Extremal Structure on Revised Edge-Szeged Index with Respect to Tricyclic Graphs

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Abstract: For a given graph $G$, $Sz^*_v(G) = \sum_{e=uv\in E(G)} \left( m_u(e) + \frac{m_u(e)}{2} \right) \left( m_v(e) + \frac{m_v(e)}{2} \right)$ is the revised edge-Szeged index of $G$, where $m_u(e)$ and $m_v(e)$ are the number of edges of $G$ lying closer to vertex $u$ than to vertex $v$ and the number of edges of $G$ lying closer to vertex $v$ than to vertex $u$, respectively, and $m_0(e)$ is the number of edges equidistant to $u$ and $v$. In this paper, we identify the lower bound of the revised edge-Szeged index among all tricyclic graphs and also characterize the extremal structure of graphs that attain the bound.

Keywords: Wiener index; revised Szeged index; revised edge-Szeged index; tricyclic graphs

MSC: 05C92; 05C12; 05C35; 05C38; 05C75; 05C76

1. Introduction

All graphs considered in this paper are finite, undirected, and simple. The molecular topological index is constructed from the molecular graph mapping, which has applications in physics, chemistry, and network theory. There are relationships between topological indices and some physical properties or some chemical properties. There are nearly a thousand kinds of topological indices that have been proposed since the Wiener index appeared in 1947. The Wiener index is one of the most important chemical topological indexes. For $u, v \in V(G)$, $d(u, v)$ denotes the distance between $u$ and $v$ in $G$. The Wiener index of graph $G$ is defined as follows:

$$W(G) = \sum_{\{u, v\} \subseteq V(G)} d(u, v).$$

It has been extensively studied, see [1–4]. The properties and applications of topological indices have been reported in [5–16]. Let $e = uv$ be an edge of $G$ and define three subsets of $V(G)$ as follows:

$$N_u(e) = \{ w \in V(G) : d(u, w) < d(v, w) \},$$
$$N_v(e) = \{ w \in V(G) : d(u, w) > d(v, w) \},$$
$$N_0(e) = \{ w \in V(G) : d(u, w) = d(v, w) \}.$$

Evidently, $N_u(e), N_v(e), N_0(e)$ consists of a partition of vertices set $V(G)$ with respect to $e$. The number of vertices of $N_u(e), N_v(e), N_0(e)$ are denoted by $n_u(e), n_v(e), n_0(e)$, respectively. Note that $n_u(e) + n_v(e) + n_0(e) = n$ for $|G| = n$. In particular, $G$ is bipartite implies that $n_0(e) = 0$ holds for each edge $e \in E(G)$; so, $n_u(e) + n_v(e) = n$.

When $T$ is a tree, the Wiener index has the following formula

$$W(T) = \sum_{e=uv\in E(T)} n_u(e)n_v(e).$$
Motivated by the above formula, Gutman produced a new graph invariant named the Szeged index and defined it by

$$S_z(G) = \sum_{e=uv \in E(G)} n_u(e)n_v(e),$$

where $G$ is an arbitrary graph, not necessarily connected. Randić discovered that Gutman did not consider the contributions of these vertices, which are equidistant to $u$ and $v$. Thus, he conceived a correctional version of the Szeged index and named it the revised Szeged index. The revised Szeged index of a connected graph $G$ is defined as

$$S_{z^*}(G) = \sum_{e=uv \in E(G)} \left(n_u(e) + \frac{m_0(e)}{2}\right)\left(n_v(e) + \frac{m_0(e)}{2}\right).$$

In addition, for a connected graph $G$, it is well known that $S_{z^*}(G) \geq S_z(G) \geq W(G)$. Let an edge $e = uv \in E(G)$; the distance between the edge $e$ and the vertex $x$, denoted by $d(e, x)$, is defined as $d(e, x) = \min\{d(u, x), d(v, x)\}$. Similarly, $M_0(e)$ is the set of edges equidistant from $u$ and $v$, $M_u(e)$ is the set of edges whose distance to vertex $u$ is smaller than the distance to vertex $v$, and $M_v(e)$ is the set of edges closer to $v$ than $u$. The number of edges of $M_0(e)$, $M_u(e)$, $M_v(e)$ are marked as $m_0(e), m_u(e), m_v(e)$, respectively. Thus, the edge-Szeged index [17] and the revised edge-Szeged index [18] of $G$ are defined below:

$$S_{z_e}(G) = \sum_{e=uv \in E(G)} m_u(e)m_v(e),$$

$$S_{z^*_e}(G) = \sum_{e=uv \in E(G)} \left(m_u(e) + \frac{m_0(e)}{2}\right)\left(m_v(e) + \frac{m_0(e)}{2}\right).$$

The results of the edge-Szeged index can be found in [19–22]. In [7,10,18], the authors determined the maximum values regarding the revised edge-Szeged index for unicyclic graphs, bicyclic graphs, and tricyclic graphs, respectively. In addition, Ref. [9] determined unicyclic graphs with the minimum value of the revised edge-Szeged index. Ref. [23] identified those graphs having the minimum value of the revised edge-Szeged index among all bicyclic graphs. Motivated by these results, we study the revised edge-Szeged index on tricyclic graphs. In this paper, we obtain a lower bound of the revised edge-Szeged index for connected tricyclic graphs.

2. Preliminaries

In the section, we will present some notations and results. If a graph $H$ is obtained by getting rid of as many pendants of $G$ as possible, we say that $H$ is the brace of $G$. All braces of tricyclic graphs are shown in Figure 1. Let $B_m$ be the set of tricyclic graphs of order $m$ and $B_m^i$ be the set whose element contains $a_i$ as its brace for $i = 1, 2, \ldots, 15$. Clearly, $B_m = \bigcup_{i=1}^{15} B_m^i$. For convenience, let $B_1 = \bigcup_{i=5}^{15} B_m^i$. We call an edge $e = uv$ a pendant edge if $d(u) \geq 3$ and $d(v) = 1$. $G \cong G_1 \cdot G_2$ is the graph obtained by fusing two vertices as a new vertex from $G_1$ and $G_2$, respectively, and name it $w$. $w$ is the fusing vertex of $G$. Clearly, $w$ is a cut vertex of $G$. For every $e = uv \in E(G_1)$, $w$ belongs to one of the three sets $N_u(e), N_v(e), N_0(e)$. Since every path from $u$ (or $v$) to each vertex in $V(G_2)$ is via $w$, all edges of $G_2$ must be contained in one of the three sets $M_u(e), M_v(e), M_0(e)$ (they are similar to $w$). Therefore, the contribution of $G_2$ to the item $(m_u(e) + \frac{m_0(e)}{2})(m_v(e) + \frac{m_0(e)}{2})$ completely relies on the number of the edges in $G_2$; in other words, changing the structure of $G_2$ cannot alter the value of $\sum_{e=uv \in E(G_1)} (m_u(e) + \frac{m_0(e)}{2})(m_v(e) + \frac{m_0(e)}{2})$. 

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Let $G$ be a connected tricyclic graph of size $m$. Then,

**Theorem 2.** Suppose $G$ is a connected bicyclic graph with size $m$. Then,

$$Sz_e^*(G) \geq \begin{cases} 
\frac{1}{2}m^2 + \frac{23}{4}m - 15, & \text{if } m \geq 15, \\
\frac{3}{4}m^2 + \frac{5}{4}m - \frac{15}{4}, & \text{if } m \leq 15 
\end{cases}$$

with equality, if and only if $G \cong S_{m,3}$ for $m \leq 14$, $G \cong S_{m,3}$, $S_{m,4}$ for $m = 15$, and $G \cong S_{m,4}$ for $m \geq 16$.

Soon after, Liu et al. [23] showed the lower bound of bicyclic graphs with the revised edge-Szeged index.

**Theorem 3.** Let $G$ be a connected bicyclic graph with size $m$. Then,

$$Sz_e^*(G) \geq \begin{cases} 
\frac{1}{2}m^2 + \frac{47}{4}m - 30, & \text{if } m \geq 17, \text{and } \sim' \text{ holds iff } G \cong A_1, \\
\frac{3}{4}m^2 + \frac{29}{4}m - \frac{99}{4}, & \text{if } 13 \leq m \leq 16, \text{and } \sim' \text{ holds iff } G \cong A_2, \\
m^2 + \frac{11}{4}m - \frac{15}{2}, & \text{if } 7 \leq m \leq 12, \text{and } \sim' \text{ holds iff } G \cong A_3, \\
121, & \text{if } m = 6, \text{and } \sim' \text{ holds iff } G \cong A_3, A_0, \\
\frac{5}{4}m^2 + \frac{35}{4}m - \frac{57}{2}, & \text{if } 11 \leq m \leq 12, \text{and } \sim' \text{ holds iff } G \cong A_6, \\
\frac{5}{4}m^2 + \frac{23}{4}m - \frac{51}{2}, & \text{if } m \leq 10, \text{and } \sim' \text{ holds iff } G \cong B_1.
\end{cases}$$

In this paper, for tricyclic graphs, the minimum value of $Sz_e^*$ is determined.
By $m_u(e) + m_v(e) + m_0(e) = m$, we have an equivalent formula of the revised edge-Szeged index.

$$Sz^*_v(G) = \sum_{e \in E(G)} (m_u(e) + m_0(e)) (m_v(e) + m_0(e))$$

$$= \sum_{e \in E(G)} \left( \frac{m + m_u(e) - m_v(e)}{2} \right) \left( \frac{m - m_u(e) + m_v(e)}{2} \right)$$

$$= \sum_{e \in E(G)} \frac{m^2 - (m_u(e) - m_v(e))^2}{4}$$

$$= \frac{m^3}{4} - \frac{1}{4} \sum_{e \in E(G)} (m_u(e) - m_v(e))^2. \quad (1)$$

For brevity, let $\delta(e) = |m_u(e) - m_v(e)|$. Equation (1) is rewritten as

$$Sz^*_v(G) = \frac{m^3}{4} - \frac{1}{4} \sum_{e \in E} (\delta(e))^2. \quad (2)$$

Based on the property of pendant edge, we have the following two elementary results. Graphs are shown in Figure 2.

![Graphs](image)

**Figure 2.** The graphs are used in Theorem 2 and Theorem 3.

**Lemma 1.** Let $e \in E(G)$. Then,

\[ \delta(e) \leq m - 1 \]

with equality, if and only if $e$ is a pendant edge.

**Lemma 2.** Let $G_2$ be a connected graph of size $m$. Then,

\[ Sz^*_v(G_1 \cdot S_m) < Sz^*_v(G_1 \cdot G_2), \]

where the fusing vertex of $G_1 \cdot S_m$ is the center vertex of $S_m$.

### 3. Lower Bound of $Sz^*_v(G)$ on $\bigcup_{i=5}^{15} B^i_m$

In this section, we identify a lower bound of $Sz^*_v(G)$ on $B^i_m = \bigcup_{i=5}^{15} B^i_m$. Before listing the proof, some preparations are necessary. The next conclusion holds due to Theorem 1.

**Lemma 3.** Let $H_1$ be a connected graph and $H_2$ be a unicyclic graph with $|E(H_1)| = m_1$, and $|E(H_2)| = m_2$. Then, $Sz^*_v(H_1 \cdot H_2) \geq Sz^*_v(H_1 \cdot S_{m_2,3})$ for $m_1 + m_2 \leq 14$, $Sz^*_v(H_1 \cdot H_2) \geq Sz^*_v(H_1 \cdot S_{m_2,3}) = Sz^*_v(H_1 \cdot S_{m_2,4})$ for $m = 15$, and $Sz^*_v(H_1 \cdot H_2) \geq Sz^*_v(H_1 \cdot S_{m_2,4})$ for $m_1 + m_2 \geq 16$, where the fusing vertex of $H_1 \cdot S_{m_2,3}$ (or $H_1 \cdot S_{m_2,4}$) is the center vertex of $S_{m_2,3}$ (or $S_{m_2,4}$).
By means of Theorem 2 and the above result, the next two conclusions are obtained. Note that the fusing vertex of $H_1 \cdot S_{m,3}^*$ (or $H_1 \cdot S_{m,4}^*$) is the center of $S_{m,3}^*$ (or $S_{m,4}^*$).

**Lemma 4.** Let $G$ be a tricyclic graph on order $(m \leq 14)$ and $H$ be a bicyclic graph on order $m_1 \leq m - 3$. If $G = H \cdot S_{m,3}^*$. Then $S_{m,3}^* (G) \geq S_{m,3}^* (A_2 \cdot S_{m,3}^*)$ for $13 \leq m \leq 14$ and equality holds only if $H \cong A_2$, $S_{m,3}^* (G) \geq S_{m,3}^* (A_3 \cdot S_{m,3}^*)$ for $m \leq 12$ and equality holds only if $H \cong A_3$.

**Lemma 5.** Let $G$ be a tricyclic graph on order $m \geq 16$ and $H$ be a bicyclic graph on order $m_1 \leq m - 4$. If $G = H \cdot S_{m,4}^*$, then $S_{m,4}^* (G) \geq S_{m,4}^* (A_1 \cdot S_{m,4}^*)$ for $m \geq 17$ with equality, only if $H \cong A_1$, and $S_{m,4}^* (G) \geq S_{m,4}^* (A_2 \cdot S_{m,4}^*)$ for $m = 16$ with equality, only if $H \cong A_2$. In particular, $S_{m,4}^* (A_2 \cdot S_{m,4}^*) = S_{m,4}^* (A_1 \cdot S_{m,4}^*)$ for $m = 15$.

We now exhibit the proof of Lemma 5.

**Proof.** Assume that $m_1 + m_2 \geq 17$. $S_{m,3}^* (A_1) \leq S_{m,3}^* (H)$ by Theorem 2. The fusing vertex of $A_1 \cdot S_{m,4}^*$ is the center of $S_{m,3}^*$, and we have

$$S_{m,3}^* (H \cdot S_{m,4}^*) = \sum_{e=uv \in E(H \cdot S_{m,4}^*)} \left( m_1(e) + m_2(e) \right) \left( m_1(e) + m_2(e) \right)$$

$$= \sum_{e=uv \in E(H)} \left( m_1(e) + m_2(e) \right) \left( m_1(e) + m_2(e) \right)$$

$$= \sum_{e=uv \in E(S_{m,4}^*)} \left( m_1(e) + m_2(e) \right) \left( m_1(e) + m_2(e) \right)$$

$$\geq S_{m,3}^* (A_1 \cdot S_{m,4}^*) - \frac{1}{2} (m_2 - 1)(m - \frac{1}{2})$$

$$+ \sum_{e=uv \in E(S_{m,4}^*)} \left( m_1(e) + m_2(e) \right) \left( m_1(e) + m_2(e) \right)$$

$$= S_{m,4}^* (A_1 \cdot S_{m,4}^*)$$

In the same way, we obtain $S_{m,4}^* (G) \geq S_{m,4}^* (A_2 \cdot S_{m,4}^*)$ for $15 \leq m \leq 16$. □

**Theorem 4.** Let $G \in B_1$ with $m$ edges. Then, $S_{m,3}^* (G) \geq S_{m,3}^* (A_4) = \frac{1}{2}(m^2 + \frac{m}{2} - 45)$ for $m \geq 17$, $S_{m,3}^* (G) \geq S_{m,4}^* (A_5)$ for $m = 16$, $S_{m,3}^* (G) \geq S_{m,4}^* (A_6)$ for $13 \leq m \leq 14$, and $S_{m,4}^* (G) \geq S_{m,4}^* (A_7)$ for $m \leq 12$. In particular, $S_{m,4}^* (G) \geq S_{m,4}^* (A_8) = S_{m,4}^* (A_9)$ for $m = 15$.

**Proof.** Suppose $G$ belongs to $B_1$, and it contains some $a_i (i = 5, 6, \ldots, 15)$ as its brace. It is clear to find a vertex $x \in V(G)$ such that $G = H_1 \cdot H_2$ with $V(H_1) \cap V(H_2) = x$ and $|H_1| + |H_2| = m - 1$, where $H_1$ is a bicyclic subgraph of $G$, and $H_2$ is an unicyclic subgraph of $G$. By means of Lemma 3, we have $S_{m,3}^* (G) \geq S_{m,3}^* (H_1 \cdot S_{m,4}^*)$ for $m \geq 16$, and $S_{m,4}^* (G) \geq S_{m,4}^* (H_1 \cdot S_{m,4}^*)$ for $m \leq 14$. In particular, $S_{m,4}^* (G) \geq S_{m,4}^* (H_1 \cdot S_{m,4}^*) = S_{m,4}^* (H_1 \cdot S_{m,4}^*)$ for $m = 15$. 


For \( m \geq 15 \), from Lemma 5, we deduce that
\[
S^*_v(H_1 \cdot S_{|H_2|,4}) \geq S^*_v(A_1 \cdot S_{|H_2|,4}) = S^*_v(A_4)
\]
\[
S^*_v(H_1 \cdot S_{|H_2|,4}) \geq S^*_v(A_2 \cdot S_{|H_2|,4}) = S^*_v(A_5)
\]
\[
S^*_v(H_1 \cdot S_{|H_2|,4}) \geq S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_4)
\]
\[
S^*_v(H_1 \cdot S_{|H_2|,4}) \geq S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_5)
\]
\[
S^*_v(H_1 \cdot S_{|H_2|,3}) \geq S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_5)
\]
\[
S^*_v(H_1 \cdot S_{|H_2|,3}) \geq S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_5)
\]
Thus, we deduce that \( S^*_v(G) = S^*_v(A_5) \) by \( S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_5) \) for \( m = 15 \).

For \( m \leq 14 \), Lemma 4 results in
\[
S^*_v(H_1 \cdot S_{|H_2|,3}) \geq S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_4)
\]
\[
S^*_v(H_1 \cdot S_{|H_2|,3}) \geq S^*_v(A_2 \cdot S_{|H_2|,3}) = S^*_v(A_4)
\]
Thus, the proof is finished.

**4. Lower Bound of \( S^*_v(G) \) on \( \cup_{i=1}^4 B_m^i \)**

In this section, we will present the lower bound of \( S^*_v(G) \) on \( \cup_{i=1}^4 B_m^i \). If \( G \in B_m^i \) \((i = 1, 2, 3, 4)\), then, it has a subgraph \( a_i \). Obviously, \( a_i \) \((i = 1, 2, 3, 4)\) is 2-connected. In view of Equation (2), in order to determine the minimum \( S^*_v(G) \), it is sufficient to ensure \( \sum_{e \in E(G)} \delta(e)^2 \) is as large as possible. Thus, we assume that the all vertices of \( G \) outside its brace are pendant edges. Let \( G_i, G_j \in B_m \) with edge sets \( E_i \) and \( E_j \), respectively. From Equation (2), \( S^*_v(G_i) - S^*_v(G_j) = \sum_{e \in E_i} \delta(e)^2 - \sum_{e \in E_j} \delta(e)^2 \). For the sake of brevity, let \( t_{ij} = \sum_{e \in E_i} \delta(e)^2 - \sum_{e \in E_j} \delta(e)^2 \). Before showing the main result of the section, we need some preparation. Graphs are shown in Figure 3.

![Figure 3](image_url)

**Figure 3.** Labeling the vertices of some braces.

**Lemma 6.** Suppose \( G \) is a tricyclic graph and contains a brace \( a_i \) \((1, 1, 1, 2, 1)\). Then, \( S^*_v(G) \geq S^*_v(B_2) \) for \( m \geq 20 \), and \( S^*_v(G) \geq S^*_v(B_3) \) for \( 8 \leq m \leq 19 \). Moreover, \( S^*_v(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45 \).

**Proof.** Suppose \( G (= G_0) \in B_m^1 \) with a brace \( a_1 \) \((1, 1, 1, 2, 1)\). Let \( x_1, x_2, x_3, x_4, x_5 \) be the five vertices of \( a_1 \), as shown in Figure 3, and \( l_i \) be the number of pendants connecting to \( x_i \). Assume that \( l_1 + l_2 \geq l_3 + l_4 \geq 1 \), let \( G_1 \) be the graph obtained from \( G_0 \) by deleting the pendant vertices of \( x_2 \) and \( x_4 \) and adding them to \( x_1 \) and \( x_3 \), respectively. We hence have
$t_{1,0} = (l_1 + l_2 - l_3 - l_4 - l_5)^2 - (l_1 + l_4 - l_3 - l_5)^2 + (l_3 + l_4 + l_5 + 2 - 3)^2$
$- (l_2 + l_4 + 3 - l_3 - l_5 - 2)^2 + (l_1 + l_2 + l_3 + l_4 - l_5)^2 - (l_1 + l_3 - l_4 - l_5)^2$
$+ (l_3 + l_4 + 3 - l_5 - 2)^2 - (l_2 + l_3 + 3 - l_4 - l_5 - 2)^2 + (l_1 + l_2)^2 - (l_1 - l_2)^2$
$+ (l_1 + l_2 + l_3 + l_4 + 3 - l_5 - 1)^2 - (l_1 + l_2 + l_3 + 3 - l_4 - l_5 - 1)^2$
$+ (l_1 + l_2 + 3 - l_3 - l_4 - l_5 - 1)^2 - (l_1 + l_2 + l_4 + 3 - l_3 - l_5 - 1)^2$
$= 8l_1l_2 - 4l_2 + 24l_3l_4 > 0.$

Let $G_2$ be the graph from $G_1$ by shifting the pendant vertices of $x_5$ to $x_3$. Then, we deduce that

$t_{2,1} = (l_1 - l_3 - l_5)^2 - (l_1 - l_3 - l_5)^2 + (l_1 + l_3 + l_5)^2 - (l_1 + l_3 - l_5)^2$
$+ (3 - l_3 - l_5 - 2)^2 - (3 - l_3 - l_5 - 2)^2 + (l_3 + l_5 + 3 - 2)^2 - (l_1 + l_3 - l_5 - 2)^2$
$+ l_1^2 - l_1^2 + (l_1 + l_3 + l_5 + 3 - 1)^2 - (l_1 + l_3 + 3 - l_5 - 1)^2$
$+ (l_1 + 3 - l_3 - l_5 - 1)^2 - (l_1 + 3 - l_3 - l_5 - 1)^2$
$= 8l_1l_5 + 12l_3l_5 + 12l_5 > 0.$

If $l_3 > 0$, let $G_3$ be the graph obtained from $G_2$ by switching $l_1$ pendant vertices from $x_1$ to $x_3$. Thus, we verify that

$t_{3,2} = (l_1 + l_3)^2 - (l_3 - l_1)^2 + (l_1 + l_3)^2 - (l_1 + l_3 - 2)^2$
$- (l_1 + 2 - 2)^2 + (l_1 + l_3 + 3 - 2)^2 - (l_3 + 3 - 2)^2 + (2 - 2)^2 - l_1^2$
$+ (l_1 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - 1)^2 + (l_1 + l_3 + 1 - 3)^2 - (l_3 + 1 - l_1 - 3)^2$
$= l_1^2 + 12l_1l_3 - 8l_1 > 0.$

Combining Equation (2) with the above three relations, we show that $Sz^*_x(G_0) > Sz^*_x(G_1) > Sz^*_x(G_2) > Sz^*_x(G_3), G_3 \cong B_3$. Clearly, $G_2 \cong B_2$ for $l_3 = 0$. Observe that

$$Sz^*_x(B_2) = m^2 + \frac{47}{4}m - \frac{190}{4} > \frac{1}{4}m^2 + \frac{71}{4}m - 45,$$
$$Sz^*_x(B_3) = \frac{5}{4}m^2 + \frac{23}{4}m - \frac{102}{4} > \frac{1}{2}m^2 + \frac{71}{4}m - 45.$$

Hence, the result holds.

Graphs are shown in Figure 4.

Figure 4. The graphs used in Lemma 6 and Theorem 5.

**Lemma 7.** Suppose $G$ includes the brace $a_2(3,1,1,2,1)$. Then, $Sz^*_x(G) \geq Sz^*_x(C_4)$ with equality only if $G \cong C_4$. Moreover, $Sz^*_x(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.

**Proof.** Let $G (= G_0) \in B^2_m$ with a brace $a_2(3,1,1,2,1)$. Label the six vertices $a_2(3,1,1,2,1)$ $x_1,x_2,\ldots,x_6$, as shown in Figure 3. $l_i$ denotes the number of pendant vertices connecting to $x_i$ for $i = 1,2,\ldots,6$. Let $G_1$ denote the graph obtained from $G_0$ by deleting the pendant
vertices of $x_5$ and $x_6$ and adding them to $x_1$. If $l_3 + l_4 \geq 1$, let $G_2$ denote the graph obtained from $G_1$ by moving the pendant vertices of $x_3$ and $x_4$ to $x_1$ and $x_2$, respectively. Thus, we obtain

$$t_{1,0} = (l_1 + l_3 + l_5 + l_6 + 3 - 2 - l_2)^2 - (l_1 + l_3 + l_5 + 3 - l_2 - l_6 - 2)^2$$

$$+ (l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + 5 - 1 - l_5 - l_6)^2$$

$$+ (l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + 5 - 1)^2 - (l_1 + l_2 + l_3 + l_4 + 5 - 1 - l_5 - l_6)^2$$

$$+ (l_1 + l_3 + l_5 + l_6 + 3 - 2 - l_2)^2 - (l_1 + l_3 + l_5 + 3 - l_2 - l_6)^2$$

$$+ (l_1 + l_2 + l_3 + l_5 + l_6 + 5 - 1 - l_3 - l_4)^2 - (l_1 + l_2 + l_3 + l_5 + 5 - 1 - l_3 - l_4)^2$$

$$+ (l_1 + l_2 + l_3 + l_5 + 3 - l_3 - l_4 - 2)^2 - (l_2 + l_4 + l_5 + 3 - l_3 - l_4 - 2)^2$$

$$- (l_2 + l_6 + 3 - l_3 - l_4 - 2)^2 + (l_1 + l_5 + l_6 + 4 - 2 - l_4)^2 - (l_1 + l_5 + 4 - l_4 - 2)^2$$

$$= 18l_1l_6 + 20l_3l_6 + 10l_5l_6 + 38l_6 + 8l_1l_5 + 8l_2l_5 + 8l_3l_5 + 8l_4l_5 + 32l_5 + 6l_4l_6 > 0,$$

$$t_{2,1} = (l_1 + l_2 + l_3 + l_4 + 2 - 3)^2 - (l_1 + l_3 + 3 - l_2 - 2)^2 + (l_1 + l_2 + l_3 + l_4 + 5 - 1)^2$$

$$- (l_1 + l_2 + l_3 + l_4 + 5 - 1)^2 + (l_1 + l_2 + l_3 + l_4 + 5 - 1 - l_5 - l_6)^2$$

$$+ (l_1 + l_2 + l_3 + l_4 + 2 - 3)^2 - (l_1 + l_3 + 3 - l_2 - 2)^2 + (l_1 + l_2 + l_3 + l_4 + 5 - 1)^2$$

$$- (l_1 + l_2 + l_3 + l_4 + 5 - 1)^2 + (l_1 + l_2 + l_3 + l_4 + 3 - 1)^2 - (l_2 + l_4 + 3 - l_3 - l_4 - 2)^2$$

$$+ (l_1 + l_2 + l_3 + l_4 + 3 - 3)^2 - (l_2 + 3 - l_3 - l_4 - 2)^2 - (l_1 + 4 - l_4 - 2)^2$$

$$= l_1^2 + 2l_4^2 - 6l_1 + 20l_3 + 12l_4 + 12l_1l_2 + 20l_2l_3 + 10l_2l_4 + 12l_1l_4 + 10l_3l_4 + 8l_1l_5 > 0.$$  

For $l_1 > 0$, $G_3$ denotes the graph from $G_2$ by switching all pendant vertices from $x_1$ to $x_2$. So, we have

$$t_{3,2} = (l_1 + l_2 + 2 - 3)^2 - (3 + l_1 - 2 - l_2)^2 + (l_1 + l_2 + 5 - 1)^2 - (l_1 + l_2 + 5 - 1)^2$$

$$+ (l_1 + l_2 + 5 - 1)^2 - (l_1 + l_2 + 5 - 1)^2 + (l_1 + l_2 + 2 - 3)^2 - (l_2 + 2 - l_1 - 3)^2$$

$$+ (l_1 + l_2 + 5 - 1)^2 - (l_1 + l_2 + 5 - 1)^2 + (l_1 + l_2 + 2 - 3)^2 - (l_2 + 2 - l_1 - 3)^2$$

$$+ (l_1 + l_2 + 3 - 2)^2 - (l_2 + 3 - 2)^2 + (4 - 2)^2 - (l_1 + 4 - 2)^2$$

$$= 3l_1^2 + 2l_2 + 12l_2l_2 > 0.$$  

The above three relations with Equation (2) infer that $Sz^*_e(G) > Sz^*_e(G_1) > Sz^*_e(G_2) > Sz^*_e(G_3)$. Clearly, $G_3 \cong C_4$, and $Sz^*_e(C_4) = \frac{3}{4}m^2 + \frac{60}{4}m - \frac{29}{4} > \frac{1}{2}m^2 + \frac{71}{4}m - 45$. Hence, we finish the proof. \[\square\]

Graphs are shown in Figure 5.

![Graphs C1, C2, C3, C4](image)

Figure 5. The graphs used in Lemmas 7-9 and Theorem 6.

**Lemma 8.** If $G$ has the brace $e_2(2, 1, 1, 2, 2)$, then $Sz^*_e(G) \geq Sz^*_e(C_3)$, and $Sz^*_e(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.

**Proof.** Suppose $G (= G_0) \in B^2_m$ with a brace $e_2(2, 1, 1, 2, 2)$. The six vertices of $e_2(2, 1, 1, 2, 2)$ are labeled as $x_1, x_2, \ldots, x_6$, see Figure 3. Let $l_i$ denote the number of pendant vertices connecting
to $x_i$. For $l_6 > 0$, the new graph $G_1$ is from $G_0$ by attaching $l_6$ pendant vertices to $x_1$ from $x_6$. It is deduced that

$$t_{1,0} = (5 + l_1 + l_3 + l_5 + l_6 - 1 - l_2)^2 - (5 + l_3 + l_5 + l_1 - 1 - l_2)^2$$
$$+ (5 + l_2 + l_1 + l_6 + l_4 - l_3 - 1)^2 - (5 + l_1 + l_2 + l_4 - l_3 - 1)^2$$
$$+ (4 + l_3 + l_1 + l_6 + l_5 - 2 - l_4)^2 - (4 + l_1 + l_3 + l_5 - l_4 - l_6 - 2)^2$$
$$+ (4 + l_1 + l_6 + l_2 + l_4 - l_5 - 2)^2 - (4 + l_1 + l_2 + l_4 + l_5 - l_6 - 2)^2$$
$$+ (4 + l_4 + l_2 + l_1 + l_6 - 2 - l_5)^2 - (4 + l_1 + l_2 + l_4 - l_5 - l_6 - 2)^2$$
$$+ (4 + l_3 + l_5 + l_1 - 2 - l_4)^2 - (4 + l_1 + l_3 + l_5 - l_4 - l_6 - 2)^2$$
$$+ (1 + l_2 - 3 - l_4)^2 - (l_2 + 1 - l_4 - l_6 - 3)^2$$
$$+ (l_3 + 1 - l_5 - l_3)^2 - (l_3 + 1 - l_5 - l_6 - 3)^2$$
$$= 24l_6 - 20l_1l_6 + 10l_2l_6 + 9l_5l_6 > 0.\)

For $l_2 + l_4 > l_3 + l_5 = 1$, let $G_2$ denote the graph from $G_1$ by shifting the pendant vertices from $x_3$ and $x_3$ to $x_2$ and $x_4$, respectively. We deduce that

$$t_{2,1} = (l_2 + l_3 + 1 - l_1 - 5)^2 - (l_1 + 5 + l_3 + l_3 - l_2 - 1)^2 + (l_1 + l_2 + l_4 + l_5 + 5 - 1)^2$$
$$+ (l_1 + l_2 + l_4 + l_5 - l_5 - l_3 - 1)^2 - (l_1 + l_2 + l_4 + 5 - l_4 - 2)^2$$
$$+ (4 + l_1 + l_2 + l_3 + l_4 + l_5 - 2)^2 - (4 + l_1 + l_2 + l_4 - l_5 - 2)^2 + (l_2 + l_3 + 1 - l_4 - l_5 - 3)^2$$
$$+ (l_2 + 1 - l_4 - 3)^2 + (3 - 1)^2 - (l_3 + 1 - l_5 - 3)^2 + (4 + l_1 + l_2 + l_5 + l_4 + l_5 - 2)^2$$
$$+ (4 + l_1 + l_2 - 2 - l_5)^2 + (4 + l_1 - 2 - l_4 - l_5)^2 - (l_5 + 4 + l_3 + 2 - l_4)^2$$
$$= 14l_2l_3 + 30l_3l_4 + 10l_2l_5 + 20l_4l_5 > 0.\)

Now, assume that $l_2 + l_4 \geq 1$, and $l_3 = l_5 = 0$. Let $G_3$ be the graph obtained from $G_2$ by transferring all pendant vertices from $x_2$ and $x_4$ to $x_1$.

$$t_{3,2} = (5 + l_1 + l_2 + l_4 - 1)^2 - (5 + l_1 - 1 - l_2)^2 + (5 + l_1 + l_2 + l_4 - 1)^2 - (5 + l_1 - 1 - l_2)^2$$
$$+ (4 + l_1 + l_2 + l_4 - 2)^2 - (4 + l_1 - l_4 - 2)^2 + (4 + l_1 + l_2 + l_4 - 2)^2 - (l_1 + l_2 + l_4 + 4 - 2)^2$$
$$+ (1 - 3)^2 - (l_2 + 1 - l_4 - 3)^2 + (1 - 3)^2 - (1 - 3)^2 + (l_1 + l_2 + l_4 + 4 - 2)^2$$
$$- (l_1 + l_2 + l_4 - 2 - l_2)^2 + (4 + l_1 + l_2 + l_4 - 2)^2 - (4 + l_1 - 2 - l_4)^2$$
$$= l_2^2 + l_4^2 + 12l_2 + 12l_4 + 4l_1l_2 + 8l_1l_4 + 6l_2l_4 > 0.\)

The above relations together with Equation (2) imply that $Sz^*_{c}(G) > Sz^*_{c}(G_2) > Sz^*_{c}(G_3)$. Observe that $G_3 \cong C_3$, and $Sz^*_{c}(C_3) = m^2 + \frac{47}{4}m - 44 > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.

Consequently, the proof is obtained. \[\square\]

**Lemma 9.** If $G$ includes a brace $a_2(2, 1, 1, 2, 1)$, then $Sz^*_{c}(G) \geq Sz^*_{c}(C_1)$ for $m \geq 19$, $Sz^*_{c}(G) \geq Sz^*_{c}(C_2)$ for $8 \leq m \leq 17$, and $Sz^*_{c}(C_1) = Sz^*_{c}(C_2)$ for $m = 18$. Moreover, $Sz^*_{c}(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.

**Proof.** Suppose $G (= G_0) \in B_{a_2}^3$ with a brace $a_2(2, 1, 1, 2, 1)$. We label the five vertices of $a_2(2, 1, 1, 2, 1)$ as $x_1, x_2, x_3, x_4, x_5$ (see, Figure 3) and denote $l_i$ the number of pendant vertices connecting to $x_i$. Assume that $l_2 + l_4 \geq l_3 + l_5 \geq 1$ by symmetry, let $G_1$ denote the graph obtained from $G_0$ by shifting the pendant vertices from $x_3$ and $x_5$ to $x_2$ and $x_4$, respectively. We obtain
\[ t_{1,0} = (l_2 + l_3 + l_1 - 4) - (l_1 + l_3 + l_5 + 4 - l_2 - 1) + (1 + l_2 + l_3 - 3 - l_4 - l_5)^2 \]
\[ - (l_2 + l_4 - 4 - l_5 - 3)^2 + (l_1 + l_3 + l_5 + 3 - 2)^2 - (l_1 + l_3 + l_5 + 2)^2 \]
\[ + (l_2 + l_4 + 4 + l_5 + 3 - 3)^2 - (l_2 + l_4 + 3 - l_5 - 3)^2 + (l_1 + l_2 + l_3 + l_4 + l_5 + 4 - 1)^2 \]
\[ - (l_1 + l_2 + l_4 + 4 - l_5 - 1)^2 + (l_4 + l_5 + 3 - 3)^2 - (l_4 + l_5 + 3 - 1 - l_5)^2 \]
\[ + (l_1 + l_2 + l_3 + 3 - 2)^2 - (l_1 + l_2 - 3 - l_5 - 2)^2 \]
\[ = 16l_2l_3 + 10l_4l_5 + 10l_2l_5 + 8l_4l_5 > 0. \]

For \( l_4 \geq 1 \), let \( G_2 \) be the graph obtained from \( G_1 \) by deleting pendant vertices of \( x_4 \) and adding them to \( x_1 \). Then, we arrive at
\[ t_{2,1} = (l_1 + l_4 + 4 - l_2 - 1)^2 - (4 + l_1 - 1 - l_2)^2 + (l_1 + l_4 + 3 - 2)^2 - (l_1 + 3 - l_4 - 2)^2 \]
\[ + (l_2 + l_4 + 3 - 2)^2 - (l_1 + l_2 + 3 - 2)^2 + (l_1 + l_2 + 4 - 1)^2 - (l_1 + l_2 + l_4 + 4 - 1)^2 \]
\[ + (l_2 + 1 - 3)^2 - (l_2 + 1 - 3 - l_4)^2 + (l_2 + 3 - 3)^2 - (l_2 + l_4 + 3 - 3)^2 + (3 - l_1 - 2)^2 - (l_1 + 4 - 1)^2 \]
\[ = 6l_2^2 + 8l_4l_5 + 4l_4^2 + 4l_2l_4 > 0. \]

For \( l_2 > 0 \), \( l_1 \geq l_2 \), let \( G_3 \) be the graph obtained from \( G_2 \) by moving the pendant vertices from \( x_2 \) to \( x_1 \). Thus, we obtain
\[ t_{3,2} = (l_1 + l_2 + 4 - 1)^2 - (4 + l_1 - 1 - l_2)^2 + (l_1 + l_2 + 3 - 2)^2 - (l_1 + 3 - 2)^2 \]
\[ + (l_1 + l_2 + 3 - 2)^2 - (l_1 + l_2 + 3 - 2)^2 + (l_1 + l_2 + 4 - 1)^2 - (l_1 + l_2 + 4 - 1)^2 \]
\[ + (1 - 3)^2 - (l_2 + 1 - 3)^2 + (3 - 3)^2 - (l_2 + 3 - 3)^2 + (3 - 1)^2 - (3 - 1)^2 \]
\[ = 6l_1l_2 + 18l_2 - l_2^2 > 0. \]

For \( l_1 > 0 \), \( l_1 \leq l_2 - 3 \), let \( G_4 \) be the graph obtained from \( G_2 \) by shifting the pendant vertices from \( x_1 \) to \( x_2 \). We deduce that
\[ t_{4,2} = (l_1 + l_2 + 1 - 4)^2 - (4 + l_1 - 1 - l_2)^2 + (3 - 2)^2 - (l_1 + 3 - 2)^2 + (3 - 1)^2 - (3 - 1)^2 \]
\[ + (l_1 + l_2 + 3 - 2)^2 - (l_1 + l_2 + 3 - 2)^2 + (l_1 + l_2 + 4 - 1)^2 - (l_1 + l_2 + 4 - 1)^2 \]
\[ + (l_1 + l_2 + 1 - 3)^2 - (l_2 + 1 - 3)^2 + (l_1 + l_2 + 3 - 3)^2 - (l_2 + 3 - 3)^2 \]
\[ = 6l_2^2 + 8l_4l_5 - 18l_1 > 0. \]

The four relations with Equation (2) infer that \( Sz^*_G(G) > Sz^*_G(G_1) > Sz^*_G(G_2) > Sz^*_G(G_3) \) or \( Sz^*_G(G_4) \), where \( G_3 \cong C_2 \) for \( l_1 \geq l_2 - 2 \), and \( G_4 \cong C_1 \) for \( l_1 \leq l_2 - 3 \). Clearly,
\[ Sz^*_G(C_1) = \frac{1}{2} m^2 + \frac{25}{4} m - \frac{105}{4} > 1 \frac{1}{2} m^2 + \frac{71}{4} m - 45, \]
\[ Sz^*_G(C_2) = \frac{3}{4} m^2 + \frac{75}{4} m - \frac{357}{4} > 1 \frac{1}{2} m^2 + \frac{71}{4} m - 45. \]

Hence, the result holds. \( \square \)

**Lemma 10.** If \( G \) has a brace \( a_3(2, 2, 2, 2) \), then \( Sz^*_G(G) \geq Sz^*_G(D_1) \) with equality only if \( G \cong D_1 \). Furthermore, \( Sz^*_G(G) > \frac{1}{2} m^2 + \frac{71}{4} m - 45. \)

**Proof.** Suppose \( G (= G_0) \in B^3_{2,6} \) with a brace \( a_3(2, 2, 2, 2) \). \( x_1, x_2, \ldots, x_6 \) denote the six vertices of \( a_3(2, 2, 2, 2) \) with \( d(x_1) = d(x_2) = 4 \) and \( d(x_i) = 2 \) for \( 3 \leq i \leq 6 \), see Figure 3. Let \( l_i \) be the number of pendant vertices connecting to \( x_i \). Assume that \( l_3 \geq l_4 \geq l_5 \geq l_6 \) by symmetry. For \( l_4 + l_3 + l_6 \geq 1 \), let \( G_4 \) denote the graph, which is obtained from \( G \) by deleting all pendant vertices of \( x_i (i \geq 4) \) and adding them to \( x_3 \). Thus, we verify that

...
\[ t_{1,0} = (l_2 + l_3 + l_4 + l_5 + l_6 + 1 - l_1 - 3)^2 - (l_1 + l_4 + l_5 + l_6 + 3 - l_2 - l_3 - 1)^2 \\
+ (1 + l_1 + l_3 + l_4 + l_5 + l_6 + 3 - l_2)^2 - (l_2 + l_4 + l_5 + l_6 + 3 - l_1 - l_3 - 1)^2 \\
+ (l_1 + l_3 + l_4 + l_5 + l_6 + 3 - l_2 - 1)^2 - (l_1 + l_3 + l_5 + l_6 + 3 - l_4 - l_2 - 1)^2 \\
+ (l_2 + l_3 + l_4 + l_5 + l_6 + 3 - l_1 - 1)^2 - (l_2 + l_3 + l_5 + l_6 + 3 - l_1 - l_4 - 1)^2 \\
+ (l_1 + l_3 + l_4 + l_5 + l_6 + 3 - l_2 - 1)^2 - (l_1 + l_3 + l_4 + l_5 + 3 - l_2 - l_1 - 1)^2 \\
+ (l_2 + l_3 + l_4 + l_5 + l_6 + 3 - l_1 - 1)^2 - (l_2 + l_3 + l_4 + l_5 + 3 - 1 - l_1 - l_3)^2 \\
+ (l_1 + l_3 + l_4 + l_5 + l_6 + 3 - l_2 - 1)^2 - (l_1 + l_3 + l_4 + l_5 + 3 - l_6 - l_2 - 1)^2 \\
+ (l_2 + l_3 + l_4 + l_5 + l_6 + 3 - l_1 - 1)^2 - (l_2 + l_3 + l_4 + l_5 + 3 - l_1 - l_6 - 1)^2 \\
= 16l_3l_4 + 16l_3l_5 + 16l_3l_6 + 16l_4l_5 + 16l_4l_6 + 16l_5l_6 > 0. \\
\]

For \( l_1 + l_2 \geq 1 \), let \( G_2 \) be the graph obtained from \( G_1 \) by shifting the pendant vertices from \( x_i \) \((i \leq 2) \) to \( x_3 \). Then, we have that

\[ t_{2,1} = (l_1 + l_2 + l_3 + 1 - 3)^2 - (l_2 + l_3 + 1 - l_1 - 3)^2 + (l_1 + l_2 + l_3 + 1 - 3)^2 \\
- (l_2 + 3 - l_1 - l_3 - 1)^2 + (l_1 + l_2 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - l_2 - 1)^2 \\
+ (l_1 + l_2 + l_3 + 3 - 1)^2 - (l_2 + l_3 + 3 - l_1 - 1)^2 + (l_1 + l_2 + l_3 + 3 - 1)^2 \\
- (l_1 + l_3 + 3 - l_2 - 1)^2 + (l_1 + l_2 + l_3 + 3 - 1)^2 - (l_2 + l_3 + 3 - l_1 - 1)^2 \\
+ (l_1 + l_2 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - l_2 - 1)^2 + (l_1 + l_2 + l_3 + 3 - 1)^2 \\
- (l_2 + l_3 + 3 - l_1 - 1)^2 \\
= 32l_1l_2 + 16l_1l_3 + 8l_2l_3 + 16l_1 + 16l_2 > 0. \]

The two relations with Equation (2) result in \( Sz^*_m(G) > Sz^*_m(G_1) > Sz^*_m(G_2) \), and \( G_2 \cong D_1 \). Furthermore,

\[ Sz^*_m(D_1) = \frac{1}{2} m^2 + \frac{95}{4} m - 102 > \frac{1}{2} m^2 + \frac{71}{4} m - 45. \]

Thus, we complete the proof. \( \square \)

Graphs are shown in Figure 6.

---

**Figure 6.** The graphs used in Lemmas 10–12 and Theorem 7.

**Lemma 11.** Suppose \( G \) contains a brace \( \alpha_3(1, 2, 2, 2) \). Then, \( Sz^*_m(G) \geq Sz^*_m(D_2) \) for \( m = 8 \), and \( Sz^*_m(G) \geq Sz^*_m(D_3) \) for \( m \geq 10 \), \( Sz^*_m(D_2) = Sz^*_m(D_1) \) for \( m = 9 \). In addition, \( Sz^*_m(G) > \frac{1}{2} m^2 + \frac{21}{4} m - 45 \).

**Proof.** Suppose \( G (= G_0) \in B^2_2 \) with a brace \( \alpha_3(1, 2, 2, 2) \). We label the vertices of \( \alpha_3(1, 2, 2, 2) \) as \( x_1, x_2, x_3, x_4, x_5 \) with \( d(x_1) = d(x_2) = 4 \) and \( d(x_i) = 2 \) for \( i = 3, 4, 5 \) and denote \( l_i \) the number of pendant vertices connecting to \( x_i \). Assume that \( l_3 \geq l_4 \geq l_5 \) by symmetry. For \( l_4 + l_5 \geq 1 \), let \( G_1 \) denote the graph obtained from \( G_0 \) by shifting the pendant vertices from \( x_i \) \((i \geq 4) \) to \( x_3 \). Thus, we obtain
From the two graphs, we deduce that
\[ t_{1,0} = (l_3 + l_4 + l_5 + 1 - l_1 - 3)^2 - (l_1 + l_4 + l_5 + 3 - l_3 - 1)^2 \\
+ (1 + l_3 + l_4 + l_5 - 3 - l_2)^2 - (l_2 + l_4 + l_5 + 3 - l_3 - 1)^2 \\
+ (l_1 + l_3 + l_4 + l_5 + 3 - 1)^2 - (l_1 + l_3 + l_5 + 3 - l_4 - 1)^2 \\
+ (l_2 + l_3 + l_4 + l_5 + 3 - 1)^2 - (l_1 + l_3 + l_5 + 3 - l_4 - 1)^2 \\
+ (l_1 + 3 - l_2 - 3)^2 - (l_1 + 3 - l_2 - 3)^2 \\
+ (l_1 + l_3 + l_4 + l_5 + 3 - 1)^2 - (l_1 + l_3 + l_4 + 3 - 1 - l_5)^2 \\
+ (l_2 + l_3 + l_4 + l_5 + 3 - 1)^2 - (l_2 + l_3 + l_4 + 3 - l_5 - 1)^2 \\
= 2l_2l_4 + 16l_3l_4 + 16l_3l_5 + 16l_4l_5 \geq 0.
\]

For \( l_2 \geq 1 \), let \( G_2 \) be the graph obtained from \( G_1 \) by shifting the pendant vertices from \( x_2 \) to \( x_1 \). Then, we have
\[ t_{2,1} = (l_1 + l_2 + 3 - 1 - l_3)^2 - (l_1 + 3 - l_3 - 1)^2 + (l_3 + 1 - 3 - l_1)^2 - (l_2 + 3 - l_5 - 1)^2 \\
+ (l_1 + l_2 + 3 - 3 - l_2)^2 - (l_1 + 3 - l_2 - 3)^2 + (l_1 + l_2 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - 1)^2 \\
+ (l_3 + 3 - 1)^2 - (l_2 + l_3 + 3 - 1)^2 + (l_1 + l_2 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - 1)^2 \\
+ (l_3 + 3 - 1)^2 - (l_2 + l_3 + 3 - 1)^2 \\
= 2l_2^2 + 10l_1l_2 + 4l_2l_3 + 8l_2 > 0.
\]

Note that \( G_2 \cong D_2 \) for \( l_1 = 0, l_3 > 0 \), and \( G_2 \cong D_3 \) for \( l_1 > 0, l_3 = 0 \). Assume that
\( l_1 > 0, l_3 > 0 \). \( G_3 \) is the graph from \( G_2 \) by switching the pendant vertices from \( x_1 \) to \( x_3 \). Then, we have
\[ t_{3,2} = (l_1 + l_3 + 1 - 3)^2 - (l_1 + 3 - l_3 - 1)^2 + (l_1 + l_3 + 1 - 3)^2 - (l_3 + 1 - 3)^2 \\
+ (3 - 3)^2 - (l_1 + 3 - 3)^2 + (l_1 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - 1)^2 \\
+ (l_1 + l_3 + 3 - 1)^2 - (l_3 + 3 - 1)^2 + (l_1 + l_3 + 3 - 1)^2 - (l_1 + l_3 + 3 - 1)^2 \\
+ (l_1 + l_3 + 3 - 1)^2 - (l_3 + 3 - 1)^2 \\
= 2l_2^2 + 10l_1l_3 - 4l_1 > 0.
\]

The above relations, combined with Equation (2), infer that \( Sz_e^*(G) > Sz_e^*(G_1) > Sz_e^*(G_2) > Sz_e^*(G_3) \). Observe that \( Sz_e^*(G_3) \cong D_2 \). Furthermore, we deduce that
\[ Sz_e^*(D_2) = \frac{3}{4}m^2 + \frac{61}{4}m - \frac{255}{4} > \frac{1}{2}m^2 + \frac{71}{4}m - 45, \]
\[ Sz_e^*(D_3) = \frac{5}{4}m^2 + \frac{29}{4}m - \frac{129}{4} > \frac{1}{2}m^2 + \frac{71}{4}m - 45. \]

Hence, the conclusion holds. \( \square \)

**Lemma 12.** Suppose \( G \) includes the subgraph \( K_3(1, 2, 2, 3) \). Then \( Sz_e^*(G) \geq Sz_e^*(D_4) \).

**Proof.** Suppose \( G (= G_0) \in B_3^m \) with a brace \( K_3(1, 2, 2, 3) \). We label the vertices of \( K_3(1, 2, 2, 3) \) as \( x_1, x_2, \ldots, x_6 \) with \( d(x_1) = d(x_2) = 4 \) and \( d(x_i) = 2 \) for \( 3 \leq i \leq 5 \). \( l_i \) denotes the number of pendant vertices connecting to \( x_i \). Assume that \( l_4 + l_5 + l_6 \geq 1 \), and let \( G_1 \) denote the graph from \( G_0 \) by transferring the pendant vertices from \( x_i (i \geq 4) \) to \( x_3 \). From the two graphs, we deduce that
For $l_1 + l_2 \geq 1$, let $G_2$ be the graph obtained from $G_1$ by deleting the pendant vertices of $x_i$ ($i \leq 2$) and adding them to $x_3$. We deduce that

$$t_{2,1} = (l_1 + l_2 + l_3 + 1 - 4)^2 - (l_1 + l_2 + l_3 + 1 - 4)^2 - (l_1 + l_2 + l_3 + 1 - 4)^2$$

Combining Equation (2) and the above two relations, we infer that $Sz^*_c(G) > Sz^*_c(G_1) > Sz^*_c(G_2)$ and $G_2 \cong D_4$. Clearly, $Sz^*_c(D_4) = m^2 + \frac{316}{4} > \frac{1}{4}m^2 + \frac{1}{2}m^2 + 45$. Hence, the proof is complete.}

Graphs are shown in Figure 7.

![Graphs](image)

**Figure 7.** Labeling some edges in the four braces $a_i (i = 1, 2, 3, 4)$.

**Theorem 5.** Let $G \in G^1_m$ with $m$ edges. We have $Sz^*_c(G) > \frac{1}{2}m^2 + \frac{21}{4}m - 45$ for $m \geq 15$, and $Sz^*_c(G) \geq Sz^*_c(B_1)$ for $m \leq 14$.

**Proof.** Suppose $G \in G^1_m$; then, $G$ has brace $a_3 (a, b, c, d, f, g)$, as shown in Figure 2. In order to deduce the conclusion, it is sufficient to show $\sum_{e \in E(G)} \delta(e)^2 < m^3 - 2m^2 - 71m + 180$ from Equation (2).

**Case 1.** $a_1$ has at least three paths with lengths no less than 2.

**Subcase 1.1** The three paths enclose a cycle. Assume that the three paths are $P(a), P(b)$, and $P(g)$ by the symmetry of $a_1$. Choose the nine edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$, see Figure 7, and count the value $\delta(e)$ of the nine edges, respectively. Evidently, they are no less than $m - 7$. It follows that $\sum_{e \in E} \delta(e)^2 \leq (m - 7)^2 + (m - 9)(m - 1)^2 < m^3 - 2m^2 - 71m + 180$. 

**Subcase 1.2** The three paths consist of a new path. Assume that the three paths are $P(a), P(b),$ and $P(d)$ by the symmetry of $a_1$. Choose the nine edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$, see Figure 7. Thus, we deduce that $\sum_{e \in E} \delta(e)^2 \leq (m - 6)^2 + 4(m - 7)^2 + 7(m - 9)^2 + (m - 9)(m - 1)^2 < m^3 - 2m^2 - 71m + 180$. 


Theorem 6. The three paths have a common vertex. Assume that the three paths are $P(a), P(b),$ and $P(c)$ by the symmetry of $\alpha_1$. Choose the nine edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$, see Figure 7, and count the value $\delta(e)$ of the nine edges. In fact, they are no less than $m - 7$, which brings about $\sum_{e \in E} \delta(e)^2 \leq 3(m - 7)^2 + 3(m - 8)^2 + 3(m - 9)^2 + (m - 9)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

Case 2. $\alpha_1$ contains just two paths with lengths no less than 2.

Subcase 2.1. The two paths belong to the same cycle in $\alpha_1$. Assume that the two paths are $P(a), P(b)$ by the symmetry of $\alpha_1$. Choose eight edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$, see Figure 7, and count the $\delta(e)$ of these edges. Thus, we obtain $\sum_{e \in E} \delta(e)^2 \leq 4(m - 6)^2 + 3(m - 7)^2 + (m - 8)^2 + (m - 8)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

Subcase 2.2. The two paths belong to two distinct cycles in $\alpha_1$.

In the same way as subcase 2.1, we also verify that $\sum_{e \in E} \delta(e)^2 \leq 4(m - 5)^2 + 4(m - 8)^2 + 2(m - 8)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

Case 3. $\alpha_1$ just possesses one path with length no less than 2.

By symmetry, assume the path is $P(d)$ with $d \geq 2$. We claim that $d = 2$ (If not, $d \geq 3$, we obtain $\sum_{e \in E} \delta(e)^2 \leq 6(m - 8)^2 + 2(m - 5)^2 + (m - 8)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$ It follows from Lemma 6 that $Sz^*(G) \geq Sz^*(B_2)$ (or $B_3$) $\geq \frac{1}{2}m^2 + \frac{71}{4}m - 45.$

Case 4. The six paths with length one.

Notice that $\alpha_1 \cong K_4$. Denote its vertices as $x_1, x_2, x_3, x_4$ and $I_l$ the number of $x_l$'s pendant vertices. Let $G_1$ be the graph from $G (= G_0)$ by shifting the pendant vertices from the other three vertices to $x_1$. We obtain $t_{1,0} = 3(l_1 + l_2 + l_3 + l_4)^2 - (l_1 - l_2)^2 - (l_1 - l_3)^2 - (l_1 - l_4)^2 - (l_2 - l_3)^2 - (l_2 - l_4)^2 - (l_3 - l_4)^2 = 8(l_1 + l_2 + l_3 + l_4)^2 + l_1 + l_1 + l_3 + l_4) > 0.$ By Equation (2), $Sz^*(G) \geq Sz^*(G_1) = Sz^*(B_1) = \frac{1}{2}m^2 + \frac{71}{4}m - 45.$ In addition, $Sz^*(B_1) < \frac{1}{2}m^2 + \frac{71}{4}m - 45$ for $m \leq 14$, and $Sz^*(B_1) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$ otherwise. □

Theorem 6. Let $G \in G^*_m$ with $m$ edges. Then, $Sz^*_m(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45.$

Proof. Suppose $G \in G^*_m$, which implies $G$ has the subgraph $a_2(a, b, c, d, f)$ as its brace. For symmetry, assume that $a, d \geq 2.$ From Equation (2), it is sufficient to confirm that $\sum_{e \in E(G)} \delta(e)^2 < m^3 - 2m^2 - 71m + 180.$ We will divide the process into four cases to verify the conclusion.

Case 1. $a, d \geq 3.$

Subcase 1.1. $b = c = f = 1$. Selecting nine edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ in $a_2$, see Figure 7, we deduce that $\sum_{e \in E} \delta(e)^2 \leq 4(m - 4)^2 + 4(m - 7)^2 + (m - 9)^2 + (m - 9)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

Subcase 1.2. At least one of $b, c, f$ is more than 1. If $b \geq 2(c \geq 2)$, we choose 10 edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$, as shown in Figure 7, and deduce $\sum_{e \in E} \delta(e)^2 \leq 2(m - 4)^2 + (m - 5)^2 + 2(m - 6)^2 + 2(m - 8)^2 + 3(m - 9)^2 + (m - 10)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

If $f \geq 2$, choose 10 edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$, see Figure 7. Thus, we verify that $\sum_{e \in E} \delta(e)^2 \leq 4(m - 4)^2 + 6(m - 7)^2 + (m - 10)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

Case 2. $a, d \geq 3$.

Subcase 2.1. $a \geq 4, d = 2$, and $b = c = f = 1$. We choose 9 edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ as shown in Figure 7, and count $\delta(e)$ of these edges. Thus, we have $\sum_{e \in E} \delta(e)^2 \leq 4(m - 5)^2 + 2(m - 6)^2 + 2(m - 7)^2 + 3(m - 8)^2 + (m - 9)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

Subcase 2.2. $a = d, d = 2$, and $b = c = f = 1$. The subcase is confirmed by Lemma 6.

Subcase 2.3. $a \geq 3, d = 2$, and at least one of $b, c, f$ is more than 1. The proof of Subcase 2.3 is similar to Subcase 2.1.

Case 3. $a = d = 2$.

Subcase 3.1. At least one of the numbers $b, c, f$ is more than one. If $b \geq 2$ (or $c \geq 2$), we choose $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ (see Figure 7) and deduce that $\sum_{e \in E} \delta(e)^2 \leq 4(m - 5)^2 + 4(m - 7)^2 + (m - 8)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$

If $f \geq 3$, we choose $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ (see Figure 7) and show that $\sum_{e \in E} \delta(e)^2 \leq 2(m - 5)^2 + 2(m - 6)^2 + 2(m - 7)^2 + 2(m - 8)^2 + (m - 9)(m - 1)^2 < m^3 - 2m^2 - 71m + 180.$ If $f = 2$, Lemma 8 means that $Sz^*_m(G) \geq Sz^*_m(C_4) > \frac{1}{2}m^2 + \frac{71}{4}m - 45.$
Subcase 3.2 $b = c = f = 1$.
Applying Lemma 9, we have $Sz^*_e(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.
Thus, we confirm the conclusion. \hfill \square

Theorem 7. Let $G \in \mathcal{G}_m^3$ with $m$ edges. Then, $Sz^*_e(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.

Proof. Since $G \in \mathcal{G}_m^3$, then there is some $a_3(a, b, c, d)$ as its brace. Without loss of generality, suppose $1 \leq a \leq b \leq c \leq d$. We now divide this into three cases to show the result.

Case 1. $3 \leq a \leq b \leq c \leq d$.
Choose the 12 edges $e_1^1, e_2^1, e_3^1, e_2^2, e_1^3, e_3^3, e_1^4, e_2^4, e_3^4$ (see Figure 7). We deduce that $\sum_{e \in E} \delta(e)^2 \leq 8(m - 8)^2 + 4(m - 12)^2 + (m - 12)^2 < m^3 - 2m^2 - 71m + 180$.

Case 2. $a = 2$.

Subcase 2.1. $3 \leq b \leq c \leq d$. We choose the 11 edges $e_1^1, e_2^1, e_3^1, e_3^2, e_1^4, e_2^4, e_3^4$ (see Figure 7) and show that $\sum_{e \in E} \delta(e)^2 \leq 2(m - 9)^2 + 3(m - 11)^2 + (m - 8)(m - 12) < m^3 - 2m^2 - 71m + 180$.

Subcase 2.2. $b = c = d = 2$. The subcase can be verified by Lemma 10.

Case 3. $a = 1$.

Subcase 3.1. $3 \leq b \leq c \leq d$. We deduce that $\sum_{e \in E} \delta(e)^2 \leq 6(m - 4)^2 + 4(m - 10)^2 + (m - 10)(m - 12) < m^3 - 2m^2 - 71m + 180$, through selecting the 10 edges $e_1^1, e_2^1, e_2^2, e_1^3, e_2^3, e_1^4, e_2^4$ (see Figure 7) in $a_3$ and figuring out their $\delta(e)$.

Subcase 3.2. $b = c = d = 2$. The proof of Subcase 3.2 is similar to that of Subcase 3.1. So, the process is omitted here.

Subcase 3.3. $b = c = 2, d \geq 3$. If $d \geq 4$, we choose 9 edges $e_1^1, e_2^1, e_3^1, e_3^2, e_1^4, e_2^4, e_3^4$ (see Figure 7) in $a_3$ and count $\delta(e)$ of these edges. We deduce that $\sum_{e \in E} \delta(e)^2 \leq (m - 9)^2 + 2(m - 5)^2 + 2(m - 7)^2 + 2(m - 8)^2 < m^3 - 2m^2 - 71m + 180$. We now assume that $d = 3$. The assertion holds from Lemma 12.

Subcase 3.4. $b = c = d = 2$.
The subcase is verified through Lemma 11.
Together with the above discussion, we complete the proof. \hfill \square

Theorem 8. Let $G \in \mathcal{G}_m^4$ with $m$ edges. Then, $Sz^*_e(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$.

Proof. Suppose $G \in \mathcal{G}_m^4$. Clearly, $G$ has a subgraph $a_4$. We choose 8 edges $e_1^1, e_2^1, e_2^2, e_1^4, e_2^4, e_3^4$ (see Figure 7) and count $\delta(e)$ of these edges. We show that $\sum_{e \in E} \delta(e)^2 \leq 4(m - 5)^2 + 4(m - 8)^2 + (m - 8)(m - 1)^2 < m^3 - 2m^2 - 71m + 180$. Combined with Equation (2), $Sz^*_e(G) > \frac{1}{2}m^2 + \frac{71}{4}m - 45$. \hfill \square

Theorems 4–8 imply Theorem 3.

5. Conclusions
The upper bound of the revised edge-Szeged index on tricyclic graphs was obtained by Liu et al. [7]. This work motivated us to further research these graphs to obtain a sharp lower bound and thereby characterize all graphs that meet the bound with the utilization of the graph operation and asymmetry of graphs. The result completes the extremal value study of tricyclic graphs and enriches the conclusions on the revised edge-Szeged index.

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