistency definition of FLPRs [30], Feng et al. [22] proposed an additively consistency definition of HFLPRs.

Definition 9 [22]**.** *An HFLPR $H = \left(h_{ij}\right)_{n\times n}$ is consistent if there exists a consistent FLPR $A = \left(a_{ij}\right)_{n\times n}$ with $a_{ij} \in h_{ij}$, for $i, j = 1, 2, \ldots, n$.*

As the sums of elements are not necessarily the same in HFLTSs of HFLPRs, Zhu and Xu [20] made each HFLTSs have the same length by adding or reducing some specified linguistic variables, and defined such HFLPRs as normalized HFPRs (NHFLPRs).

Xu and Wang [21] proposed the definition of additively consistency of HFLPR based on the additively consistency definition of Zhu and Xu [20].

Definition 10 [21]**.** *Given an HFLPR $H = \left(h_{ij}\right)_{n\times n}$ and its NHFLPR $\overline{H} = \left(\overline{h}_{ij}\right)_{n\times n}$, if*

$$\Delta^{-1}\left(\overline{h_{ij}^{(r)}}\right) + \Delta^{-1}\left(s_{\frac{g}{2}}\right) = \Delta^{-1}\left(\overline{h_{ik}^{(r)}}\right) + \Delta^{-1}\left(\overline{h_{kj}^{(r)}}\right)(i, j, k = 1, 2, \ldots, n) \tag{15}$$

then $\overline{H}$ is a consistent NHFLPR.

Furthermore, Xu and Wang [21] proposed a method to derive the corresponding additively consistency HFLPR for an original inconsistent one.

Definition 11 [21]**.** *Assume an HFLPR $H = \left(h_{ij}\right)_{n\times n}$ and its additively consistency NHFLPR $\overline{R} = \left(\overline{r}_{ij}\right)_{n\times n}$, if*

$$\overline{r_{ij}^{(r)}} = \left(\frac{1}{n}\sum_{k=1}^{n}\left(\Delta^{-1}\left(\overline{h_{ik}^{(r)}}\right) + \Delta^{-1}\left(\overline{h_{kj}^{(r)}}\right)\right) - \frac{g}{2}\right)(i, j, k = 1, 2, \ldots, n) \tag{16}$$

then $\overline{R} = \left(\overline{r}_{ij}\right)_{n\times n}$ is a consistent NHFLPR.

3. Models on Additive Consistency of HFLPRs

In this section, new additive consistency definitions of HFLPRs are introduced, together with the corresponding consistency improvement method. For additively consistent HFLPRs, a programming model is proposed to obtain the priority weights.

3.1. New Additive Consistency Definition of HFLPRs

New consistency definition takes all linguistic elements into account and requires linguistic elements at any position to satisfy additively consistency condition.

Definition 12. *An HFLPR $H = \left(h_{ij}\right)_{n\times n}$ with $h_{ij} = \left\{h_{ij}^r | r = 1, 2, \ldots, \# h_{ij}\right\}$ ($\# h_{ij}$ is the number of linguistic terms in h_{ij}) is called an additively consistent HFLPR, if for $\forall h_{ij}^r \in h_{ij}$, $\exists a_{ik} \in h_{ik}$, $\exists a_{kj} \in h_{kj}$ such that*

$$\Delta^{-1}\left(h_{ij}^r\right) = \Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) - \frac{g}{2}, i, j, k = 1, 2, \ldots, n \tag{17}$$

where h_{ij}^r is the rth linguistic terms of h_{ij}.

Remark 1. *It is easy to know that **Definition 12** does not change with the different ordering of comparison objects in the HFLPRs.*

Based on Definition 12, the following programming model can be constructed to judge if an HFLPR $H = \left(h_{ij}\right)_{n\times n}$ is additively consistent:

$$(M-1) \quad \min F = \sum_{i,j=1}^{n}\sum_{r=1}^{\# h_{ij}}\left|\Delta^{-1}\left(h_{ij}^r\right) - \left(\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) - \frac{g}{2}\right)\right|$$

s.t

$$\begin{cases} h_{ij}^r \in h_{ij}, a_{ik} \in h_{ik}, a_{kj} \in h_{kj} \\ r = 1, 2, \ldots, \# h_{ij} \\ i, j, k = 1, 2, \ldots, n \end{cases}$$

$H = \left(h_{ij}\right)_{n \times n}$ is consistent according to Definition 12 if the target function value of Model 1 is equal to 0. Otherwise, it is not consistent.

Since all the elements in HFLTSs are taken into account in Definition 12, M-1 is always fairly complicated. For simplicity, another simpler definition on additive consistency of HFLPRs is introduced, which only considers the boundary element.

Definition 13. *An HFLPR $H = \left(h_{ij}\right)_{n \times n}$ with $h_{ij} = \left\{h_{ij}^r \middle| r = 1, 2, \ldots, \# h_{ij}\right\}$ ($\# h_{ij}$ is the number of linguistic terms in h_{ij}) is called an additively consistent HFLPR, if for $\forall h_{ij}^-, h_{ij}^+ \in h_{ij}$, $\exists a_{ik}, b_{ik} \in h_{ik}$ and $\exists a_{kj}, b_{kj} \in h_{kj}$, such that*

$$\begin{cases} \Delta^{-1}\left(h_{ij}^-\right) = \Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) - \frac{g}{2} \\ \Delta^{-1}\left(h_{ij}^+\right) = \Delta^{-1}(b_{ik}) + \Delta^{-1}\left(b_{kj}\right) - \frac{g}{2} \\ i, j, k = 1, 2, \ldots, n \end{cases} \quad (18)$$

where h_{ij}^- and h_{ij}^+ denote the upper and lower bounds of h_{ij}, respectively.

Theorem 1. *Definition 12 is equivalent to Definition 13.*

Proof of Theorem 1: It is easy to know that if an HFLPR is additively consistent according to Definition 12 then it must also be additively consistent according to Definition 13. Thus, in what follows, only the inverse proposition needs to be proved.

Since $\Delta^{-1}\left(h_{ij}^-\right) \leq \Delta^{-1}\left(h_{ij}^+\right)$, $\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) \leq \Delta^{-1}(b_{ik}) + \Delta^{-1}\left(b_{kj}\right)$ is obtained. Thus, there only exist the following three cases:

(1) $\Delta^{-1}(a_{ik}) \leq \Delta^{-1}(b_{ik})$ and $\Delta^{-1}\left(a_{kj}\right) \leq \Delta^{-1}\left(b_{kj}\right)$. This case means the value domains of $\Delta^{-1}(h_{ik}^r)$ and $\Delta^{-1}\left(h_{kj}^r\right)$ ($1 \leq r \leq \# h_{ij}$) contain all the integers in the intervals $\left[\Delta^{-1}(a_{ik}), \Delta^{-1}(b_{ik})\right]$ and $\left[\Delta^{-1}\left(a_{kj}\right), \Delta^{-1}\left(b_{kj}\right)\right]$, respectively.

Let I_{ik} and I_{kj} be the corresponding integers in the intervals $\left[\Delta^{-1}(a_{ik}), \Delta^{-1}(b_{ik})\right]$ and $\left[\Delta^{-1}\left(a_{kj}\right), \Delta^{-1}\left(b_{kj}\right)\right]$, respectively. Thus, the value domain of the sum of I_{ik} and I_{kj} contains all the integers in the intervals $\left[\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right), \Delta^{-1}(b_{ik}) + \Delta^{-1}\left(b_{kj}\right)\right]$, i.e., $\left[\Delta^{-1}\left(h_{ij}^-\right) + \frac{g}{2}, \Delta^{-1}\left(h_{ij}^+\right) + \frac{g}{2}\right]$, which ensures that Equation (17) always holds.

(2) $\Delta^{-1}(a_{ik}) \leq \Delta^{-1}(b_{ik})$ and $\Delta^{-1}\left(a_{kj}\right) \geq \Delta^{-1}\left(b_{kj}\right)$. This case means the value domains of $\Delta^{-1}(h_{ik}^r)$ and $\Delta^{-1}\left(h_{kj}^r\right)$ ($1 \leq r \leq \# h_{ij}$) contain all the integers in the intervals $\left[\Delta^{-1}(a_{ik}), \Delta^{-1}(b_{ik})\right]$ and $\left[\Delta^{-1}\left(b_{kj}\right), \Delta^{-1}\left(a_{kj}\right)\right]$, respectively. Similarly, the value domain of the sum of I_{ik} and I_{kj} contains all the integers in the intervals $\left[\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(b_{kj}\right), \Delta^{-1}(b_{ik}) + \Delta^{-1}\left(a_{kj}\right)\right]$, which contains the interval $\left[\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right), \Delta^{-1}(b_{ik}) + \Delta^{-1}\left(b_{kj}\right)\right]$. Thus, Equation (17) always holds.

(3) $\Delta^{-1}(a_{ik}) \geq \Delta^{-1}(b_{ik})$ and $\Delta^{-1}\left(a_{kj}\right) \leq \Delta^{-1}\left(b_{kj}\right)$. The proof is similar to (2) and is omitted.

In summary, the inverse proposition abovementioned is proved, and Definition 12 is equivalent to Definition 13.

According to Definition 13, the following integer programming model can be constructed to judge if an HFLPR is additively consistent:

$$(\mathrm{M}-2)\quad \begin{array}{c} \min F = \sum_{i,j=1}^{n} \left| \Delta^{-1}\left(h_{ij}^{-}\right) - \left(\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) - \frac{g}{2}\right) \right| + \\ \sum_{i,j=1}^{n} \left| \Delta^{-1}\left(h_{ij}^{+}\right) - \left(\Delta^{-1}(b_{ik}) + \Delta^{-1}\left(b_{kj}\right) - \frac{g}{2}\right) \right| \end{array}$$

s.t

$$\begin{cases} a_{ik} \in h_{ik}, a_{kj} \in h_{kj} \\ b_{ik} \in h_{ik}, b_{kj} \in h_{kj} \\ i, j, k = 1, 2, \ldots, n \end{cases}$$

□

Remark 2. *The construction and solution of Model 2 is simpler and more convenient than that of Model 1.* $H = \left(h_{ij}\right)_{n\times n}$ *is consistent according to Definition 13 if the target function value of Model 2 is equal to 0. Otherwise, it is not consistent.*

3.2. Consistency Improvement for HLFPRs

In real life, due to the complexity of decision-making, DMs cannot guarantee that the original preference relations given are consistent. Therefore, in order to ensure effectiveness and reasonability of decision-making, an important process is to improve the consistency of the original preference relation not satisfying consistency. In this section, a programming model is built to adjust an inconsistent HFLPR to a consistent one, which retains the original information as much as possible.

To facilitate calculation, only the upper triangle elements of HFLPRs are used to construct the model, in view of the symmetry of HFLPRs.

$$(\mathrm{M}-3)\quad \min F = \sum_{i<j}^{n} \left(\left| \Delta^{-1}\left(r_{ij}^{-}\right) - \Delta^{-1}\left(h_{ij}^{-}\right) \right| + \left| \Delta^{-1}\left(r_{ij}^{+}\right) - \Delta^{-1}\left(h_{ij}^{+}\right) \right| \right)$$

s.t

$$\begin{cases} \sum_{i<j}^{n} \left(\left| \Delta^{-1}\left(r_{ij}^{-}\right) - \left(\Delta^{-1}(a_{ik}) + \Delta^{-1}\left(a_{kj}\right) - \frac{g}{2}\right) \right| + \left| \Delta^{-1}\left(r_{ij}^{+}\right) - \left(\Delta^{-1}(b_{ik}) + \Delta^{-1}\left(b_{kj}\right) - \frac{g}{2}\right) \right| \right) = 0 \\ \Delta^{-1}\left(r_{ij}^{-}\right), \Delta^{-1}\left(r_{ij}^{+}\right) \in [o, g] \\ \Delta^{-1}\left(r_{ij}^{-}\right) < \Delta^{-1}\left(r_{ij}^{+}\right) \\ a_{ik} \in r_{ik}, a_{kj} \in r_{kj} \\ b_{ik} \in r_{ik}, b_{kj} \in r_{kj} \\ i, j, k = 1, 2, \ldots, n \end{cases}$$

where $R = \left(r_{ij}\right)_{n\times n}$ represents the adjusted HFLPR, and r_{ij}^{-} and r_{ij}^{+} denote the upper and lower bounds of $R = \left(r_{ij}\right)_{n\times n}$, respectively.

In what follows, an example is provided to illustrate the above procedure.

Example 1 [9]**.** *Let S be a linguistic term set defined as follows:*

$$S = \left\{ \begin{array}{l} s_0 = extremely\ poor, s_1 = very\ poor, s_2 = poor, s_3 = slightly\ poor, \\ s_4 = fair, s_5 = slight\ good, s_6 = good, s_7 = very\ good, s_8 = extremely\ good \end{array} \right\}$$

Consider the following HFLPR H:

$$H = \begin{pmatrix} s_4 & \{s_6, s_7\} & \{s_5, s_6\} & \{s_2, s_3, s_4\} \\ \{s_2, s_1\} & s_4 & \{s_7\} & \{s_6\} \\ \{s_3, s_2\} & \{s_1\} & s_4 & \{s_4, s_5, s_6\} \\ \{s_6, s_5, s_4\} & \{s_2\} & \{s_4, s_3, s_2\} & s_4 \end{pmatrix}$$

According to Model 2, the following model is constructed:

$$\begin{aligned}
\min = & \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(a_{13}) + \Delta^{-1}(a_{32}) - 4\right)\right| + \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(a_{14}) + \Delta^{-1}(a_{42}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_7) - \left(\Delta^{-1}(b_{13}) + \Delta^{-1}(b_{32}) - 4\right)\right| + \left|\Delta^{-1}(s_7) - \left(\Delta^{-1}(b_{14}) + \Delta^{-1}(b_{42}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_5) - \left(\Delta^{-1}(a_{12}) + \Delta^{-1}(a_{23}) - 4\right)\right| + \left|\Delta^{-1}(s_5) - \left(\Delta^{-1}(a_{14}) + \Delta^{-1}(a_{43}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{12}) + \Delta^{-1}(b_{23}) - 4\right)\right| + \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{14}) + \Delta^{-1}(b_{43}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{12}) + \Delta^{-1}(a_{24}) - 4\right)\right| + \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{13}) + \Delta^{-1}(a_{34}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(b_{12}) + \Delta^{-1}(b_{24}) - 4\right)\right| + \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(b_{13}) + \Delta^{-1}(b_{34}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_7) - \left(\Delta^{-1}(a_{21}) + \Delta^{-1}(a_{13}) - 4\right)\right| + \left|\Delta^{-1}(s_7) - \left(\Delta^{-1}(a_{24}) + \Delta^{-1}(a_{43}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_7) - \left(\Delta^{-1}(b_{21}) + \Delta^{-1}(b_{13}) - 4\right)\right| + \left|\Delta^{-1}(s_7) - \left(\Delta^{-1}(b_{24}) + \Delta^{-1}(b_{43}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(a_{21}) + \Delta^{-1}(a_{14}) - 4\right)\right| + \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(a_{23}) + \Delta^{-1}(a_{34}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{21}) + \Delta^{-1}(b_{14}) - 4\right)\right| + \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{23}) + \Delta^{-1}(b_{34}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(a_{31}) + \Delta^{-1}(a_{14}) - 4\right)\right| + \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(a_{32}) + \Delta^{-1}(a_{24}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{31}) + \Delta^{-1}(b_{14}) - 4\right)\right| + \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{32}) + \Delta^{-1}(b_{24}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_1) - \left(\Delta^{-1}(a_{23}) + \Delta^{-1}(a_{31}) - 4\right)\right| + \left|\Delta^{-1}(s_1) - \left(\Delta^{-1}(a_{24}) + \Delta^{-1}(a_{41}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(b_{23}) + \Delta^{-1}(b_{31}) - 4\right)\right| + \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(b_{24}) + \Delta^{-1}(b_{41}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{32}) + \Delta^{-1}(a_{21}) - 4\right)\right| + \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{34}) + \Delta^{-1}(a_{41}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_3) - \left(\Delta^{-1}(b_{32}) + \Delta^{-1}(b_{21}) - 4\right)\right| + \left|\Delta^{-1}(s_3) - \left(\Delta^{-1}(b_{34}) + \Delta^{-1}(b_{41}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(a_{42}) + \Delta^{-1}(a_{21}) - 4\right)\right| + \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(a_{43}) + \Delta^{-1}(a_{31}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{42}) + \Delta^{-1}(b_{21}) - 4\right)\right| + \left|\Delta^{-1}(s_6) - \left(\Delta^{-1}(b_{43}) + \Delta^{-1}(b_{31}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_1) - \left(\Delta^{-1}(a_{31}) + \Delta^{-1}(a_{12}) - 4\right)\right| + \left|\Delta^{-1}(s_1) - \left(\Delta^{-1}(a_{34}) + \Delta^{-1}(a_{42}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_1) - \left(\Delta^{-1}(b_{31}) + \Delta^{-1}(b_{12}) - 4\right)\right| + \left|\Delta^{-1}(s_1) - \left(\Delta^{-1}(b_{34}) + \Delta^{-1}(b_{42}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{41}) + \Delta^{-1}(a_{12}) - 4\right)\right| + \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{43}) + \Delta^{-1}(a_{32}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(b_{41}) + \Delta^{-1}(b_{12}) - 4\right)\right| + \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(b_{43}) + \Delta^{-1}(b_{32}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{41}) + \Delta^{-1}(a_{13}) - 4\right)\right| + \left|\Delta^{-1}(s_2) - \left(\Delta^{-1}(a_{42}) + \Delta^{-1}(a_{23}) - 4\right)\right| + \\
& \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(b_{41}) + \Delta^{-1}(b_{13}) - 4\right)\right| + \left|\Delta^{-1}(s_4) - \left(\Delta^{-1}(b_{42}) + \Delta^{-1}(b_{23}) - 4\right)\right|
\end{aligned}$$

s.t

$$\begin{cases} a_{12} \in \{s_6, s_7\}, a_{13} \in \{s_5, s_6\}, a_{14} \in \{s_2, s_3, s_4\}, a_{23} \in \{s_7\}, a_{24} \in \{s_6\}, a_{34} \in \{s_4, s_5, s_6\} \\ a_{21} \in \{s_2, s_1\}, a_{31} \in \{s_3, s_2\}, a_{41} \in \{s_6, s_5, s_4\}, a_{32} \in \{s_1\}, a_{42} \in \{s_2\}, a_{43} \in \{s_4, s_3, s_2\} \\ b_{12} \in \{s_6, s_7\}, b_{13} \in \{s_5, s_6\}, b_{14} \in \{s_2, s_3, s_4\}, b_{23} \in \{s_7\}, b_{24} \in \{s_6\}, b_{34} \in \{s_4, s_5, s_6\} \\ b_{21} \in \{s_2, s_1\}, b_{31} \in \{s_3, s_2\}, b_{41} \in \{s_6, s_5, s_4\}, b_{32} \in \{s_1\}, b_{42} \in \{s_2\}, b_{43} \in \{s_4, s_3, s_2\} \end{cases}$$

By solving this model, min$F$ = 132 is obtained, which means that HFLPR $H = \left(h_{ij}\right)_{n \times n}$ is not additively consistent. Thus, according to Model 3, the following model is constructed:

$$\begin{aligned}
\min = & \left|\Delta^{-1}\left(r_{12}^{-}\right) - \Delta^{-1}(s_6)\right| + \left|\Delta^{-1}\left(r_{12}^{+}\right) - \Delta^{-1}(s_7)\right| + \left|\Delta^{-1}\left(r_{13}^{-}\right) - \Delta^{-1}(s_5)\right| + \left|\Delta^{-1}\left(r_{13}^{+}\right) - \Delta^{-1}(s_6)\right| + \\
& \left|\Delta^{-1}\left(r_{14}^{-}\right) - \Delta^{-1}(s_2)\right| + \left|\Delta^{-1}\left(r_{14}^{+}\right) - \Delta^{-1}(s_4)\right| + \left|\Delta^{-1}\left(r_{23}^{-}\right) - \Delta^{-1}(s_7)\right| + \left|\Delta^{-1}\left(r_{23}^{+}\right) - \Delta^{-1}(s_7)\right| + \\
& \left|\Delta^{-1}\left(r_{24}^{-}\right) - \Delta^{-1}(s_6)\right| + \left|\Delta^{-1}\left(r_{24}^{+}\right) - \Delta^{-1}(s_6)\right| + \left|\Delta^{-1}\left(r_{34}^{-}\right) - \Delta^{-1}(s_4)\right| + \left|\Delta^{-1}\left(r_{34}^{+}\right) - \Delta^{-1}(s_6)\right|
\end{aligned}$$

s.t

$$\begin{cases} \left|\Delta^{-1}(r_{12}^-) - \left(\Delta^{-1}(a_{13}) + \left(8 - \Delta^{-1}(a_{23})\right) - 4\right)\right| + \left|\Delta^{-1}(r_{12}^-) - \left(\Delta^{-1}(a_{14}) + \left(8 - \Delta^{-1}(a_{24})\right) - 4\right)\right| + \\ \left|\Delta^{-1}\ (r_{12}^+) - \left(\Delta^{-1}\ (b_{13}) + \left(8 - \Delta^{-1}\ (b_{23})\right) - 4\right)\right| + \left|\Delta^{-1}\ (r_{12}^+) - \left(\Delta^{-1}\ (b_{14}) + \left(8 - \Delta^{-1}\ (b_{24})\right) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{13}^-) - \left(\Delta^{-1}(a_{12}) + \Delta^{-1}(a_{23}) - 4\right)\right| + \left|\Delta^{-1}(r_{13}^-) - \left(\Delta^{-1}(a_{14}) + \left(8 - \Delta^{-1}(a_{34})\right) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{13}^+) - \left(\Delta^{-1}(b_{12}) + \Delta^{-1}(b_{23}) - 4\right)\right| + \left|\Delta^{-1}(r_{13}^+) - \left(\Delta^{-1}(b_{14}) + \left(8 - \Delta^{-1}(b_{34})\right) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{14}^-) - \left(\Delta^{-1}(a_{12}) + \Delta^{-1}(a_{24}) - 4\right)\right| + \left|\Delta^{-1}(r_{14}^-) - \left(\Delta^{-1}(a_{13}) + \Delta^{-1}(a_{34}) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{14}^+) - \left(\Delta^{-1}(b_{12}) + \Delta^{-1}(b_{24}) - 4\right)\right| + \left|\Delta^{-1}(r_{14}^+) - \left(\Delta^{-1}(b_{13}) + \Delta^{-1}(b_{34}) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{23}^-) - \left(\left(8 - \Delta^{-1}(a_{12})\right) + \Delta^{-1}(a_{13}) - 4\right)\right| + \left|\Delta^{-1}(r_{23}^-) - \left(\Delta^{-1}(a_{24}) + \left(8 - \Delta^{-1}(a_{34})\right) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{23}^+) - \left(\left(8 - \Delta^{-1}(b_{12})\right) + \Delta^{-1}(b_{13}) - 4\right)\right| + \left|\Delta^{-1}(r_{23}^+) - \left(\Delta^{-1}(b_{24}) + \left(8 - \Delta^{-1}(b_{34})\right) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{24}^-) - \left(\left(8 - \Delta^{-1}(a_{12})\right) + \Delta^{-1}(a_{14}) - 4\right)\right| + \left|\Delta^{-1}(r_{24}^-) - \left(\Delta^{-1}(a_{23}) + \Delta^{-1}(a_{34}) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{24}^+) - \left(\left(8 - \Delta^{-1}(b_{12})\right) + \Delta^{-1}(b_{14}) - 4\right)\right| + \left|\Delta^{-1}(r_{24}^+) - \left(\Delta^{-1}(b_{23}) + \Delta^{-1}(b_{34}) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{34}^-) - \left(\left(8 - \Delta^{-1}(a_{13})\right) + \Delta^{-1}(a_{14}) - 4\right)\right| + \left|\Delta^{-1}(r_{34}^-) - \left(\left(8 - \Delta^{-1}(a_{23})\right) + \Delta^{-1}(a_{24}) - 4\right)\right| + \\ \left|\Delta^{-1}(r_{34}^+) - \left(\left(8 - \Delta^{-1}(b_{13})\right) + \Delta^{-1}(b_{14}) - 4\right)\right| + \left|\Delta^{-1}(r_{34}^+) - \left(\left(8 - \Delta^{-1}(b_{23})\right) + \Delta^{-1}(b_{24}) - 4\right)\right| = 0 \\ a_{12} \in \{s_6, s_7\}, a_{13} \in \{s_5, s_6\}, a_{14} \in \{s_2, s_3, s_4\}, a_{23} \in \{s_7\}, a_{24} \in \{s_6\}, a_{34} \in \{s_4, s_5, s_6\} \\ b_{12} \in \{s_6, s_7\}, b_{13} \in \{s_5, s_6\}, b_{14} \in \{s_2, s_3, s_4\}, b_{23} \in \{s_7\}, b_{24} \in \{s_6\}, b_{34} \in \{s_4, s_5, s_6\} \\ \Delta^{-1}\left(r_{ij}^-\right), \Delta^{-1}\left(r_{ij}^+\right) \in [o, g], i, j = 1, 2, \ldots, n \\ \Delta^{-1}\ \left(r_{ij}^-\right) < \Delta^{-1}\ \left(r_{ij}^+\right) \end{cases}$$

Then, the adjusted HFLPR $R = \left(r_{ij}\right)_{n \times n}$ is obtained as follows:

$$R = \begin{pmatrix} s_4 & \{s_3, s_4, s_5\} & \{s_4, s_5, s_6\} & \{s_4, s_5, s_6\} \\ \{s_5, s_4, s_3\} & s_4 & \{s_5, s_6, s_7\} & \{s_5, s_6, s_7\} \\ \{s_4, s_3, s_2\} & \{s_3, s_2, s_1\} & s_4 & \{s_4, s_5, s_6\} \\ \{s_4, s_3, s_2\} & \{s_3, s_2, s_1\} & \{s_4, s_3, s_2\} & s_4 \end{pmatrix}$$

3.3. Models to Derive Priority Weights from Additively Consistent HFLPRs

The main purpose of solving a decision problem is to find the optimal alternative according to the preference relation given by DMs, so deriving the weight vector is always performed in many existing literatures. Generally speaking, crisp and interval weight vector are always applied to solve decision-making problems based on crisp and interval preference relations, respectively. Accordingly, for decision-making problems with HFLPRs, applying weight vector denoted by HFLTSs is more natural and practical. Thus, in what follows, the definition of hesitant fuzzy linguistic weight vector (HFLWV) is introduced, which adopts several possible linguistic terms to express the importance of each alternative.

Definition 14. Suppose $S = \left\{s_0, s_1, \ldots, s_g\right\}$ is a linguistic term set. $W = (w_1, w_2, \ldots, w_n)^T$ is called a hesitant fuzzy linguistic weight vector (HFLWV), if $w_i (i = 1, 2, \ldots, n)$ is an ordered finite subset of consecutive linguistic terms of S, where $w_i = \left\{w_i^r | r = 1, 2, \ldots, \#w_i\right\}$ ($\#w_i$ is the number of linguistic terms in w_i) reflects the importance degree of the ith alternative.

According to Equation (17) in Definition 12, each element h_{ij}^r is supposed to be expressed by specified a_{ik} and a_{kj}. If a_{ik} and a_{kj} can be determined as

$$\Delta^{-1}(a_{ik}) = 0.5\left(\Delta^{-1}(w_i^a) - \Delta^{-1}(w_k^a) + g\right), w_i^a \in w_i, w_k^a \in w_k \tag{19}$$

$$\Delta^{-1}\left(a_{kj}\right) = 0.5\left(\Delta^{-1}\left(w_k^b\right) - \Delta^{-1}\left(w_j^b\right) + g\right), w_k^b \in w_k, w_j^b \in w_j \tag{20}$$

then h_{ij}^r can be expressed as follows:

$$\Delta^{-1}\left(h_{ij}^{r}\right)=0.5\left(\Delta^{-1}\left(w_{i}^{a}\right)-\Delta^{-1}\left(w_{k}^{a}\right)+g\right)+0.5\left(\Delta^{-1}\left(w_{k}^{b}\right)-\Delta^{-1}\left(w_{j}^{b}\right)+g\right)-\frac{g}{2},\quad i,j,k=1,2,\ldots,n \tag{21}$$

Based on this idea, the following model is constructed to derive the HFLWV, where its number of linguistic terms is as small as possible.

$$(\mathrm{M}-4)\ \min F=\sum_{i=1}^{n}\left|\Delta^{-1}\left(w_{i}^{+}\right)-\Delta^{-1}\left(w_{i}^{-}\right)\right|$$

s.t

$$\begin{cases}\Delta^{-1}\left(h_{ij}^{r}\right)=0.5\left(\Delta^{-1}\left(w_{i}^{a}\right)-\Delta^{-1}\left(w_{k}^{a}\right)+g\right)+0.5\left(\Delta^{-1}\left(w_{k}^{b}\right)-\Delta^{-1}\left(w_{j}^{b}\right)+g\right)-\dfrac{g}{2}\\ w_{i}^{a}\in w_{i},w_{k}^{a}\in w_{k}\\ w_{k}^{b}\in w_{k},w_{j}^{b}\in w_{j}\\ i,j,k=1,2,\ldots,n\\ r=1,2,\ldots,\#h_{ij}\end{cases}$$

In what follows, one example is provided to illustrate the above model.

Example 2. *Let S be a linguistic term set defined as follows:*

$$S=\left\{\begin{array}{l}s_0=extremely\ poor,s_1=very\ poor,s_2=poor,s_3=slightly\ poor,\\ s_4=fair,s_5=slight\ good,s_6=good,s_7=very\ good,s_8=extremely\ good\end{array}\right\}$$

Consider the following HFLPR H:

$$H=\begin{pmatrix}s_5 & \{s_4,s_5,s_6\} & \{s_3,s_4\} & \{s_5,s_6,s_7\}\\ \{s_6,s_5,s_4\} & s_5 & \{s_3\} & \{s_6,s_7,s_8\}\\ \{s_7,s_6\} & \{s_7\} & s_5 & \{s_8,s_9\}\\ \{s_5,s_4,s_3\} & \{s_4,s_3,s_2\} & \{s_2,s_1\} & s_5\end{pmatrix}$$

According to Model 2, the objective function value is not equal to 0. That is, the HFLPR $H=\left(h_{ij}\right)_{n\times n}$ is not additively consistent; according to Model 3, the adjusted HFLPR $R=\left(r_{ij}\right)_{n\times n}$ is obtained as follows:

$$R=\begin{pmatrix}s_5 & \{s_4,s_5,s_6\} & \{s_3,s_4\} & \{s_6,s_7\}\\ \{s_6,s_5,s_4\} & s_5 & \{s_3,s_4\} & \{s_6,s_7,s_8\}\\ \{s_7,s_6\} & \{s_7,s_6\} & s_5 & \{s_8,s_9\}\\ \{s_4,s_3\} & \{s_4,s_3,s_2\} & \{s_2,s_1\} & s_5\end{pmatrix}$$

Using Model 4, the following HFLWV is obtained:

$$w_1=\{s_3,s_4\},w_2=\{s_2,s_3,s_4,s_5\},w_3=\{s_6,s_7\},w_4=\{s_0\}$$

By Equation (5), the following score function can be obtained:

$$Score(w_1)=\frac{1}{\#w_1}\sum_{l=1}^{\#w_1}\Delta^{-1}\left(s_{\sigma_{(l)}}\right)=3.5,Score(w_2)=\frac{1}{\#w_2}\sum_{l=1}^{\#w_2}\Delta^{-1}\left(s_{\sigma_{(l)}}\right)=3.5$$

$$Score(w_3)=\frac{1}{\#w_3}\sum_{l=1}^{\#w_3}\Delta^{-1}\left(s_{\sigma_{(l)}}\right)=6.5,Score(w_4)=\frac{1}{\#w_4}\sum_{l=1}^{\#w_4}\Delta^{-1}\left(s_{\sigma_{(l)}}\right)=0$$

By Equation (6), the following accurate functions can be obtained:

$$H(w_1) = \Delta^{-1}(w_1^-) - \Delta^{-1}(w_1^+) + g = 9, H(w_2) = \Delta^{-1}(w_2^-) - \Delta^{-1}(w_2^+) + g = 7$$

Thus, the ranking results should be

$$x_3 \succ x_1 \succ x_2 \succ x_4$$

Alternative x_3 is the best option.

4. Examples and Comparative Analysis

In this section, some numerical examples are presented to illustrate the proposed methods.

Example 3 [26]. *Let $S = \{s_0, s_1, \ldots, s_6\}$ be a linguistic term set, and consider the following HFLPR H:*

$$H = \begin{pmatrix} s_3 & \{s_5\} & \{s_0, s_1, s_2\} & \{s_2, s_3\} \\ \{s_1\} & s_3 & \{s_0, \ldots, s_6\} & \{s_1, s_2\} \\ \{s_6, s_5, s_4\} & \{s_6, \ldots, s_0\} & s_3 & \{s_4\} \\ \{s_4, s_3\} & \{s_5, s_4\} & \{s_2\} & s_3 \end{pmatrix}$$

Liu et al. [26] improved the additive consistency of the HFLPR to an acceptable level (consistency threshold is 0.9), and the improved HFLPR is listed as follows:

$$R^{\text{Liu et al.}} = \begin{pmatrix} s_3 & \{s_5\} & \{s_2\} & \{s_3\} \\ \{s_1\} & s_3 & \{s_0, s_1\} & \{s_1\} \\ \{s_4\} & \{s_6, s_5\} & s_3 & \{s_4\} \\ \{s_3\} & \{s_5\} & \{s_2\} & s_3 \end{pmatrix}$$

By Model 3, the following consistent HFLPR is derived:

$$R = \begin{pmatrix} s_3 & \{s_4, s_5\} & \{s_1, s_2\} & \{s_2, s_3\} \\ \{s_2, s_1\} & s_3 & \{s_0, s_1\} & \{s_1, s_2\} \\ \{s_5, s_4\} & \{s_6, s_5\} & s_3 & \{s_4\} \\ \{s_4, s_3\} & \{s_5, s_4\} & \{s_2\} & s_3 \end{pmatrix}$$

By Equation (7), distance between the original and adjusted HFLPRs can be obtained, where only the HFLTSs in the upper triangular matrix are considered. The results are showed as D_1 in Table 1. In addition, D_2 in Table 1 describes the sum of HFLTSs adjusted in the upper triangular matrix. For example, $H_{14} = \{s_2, s_3\} \neq R_{14}^{\text{Liu et al.}} = \{s_3\}$, which means that original HFLTS H_{14} is adjusted by Liu et al. [26]. There are altogether four HFLTSs that are adjusted in $R^{\text{Liu et al.}}$. The comparative results are given in Table 1.

Table 1. Comparison results for different methods.

Improved HFLPRs	D1	D2
$R^{\text{Liu et al.}}$	9	4
R	7	3

As shown in Table 1, the distance between H and $R^{\text{Liu et al.}}$ is larger than that between H and R. Fewer HFLTSs are adjusted in R, compared with $R^{\text{Liu et al.}}$. These results show that the adjusted HFLPR derived by this paper contains much more information of the original HFLPR.

Example 4 [9]. *Let $S = \{s_0, s_1, \ldots, s_6\}$ be a linguistic term set, and consider the following HFLPR H:*

$$H = \begin{pmatrix} s_4 & \{s_2, s_3, s_4\} & \{s_5, s_6\} & \{s_4\} \\ \{s_6, s_5, s_4\} & s_4 & \{s_1, s_2, s_3\} & \{s_6, s_7\} \\ \{s_3, s_2\} & \{s_7, s_6, s_5\} & s_4 & \{s_4, s_5\} \\ \{s_4\} & \{s_2, s_1\} & \{s_4, s_3\} & s_4 \end{pmatrix}$$

By the method of Chen et al. [9], the additive consistency of the HFLPR is improved to an acceptable level (consistency threshold is 0.95), and the improved HFLPR is listed as follows:

$$R^{\text{Chen et al.}} = \begin{pmatrix} s_4 & \{s_3, s_4\} & \{s_4\} & \{s_4\} \\ \{s_5, s_4\} & s_4 & \{s_4\} & \{s_5\} \\ \{s_4\} & \{s_4\} & s_4 & \{s_4, s_5\} \\ \{s_4\} & \{s_3\} & \{s_4, s_3\} & s_4 \end{pmatrix}$$

By Model 3, the following consistent HFLPR is derived:

$$R = \begin{pmatrix} s_4 & \{s_2, s_3, s_4\} & \{s_4, s_5\} & \{s_4, s_5\} \\ \{s_6, s_5, s_4\} & s_4 & \{s_4, s_5, s_6\} & \{s_5, s_6, s_7\} \\ \{s_4, s_3\} & \{s_4, s_3, s_2\} & s_4 & \{s_4, s_5\} \\ \{s_4, s_3\} & \{s_3, s_2, s_1\} & \{s_4, s_3\} & s_4 \end{pmatrix}$$

Similar to Example 3, D_1 and D_2 are also applied, and the comparative results are given in Table 2:

Table 2. Comparison results for different methods.

Improved HFLPRs	D1	D2
$R^{\text{Chen et al.}}$	11	4
R	10	4

As shown in Table 2, the distance between H and $R^{\text{Chen et al.}}$ is larger than that between H and R. These results show that the adjusted HFLPR derived by this paper contains much more information of the original HFLPR.

Example 5 [19]. *Let S be a linguistic term set defined as follows,* $S = \{s_0, s_1, \ldots, s_8\}$ *be a linguistic term set, and consider the following HFLPR H:*

$$H = \begin{pmatrix} s_4 & \{s_2, s_3, s_4\} & \{s_5, s_6\} & \{s_1, s_2\} \\ \{s_6, s_5, s_4\} & s_4 & \{s_7\} & \{s_6\} \\ \{s_3, s_2\} & \{s_1\} & s_4 & \{s_4, s_5, s_6\} \\ \{s_7, s_6\} & \{s_2\} & \{s_4, s_3, s_2\} & s_4 \end{pmatrix}$$

According to the methods of Zhang and Wu [19] and Zhu and Xu [20], the corresponding multiplicative consistency HFLPR $R^{\text{Zhang and Wu}}$ and additively consistent HFLPR $R^{\text{Zhu and Xu}}$ are obtained as follows:

$$R^{\text{Zhang and Wu}} = \begin{pmatrix} s_4 & \{s_{1.27}, s_{2.13}, s_{2.55}\} & \{s_{3.96}, s_{4.99}, s_{4.95}\} & \{s_{2.40}, s_{4.00}, s_{4.55}\} \\ \{s_{6.73}, s_{5.87}, s_{5.45}\} & s_4 & \{s_{6.71}, s_{6.57}, s_{6.22}\} & \{s_{5.56}, s_{5.87}, s_{5.90}\} \\ \{s_{4.04}, s_{3.05}, s_{3.01}\} & \{s_{1.29}, s_{1.43}, s_{1.78}\} & s_4 & \{s_{2.44}, s_{3.01}, s_{3.58}\} \\ \{s_{5.60}, s_{4.00}, s_{3.45}\} & \{s_{2.44}, s_{2.13}, s_{2.10}\} & \{s_{5.56}, s_{4.99}, s_{4.42}\} & s_4 \end{pmatrix}$$

$$R^{\text{Zhu and Xu}} = \begin{pmatrix} s_4 & \{s_{1.25}, s_{2.25}, s_{2.75}\} & \{s_{4.00}, s_{4.75}, s_{4.75}\} & \{s_{2.75}, s_{4.00}, s_{4.50}\} \\ \{s_{6.75}, s_{5.75}, s_{5.25}\} & s_4 & \{s_{6.00}, s_{6.50}, s_{6.75}\} & \{s_{5.50}, s_{5.75}, s_{5.75}\} \\ \{s_{4.00}, s_{3.25}, s_{3.25}\} & \{s_{2.00}, s_{1.50}, s_{1.25}\} & s_4 & \{s_{2.75}, s_{3.25}, s_{3.75}\} \\ \{s_{5.25}, s_{4.00}, s_{3.50}\} & \{s_{2.50}, s_{2.25}, s_{2.25}\} & \{s_{5.25}, s_{4.75}, s_{4.25}\} & s_4 \end{pmatrix}$$

By Model 3, the consistency HFLPR is derived as follows:

$$R = \begin{pmatrix} s_4 & \{s_2, s_3\} & \{s_5, s_6\} & \{s_3, s_4\} \\ \{s_6, s_5\} & s_4 & \{s_7\} & \{s_5, s_6\} \\ \{s_3, s_2\} & \{s_1\} & s_4 & \{s_2, s_3\} \\ \{s_5, s_4\} & \{s_3, s_2\} & \{s_6, s_5\} & s_4 \end{pmatrix}$$

Both Zhang and Wu [19] and Zhu and Xu [20] performed the normalization process, where some specified linguistic terms are added to HFLTSs with fewer elements until all the HFLTSs in HFLPR H have the same sum of linguistic terms. Such a process not only increases the burden of DMs but also easily distorts the original preference information. In addition, in Zhang and Wu [19] and Zhu and Xu [20], most of the original linguistic terms in the HFLPRs are adjusted and the modified elements no longer belong to the original linguistic term set, which may not be agreed to by DMs. By contrast, the method proposed by this paper does not have the above related problems.

Example 6. *Let* $S = \{s_0, s_1, \ldots, s_8\}$ *be a linguistic term set, and consider the following HFLPR* H*:*

$$H = \begin{pmatrix} s_4 & \{s_5, s_6, s_7\} & \{s_7, s_8\} & \{s_6, s_7\} \\ \{s_3, s_2, s_1\} & s_4 & \{s_0\} & \{s_1\} \\ \{s_1, s_0\} & \{s_8\} & s_4 & \{s_6\} \\ \{s_2, s_1\} & \{s_7\} & \{s_2\} & s_4 \end{pmatrix}$$

According to the methods of Xu and Wang [21], a normalization procedure is also performed. If the corresponding upper bounds of HFLTSs are added in Xu and Wang [21], the additively consistent HFLPR is obtained as follows:

$$R^{\text{Xu and Wang}} = \begin{pmatrix} s_4 & \{s_{7.50}, s_{8.50}, s_{9.00}\} & \{s_{4.75}, s_{5.75}, s_{6.00}\} & \{s_{5.75}, s_{6.75}, s_{7.00}\} \\ \{s_{0.50}, s_{-0.50}, s_{-1.00}\} & s_4 & \{s_{1.25}, s_{1.25}, s_{1.00}\} & \{s_{2.25}, s_{2.25}, s_{2.00}\} \\ \{s_{3.25}, s_{2.25}, s_{2.00}\} & \{s_{6.75}, s_{6.75}, s_{7.00}\} & s_4 & \{s_{5.00}, s_{5.00}, s_{5.00}\} \\ \{s_{2.25}, s_{1.25}, s_{1.00}\} & \{s_{5.75}, s_{5.75}, s_{6.00}\} & \{s_{3.00}, s_{3.00}, s_{3.00}\} & s_4 \end{pmatrix}$$

By Model 3, the following additively consistent HFLPR is derived:

$$R = \begin{pmatrix} s_4 & \{s_7, s_8\} & \{s_4, s_5\} & \{s_6, s_7\} \\ \{s_1, s_0\} & s_4 & \{s_0, s_1\} & \{s_2, s_3\} \\ \{s_4, s_3\} & \{s_8, s_7\} & s_4 & \{s_6\} \\ \{s_2, s_1\} & \{s_6, s_5\} & \{s_2\} & s_4 \end{pmatrix}$$

In the method of Xu and Wang [21], the normalization process is indispensable to ensure each HFLTS has the same length. In addition, it is worth mentioning that some modified elements, such as $s_{9.00}$ and $s_{-1.00}$, are out of the range of the original linguistic set S. Although such elements can be transformed into certain elements in the range of S, this transformation process may distort the preference information. On the contrary, the method proposed by this paper does not need the normalization process and maintains the objective of decision-making.

Example 7. *Let $S = \{s_0, s_1, \dots, s_8\}$ be a linguistic term set, and consider the following HFLPR H:*

$$H = \begin{pmatrix} s_4 & \{s_3, s_4, s_5\} & \{s_5, s_6\} & \{s_7\} \\ \{s_5, s_4, s_3\} & s_4 & \{s_5, s_6\} & \{s_6, s_7, s_8\} \\ \{s_3, s_2\} & \{s_3, s_2\} & s_4 & \{s_3, s_4, s_5, s_6\} \\ \{s_1\} & \{s_2, s_1, s_0\} & \{s_5, s_4, s_3, s_2\} & s_4 \end{pmatrix}$$

By the method of Xu and Wang [21], two NHFLPRs can be derived by normalization procedure, where the lower and upper bounds of HFLTS are added, respectively.

$$R_1^{\text{Xu and Wang}} = \begin{pmatrix} s_4 & \{s_{3.75}, s_{3.75}, s_{4.00}, s_{4.25}\} & \{s_{5.50}, s_{5.25}, s_{5.25}, s_{6.00}\} & \{s_{5.75}, s_{6.00}, s_{6.75}, s_{7.75}\} \\ \{s_{4.25}, s_{4.25}, s_{4.00}, s_{3.75}\} & s_4 & \{s_{5.75}, s_{5.50}, s_{5.25}, s_{5.75}\} & \{s_{6.00}, s_{6.25}, s_{6.75}, s_{7.50}\} \\ \{s_{2.50}, s_{2.75}, s_{2.75}, s_{2.00}\} & \{s_{2.25}, s_{2.50}, s_{2.75}, s_{2.25}\} & s_4 & \{s_{4.25}, s_{4.75}, s_{5.50}, s_{5.75}\} \\ \{s_{2.25}, s_{2.00}, s_{1.25}, s_{0.25}\} & \{s_{2.00}, s_{1.75}, s_{1.25}, s_{0.50}\} & \{s_{3.75}, s_{3.25}, s_{2.50}, s_{2.25}\} & s_4 \end{pmatrix}$$

$$R_2^{\text{Xu and Wang}} = \begin{pmatrix} s_4 & \{s_{3.75}, s_{4.00}, s_{4.25}, s_{4.25}\} & \{s_{5.50}, s_{6.25}, s_{6.25}, s_{6.00}\} & \{s_{5.75}, s_{6.75}, s_{7.50}, s_{7.75}\} \\ \{s_{4.25}, s_{4.00}, s_{3.75}, s_{3.75}\} & s_4 & \{s_{5.75}, s_{6.25}, s_{6.00}, s_{5.75}\} & \{s_{6.00}, s_{6.75}, s_{7.25}, s_{7.50}\} \\ \{s_{2.50}, s_{1.75}, s_{1.75}, s_{2.00}\} & \{s_{2.25}, s_{1.75}, s_{2.00}, s_{2.25}\} & s_4 & \{s_{4.25}, s_{4.50}, s_{5.25}, s_{5.75}\} \\ \{s_{2.25}, s_{1.25}, s_{0.50}, s_{0.25}\} & \{s_{2.00}, s_{1.25}, s_{0.75}, s_{0.50}\} & \{s_{3.75}, s_{3.5}, s_{2.75}, s_{2.25}\} & s_4 \end{pmatrix}$$

In the method of Xu and Wang [21], the hesitant fuzzy linguistic averaging (HFLA) operator and a hesitant fuzzy linguistic geometric (HFLG) operator [5] are used to obtain the aggregated HFLTSs. Then, the ranking order can be determined by score function, which is listed in Table 3.

Table 3. Results and comparisons of different models.

Methods	Consistency	Terms Added	Selection Process	Ranking Values	Ranking Order
Xu and Wang [21]	Additive	Low bounds	HFLA	0.630.63 0.44 0.30	$x_2 \succ x_1 \succ x_3 \succ x_4$
Xu and Wang [21]	Additive	Low bounds	HFLG	0.61 0.62 0.42 0.27	$x_2 \succ x_1 \succ x_3 \succ x_4$
Xu and Wang [21]	Additive	Up bounds	HFLA	0.66 0.65 0.41 0.29	$x_1 \succ x_2 \succ x_3 \succ x_4$
Xu and Wang [21]	Additive	Up bounds	HFLG	0.64 0.63 0.37 0.23	$x_1 \succ x_2 \succ x_3 \succ x_4$
This method	Additive	Not added	Priority weights	6.50 7.00 3.50 1.50	$x_2 \succ x_1 \succ x_3 \succ x_4$

By Model 3, the consistent HFLPR is obtained as follows:

$$R = \begin{pmatrix} s_4 & \{s_3, s_4, s_5\} & \{s_5, s_6, s_7\} & \{s_7\} \\ \{s_5, s_4, s_3\} & s_4 & \{s_5, s_6\} & \{s_6, s_7, s_8\} \\ \{s_3, s_2, s_1\} & \{s_3, s_2\} & s_4 & \{s_4, s_5, s_6\} \\ \{s_1\} & \{s_2, s_1, s_0\} & \{s_4, s_3, s_2\} & s_4 \end{pmatrix}$$

Then, based on Model 4, the HFLWV of R can be derived, and corresponding results are showed in Table 3.

According to Table 3, the ranking order varies with different linguistic terms added in the normalization procedure of Xu and Wang [21]. The method of determining the ranking order in this paper avoids the above problems and keeps the objective principle of decision-making.

5. Conclusions

This paper introduces a new additively consistent definition for HFLPRs. For inconsistent HFLPRs, a model to improve its consistency is constructed. To obtain the ranking order, a new definition of HFLWV is introduced and a programming model is proposed to determine the weight vector. Compared with other existing methods, the method in this paper has the following characteristics:

(1) The proposed additively consistent definition for HFLPRs takes all the elements of HFLTSs into account. Some existing additively consistent definitions [22,26] only consider local elements in HFLPR, which easily results in the loss of information. In addition, the proposed additively consistent definition does not require the HFLTSs in the same HFLPR to have the same length. In the normalization process in [19–21], the original opinions of DMs are inclined to be distorted.

(2) Model 3, which proposed to improve additive consistency, takes all the elements of HFLTSs into account. Of course, in actual application, an equivalent Model 4 can be easily constructed and solved. The improved HFLPRs keep the original information of HFLPRs as much as possible. In the additively consistent HFLPRs obtained, all the elements are derived from the original linguistic term set. In the method of Zhang and Wu [19], Zhu and Xu [20], Xu and Wang [21], most of the original linguistic terms in the HFLPR are adjusted and no longer belong to the original linguistic term set. Even some the modified elements are beyond the range of the original linguistic set.

(3) The HFLWV is based on the additively consistent definition proposed in this paper. The HFLWV can be directly obtained from a simple programming model, and the ranking order can be finally determined by score and accurate functions. In hesitant fuzzy linguistic decision-making, HFLWV may be more suitable and accepted by DMs.

This paper mainly takes additive consistency of HFLPRs into account. In light of the fact that consensus issue is also an important topic in group decision-making, group decision-making modeling involving both consistency and consensus issues of HFLPRs will be carried out in our future studies.

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References

1. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [CrossRef]
2. Rodríguez, R.M.; Martinez, L.; Herrera, F. Hesitant fuzzy linguistic term sets for decision making. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 109–119. [CrossRef]
3. Dong, Y.C.; Chen, X.; Herrera, F. Minimizing adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making. *Inf. Sci.* **2015**, *297*, 95–117. [CrossRef]
4. Song, Y.M.; Li, G.X. A mathematical programming approach to manage group decision making with incomplete hesitant fuzzy linguistic preference relations. *Comput. Ind. Eng.* **2019**, *135*, 467–475. [CrossRef]
5. Tang, J.; Meng, F.Y. Decision making with multiplicative hesitant fuzzy linguistic preference relations. *Neural Comput. Appl.* **2019**, *31*, 2749–2761. [CrossRef]
6. Wu, P.; Zhu, J.M.; Zhou, L.G.; Chen, H.Y. Local feedback mechanism based on consistency derived for consensus building in group decision making with hesitant fuzzy linguistic preference relations. *Comput. Ind. Eng.* **2019**, *137*, 106001. [CrossRef]
7. Tang, J.; Meng, F.Y.; Zhang, S.L.; An, Q.X. Group decision making with interval linguistic hesitant fuzzy preference relations. *Expert Syst. Appl.* **2019**, *119*, 231–246. [CrossRef]
8. Tang, J.; Chen, S.M.; Meng, F.Y. Group decision making with multiplicative interval linguistic hesitant fuzzy preference relations. *Inf. Sci.* **2019**, *495*, 215–233. [CrossRef]
9. Chen, X.; Peng, L.J.; Wu, Z.B.; Pedrycz, W. Controlling the worst consistency index for hesitant fuzzy linguistic preference relations in consensus optimization models. *Comput. Ind. Eng.* **2020**, *143*, 106423. [CrossRef]
10. Zhang, Z.M.; Chen, S.M. Group decision making based on acceptable multiplicative consistency and consensus of hesitant fuzzy linguistic preference relations. *Inf. Sci.* **2020**, *541*, 531–550. [CrossRef]

11. Ren, P.J.; Xu, Z.S.; Wang, X.X.; Zeng, X.J. Group decision making with hesitant fuzzy linguistic preference relations based on modified extent measurement. *Expert Syst. Appl.* **2021**, *171*, 114235. [CrossRef]
12. Zheng, C.L.; Zhou, Y.Y.; Zhou, L.G.; Chen, H.Y. Clustering and compatibility-based approach for large-scale group decision making with hesitant fuzzy linguistic preference relations: An application in e-waste recyclin. *Expert Sys. Appl.* **2022**, *197*, 116615. [CrossRef]
13. Xu, Y.; Liu, S.F.; Wang, J.; Shang, X.P. Consensus checking and improving methods for AHP with q-rung dual hesitant fuzzy preference relations. *Expert Syst. Appl.* **2022**, *208*, 117902. [CrossRef]
14. Li, Z.L.; Zhang, Z.; Yu, W.Y. Consensus reaching with consistency control in group decision making with incomplete hesitant fuzzy linguistic preference relations. *Comput. Ind. Eng.* **2022**, *170*, 108311. [CrossRef]
15. Xu, Y.; Herrera, F. Visualizing and rectifying different inconsistencies for fuzzy reciprocal preference relations. *Fuzzy Set. Syst.* **2019**, *362*, 85–109. [CrossRef]
16. Xu, Y.; Wang, Q.; Cabrerizo, F. Methods to improve the ordinal and multiplicative consistency for reciprocal preference relations. *Appl. Soft Comput.* **2018**, *67*, 479–493. [CrossRef]
17. Ren, P.J.; Liu, Z.X.; Zhang, W.G.; Wu, X.L. Consistency and consensus driven for hesitant fuzzy linguistic decision making with pairwise comparisons. *Expert Sys. Appl.* **2022**, *202*, 117307. [CrossRef]
18. Xu, Y.J.; Dai, W.J.; Huang, J.; Li, M.Q.; Herrera, E. Some models to manage additive consistency and derive priority weights from hesitant fuzzy preference relations. *Inf. Sci.* **2022**, *586*, 450–467. [CrossRef]
19. Zhang, Z.M.; Wu, C. On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations. *Knowl.-Based Syst.* **2014**, *72*, 13–27. [CrossRef]
20. Zhu, B.; Xu, Z.S. Consistency measures for hesitant fuzzy linguistic preference relations. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 35–45. [CrossRef]
21. Xu, Y.J.; Wang, H.M. A group consensus decision support model for hesitant 2-tuple fuzzy linguistic preference relations with additive consistency. *J. Intell. Fuzzy Syst.* **2017**, *33*, 41–54. [CrossRef]
22. Feng, X.Q.; Zhang, L.; Wei, C.P. The consistency measures and priority weights of hesitant fuzzy linguistic preference relations. *Appl. Soft Comput.* **2018**, *65*, 79–90. [CrossRef]
23. Li, C.C.; Rodríguez, R.M.; Martínez, L.; Dong, Y.C.; Herrera, F. Consistency of hesitant fuzzy linguistic preference relations: An interval consistency index. *Inf. Sci.* **2018**, *432*, 347–361. [CrossRef]
24. Zhang, Z.M.; Chen, S.M. A consistency and consensus-based method for group decision making with hesitant fuzzy linguistic preference relations. *Inf. Sci.* **2019**, *501*, 317–336. [CrossRef]
25. Liu, N.N.; He, Y.; Xu, Z.S. A new approach to deal with consistency and consensus issues for hesitant fuzzy linguistic preference relations. *Appl. Soft Comput.* **2019**, *76*, 400–415. [CrossRef]
26. Liu, H.B.; Ma, Y.; Jiang, L. Managing incomplete preferences and consistency improvement in hesitant fuzzy linguistic preference relations with applications in group decision making. *Inf. Fusion.* **2019**, *51*, 19–29. [CrossRef]
27. Herrera, F.; Martínez, L. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 746–752.
28. Meng, F.Y.; Chen, S.M.; Zhang, S.L. Group decision making based on acceptable consistency analysis of interval linguistic hesitant fuzzy preference relations. *Inf. Sci.* **2020**, *530*, 66–84. [CrossRef]
29. Liu, H.B.; Jiang, L. Optimizing consistency and consensus improvement process for hesitant fuzzy linguistic preference relations and the application in group decision making. *Inf. Fusion.* **2020**, *56*, 114–127. [CrossRef]
30. Alonso, S.; Cabrerizo, F.; Chiclana, F.; Herrera, F.; Herrera-Viedma, E. Group decision making with incomplete fuzzy linguistic preference relations. *Int. J. Intell. Syst.* **2009**, *24*, 201–222. [CrossRef]