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Abstract: This study examines approximate long wave and the modified Boussinesq equations, as well as their complexes with the Atangana–Baleanu fractional derivative operator in the Caputo sense. The analytical solution of the aforementioned model is discussed using the Elzaki transform and the Adomian decomposition method. These problems are indispensable for defining the characteristics of surface water waves by applying a particular relationship of dispersion. We used Elzaki transformation on time-fractional approximate long wave and modified Boussinesq equations, followed by inverse Elzaki transformation, to achieve the results of the equations. To validate the methodology, we concentrated on two systems and compared them to the actual solutions. The numerical and graphical results demonstrate that the proposed method is computationally precise and straightforward for investigating and resolving fractionally coupled nonlinear phenomena that occur in scientific and technological.

Keywords: fractional approximate long wave equations; Atangana–Baleanu operator; fractional modified Boussinesq equations; Elzaki transform; Adomian decomposition method

1. Introduction

In recent years, fractional partial differential equations (PDEs) have achieved popularity and validation due to their demonstrated applicability in a broad range of applied science fields of study. Evaluating that the nonlinear oscillation of fractional derivatives can be used to model earthquakes, the fractional fluid-dynamic traffic model can be used to compensate for the shortcoming caused by the presumption of continued traffic flow. Numerous researchers, such as Coimbra, Riemann–Liouville, Riesz, Weyl, Hadamard, Caputo–Fabrizio, Atangana–Baleanu, Jumarie, Grunwald–Letnikov, Liouville, and Caputo, have proposed various fractional operator formulations and concepts. In addition, fractional partial differential equations are used to model numerous physical phenomena, such as population dynamics, chemical reaction, image processing, virology, thermodynamics, bifurcation, porous media, Levy statistics, physics, and engineering problems [1–6].
In the 20th century, Whitham [14], Broer [15], and Kaup [16] investigated the Whitham–Broer–Kaup (WBK) equations, which explain the propagation of shallow water with a different dispersion comparison. Consider the system of fractional-order Whitham–Broer–Kaup equations [17]:

\[
\begin{align*}
\frac{\partial^\beta u}{\partial \tau^\beta} + u \frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial \psi} + b \frac{\partial^2 u}{\partial \psi^2} &= 0, \\
\frac{\partial^\beta v}{\partial \tau^\beta} + u \frac{\partial v}{\partial \psi} + v \frac{\partial u}{\partial \psi} + a \frac{\partial^3 u}{\partial \psi^3} - b \frac{\partial^2 v}{\partial \psi^2} &= 0, \quad 0 < \beta \leq 1.
\end{align*}
\]

(1)

where \( \frac{\partial^\beta}{\partial \tau^\beta} \) is the fractional derivative, \( u = u(\psi, \tau) \) is the horizontal velocity and \( v = v(\psi, \tau) \) is the height that is different from the liquid’s equilibrium position. In addition, since \( a \) and \( b \) are constants representing various diffusion strengths, Equation (1) becomes a modified Boussinesq equation when \( a = 1 \) and \( b = 0 \). Similarly, for \( a = 0 \) and \( b = 1 \), the scheme represents the standard long wave model. These relations are derived from fluid mechanics to demonstrate the propagation of waves in dissipative and nonlinear media. They are suggested for models via the water leakage through porous groundwater strata and are broadly used in coastal and marine ecosystems technology [18–20]. In addition, Equation (1) is the basis for different designs used to represent the unconfined groundwater, such as drains and surface water fluid models.

The Adomian decomposition method was developed by the American scientist G. Adomian. It emphasizes the search for a set of solutions and the decomposition of the non-linear operator into a sequence in which Adomian polynomials are repeatedly computed using the terms. This method is enhanced by the Elzaki transformation, and the enhanced method is called the Elzaki decomposition method [21–24]. Elzaki transform is a contemporary integral transform that Tarig Elzaki introduced in 2010. Elzaki transformation is an altered combination of the Sumudu and Laplace transforms [25–27].

2. Preliminaries

**Definition 1.** The fractional Caputo derivative (CFD) is defined as [28–31]:

\[
D^\beta_C(\ell(\varphi)) = \begin{cases}
\frac{1}{\Gamma(m-\beta)} \int_0^\varphi \frac{\ell^m(\eta)}{(\varphi-\eta)^{\beta+m}} d\eta, & m-1 < \beta < m, \\
\frac{\partial^\beta}{\partial \varphi^\beta} \ell(\varphi), & \beta = m.
\end{cases}
\]

(2)

**Definition 2.** The derivative in terms of Atangana–Baleanu Caputo manner (ABC) is defined as [28–31]:

\[
D^\beta_A(\ell(\varphi)) = \frac{N(\beta)}{1-\beta} \int_m^\varphi \ell(\eta) E_\beta \left[ -\frac{\beta(\varphi - \eta)^\beta}{1-\beta} \right] d\eta,
\]

(3)

\( N(\beta) \) is a normalization function equal to 1 when \( \beta = 0 \) and \( \beta = 1 \) is represented by \( N(\beta) \) in Equation (3).

**Definition 3.** The ABC fractional integral operator is as [28–31]

\[
I^\beta_A(\ell(\varphi)) = \frac{1 - \beta}{N(\beta)} \ell(\varphi) + \frac{\beta}{\Gamma(\beta)} \int_m^\varphi \ell(\eta)(\varphi - \eta)^{\beta-1} d\eta.
\]

(4)

**Definition 4.** For the function \( \ell(\varphi) \), the transform in term of Elzaki is as [28–31]

\[
\mathcal{E}\{\ell(\varphi)\}(\omega) = \hat{\omega}(\omega) = \omega \int_0^\infty e^{-\omega \varphi} \ell(\varphi) d\varphi,
\]

(5)

where \( \varphi \geq 0, p_1 \leq \omega \leq p_2 \).
Theorem 1. Elzaki transformation convolution theorem, The following equality holds:

\[ E\{\ell * v\} = \frac{1}{\omega} E(\ell)E(v), \]  

(6)

where the Elzaki transform is indicated by \( E\{\cdot\} \).

Definition 5. The Elzaki transform of the CFD operator \( D^\beta_{\psi}\ell(\psi) \) is given by [28–31]

\[ E\{D^\beta_{\psi}\ell(\psi)\}(\omega) = \omega^{-\beta}\bar{U}(\omega) - \sum_{k=0}^{m-1} \omega^{2-\beta+k}\ell^k(0), \]  

(7)

where \( m - 1 < \beta < m \).

Theorem 2. The fractional Elzaki transform of ABC derivative \( D^\beta_{\psi}\ell(\psi) \) is defined as

\[ E\{D^\beta_{\psi}\ell(\psi)\}(\omega) = \frac{N(\beta)\omega}{\beta\omega^\beta + 1 - \beta} \left( \frac{\bar{U}(\omega)}{\omega} - \omega\ell(0) \right), \]  

(8)

where \( E\{\ell(\psi)\}(\omega) = \bar{U}(\omega) \).

Proof. From Definition 2, we have:

\[ E\{D^\beta_{\psi}\ell(\psi)\}(\omega) = E\left\{ \frac{N(\beta)}{1-\beta} \int_0^\psi \ell'(\eta)E_\beta \left[ \frac{-\beta(\psi - \eta)^\beta}{1-\beta} \right] d\eta \right\}(\omega). \]  

(9)

Then, taking into account the definition and convolution of the Elzaki transform, we get

\[ E\{D^\beta_{\psi}\ell(\psi)\}(\omega) = E\left\{ \frac{N(\beta)}{1-\beta} \int_0^\psi \ell'(\eta)E_\beta \left[ \frac{-\beta(\psi - \eta)^\beta}{1-\beta} \right] d\eta \right\}(\omega) \]

\[ = \frac{N(\beta)}{1-\beta} \omega \cdot E\{\ell'(\eta)E_\beta \left[ \frac{-\beta(\psi - \eta)^\beta}{1-\beta} \right] d\eta \} \]

\[ = \frac{N(\beta)}{1-\beta} \omega \left[ \frac{\bar{U}(\omega)}{\omega} - \omega\ell(0) \right] \int_0^\infty e^{-\frac{s}{\omega}}E_\beta \left[ \frac{-\beta\psi^\beta}{1-\beta} \right] d\sigma \]

\[ = \frac{N(\beta)\omega}{\beta\omega^\beta + 1 - \beta} \left[ \frac{\bar{U}(\omega)}{\omega} - \omega\ell(0) \right]. \]  

\[ \Box \]

3. Methodology

Consider the general fractional partial differential equation is given as

\[ D^\beta_{\tau}\nu(\psi, \tau) = \mathcal{L}(\nu(\psi, \tau)) + M(\nu(\psi, \tau)) + h(\psi, \tau), \]  

(11)

with initial condition

\[ \nu(\psi, 0) = \phi(\psi), \]  

(12)

here \( \mathcal{L} \) and \( M \) are linear and non-linear terms and source function is \( h(\psi, \tau) \).

Applying the Elzaki transform of Equation (11), we get

\[ \frac{1}{\beta\omega^\beta + 1 - \beta} \left( E[\nu(\psi, \tau)] - \frac{\phi(\psi)}{s} \right) = E[\mathcal{L}(\nu(\psi, \tau)) + M(\nu(\psi, \tau)) + h(\psi, \tau)], \]  

(13)
Using inverse Elzaki transform (13), we have
\[ \nu(\psi, \tau) = E^{-1} \left( s^2 \phi(\psi) + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E[\mathcal{L}(\nu(\psi, \tau)) + M(\nu(\psi, \tau)) + h(\psi, \tau)] \right). \] (14)

The nonlinear term \( M(\nu(\psi, \tau)) \) is defined as
\[ M(\nu(\psi, \tau)) = \sum_{i=0}^{\infty} A_i, \] (15)
where \( A_i \) represents the Adomian polynomials and they are expressed as
\[ \nu(\psi, \tau) = \sum_{i=0}^{\infty} \nu_i(\psi, \tau). \] (16)

By putting Equations (15) and (16) into (14), we get
\[ \sum_{i=0}^{\infty} \nu_i(\psi, \tau) = E^{-1} \left( s^2 \phi(\psi) + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E[\mathcal{L}(\nu(\psi, \tau)) + A_i] \right) \] (17)

From (8), we get
\[ \nu_{ABC}^0(\psi, \tau) = E^{-1} \left( s^2 \phi(\psi) + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E[\mathcal{L}(\nu_0(\psi, \tau)) + A_0] \right), \]
\[ \nu_{ABC}^1(\psi, \tau) = E^{-1} \left( \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E[\mathcal{L}(\nu_1(\psi, \tau)) + A_1] \right), \] (18)
\[ \vdots \]
\[ \nu_{ABC}^{l+1}(\psi, \tau) = E^{-1} \left( \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E[\mathcal{L}(\nu_l(\psi, \tau)) + A_l] \right), \] \( l = 1, 2, 3, \ldots \)
we achieved the EDM result of (11) by put (18) into (16),
\[ \nu_{ABC}(\psi, \tau) = \nu_{ABC}^0(\psi, \tau) + \nu_{ABC}^1(\psi, \tau) + \nu_{ABC}^2(\psi, \tau) + \cdots \] (19)

4. Numerical Examples

Example 1. Consider the fractional modified Boussinesq equations
\[ D_{\tau}^{\beta} u = -u \frac{\partial u}{\partial \psi} - \frac{\partial v}{\partial \phi}, \]
\[ D_{\tau}^{\beta} v = -v \frac{\partial v}{\partial \phi} - u \frac{\partial u}{\partial \phi} - \frac{\partial^3 u}{\partial \phi^3}, \quad 0 < \beta \leq 1, \] (20)
with the initial conditions
\[ u(\psi, 0) = \theta - 2\kappa \coth[\kappa(\psi + c)], \]
\[ v(\psi, 0) = -2\kappa^2 \text{csch}^2[\kappa(\psi + c)]. \] (21)
Using the Elzaki transformation to Equation (20), we get

\[ E[ D^\beta \tau u(\psi, \tau)] = -E \left\{ u \frac{\partial u}{\partial \psi} \right\} - E \left\{ v \frac{\partial u}{\partial \psi} \right\}, \]

\[ E[ D^\beta \tau v(\psi, \tau)] = -E \left\{ \frac{\partial v}{\partial \psi} \right\} - E \left\{ \frac{\partial u}{\partial \psi} \right\} - E \left\{ \frac{\partial^3 u}{\partial \psi^3} \right\}. \] (22)

The nonlinear term can be expressed as

\[ \frac{N(\beta)}{\beta s^\beta + 1 - \beta} E[ u(\psi, \tau) - s^2 u(\psi, 0)] = E \left[ -u \frac{\partial u}{\partial \psi} - \frac{\partial v}{\partial \psi} \right], \]

\[ \frac{N(\beta)}{\beta s^\beta + 1 - \beta} E[ v(\psi, \tau) - s^2 v(\psi, 0)] = E \left[ -v \frac{\partial v}{\partial \psi} - \frac{\partial u}{\partial \psi} - \frac{\partial^3 u}{\partial \psi^3} \right]. \] (23)

By making it easier, the above equation can be written as

\[ E[ u(\psi, \tau)] = s^2 \left[ \theta - 2\kappa \coth[\kappa(\psi + c)] \right] + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E \left[ -u \frac{\partial u}{\partial \psi} - \frac{\partial v}{\partial \psi} \right], \]

\[ E[ v(\psi, \tau)] = s^2 \left[ -2\kappa^2 \text{csch}^2[\kappa(\psi + c)] \right] + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E \left[ -v \frac{\partial v}{\partial \psi} - \frac{\partial u}{\partial \psi} - \frac{\partial^3 u}{\partial \psi^3} \right]. \] (24)

We obtain by using inverse Elzaki transform to Equation (24), we get

\[ u(\psi, \tau) = \left[ \theta - 2\kappa \coth[\kappa(\psi + c)] \right] \]

\[ + E^{-1} \left[ \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E \left\{ -u \frac{\partial u}{\partial \psi} - \frac{\partial v}{\partial \psi} \right\} \right], \] (25)

\[ v(\psi, \tau) = \left[ -2\kappa^2 \text{csch}^2[\kappa(\psi + c)] \right] \]

\[ + E^{-1} \left[ \frac{\beta s^\beta + 1 - \beta}{N(\beta)} E \left\{ -v \frac{\partial v}{\partial \psi} - \frac{\partial u}{\partial \psi} - \frac{\partial^3 u}{\partial \psi^3} \right\} \right]. \]

Now we are implementing EDM Suppose that the unknown terms \( u(\psi, \tau) \) and \( v(\psi, \tau) \) obtain the infinite series solution is given as

\[ u(\psi, \tau) = \sum_{i=0}^{\infty} u_i(\psi, \tau) \text{ and } v(\psi, \tau) = \sum_{i=0}^{\infty} v_i(\psi, \tau) \] (26)

The Adomian polynomials represented the nonlinear functions and are expressed by \( uu_\psi = \sum_{i=0}^{\infty} A_i, uv_\psi = \sum_{i=0}^{\infty} B_i, \) and \( vv_\psi = \sum_{i=0}^{\infty} C_i. \)
\[ \sum_{i=0}^{\infty} u_{i+1}(\psi, \tau) = \theta - 2 \kappa \coth[k(\psi + c)] + \mathcal{E}^{-1} \left[ \frac{\beta s^2 + 1 - \beta}{N(\beta)} \mathcal{E} \left\{ - \sum_{i=0}^{\infty} A_i - \sum_{i=0}^{\infty} v_{i\psi} \right\} \right], \]

\[ \sum_{i=0}^{\infty} v_{i+1}(\psi, \tau) = -2 \kappa^2 \operatorname{csch}^2[k(\psi + c)] + \mathcal{E}^{-1} \left[ \frac{\beta s^2 + 1 - \beta}{N(\beta)} \mathcal{E} \left\{ - \sum_{i=0}^{\infty} B_i - \sum_{i=0}^{\infty} C_i - \sum_{i=0}^{\infty} u_{i\psi\psi} \right\} \right]. \tag{27} \]

On both sides, comparing Equation (27)

\[ u_0(\psi, \tau) = \theta - 2 \kappa \coth[k(\psi + c)], \]
\[ v_0(\psi, \tau) = -2 \kappa^2 \operatorname{csch}^2[k(\psi + c)]. \]

\[ u_1(\psi, \tau) = -2 \kappa^2 \theta \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right), \]
\[ v_1(\psi, \tau) = -4 \kappa^3 \theta \coth[k(\psi + c)] \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right). \tag{28} \]

\[ u_2(\psi, \tau) = 2 \kappa^2 \theta \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right) - 4 \kappa^3 \theta^2 \coth[k(\psi + c)] \operatorname{csch}^2[k(\psi + c)] \]
\[ \left[ \frac{\beta^2 \tau^2}{\Gamma(2\beta + 1)} + 2 \beta (1 - \beta) \frac{\tau^2}{\Gamma(\beta + 1)} + (1 - \beta)^2 \right], \tag{29} \]
\[ v_2(\psi, \tau) = 4 \kappa^3 \theta \coth[k(\psi + c)] \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right) - 4 \kappa^4 \theta^2 (2 + \cosh[2\kappa(\psi + c)]) \]
\[ \operatorname{csch}^4[k(\psi + c)] \left[ \frac{\beta^2 \tau^2}{\Gamma(2\beta + 1)} + 2 \beta (1 - \beta) \frac{\tau^2}{\Gamma(\beta + 1)} + (1 - \beta)^2 \right]. \]

The remaining terms of the EDM solution \( u_i \) and \( v_i \) for \( i \geq 3 \) can be achieved simply.

The series type solution is determined as

\[ u(\psi, \tau) = \sum_{i=0}^{\infty} u_i(\psi, \tau) = u_0(\psi, \tau) + u_1(\psi, \tau) + u_2(\psi, \tau) + \cdots, \]

\[ u(\psi, \tau) = \theta - 2 \kappa \coth[k(\psi + c)] - 2 \kappa^2 \theta \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right) + 2 \kappa^2 \theta \operatorname{csch}^2[k(\psi + c)] \]
\[ \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right) - 4 \kappa^3 \theta^2 \coth[k(\psi + c)] \operatorname{csch}^2[k(\psi + c)] \left[ \frac{\beta^2 \tau^2}{\Gamma(2\beta + 1)} + 2 \beta (1 - \beta) \frac{\tau^2}{\Gamma(\beta + 1)} + (1 - \beta)^2 \right] + \cdots. \tag{30} \]

\[ v(\psi, \tau) = \sum_{i=0}^{\infty} v_i(\psi, \tau) = v_0(\psi, \tau) + v_1(\psi, \tau) + v_2(\psi, \tau) + \cdots, \]

\[ v(\psi, \tau) = -2 \kappa^2 \operatorname{csch}^2[k(\psi + c)] - 4 \kappa^3 \theta \coth[k(\psi + c)] \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right) + 4 \kappa^3 \theta \coth[k(\psi + c)] \]
\[ \operatorname{csch}^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^2}{\Gamma(\beta + 1)} \right) - 4 \kappa^4 \theta^2 (2 + \cosh[2\kappa(\psi + c)]) \operatorname{csch}^4[k(\psi + c)] \]
\[ \left[ \frac{\beta^2 \tau^2}{\Gamma(2\beta + 1)} + 2 \beta (1 - \beta) \frac{\tau^2}{\Gamma(\beta + 1)} + (1 - \beta)^2 \right] + \cdots. \tag{31} \]
For Equation (20), the exact solutions are achieved at $\beta = 1$

\[
\begin{align*}
    u(\psi, \tau) &= \theta - 2\kappa \coth[\kappa(\psi + c - \theta\tau)], \\
    v(\psi, \tau) &= -2\kappa^2 \csc h^2[\kappa(\psi + c - \theta\tau)].
\end{align*}
\] 

(32)

In Figure 1, the actual and analytical results for $u(\psi, \tau)$ at $\beta = 1$ of Example 1 with show that they closed contact with each other. Figure 2, analytic result figure for $u(\psi, \tau)$ at $\beta = 0.8$ and 0.6 and Figure 3, analytic result figure for $u(\psi, \tau)$ at different value of $\beta$ of Example 1. Figure 4, the actual and analytic result of $v(\psi, \tau)$ at $\beta = 1$ of Example 1 with a show that they closed contact with each other and Figure 5, the analytic result graph for $v(\psi, \tau)$ at different value of $\beta$ of Example 1. In Tables 1 and 2 the actual and analytical solutions of $u(\psi, \tau)$ and $v(\psi, \tau)$ of Example 1 at a different fractional-order of $\beta$ having various value of $\psi$ and $\tau$. 

![Figure 1](image1.png)

**Figure 1.** The actual and analytical results for $u(\psi, \tau)$ at $\beta = 1$ of Example 1.

![Figure 2](image2.png)

**Figure 2.** The analytic result figure for $u(\psi, \tau)$ at $\beta = 0.8$ and 0.6.
Figure 3. The analytic result figure for $u(\psi, \tau)$ at a different value of $\beta$ of Example 1.

Figure 4. The actual and analytic result of $v(\psi, \tau)$ at $\beta = 1$ of Example 1.

Figure 5. The analytic result figure for $v(\psi, \tau)$ at a different value of $\beta$ of Example 1.
Table 1. The actual and analytical solutions of \( u(\psi, \tau) \) of Example 1 at a different fractional-order of \( \beta \) having various values of \( \psi \) and \( \tau \).

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<th>( \psi )</th>
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<th>( \beta = 0.6 )</th>
<th>( \beta = 0.8 )</th>
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Table 2. The actual and analytical solutions of \( v(\psi, \tau) \) of Example 1 at a different fractional-order of \( \beta \) having various values of \( \psi \) and \( \tau \).

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Example 2. Consider the fractional order approximate long wave equations

\[ D_\tau^\beta u = -u \frac{\partial u}{\partial \tau} - v \frac{\partial v}{\partial \tau} - \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2}, \]
\[ D_\tau^\beta v = -u \frac{\partial u}{\partial \tau} - \frac{v}{\partial \tau} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2}, \quad 0 < \beta \leq 1, \]

with the initial conditions

\[ u(\psi, 0) = \theta - \kappa \coth[\kappa(\psi + c)], \]
\[ v(\psi, 0) = -\kappa^2 \text{csch}^2[\kappa(\psi + c)]. \]  

Using the Elzaki transformation to Equation (33), we get

\[ E[D_\tau^\beta u(\psi, \tau)] = -E \left[ \frac{\partial u}{\partial \tau} \right] - E \left[ \frac{\partial v}{\partial \tau} \right] - \frac{1}{2} E \left[ \frac{\partial^2 v}{\partial \psi^2} \right], \]
\[ E[D_\tau^\beta v(\psi, \tau)] = -E \left[ \frac{\partial v}{\partial \tau} \right] + E \left[ \frac{\partial u}{\partial \tau} \right] + \frac{1}{2} E \left[ \frac{\partial^2 v}{\partial \psi^2} \right]. \]

The nonlinear term can be written as

\[ \frac{N(\beta)}{\beta s^\beta + 1 - \beta} E[u(\psi, \tau) - s^2 u(\psi, 0)] = E \left[ -u \frac{\partial u}{\partial \psi} - \frac{v}{\partial \psi} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2} \right], \]
\[ \frac{N(\beta)}{\beta s^\beta + 1 - \beta} E[v(\psi, \tau) - s^2 v(\psi, 0)] = E \left[ -u \frac{\partial v}{\partial \psi} + \frac{\partial u}{\partial \psi} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2} \right]. \]

By making it easier, the above equation can be written as

\[ E[u(\psi, \tau)] = s^2 \left[ \theta - \kappa \coth[\kappa(\psi + c)] \right] + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} \left[ -u \frac{\partial u}{\partial \psi} - \frac{v}{\partial \psi} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2} \right], \]
\[ E[v(\psi, \tau)] = s^2 \left[ -\kappa^2 \text{csch}^2[\kappa(\psi + c)] \right] + \frac{\beta s^\beta + 1 - \beta}{N(\beta)} \left[ -u \frac{\partial v}{\partial \psi} + \frac{\partial u}{\partial \psi} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2} \right]. \]

Applying inverse Elzaki transform to Equation (37), we get

\[ u(\psi, \tau) = \theta - \kappa \coth[\kappa(\psi + c)] + E^{-1} \left[ \frac{\beta s^\beta + 1 - \beta}{N(\beta)} - u \frac{\partial u}{\partial \psi} - \frac{\partial v}{\partial \psi} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2} \right], \]
\[ v(\psi, \tau) = -\kappa^2 \text{csch}^2[\kappa(\psi + c)] + E^{-1} \left[ \frac{\beta s^\beta + 1 - \beta}{N(\beta)} - \frac{\partial v}{\partial \psi} + \frac{\partial u}{\partial \psi} + \frac{1}{2} \frac{\partial^2 v}{\partial \psi^2} \right]. \]

Now we apply EDM Suppose that the unknown terms \( u(\psi, \tau) \) and \( v(\psi, \tau) \) obtain the infinite series solution as

\[ u(\psi, \tau) = \sum_{i=0}^{\infty} u_i(\psi, \tau) \quad \text{and} \quad v(\psi, \tau) = \sum_{i=0}^{\infty} v_i(\psi, \tau) \]

The nonlinear functions expressed with the help of Adomian polynomials

\[ uu_\psi = \sum_{i=0}^{\infty} A_i, \quad u_\psi = \sum_{i=0}^{\infty} B_i, \quad \text{and} \quad vv_\psi = \sum_{i=0}^{\infty} C_i. \]
\[
\sum_{i=0}^{\infty} u_i(\psi, \tau) = \left[ \theta - \kappa \coth[k(\psi + c)] \right] + e^{-1} \left[ \beta \tau^\beta + 1 - \beta \right] \left( \frac{N(\beta)}{c} \right) \left( - \sum_{i=0}^{\infty} A_i - \sum_{i=0}^{\infty} B_i - \sum_{i=0}^{\infty} C_i - \frac{1}{2} \sum_{i=0}^{\infty} v_i \right),
\]

\[
\sum_{i=0}^{\infty} v_i(\psi, \tau) = \left[ - \kappa^2 \csch^2[k(\psi + c)] \right] + e^{-1} \left[ \beta \tau^\beta + 1 - \beta \right] \left( \frac{N(\beta)}{c} \right) \left( - \sum_{i=0}^{\infty} A_i - \sum_{i=0}^{\infty} B_i + \sum_{i=0}^{\infty} C_i + \frac{1}{2} \sum_{i=0}^{\infty} v_i \right).
\]

On both sides, comparing the Equation (40), we get

\[
u_0(\psi, \tau) = \theta - \kappa \coth[k(\psi + c)],
\]

\[
u_0(\psi, \tau) = -\kappa^2 \csch^2[k(\psi + c)].
\]

\[
u_1(\psi, \tau) = -\kappa^2 \theta \csch^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^\beta}{\Gamma(\beta + 1)} \right),
\]

\[
u_1(\psi, \tau) = -2\kappa^3 \theta \coth[k(\psi + c)] \csch^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^\beta}{\Gamma(\beta + 1)} \right).
\]

\[
u_2(\psi, \tau) = -\kappa^2 \theta \csch^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^\beta}{\Gamma(\beta + 1)} \right) - 2\kappa^3 \theta^2 \coth[k(\psi + c)] \csch^2[k(\psi + c)] \left[ \frac{\beta^2 \tau^2 \tau^\beta}{\Gamma(\beta + 1)^2} + 2\beta(1 - \beta) \right],
\]

\[
u_2(\psi, \tau) = \csch^4[k(\psi + c)] \left[ \frac{\beta^2 \tau^2 \tau^\beta}{\Gamma(\beta + 1)^2} + 2\beta(1 - \beta) \right] \frac{\tau^\beta}{\Gamma(\beta + 1)} + (1 - \beta^2) \right] + \cdots.
\]

The remaining terms of the EDM solution \( u_i \) and \( v_i \) \((i \geq 3)\) can be obtained simply.

The series type solution is determined as

\[
u(\psi, \tau) = \sum_{i=0}^{\infty} v_i(\psi, \tau) = \nu_0(\psi, \tau) + \nu_1(\psi, \tau) + \nu_2(\psi, \tau) + \cdots,
\]

\[
u(\psi, \tau) = -\kappa^2 \csch^2[k(\psi + c)] - 2\kappa^3 \theta \coth[k(\psi + c)] \csch^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^\beta}{\Gamma(\beta + 1)} \right)
\]

\[
- \kappa^2 \theta \csch^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^\beta}{\Gamma(\beta + 1)} \right) - 2\kappa^3 \theta^2 \coth[k(\psi + c)] \csch^2[k(\psi + c)] \left[ \frac{\beta^2 \tau^2 \tau^\beta}{\Gamma(\beta + 1)^2} + 2\beta(1 - \beta) \right] + \cdots,
\]

\[
u(\psi, \tau) = \sum_{i=0}^{\infty} v_i(\psi, \tau) = \nu_0(\psi, \tau) + \nu_1(\psi, \tau) + \nu_2(\psi, \tau) + \cdots,
\]

\[
u(\psi, \tau) = -2\kappa^3 \theta \coth[k(\psi + c)] \csch^2[k(\psi + c)] \left( 1 - \beta + \frac{\beta \tau^\beta}{\Gamma(\beta + 1)} \right) - 2\kappa^3 \theta \coth[k(\psi + c)] \csch^2[k(\psi + c)] \left[ \frac{\beta^2 \tau^2 \tau^\beta}{\Gamma(\beta + 1)^2} + 2\beta(1 - \beta) \right] + \cdots.
\]
For Equation (33), the exact solutions is achieved at $\beta = 1$

\begin{align*}
u(\psi, \tau) &= \theta - \kappa \coth[\kappa(\psi + c - \theta \tau)] , \\
v(\psi, \tau) &= -\kappa^2 \text{csch}^2[\kappa(\psi + c - \theta \tau)].
\end{align*} \tag{42}

In Figure 6, the actual and analytical results for $u(\psi, \tau)$ at $\beta = 1$ of Example 2 with a show that they closed contact with each other. Figure 7, analytic result figure for $u(\psi, \tau)$ at $\beta = 0.8$ and 0.6 and Figure 8, analytic result figure for $u(\psi, \tau)$ at different value of $\beta$ of Example 2. Figure 9, the actual and analytic result of $v(\psi, \tau)$ at $\beta = 1$ of Example 2 with show that the closed contact with each other and Figure 10, the analytic result graph for $v(\psi, \tau)$ at different value of $\beta$ of Example 2. In Tables 3 and 4, the exact and EDM solutions of $u(\psi, \tau)$ and $v(\psi, \tau)$ of Example 2 at different fractional-order of $\beta$ having various values of $\psi$ and $\tau$.

**Table 3.** The exact and EDM solutions of $u(\psi, \tau)$ of Example 2 at different fractional-order of $\beta$ having various values of $\psi$ and $\tau$.

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**Figure 6.** The exact and analytical results for $u(\psi, \tau)$ at $\beta = 1$ of Example 2.
Table 4. The exact and EDM solutions of $u(\psi, \tau)$ of Example 2 at a different fractional-order of $\beta$ having various values of $\psi$ and $\tau$.

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Figure 7. The analytic result figure for $u(\psi, \tau)$ at $\beta = 0.8$ and 0.6 of Example 2.

Figure 8. The analytic result figure for $u(\psi, \tau)$ at different values of $\beta$ of Example 2.
5. Conclusions

In this study, the Elzaki decomposition method is applied to analyze the system of approximate long wave and modified Boussinesq equations. The two problems are investigated to calculate and validate the accuracy of the proposed method. Compared to the current analytic method for finding approximate results in the system of fractional nonlinear partial differential equations, the proposed methodology is quick and easy. The solutions have been demonstrated in the shape of tables and graphs. The suggested technique makes a list of results in the form of a given equation, which is more accurate and requires less work. For both fractionally coupled systems, calculations were done to figure out the absolute error. At $\beta = 1$, a number of mathematical results are evaluated by comparing what is known as the analytic method and the actual solutions. The current methods are better because they require less math and are more accurate. Moreover, it is shown that the presented method is very simple to use, it works, and can be used to solve a different fractional system of differential equations. In the future, we plan to extend the application of the Elzaki decomposition method to analyze higher dimensional physical applications. In particular, we will consider the recent implementations of fractional-order problems.

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