Mild Solutions for Fractional Impulsive Integro-Differential Evolution Equations with Nonlocal Conditions in Banach Spaces

Ye Li 1,2 and Biao Qu 1,*

1 Institute of Operations Research, Qufu Normal University, Jining 273165, China
2 School of Data and Computer Science, Shandong Women’s University, Jining 250062, China
* Correspondence: qubiao001@163.com

Abstract: In this paper, by using the cosine family theory, measure of non-compactness, the Mönch fixed point theorem and the method of estimate step by step, we establish the existence theorems of mild solutions for fractional impulsive integro-differential evolution equations of order $1 < \beta \leq 2$ with nonlocal conditions in Banach spaces under some weaker conditions. The results obtained herein generalizes and improves some known results. Finally, an example is presented for the demonstration of obtained results.

Keywords: fractional impulsive integro-differential evolution equations; fixed point; measure of noncompactness; existence

1. Introduction

Fractional differential equations, in comparison with classical integer order ones, have apparent advantages in modeling mechanical and electrical properties of various real materials and in some other fields. The theory of fractional differential evolution has been emerging as an important area of investigation in recent years (see [1–4]). By using semigroup theory, the properties of noncompact measures, the references [5–7] studied local and global existence of solutions of the initial value problem for a class of fractional evolution equations of order $0 < \alpha \leq 1$. The references [8–13] studied local and global existence of solutions for a class of fractional evolution equations of order $0 < \alpha \leq 1$ with nonlocal conditions. By using semigroup theory, the properties of noncompact measures or Lipschitz conditions, the references [14,15] studied local and global existence and uniqueness of mild solutions for fractional impulsive evolution differential equations of order $0 < \alpha \leq 1$. The references [16–18] studied the existence and uniqueness of mild solutions for fractional differential evolution equations of order $1 < \alpha \leq 2$ with nonlocal conditions by using semigroup theory, the properties of noncompact measures or Lipschitz conditions. The references [19–25] studied the controllability of nonlocal fractional differential evolution equations with nonlocal conditions.

In recent years, many scholars have studied the existence and uniqueness of mild solutions for fractional differential evolution equations by using semigroup theory, the properties of noncompact measures and various fixed point theorems. In [16], the authors studied the existence and uniqueness of mild solutions for semilinear fractional integro-differential equations with nonlocal conditions of order $1 < \beta \leq 2$:

$$\begin{cases}
\hat{C}D^\beta x(t) = Ax(t) + f(t, x(t)) + \int_0^t q(t-s)g(s, x(s))ds, & t \in J = [0, T], \\
x(0) + m(x) = x_0 \in X, & x'(0) + n(x) = x_1 \in X,
\end{cases}$$
The reference [17] studied the local and global existence of mild solutions for fractional integro-differential evolution equations of order $1 < \beta \leq 2$ with nonlocal conditions:

\[
\begin{cases}
\frac{D^\beta}{\Delta} x(t) = Ax(t) + f(t, x(t), (Jx)(t), (Gx)(t)), t \in J \\
x(0) + g_1(x) = x_1 \in X, x'(0) + g_2(x) = x_2 \in X,
\end{cases}
\]

where

\[
Hx(t) = \int_0^t h(t,s)x(s)ds, \quad Gx(t) = \int_0^a g(t,s)x(s)ds,
\]

Under two cases where the solution operator is compact and noncompact, respectively, the reference [26] investigated the existence and uniqueness of mild solutions of the following fractional impulsive integro-differential evolution equations of order $1 < \beta \leq 2$ with nonlocal conditions in Banach space $E$:

\[
\begin{cases}
\frac{D^\beta}{\Delta} x(t) = Ax(t) + f(t, x(t), x(\tau(t)), (Jx)(t), (Gx)(t)), t \in J, t \neq t_k, \\
\Delta x|_{t=t_k} = I_k(x(t_k)), \\
\Delta x'|_{t=t_k} = \hat{I}_k(x(t_k)), k = 1, 2, \ldots, m, \\
x(0) + g_1(x) = a_0 \in E, x'(0) + g_2(x) = a_1 \in E,
\end{cases}
\]

where $\frac{D^\beta}{\Delta}$ is fractional derivatives in the Caputo sense with values of $\beta \in (1, 2]$, $(E, \| \cdot \|)$ is a real Banach space, $x(\cdot) \in E$, $A : D(A) \subset E \to E$ is the infinitesimal generator of a uniformly continuous cosine family $\{C_\beta(t)\}_{t \geq 0}$ on $E$, $J = [0, a] \subset \mathbb{R}$ is a compact interval, $0 < t_1 < t_2 < \ldots < t_m < a < +\infty$, $f \in C[J \times E \times E \times E, E]$, $I_k \in C[E, E]$, $\hat{I}_k \in C[E, E]$, $\tau \in C[J, J]$ and $0 \leq \tau(t) \leq t, t \in J$, where

\[
Hx(t) = \int_0^t h(t,s,x(s))ds, \quad Gx(t) = \int_0^a g(t,s,x(s))ds,
\]

$\Delta = \{(t, s) \in J \times J \mid 0 \leq s \leq t \leq a\}$, $h \in C[\Delta \times E, E]$, $g \in C[J \times J \times E, E]$, $\Delta x|_{t=\tau(t)}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. They established the existence results of mild solutions by the (generalized) Darbo fixed point theorem and Schauder fixed point theorem, improving the known results, while the compactness condition is added to the impulse term.

In most of the works mentioned above, the strict conditions on the nonlinearity or impulse term and the corresponding coefficients are still imposed. Evidently, it is essential and interesting to widen or remove these conditions, which is very helpful for the applications of the problem. The purpose of this paper is to further study the existence of mild solutions of problem (1) in Banach space. Through Mönch fixed point theorem and the method of estimate step by step, under simple conditions and without restrictive ones on the impulse terms, the existence of a global mild solution for problem (1) is obtained. It should be pointed out that the compactness condition on the impulse term is removed.

The rest of this paper is organized as follows. In Section 2, we present some notations, definitions and lemmas. In Section 3, we give the existence theorems of solutions for fractional impulsive integro-differential evolution equations of order $1 < \beta \leq 2$ with nonlocal conditions. An illustrated example is presented in Section 4.

2. Preliminaries and Lemmas

Throughout this paper, let $E$ be a real Banach space with the norm $\| \cdot \|$. Let $J = [0, a]$, $0 < t_1 < \ldots < t_m < a < +\infty$, $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_k = (t_k, t_{k+1}], \ldots, J_m = (t_m, a]$, $J = J \setminus \{t_1, t_2, \ldots, t_m\}$ and
$PC[J, E] = \{ x : J \to E \mid x(t) \text{ is continuous at } t \neq t_k, \quad \text{and left continuous at } t = t_k, \quad \text{and } x(t_k^+) \text{ exists, } k = 1, 2, \ldots, m \}$,
where $x'(t_k^+)$ and $x'(t_k^-)$ represent the right and left derivatives of $x(t)$ at $t = t_k$, respectively.

Evidently, $PC[J, E]$ is a Banach space with norm $\| x \|_{PC} = \sup_{t \in I} x(t)$.

For any $S \subset PC[J, E]$, we note $S(t) = \{ x(t) \mid x \in S, t \in J \}$. For any $R > 0$, we denote

$T_R = \{ x \in PC[J, E] : \| x \|_{PC} \leq R \}, \quad B_R = \{ x \in E : \| x \| \leq R \}$.

**Definition 1 ([2]).** The Riemann–Liouville fractional derivative of order $\beta > 0$ for function $x(t)$ is defined by

$$RLD_{t_0}^{-\beta} = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} x(s) (t-s)^{1-\beta} ds. \tag{2}$$

**Definition 2 ([2]).** The Caputo fractional derivative of order $\beta > 0$ for function $x(t)$ is defined by

$$CD_{t_0}^{\beta} = \frac{1}{\Gamma(m-\beta)} \int_{t_0}^{t} x^{(m)}(s) (t-s)^{1+\beta-m} ds. \tag{3}$$

where $m = \lfloor \beta \rfloor + 1$ is an integer, $\lfloor \beta \rfloor$ is the integer portion of $m$.

**Definition 3 ([26]).** Let $\beta \in (1, 2]$. A family $\{ C_{\beta}(t) \}_{t \geq 0} \subset L(E)$ is called solution operator (or a strongly continuous fractional cosine family of order $\beta$) for the problem

$$CD_0^{\beta} x(t) = Ax(t), x(0) = \eta, x'(0) = 0$$

if the following conditions are satisfied:

(i) $\{ C_{\beta}(t) \}_{t > 0}$ is strongly continuous for $t > 0$, and $\{ C_{\beta}(0) \} = I$;

(ii) $C_{\beta}(t) D(A) \subset D(A)$ and $AC_{\beta}(t) \eta = C_{\beta}(t) A \eta$ for all $\eta \in D(A), t \geq 0$;

(iii) $C_{\beta}(t) \eta$ is a solution of $x(t) = \eta + \int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} A x(s) ds (\eta \in D(A), t \geq 0)$, where $A : D(A) \subset E \to E$ is a closed densely defined linear operator and be known as the infinitesimal generator of $C_{\beta}(t)$.

**Definition 4 ([17]).** The fractional sine family $S_{\beta} : [0, +\infty) \to L(E)$ associated with $C_{\beta}$ is defined by

$$S_{\beta}(t) = \int_{0}^{t} C_{\beta}(s) ds, \quad t \geq 0.$$

**Definition 5 ([17]).** The fractional Riemann–Liouville family $P_{\beta} : [0, +\infty) \to L(E)$ associated with $C_{\beta}$ is defined by

$$P_{\beta}(t) = RLD_{t_0}^{1-\beta} C_{\beta}(t) = \frac{1}{\Gamma(\beta-1)} \int_{0}^{t} (t-s)^{\beta-2} C_{\beta}(s) ds, \quad t \geq 0.$$

**Property 1 ([1]).** Laplace transform of Caputo derivative of order $\beta$ is

$$L(C_{\beta}(t), s) = s^\beta (L x)(s) - \sum_{j=1}^{n-1} s^{\beta-j} x^{(j)}(0), \quad n - 1 < \beta \leq n.$$

**Property 2 ([17]).** The solution operator $C_{\beta}(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\| C_{\beta}(t) \| \leq M e^{\omega t}, \quad t \geq 0.$$
Property 3 ([26]). The solution operator \( \{C_\beta(t)\}_{t \geq 0} \) is compact for \( t > 0 \) if and only if \((\lambda \beta - A)^{-1}\) is compact for all \( \lambda \in \rho(A) \).

Property 4 ([26]). If the solution operator \( C_\beta(t)(1 < \beta \leq 2) \) is exponentially bounded, then

(i) \( \{S_\beta(t)\}_{t \geq 0} \) and \( \{P_\beta(t)\}_{t \geq 0} \) are strongly continuous.

(ii) If \( \{C_\beta(t)\}_{t \geq 0} \) is compact for \( t > 0 \), then \( \{S_\beta(t)\}_{t \geq 0} \) and \( \{P_\beta(t)\}_{t \geq 0} \) are also compact for \( t > 0 \).

(iii) There exists a constant \( M^* > 0 \) such that \( \|C_\beta(t)\| \leq M^*, t \in [0,a] \) and

\[
\|S_\beta(t)\| \leq \int_0^t \|C_\beta(s)\| \, ds \leq M^* t, t \in [0, a];
\]

\[
\|P_\beta(t)\| \leq \frac{M^*}{\Gamma(\beta - 1)} \int_0^t (t - s)^{\beta - 2} \, ds = \frac{M^*}{\Gamma(\beta)} t^{\beta - 1} \leq \frac{M^*}{\Gamma(\beta)} \alpha^\beta - 1, t \in [0,a].
\]

In the sequel, let \( \alpha, \alpha_{PC} \) denote the Kuratowski measure of noncompactness in \( E, PC[J, E], \) respectively. We first give the following lemmas in order to prove our main results.

Lemma 1 ([26]). If \( H \) is a bounded subset of \( PC[J, E] \), the element of \( H \) is equicontinuous at \( J_k \) for all \( k = 1,2, \ldots, m \), then \( \alpha(\{x(t) | x \in H\}) \) is continuous with respect to \( t \in t_k \) and

\[
\alpha\left( \left\{ \int_J x(t) \, dt : x \in H \right\} \right) \leq \int_J \alpha(H(t)) \, dt.
\]

Lemma 2 ([20]). If \( H \) is a bounded subset of \( PC[J, E] \), the element of \( H \) is equicontinuous at \( J_k \) for all \( k = 1,2, \ldots, m \), then \( \overline{CoH} \subset PC[J, E] \) is bounded and equicontinuous.

Lemma 3 ([27]). For any \( R > 0, f \) is bounded and uniformly continuous on \( J \times B_R \times B_R \times B_R \times B_R, \) if \( S \subset PC[J, E] \) is bounded and equicontinuous , then \( \{f(t, x(t), x(\tau(t)), (Hx)(t), (Gx)(t)) : x \in S\} \subset PC[J, E] \) is also bounded and equicontinuous.

Lemma 4 ([27]). If \( H \subset PC[J, E] \) is a bounded and equicontinuous at \( J_k \) for all \( k = 1,2, \ldots, m \), then

\[
\alpha_{PC}(H) = \sup_{t \in J} \alpha(H(t)).
\]

Lemma 5 ([28]). If \( H \subset PC[J, E] \) is a bounded , then

\[
\alpha(H(J)) \leq \alpha_{PC}(H).
\]

Lemma 6 ([27]). Assume that \( m \in C[J, R^+] \) satisfies

\[
m(t) \leq M_1 \int_0^t m(s) \, ds + M_2 \int_0^t m(s) \, ds + M_3 t \int_0^a m(s) \, ds, t \in J,
\]

where \( M_1 > 0, M_2 \geq 0, M_3 \geq 0 \) are constants, then \( m(t) \equiv 0 \) for any \( t \in J \), provided one of following conditions holds

(i) \( aM_3(e^{(M_1 + aM_2)} - 1) < M_1 + aM_2; \)

(ii) \( a(2M_1 + aM_2 + aM_3) < 2. \)

Lemma 7 ([29]). Let \( E \) be a Banach space, \( \Omega \subset E \) is a bounded open set, \( \theta \in \Omega, F : \overline{\Omega} \to \overline{\Omega} \) is continuous and satisfy the following conditions:

(i) \( x \neq \lambda Fx, \lambda \in [0,1], x \in \partial \Omega; \)

(ii) \( \text{If } D \subset \overline{\Omega} \text{ is countable and } D \subset \overline{\Omega} \cup \{x \cup F(D)\}, \text{then } D \text{ is relative compact.} \)

Then \( F \) has at least a fixed point on \( \Omega. \)
Lemma 8 ([26]). \(x \in \text{PC}[J, E] \) is a solution of the problem (1) if and only if \(x(t) \) satisfy the following integral equations:

\[
x(t) = C_p(t)[a_0 - g_1(x)] + S_p(t)[a_1 - g_2(x)] + \int_0^t P_p(t-s)f(s, x(s), x(\tau(s)), (Hx)(s), (Gx)(s))ds + \sum_{0 < t_k < t} C_p(t-t_k)I_k(x(t_k)) + \sum_{0 < t_k < t} S_p(t-t_k)\hat{I}_k(x(t_k)), t \in J.
\]

3. Main Results

In this section, we are in a position to prove our main results concerning the solutions of fractional impulsive integro-differential evolution Equation (1) in Banach spaces.

Now, let us first list the following assumptions for convenience.

(H1) For any \(R > 0 \), \(f \) is bounded and uniformly continuous on \(J \times T_R \times T_R \times T_R \), \(I_k, \hat{I}_k \) is bounded on \(T_R \).

(H2) There exists \(M^* > 0 \) such that for any \(t \geq 0 \), we have

\[
\| C_p(t) \| \leq M^*.
\]

(H3) There exist \(h_1(t, \cdot), g_1(t, \cdot) \in L[J, R^+] \), \(R^+ = [0, +\infty) \) such that

\[
h_0 = \max_{t \in [0, a]} \int_0^t h_1(t, s)ds < +\infty, g_0 = \max_{t \in [0, a]} \int_0^a f_1(t, s)ds < +\infty,
\]

\[
\| h(t, s, x(s)) \| \leq h_1(t, s) \| x \| + e_{h_1}(t, s) \in \Delta, x \in E,
\]

\[
\| g(t, s, x(s)) \| \leq g_1(t, s) \| x \| + e_{g_1}(t, s) \in J \times J, x \in E.
\]

(H4) There exist \(b_1 \in L[J, R^+], b_2, b_3, b_4 \in L^2[J, R^+] \) such that

\[
\| f(s, x(s), u_1, u_2, u_3) \| \leq b_1(s) \| b_2(s) \| u_1 \| + b_3(s) \| u_2 \| + b_4(s) \| u_3 \| .
\]

(H5) Suppose \(g_1 \in C[PC[J, E], E] \), for any \(x \in PC[J, E] \), there exist constants \(d_i > 0, c_i > 0 \) satisfying

\[
\| g_1(x) \| \leq d_i \| x \|_{PC} + e_{i}, i = 1, 2.
\]

(H6) There exist constants \(L_h > 0, L_g > 0 \) such that for any bounded set \(S \subset E \) satisfying

\[
a(h(t, s, S)) \leq L_h a(S), a(g(t, s, S)) \leq L_g a(S).
\]

(H7) There exist non-negative constants \(L_i (i = 1, 2, 3, 4) \) satisfying one of the following two conditions:

\[
\text{(i)} \quad aL_2L_4(e^{a^2 \frac{M^*}{l_{[p]}}}L_1 + L_2 + aL_3L_h) - 1 < L_1 + L_2 + aL_3L_h;
\]

\[
\text{(ii)} \quad a^2 \frac{M^*}{l_{[p]}}(2L_1 + 2L_2 + aL_3 + aL_4) < 2,
\]

and for any bounded set \(B_i \subset E(i = 1, 2, \ldots, n + 3) \) and all \(t \in J \),

\[
a(f(t, B_1, B_2, \ldots, B_4) \leq \sum_{i=1}^4 L_i a(B_i).
\]

(H8) There exist constants \(L_{g_i} > 0 (i = 1, 2) \) such that for any bounded set \(S \subset PC[J, E] \) satisfying

\[
a(g_i(S)) \leq L_{g_i} \int_0^t a(S)ds.
\]
(H9) There exist non-negative constants $L_i (i = 1, 2, 3, 4)$ satisfying one of the following two conditions:

(i) $a^\beta \frac{M^*}{F(t)} I_3 L_4 (a^\beta \frac{M^*}{F(t)} (L_1 + L_2 + aL_3 L_h) + aM^*(L_g + aL_{g_2})) - 1 < a^\beta - 1 \frac{M^*}{F(t)} (L_1 + L_2 + L_3 L_h) + aM^*(L_g + aL_{g_2})$,

(ii) $a^\beta \frac{M^*}{F(t)} (2L_1 + 2L_2 + aL_3 L_h + aL_{g_2} L_4) + 2aM^*(L_g + aL_{g_2}) < 2$,

and for any bounded set $B_i \subset E (i = 1, 2, \ldots, n + 3)$ and all $t \in J$,

$$a(f(t, B_1, B_2, \ldots, B_4) \leq \sum_{i=1}^{4} L_i \alpha(B_i).$$

We now prove the following main result of this paper.

**Theorem 1.** Let $E$ be a Banach space, assume that conditions (H1) – (H7) hold, $g_i (i = 1, 2)$ is compact. Then (1) has at least a global mild solution on $PC(J, E)$.

**Proof.** We define $F : PC(J, E) \to PC(J, E)$ as follows:

$$F(x)(t) = C_\beta(t)[a_0 - g_1(x)] + S_\beta(t)[a_1 - g_2(x)] + \int_0^t P_\beta(t - s)f(s, x(s), x(\tau(s)), (Hx)(s), (Gx)(s))ds$$

$$+ \sum_{0 < t_k < t} C_\beta(t - t_k)I_k(x(t_k)) + \sum_{0 < t_k < t} S_\beta(t - t_k)I_k(x(t_k)), t \in J. \tag{4}$$

As $f$ is uniformly continuous and $g_1, g_2, I_k, I_\beta$ are continuous, we can get that $F$ is continuous operator.

We first prove that $\Omega_0 = \{x \in PC(J, E) \mid x = \lambda Fx, 0 \leq \lambda \leq 1\}$ is bounded set. In fact, if $x_0 \in \Omega_0$, then there exists $\lambda_0 \in [0, 1]$ satisfying $x_0(t) = \lambda_0 (Fx_0)(t), t \in J$.

Let us first consider the case: $t \in J_0 = [0, t_1]$, in this case, we have

$$x_0(t) = \lambda_0 (Fx_0)(t)$$

$$= \lambda_0 C_\beta(t)[a_0 - g_1(x_0)] + S_\beta(t)[a_1 - g_2(x_0)]$$

$$+ \lambda_0 \int_0^t P_\beta(t - s)f(s, x_0(s), x_0(\tau(s)), (Hx_0)(s), (Gx_0)(s))ds. \tag{5}$$

Let $u(t) = \|x_0(t)\|$, by assumption (H2), (H3), (H4), Property 4 and (5), we have

$$u(t) \leq M^*(\|a_0\| + \|g_1(x_0)\|) + M^* a(\|a_1\| + \|g_2(x_0)\|) + a^\beta - 1 \frac{M^*}{F(t)} \int_0^t [b_1(s) + b_2(s) \|x_0(s)\| + b_3(s) \|Hx_0(s)\| + b_4(s) \|Gx_0(s)\|)ds.$$

Since

$$\|Hx_0(t)\| \leq \int_0^t \|h(t, s, x_0(s))\|ds$$

$$\leq (h_1(t, s) \|x_0\| + e_k) t_1$$

$$\leq t_1 h_0 \|x_0\| + ae_k,$$

$$= t_1 h_0 u(t) + ae_k,$$

$$\|Gx_0(t)\| \leq \int_0^t \|g(t, s, x_0(s))\|ds$$

$$\leq (g_1(t, s) \|x_0\| + e_k) t_1$$

$$\leq t_1 g_0 \|x_0\| + ae_k,$$

$$= ag_0 u(t) + ae_k,$$
then
\[
    u(t) \leq M^*(\| a_0 \| + \| g_1(x_0) \|) + M^* a(\| a_1 \| + \| g_2(x_0) \|) + a^\beta \frac{M^*}{\Gamma(\beta)} t_1(e_h + e_g) \\
    + a^{\beta - 1} \frac{M^*}{\Gamma(\beta)} \int_0^t [b_1(s) + b_2(s)u(s) + t_1 b_3(s)h_0 + ab_4(s)u(s)g_0]ds \\
    \leq M_1 + M_2 + a^{\beta - 1} \frac{M^*}{\Gamma(\beta)} Bu(t),
\]

where
\[
    M_1 = M^*(\| a_0 \| + \| g_1(x_0) \|) + M^* a(\| a_1 \| + \| g_2(x_0) \|) + a^\beta \frac{M^*}{\Gamma(\beta)} t_1(e_h + e_g) \\
    \leq M^*(\| a_0 \| + d_1 + e_1) + M^* a(\| a_1 \| + d_2 + e_2) + a^\beta \frac{M^*}{\Gamma(\beta)} t_1(e_h + e_g),
\]

\[
    M_2 = a^{\beta - 1} \frac{M^*}{\Gamma(\beta)} \int_0^a b_1(s)ds, Bu(t) = \int_0^t [b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0]u(s)ds,
\]

then \( B : C[0, R^+] \rightarrow C[0, R^+] \) is completely continuous operator. Since \( b_2, b_3, b_4 \in L^2[0, R^+] \), then
\[
    M = \left( \int_0^a [(b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)]^2 ds \right)^{1/2} < +\infty.
\]

Next, we will prove \( r_{e'}(B) = 0 \).

Since
\[
    \| Bu(t) \| \leq \int_0^a \| (b_1(s) + b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)u(s) \| ds \\
    \leq \left( \int_0^a [(b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)]^2 ds \right)^{1/2} \left( \int_0^t \| u \|^2 ds \right)^{1/2} \\
    \leq M \| u \| t^{1/2},
\]

\[
    \| B^2 u(t) \| \leq \int_0^a \| (b_1(s) + b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)Bu(s) \| ds \\
    \leq \left( \int_0^a [(b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)]^2 ds \right)^{1/2} \left( \int_0^t (M \| u \| t^{1/2})^2 ds \right)^{1/2} \\
    \leq \frac{1}{\sqrt{2}} M^2 \| u \| t, 
\]

\[
    \| B^3 u(t) \| \leq \int_0^a \| (b_1(s) + b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)B^2 u(s) \| ds \\
    \leq \left( \int_0^a [(b_2(s) + t_1 b_3(s)h_0 + ab_4(s)g_0)]^2 ds \right)^{1/2} \left( \int_0^t \left( \frac{1}{\sqrt{2}} M^2 \| u \| t \right)^2 ds \right)^{1/2} \\
    \leq \frac{1}{\sqrt{3}} \sqrt{2} M^3 \| u \| t^{3/2},
\]

by mathematical induction, for any \( n \in N \), we can easily get
\[
    \| B^n u(t) \| \leq \frac{1}{\sqrt{n!}} M^n \| u \| t^{n/2},
\]

then
\[
    \| B^n \| \leq \frac{1}{\sqrt{n!}} M^n a^{n/2},
\]
\[ r_o(B) = \lim_{n \to +\infty} \| B^n \| \hat{=} 0, \]

there exists a constant \( C_1 > 0 \) independent of function \( u \) such that

\[ \| x_0(t) \| \leq C_1, t \in J_0, \]

by assumption \((H_1)\) and \((H_2)\), there exists a constant \( \beta_1 > 0 \) such that

\[ \| P_\beta(t-s)f(s, x(s), x(\tau(s)), (Hx)(s), (Gx)(s)) \| \leq \beta_1, \]

\[ \| I_k \| \leq \beta_1, \| \hat{I}_k \| \leq \beta_1, \forall (t, s) \in \Delta, \| x \| \leq C_1, \]

since

\[
x_0(t^+_1) = \lambda_0 C_\beta(t_1)[a_0 - g_1(x_0)] + \lambda_0 S_\beta(t_1)[a_1 - g_2(x_0)]
+ \lambda_0 \int_0^{t_1} P_\beta(t_1 - s)f(s, x(s), x(\tau(s)), (Hx)(s), (Gx)(s))ds
+ \lambda_0 C_\beta(t_1 - t_1)I_1(x(t_1)) + \lambda_0 S_\beta(t_1 - t_1)\hat{I}_1(x(t_1))
= x_0(t_1) + \lambda_0 C_\beta(0)I_1(x(t_1)),
\]

then

\[ \| x_0(t^+_1) \| = \| x_0(t_1) + \lambda_0 C_\beta(0)I_1(x(t_1)) \| \leq C_1 + \beta_1. \]

Let

\[ y_0(t) = \begin{cases} x_0(t), t \in (t_1, t_2], \\ x_0(t^+_1), t = t_1, \end{cases} \]

then \( y_0 \in C[(t_1, t_2), E] \) and

\[
y_0(t) = \lambda_0 C_\beta(t)[a_0 - g_1(y_0)] + \lambda_0 S_\beta(t)[a_1 - g_2(y_0)]
+ \lambda_0 \int_0^t P_\beta(t-s)f(s, y_0(s), y_0(\tau(s)), (Hy_0)(s), (Gy_0)(s))ds
+ \lambda_0 C_\beta(t-t_1)I_1(x(t_1)) + \lambda_0 S_\beta(t-t_1)\hat{I}_1(x(t_1))
= \lambda_0 C_\beta(t)[a_0 - g_1(y_0)] + \lambda_0 S_\beta(t)[a_1 - g_2(y_0)]
+ \lambda_0 \int_0^{t_1} P_\beta(t-s)f(s, y_0(s), y_0(\tau(s)), (Hy_0)(s), (Gy_0)(s))ds
+ \lambda_0 \int_{t_1}^t P_\beta(t-s)f(s, y_0(s), y_0(\tau(s)), (Hy_0)(s), (Gy_0)(s))ds
+ \lambda_0 C_\beta(t-t_1)I_1(x(t_1)) + \lambda_0 S_\beta(t-t_1)\hat{I}_1(x(t_1)),
\]

thus

\[ \| y_0(t) \| \leq M^*[a_0 + a_1C_1 + e_1] + M*[a_1 + d_2C_1 + e_2] + t_1\beta_1 + M^*\beta_1 + M^*a\beta_1 \]

\[ + \lambda_0 \int_0^t \| P_\beta(t-s)f(s, y_0(s), y_0(\tau(s)), (Hy_0)(s), (Gy_0)(s))ds \|, \]

by the method of the previous step, we can prove that there exists a non-negative constant \( C_2 > 0 \) such that

\[ \| y_0(t) \| \leq C_2, t \in [t_1, t_2], \]

thus

\[ \| x_0(t) \| \leq C_2, t \in [t_1]. \]

similarly, we can get that there exist non-negative constants \( C_{m+1} > 0 \) such that

\[ \| x_0(t) \| \leq C_{m+1}, t \in J_m. \]
let
\[ C = \max \{ C_i | 1 \leq i \leq m + 1 \}, \]
then
\[ \| x_0(t) \| \leq C, t \in J, \]
thus
\[ \| x_0 \|_{PC} \leq C, x_0 \in \Omega_0, \]
therefore \( \Omega_0 \) is the bounded set.

We take \( R > C \) and let \( \Omega = \{ x \in PC[J,E] | \| x \|_{PC} \leq R \} \), then \( \Omega \subset PC[J,E] \) is bounded open set, it follows from the taking of \( R \), when \( x \in \partial \Omega, \lambda \in [0,1], \)
\[ x_0(t) \neq \lambda_0(Fx_0). \]

Next, let \( D \subset \overline{\Omega} \) be a countable set and \( D \subset \overline{\Omega}(\{ \theta \} \cup F(D)) \), where \( \theta \) is an element of \( \Omega \), then \( D \) is a relative compact set.

In fact, by assumption \((H_7)\) and the strong continuity of \( C_\beta(t), S_\beta(t) \), we can get that \( F(D) \) is equicontinuous on each \( J_k, (k = 0, 1, 2, \ldots, m) \), therefore \( D \) is equicontinuous on each \( J_k, (k = 0, 1, 2, \ldots, m) \).

\( \forall t \in J_0 = [0, t_1], \) by the property of measure of noncompactness, assumption \((H_6), (H_7)\) and the compactness of \( g_k \), we can get
\[
\alpha(D(t)) \leq \alpha(FD(t))
\[
= \alpha \left( \int_0^t P_\beta(t-s) f(s, D(s), D(\tau(s)), (HD)(s), (GD)(s)) ds \right)
\[
\leq \alpha^{b-1} \frac{M^*}{F(\beta)^*} \int_0^t \left( L_1 \alpha(D(s)) + L_2 \alpha(D(\tau(s))) + L_3 \alpha((HD)(s)) + L_4 \alpha((GD)(s)) \right) ds,
\]
since
\[
\alpha(HD(t)) = \alpha \left( \int_0^t h(t, \tau, x(s)) d\tau \right) \leq \int_0^t \alpha(h(t, \tau, x(s)) d\tau) \leq \int_0^t \lambda_h \alpha(D(\tau)) d\tau
\[
= \lambda_h \int_0^t \alpha(D(\tau)) d\tau,
\]
then
\[
\int_0^t \alpha(HD(s)) ds \leq \lambda_h \int_0^t \left( \int_0^t \alpha(D(\tau)) d\tau \right) ds
\[
= \lambda_h \int_0^t \int_0^t \alpha(D(\tau)) d\tau d\tau
\[
= \lambda_h \int_0^t \alpha(D(\tau))(t - \tau) d\tau
\[
\leq \lambda_h t \int_0^t \alpha(D(\tau)) d\tau,
\]
similarly, we can get
\[
\int_0^t \alpha(GD(s)) ds \leq \lambda_g \int_0^t \left( \int_0^t D(\tau) d\tau \right) ds
\[
= \lambda_g \int_0^t ds \int_0^t \alpha(D(\tau)) d\tau
\[
\leq \lambda_g t \int_0^t \alpha(D(\tau)) d\tau,
thus
\[
\alpha(D(t)) \leq a^{β-1} \frac{M^*}{Γ(β)} (L_1 + L_2) \int_0^t \alpha(D(s)) ds \\
+ a^{β-1} \frac{M^*}{Γ(β)} L_4 t \int_0^t \alpha(D(s)) ds \\
+ a^{β-1} \frac{M^*}{Γ(β)} L_4 t \int_0^t \alpha((D(s))) ds,
\]
(6)

By Lemma 6 and assumption (H7), we get \( \alpha(D(t)) = 0, t \in J_0 = [0, t_1] \), especially \( \alpha(D(t_1)) = 0 \), that is to say \( D \) is a relative compact set on \( C[J_0, E] \). \( \forall t \in J_1 = (t_1, t_2] \), by (6), we get
\[
\alpha(D(t)) \leq a^{β-1} \frac{M^*}{Γ(β)} (L_1 + L_2) \int_{J_1}^t \alpha(D(s)) ds \\
+ a^{β-1} \frac{M^*}{Γ(β)} L_4 t \int_{J_1}^t \alpha(D(s)) ds \\
+ a^{β-1} \frac{M^*}{Γ(β)} L_4 t \int_{J_1}^t \alpha(D(s))) ds \\
+ M^* a(I_1 D(t_1)) + M^* a(I_1 D(t_1)))
\]
as \( I_1, I_1 \in C[E, E], D(t_1) \) is a relative compact set of \( E \), thus
\[
\alpha(I_1 D(t_1)) = 0, \alpha(I_1 D(t_1)) = 0,
\]
by (7), we get
\[
\alpha(D(t)) = a^{β-1} \frac{M^*}{Γ(β)} (L_1 + L_2) \int_{J_1}^t \alpha(D(s)) ds \\
+ a^{β-1} \frac{M^*}{Γ(β)} L_4 t \int_{J_1}^t \alpha(D(s)) ds \\
+ a^{β-1} \frac{M^*}{Γ(β)} L_4 t \int_{J_1}^t \alpha(D(s))) ds,
\]
therefore \( \alpha(D(t)) = 0, t \in J_1 = [t_1, t_2] \), especially \( \alpha(D(t_2)) = 0 \), so \( D \) is a relative compact set on \( C[J_1, E] \). Similarly, we can prove \( D \) is a relative compact set on \( C[J_k, E], (k = 2, 3, \ldots, m) \), then \( D \) is a relative compact set on \( C[J, E] \).

By Lemma 7, \( F \) has at least a fixed point on \( Ω \), then problem (1) has a solution \( x^* \) in \( PC[J, E] \), that is to say, \( x^* \) is a global mild solution for problem (1). □

**Theorem 2.** Let \( E \) be a Banach space, assume that conditions \((H_1) - (H_6), (H_8), (H_9)\) hold. Then (1) has at least a global mild solution on \( PC[J, E] \).

**Proof.** From the proof of Theorem 1, we get that \( Ω_0 = \{ x \in PC[J, E] || x = λFx, 0 ≤ λ ≤ 1 \} \) is bounded set. \( \forall t \in J_0 = [0, t_1] \), by the property of measure of noncompactness, assumption \((H_6)\), we can get
\[
\alpha(D(t)) = a(FD(t)) \\
\leq M^*(a(g_1(D)) + aa(g_2(D))) + a^{β-1} \frac{M^*}{Γ(β)} a(\int_0^t (L_1 a(D(s)) + L_2 a(D(τ(s)))) ds) \\
+ L_3 a((HD)(s)) + L_4 a(((GD)(s))) ds \\
\leq M^*(L_{g_1} + aL_{g_2}) \int_0^t a(D(s)) ds + a^{β-1} \frac{M^*}{Γ(β)} a(\int_0^t (L_1 a(D(s))) ds) \\
+ L_2 a(D(τ(s)) + L_3 a((HD)(s)) + L_4 a(((GD)(s))) ds),
\]
by the proof of Theorem 1, we get that
\[
\int_0^t \alpha(HD(s))ds \leq L_h t \int_0^t \alpha(D(\tau))d\tau
\]
\[
\int_0^t \alpha(GD(s))ds \leq L_g t \int_0^t \alpha(D(\tau))d\tau
\]
thus
\[
\alpha(D(t)) \leq a^{-1} \frac{M^*}{\Gamma(\beta)} (L_1 + L_2) \int_0^t \alpha(D(s))ds
\]
\[
+ a^{-1} \frac{M^*}{\Gamma(\beta)} L_h L_3 t \int_0^t \alpha(D(s))ds
\]
\[
+ a^{-1} \frac{M^*}{\Gamma(\beta)} L_g L_4 t \int_0^a \alpha(D(s))ds
\]
(8)

By Lemma 6 and assumption (H_0), we get \(\alpha(D(t)) = 0, t \in J_0 = [0, t_1]\), especially \(\alpha(D(t_1)) = 0\), that is to say \(D\) is a relative compact set on \(C[J_0, E]\). \(\forall t \in J_1 = (t_1, t_2]\), by (8), we get
\[
\alpha(D(t_1)) = 0, \alpha(I_1 D(t_1)) = 0,
\]
by (9), we get
\[
\alpha(D(t)) \leq (a^{-1} \frac{M^*}{\Gamma(\beta)} (L_1 + L_2) + M^*(L_{g_1} + aL_{g_2})) \int_0^t \alpha(D(s))ds
\]
\[
+ a^{-1} \frac{M^*}{\Gamma(\beta)} L_h L_3 t \int_0^t \alpha(D(s))ds
\]
\[
+ a^{-1} \frac{M^*}{\Gamma(\beta)} L_g L_4 t \int_0^a \alpha(D(s))ds
\]
therefore \(\alpha(D(t)) = 0, t \in J_1 = [t_1, t_2]\), especially \(\alpha(D(t_2)) = 0\), so \(D\) is a relative compact set on \(C[J_1, E]\). Similarly, we can prove \(D\) is a relative compact set on \(C[J_k, E]\), \((k = 2, 3, \ldots, m)\), then \(D\) is a relative compact set on \(C[J, E]\).

By Lemma 7, \(F\) has at least a fixed point on \(\Omega\), then problem (1) have a solution \(x^*\) in \(PC[J, E]\), that is to say, \(x^*\) is a global mild solution for problem (1). \(\square\)

**Remark 1.** Theorem 1 is given when operator \(g_1 : PC[J, E] \rightarrow E\) is compact, and Theorem 2 does not require operator \(g_1 : PC[J, E] \rightarrow E\) to be compact, but requires assumption \((H_0)\).

**Remark 2.** In [26], the authors used the following conditions to the impulse term:
\[
\| I_k(x) \| \leq d_k \| x \|_{PC} + e_k, \| \hat{I}_k(x) \| \leq d_k \| x \|_{PC} + e_k,
\]
\[
\| \alpha(I_k(S)) \| \leq L_k \alpha(S), \| \alpha(\hat{I}_k(S)) \| \leq L_k \alpha(S),
\]
\[ M^* \sum k = 1 m(L_{k} + TL_{k}) < 1, \]

through the method of estimate step by step, we remove the compactness condition on the impulsive term and obtain the existence of a mild solution for problem (1).

Remark 3. When \( L_4 = 0 \), the condition (i) in (\( H_7 \)) is obviously true. In this case, we can get the result of Theorem 1 in [26], while the condition is weaker.

4. Example

In this section, we give an example to illustrate our main results.

Example 1. We consider the following fractional integro-differential equation:

\[
\begin{cases}
\frac{C^\beta}{\Gamma(\beta)} x(\xi, t) = \frac{\partial^2}{\partial \xi^2} x(\xi, t) + \frac{x(\xi, t)}{4(1 + \sqrt{t})} + \frac{e^{-t}}{4(1 + e^t)} \sin(x(\xi, t - \tau)) \\
+ \frac{1}{8(1 + t)} \int_0^t tx(\xi, s) \cos ds + \frac{1}{8(1 + e^t)} \int_0^1 \frac{s \sin t}{2} x(\xi, s) ds,
\end{cases}
\]

\( t \in [0, 1], \xi \in [0, \pi], t \neq \frac{1}{2} \) \( \quad (10) \)

\( x(\frac{1}{2}, 0) - x(\frac{1}{2}, -1) = \frac{1}{8} x(\frac{1}{2}), \)

\( x(0, t) = x(0, \pi) = 0, \)

\( x(\xi, 0) + \frac{1}{8}(x + \cos x) = x_0 \in E, \)

\( \frac{\partial}{\partial \xi} x(\xi, 0) + \frac{1}{16}(x + \sin x) = x_1 \in E, \)

where \( \beta = \frac{3}{2}, a = 1. \)

Take \( E = L^2([0, \pi]) \), and we consider the operator \( A : D(A) \subseteq E \rightarrow E \) defined by \( Ax = \frac{\partial^2}{\partial \xi^2} x(\xi, t) \) with domain \( D(A) = \{ x \in E : x'' \in E, x(0) = x(\pi) = 0 \} \). It is well known that \( A \) is the infinitesimal generator of a strongly continuous cosine family \( \{ C_\beta(t) \}_{t \in \mathbb{R}} \) on \( E \). Furthermore, we can get \( ||C_\beta(t)|| = 1 \) for all \( t \in \mathbb{R} \).

Define

\( x(t) = x(\xi, t), \tau(t) = t - \tau \leq t, (Hx)(t) = \int_0^t t \cos x(\xi, s) ds, \)

\( (Gx)(t) = \int_0^1 \frac{s \sin t}{2} x(\xi, s) ds, h(t, s, x) = tx \cos, g(t, s, x) = \frac{s \sin t}{2} x, \)

\( f(t, x_1, x_2, x_3, x_4) = \frac{x_1}{4(1 + \sqrt{t})} + \frac{e^{-t}}{4(1 + e^t)} \sin x_2 + \frac{1}{8(1 + t)} x_3 + \frac{1}{8(1 + e^t)} x_4, \)

\( g_1(x) = \frac{1}{8}(x + \cos x), g_2(x) = \frac{1}{16}(x + \sin x), I(x) = \frac{1}{8} x, \)

then the fractional impulsive integro-differential system (10) can be transformed into the abstract form:

\[
\begin{cases}
\frac{C^\beta}{\Gamma(\beta)} x(t) = Ax(t) + f(t, x(t), x(\tau(t)), (Hx)(t), (Gx)(t)), t \in [0, 1], t \neq \frac{1}{2} , \\
\Delta x|_{t = \frac{1}{2}} = I(x(\frac{1}{2})), \\
x(0) + g_1(x) = a_0 \in E, x'(0) + g_2(x) = a_1 \in E,
\end{cases}
\]

\( \quad (11) \)
For $\forall u_{1}, v_{1} \in E$, $t \in [0, 1]$, we have

$$\|f(t, u_{1}, u_{2}, u_{3}, u_{4}) - f(t, v_{1}, v_{2}, v_{3}, v_{4})\| \leq \frac{1}{4(1 + \sqrt{t})} \| u_{1} - v_{1}\| + \frac{e^{-t}}{4(1 + e^{t})} \| u_{2} - v_{2}\|
+ \frac{1}{8(1 + t)} \| u_{3} - v_{3}\| + \frac{1}{8(1 + e^{t})} \| u_{4} - v_{4}\|,$$

$$\|h(t, s, u) - h(t, s, v)\| \leq \|u - v\|, \|g(t, s, u) - g(t, s, v)\| \leq \frac{1}{2}\|u - v\|.$$

It is easy to see that $L_{1} = \frac{1}{4}, L_{2} = \frac{1}{4}, L_{3} = \frac{1}{8}, L_{4} = \frac{1}{8}, L_{h} = 1, L_{g} = \frac{1}{2}$, by computation, the condition in (H2) are satisfying and other assumptions given in Theorem 1 are also satisfied. Therefore, the problem (10) has a mild solution.

5. Conclusions

In this paper, by using the Mönch fixed point theorem and the method of estimate step by step, we remove the compactness condition on the impulsive term and obtain the existence of a mild solution for problem (1). The results obtained herein generalize and improve some known results. In our future work, we aim to study the fractional non-autonomous evolution equations with impulses and delay.

Author Contributions: Y.L. and B.Q. authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Science Foundation of Shandong Province grant number ZR2018MA019.

Institutional Review Board Statement: Exclude this statement.

Informed Consent Statement: Exclude this statement.

Data Availability Statement: Not applicable.

Acknowledgments: The authors express their sincerely thanks to the editors and reviewers for the careful reading of the manuscript and thoughtful comments.

Conflicts of Interest: The authors declare that they have no competing interest.

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