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Positive Solutions for a Fractional Differential Equation with Sequential Derivatives and Nonlocal Boundary Conditions

Alexandru Tudorache [1] and Rodica Luca [2,*]

[1] Department of Computer Science and Engineering, Gh. Asachi Technical University, 700050 Iasi, Romania
[2] Department of Mathematics, Gh. Asachi Technical University, 700506 Iasi, Romania
* Correspondence: rluca@math.tuiasi.ro

Abstract: We study the existence of positive solutions for a Riemann–Liouville fractional differential equation with sequential derivatives, a positive parameter and a sign-changing singular nonlinearity, subject to nonlocal boundary conditions containing varied fractional derivatives and general Riemann–Stieltjes integrals. We also present the associated Green functions and some of their properties. In the proof of the main results, we apply the Guo–Krasnosel'skii fixed point theorem. Two examples are finally given that illustrate our results.

Keywords: Riemann–Liouville fractional differential equation; nonlocal boundary conditions; sign-changing functions; singular functions; positive solutions

MSC: 34A08; 34B10; 34B16; 34B18

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1. Introduction

We consider the nonlinear ordinary fractional differential equation with sequential derivatives

$$D_{0+}^{\beta}\big(\mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{u}(t)\big) = \lambda\mathfrak{f}(t,\mathfrak{u}(t)), \quad t \in (0,1), \tag{1}$$

subject to the nonlocal boundary conditions

$$\begin{cases} \mathfrak{u}^{(j)}(0) = 0, \quad j = 0,\ldots,n-2, \quad D_{0+}^{\gamma}\mathfrak{u}(0) = 0, \\ \mathfrak{q}(1)D_{0+}^{\gamma}\mathfrak{u}(1) = \displaystyle\int_0^1 \mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{u}(t)\,d\mathfrak{H}_0(t), \quad D_{0+}^{\alpha_0}\mathfrak{u}(1) = \sum_{i=1}^{p}\int_0^1 D_{0+}^{\alpha_i}\mathfrak{u}(t)\,d\mathfrak{H}_i(t), \end{cases} \tag{2}$$

where $\beta \in (1,2]$, $\gamma \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $i = 0,\ldots,p$, $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_p \leq \alpha_0 < \gamma - 1$, $\alpha_0 \geq 1$, $\lambda > 0$, $\mathfrak{q} : [0,1] \to (0,\infty)$ is a continuous function, $\mathfrak{f} : (0,1) \times [0,\infty) \to \mathbb{R}$ is a continuous function which may be singular at $t = 0$ and/or $t = 1$, D_{0+}^{κ} denotes the Riemann–Liouville fractional derivative of order κ, for $\kappa = \beta, \gamma, \alpha_0, \alpha_1, \ldots, \alpha_p$, and the integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with $\mathfrak{H}_i$, $i = 0,\ldots,p$ functions of bounded variation. The last two conditions from (2) contain symmetric cases for the fractional derivatives of $\mathfrak{u}$. For example, if $p = 1$, $\mathfrak{H}_1(t) = \{0,\ t = 0;\ 1,\ t \in (0,1]\}$ and $\alpha_1 = \alpha_0$, then the last condition from (2) becomes a symmetric one, namely $D_{0+}^{\alpha_0}\mathfrak{u}(1) = D_{0+}^{\alpha_0}\mathfrak{u}(0)$. If in addition $\alpha_0 = 1$, we obtain the periodic condition for the first derivative of $\mathfrak{u}$, that is, $\mathfrak{u}'(1) = \mathfrak{u}'(0)$.

We present some assumptions for the function $\mathfrak{f}$ and give intervals for the parameter λ such that there exists at least one positive solution of problem (1) and (2). A positive solution of (1) and (2) is a function $\mathfrak{u} \in C[0,1]$ satisfying (1) and (2) with $\mathfrak{u}(t) > 0$ for all $t \in (0,1]$. In the proofs of our main results, we apply the Guo–Krasnosel'skii fixed point theorem (see [1]). We present now some recent results connected to our problem (1) and (2). In [2], the authors investigated the existence of positive solutions for the fractional differential equation with sequential derivatives

$$D_{0+}^{\beta}\big(\mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{u}(t)\big) = \lambda\mathfrak{r}(t)\mathfrak{g}(t,\mathfrak{u}(t)),\ \ t \in (0,1), \tag{3}$$

with the nonlocal boundary conditions

$$\begin{cases} \mathfrak{u}^{(j)}(0) = 0,\ \ j = 0,\dots,n-2,\ \ D_{0+}^{\gamma}\mathfrak{u}(0) = 0, \\ \mathfrak{q}(1)D_{0+}^{\gamma}\mathfrak{u}(1) = a\mathfrak{q}(\xi)D_{0+}^{\gamma}\mathfrak{u}(\xi),\ \ D_{0+}^{\alpha_0}\mathfrak{u}(1) = \displaystyle\sum_{i=1}^{p}\int_0^1 D_{0+}^{\alpha_i}\mathfrak{u}(t)\,d\mathfrak{H}_i(t), \end{cases} \tag{4}$$

where $\beta \in (1,2]$, $\gamma \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $i = 0,\dots,p$, $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_p \leq \alpha_0 < \gamma - 1$, $\alpha_0 \geq 1$, $\lambda > 0$, $a \geq 0$, $\xi \in (0,1)$, $\mathfrak{q} : [0,1] \to (0,\infty)$ is a continuous function, the function $\mathfrak{g} : [0,1] \times (0,\infty) \to [0,\infty)$ is continuous and may have a singularity at the second variable in the point 0, the function $\mathfrak{r} : (0,1) \to [0,\infty)$ is continuous and may be singular at $t = 0$ and/or $t = 1$, and $\mathfrak{H}_i$, $i = 1,\dots,p$ are bounded variation functions. In the proof of the main results, they used some theorems from the fixed point index theory. In comparison with our problem (1) and (2) in which the nonlinearity $\mathfrak{f}$ has arbitrary values, the nonlinearity $\mathfrak{g}$ in (3) is nonnegative; in addition, the first condition from the second line of (4) is a particular case of the condition from (2), with $\mathfrak{H}_0(t) = \{0,\ t \in [0,\xi);\ a,\ t \in [\xi,1]\}$. Besides this, in this paper, we use for the function $\mathfrak{f}$ from (1) different assumptions than those used for the function $\mathfrak{g}$ in [2]. In [3], the authors studied the existence of at least one or two positive solutions for the fractional differential equation

$$D_{0+}^{\alpha}\mathfrak{u}(t) + \mathfrak{f}(t,\mathfrak{u}(t)) = 0,\ \ t \in (0,1), \tag{5}$$

supplemented with the boundary conditions

$$\mathfrak{u}(0) = \mathfrak{u}'(0) = \dots = \mathfrak{u}^{(n-2)}(0) = 0,\ \ D_{0+}^{\beta_0}\mathfrak{u}(1) = \sum_{i=1}^{m}\int_0^1 D_{0+}^{\beta_i}\mathfrak{u}(t)\,d\mathfrak{H}_i(t), \tag{6}$$

where $\alpha \in \mathbb{R}$, n, $m \in \mathbb{N}$, $n \geq 3$, $\alpha \in (n-1,n]$, $\beta_i \in \mathbb{R}$ for all $i = 0,\dots,m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \beta_0 < \alpha - 1$, $\beta_0 \geq 1$, and the nonlinearity $\mathfrak{f}(t,u)$ may change sign and may be singular at the points $t = 0$, $t = 1$ and/or $u = 0$. The authors used in [3] various height functions of $\mathfrak{f}$ defined on special bounded sets and the Leggett–Williams and the Krasnosel'skii fixed point index theorems. For other recent results related to the existence, nonexistence and multiplicity of positive solutions for fractional differential equations and systems with or without p-Laplacian operators, subject to varied nonlocal boundary conditions and their applications, we mention the books [4–15] and their references. Some fixed point results for a pair of fuzzy dominated mappings are presented in [11,12].

The organization of our paper is as follows. In Section 2, we investigate a linear fractional boundary value problem that is associated to problem (1) and (2) and present the associated Green functions with their properties. Section 3 is concerned with the main existence theorems for (1) and (2), and in Section 4, we give two examples that illustrate our results. Finally, Section 5 contains the conclusions for this paper.

2. Auxiliary Results

We consider the fractional differential equation

$$D_{0+}^{\beta}\big(\mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{u}(t)\big) = \mathfrak{h}(t),\ \ t \in (0,1), \tag{7}$$

with the boundary conditions (2), where $\mathfrak{h} \in C(0,1) \cap L^1(0,1)$. We denote by

$$\mathfrak{d}_1 = 1 - \int_0^1 \tau^{\beta-1}\,d\mathfrak{H}_0(\tau),\ \ \mathfrak{d}_2 = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha_0)} - \sum_{i=1}^{p}\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha_i)}\int_0^1 \tau^{\gamma-\alpha_i-1}\,d\mathfrak{H}_i(\tau). \tag{8}$$

Lemma 1. *If $\mathfrak{d}_1 \neq 0$ and $\mathfrak{d}_2 \neq 0$, then, the unique solution $\mathfrak{u} \in C[0,1]$ of problem (2) and (7) is*

$$\mathfrak{u}(t) = \int_0^1 \mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{G}_1(s,\tau)\mathfrak{h}(\tau)\,d\tau\right)ds, \;\; t \in [0,1], \tag{9}$$

where

$$\mathfrak{G}_1(t,s) = \mathfrak{g}_1(t,s) + \frac{t^{\beta-1}}{\mathfrak{d}_1}\int_0^1 \mathfrak{g}_1(\tau,s)\,d\mathfrak{H}_0(\tau), \;\; (t,s) \in [0,1]\times[0,1], \tag{10}$$

with

$$\mathfrak{g}_1(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{11}$$

and

$$\mathfrak{G}_2(t,s) = \mathfrak{g}_2(t,s) + \frac{t^{\gamma-1}}{\mathfrak{d}_2}\sum_{i=1}^{p}\left(\int_0^1 \mathfrak{g}_{3i}(\tau,s)\,d\mathfrak{H}_i(\tau)\right), \;\; (t,s) \in [0,1]\times[0,1], \tag{12}$$

with

$$\begin{aligned} &\mathfrak{g}_2(t,s) = \frac{1}{\Gamma(\gamma)}\begin{cases} t^{\gamma-1}(1-s)^{\gamma-\alpha_0-1} - (t-s)^{\gamma-1}, & 0 \le s \le t \le 1, \\ t^{\gamma-1}(1-s)^{\gamma-\alpha_0-1}, & 0 \le t \le s \le 1, \end{cases} \\ &\mathfrak{g}_{3i}(t,s) = \frac{1}{\Gamma(\gamma-\alpha_i)}\begin{cases} t^{\gamma-\alpha_i-1}(1-s)^{\gamma-\alpha_0-1} - (t-s)^{\gamma-\alpha_i-1}, & 0 \le s \le t \le 1, \\ t^{\gamma-\alpha_i-1}(1-s)^{\gamma-\alpha_0-1}, & 0 \le t \le s \le 1. \end{cases} \\ &i = 1,\ldots,p. \end{aligned} \tag{13}$$

Proof. We denote by $\mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{u}(t) = \mathfrak{v}(t)$. Then, problem (2) and (7) is equivalent to the following two boundary value problems:

$$(\mathrm{I}) \qquad \begin{cases} D_{0+}^{\beta}\mathfrak{v}(t) = \mathfrak{h}(t), \;\; t \in (0,1), \\ \mathfrak{v}(0) = 0, \;\; \mathfrak{v}(1) = \displaystyle\int_0^1 \mathfrak{v}(\tau)\,d\mathfrak{H}_0(\tau), \end{cases}$$

and

$$(\mathrm{II}) \qquad \begin{cases} D_{0+}^{\gamma}\mathfrak{u}(t) = \mathfrak{v}(t)/\mathfrak{q}(t), \;\; t \in (0,1), \\ \mathfrak{u}^{(j)}(0) = 0, \;\; j = 0,\ldots,n-2, \;\; D_{0+}^{\alpha_0}\mathfrak{u}(1) = \displaystyle\sum_{i=1}^{p}\int_0^1 D_{0+}^{\alpha_i}\mathfrak{u}(t)\,d\mathfrak{H}_i(t). \end{cases}$$

By Lemma 4.1.5 from [7], the unique solution $\mathfrak{v} \in C[0,1]$ of problem (I) is

$$\mathfrak{v}(t) = -\int_0^1 \mathfrak{G}_1(t,s)\mathfrak{h}(s)\,ds, \;\; t \in [0,1], \tag{14}$$

where $\mathfrak{G}_1$ is given by (10). By Lemma 2.2 from [3], the unique solution $\mathfrak{u} \in C[0,1]$ of problem (II) is

$$\mathfrak{u}(t) = -\int_0^1 \mathfrak{G}_2(t,s)\mathfrak{v}(s)/\mathfrak{q}(s)\,ds, \;\; t \in [0,1], \tag{15}$$

where $\mathfrak{G}_2$ is given by (12). By using now (14) and (15) we obtain the solution $\mathfrak{u}$ of problem (2) and (7) given by relation (9). □

We present now some properties of functions $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_{3i}$, $i = 1,\ldots,p$ from [7,16].

Lemma 2. *The functions $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_{3i}$, $i = 1, \ldots, p$ given by (11) and (13) have the properties*

(a) $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_{3i} : [0,1] \times [0,1] \to [0,\infty)$, $i = 1, \ldots, p$ *are continuous functions, and* $\mathfrak{g}_1(t,s) > 0$, $\mathfrak{g}_2(t,s) > 0$ *and* $\mathfrak{g}_{3i}(t,s) > 0$ *for all* $(t,s) \in (0,1) \times (0,1)$;

(b) $\mathfrak{g}_1(t,s) \leq \mathfrak{h}_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\mathfrak{h}_1(s) = \frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}$, $s \in [0,1]$;

(c) $\mathfrak{g}_2(t,s) \leq \mathfrak{h}_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\mathfrak{h}_2(s) = \frac{1}{\Gamma(\gamma)}(1-s)^{\gamma-\alpha_0-1}(1-(1-s)^{\alpha_0})$, $s \in [0,1]$;

(d) $\mathfrak{g}_2(t,s) \geq t^{\gamma-1}\mathfrak{h}_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$;

(e) $\mathfrak{g}_{3i}(t,s) \leq \frac{1}{\Gamma(\gamma-\alpha_i)} t^{\gamma-\alpha_i-1}$, $\mathfrak{g}_{3i}(t,s) \leq \frac{1}{\Gamma(\gamma-\alpha_i)}(1-s)^{\gamma-\alpha_0-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, $i = 1, \ldots, p$;

(f) $\mathfrak{g}_{3i}(t,s) \geq t^{\gamma-\alpha_i-1}\mathfrak{h}_{3i}(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\mathfrak{h}_{3i}(s) = \frac{1}{\Gamma(\gamma-\alpha_i)}(1-s)^{\gamma-\alpha_0-1}(1-(1-s)^{\alpha_0-\alpha_i})$, $s \in [0,1]$, $i = 1, \ldots, p$.

By using the properties of the functions $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_{3i}$, $i = 1, \ldots, p$ presented in Lemma 2, we obtain the following result.

Lemma 3. *If* $\mathfrak{d}_1 > 0$, $\mathfrak{d}_2 > 0$, $\mathfrak{H}_i$, $i = 0, \ldots, p$ *are nondecreasing functions, then the functions* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *given by (10) and (12) have the properties:*

(a) $\mathfrak{G}_1, \mathfrak{G}_2 : [0,1] \times [0,1] \to [0,\infty)$ *are continuous functions;*

(b) $\mathfrak{G}_1(t,s) \leq \mathfrak{J}_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\mathfrak{J}_1(s) = \mathfrak{h}_1(s) + \frac{1}{\mathfrak{d}_1}\int_0^1 \mathfrak{g}_1(\tau,s)\, d\mathfrak{H}_0(\tau), \ \forall s \in [0,1];$$

(c) $\mathfrak{G}_2(t,s) \leq \mathfrak{J}_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\mathfrak{J}_2(s) = \mathfrak{h}_2(s) + \frac{1}{\mathfrak{d}_2}\sum_{i=1}^{p}\int_0^1 g_{3i}(\tau,s)\, d\mathfrak{H}_i(\tau), \ \forall s \in [0,1];$$

(d) $\mathfrak{G}_2(t,s) \geq t^{\gamma-1}\mathfrak{J}_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$;

(e) $\mathfrak{G}_2(t,s) \leq \sigma t^{\gamma-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\sigma = \frac{1}{\Gamma(\gamma)} + \frac{1}{\mathfrak{d}_2}\sum_{i=1}^{p}\frac{1}{\Gamma(\gamma-\alpha_i)}\int_0^1 \tau^{\gamma-\alpha_i-1}\, d\mathfrak{H}_i(\tau).$$

Lemma 4. *If* $\mathfrak{d}_1 > 0$, $\mathfrak{d}_2 > 0$, $\mathfrak{H}_i$, $i = 0, \ldots, p$ *are nondecreasing functions, and* $\mathfrak{h} \in C(0,1) \cap L^1(0,1)$ *with* $\mathfrak{h}(t) \geq 0$ *for all* $t \in (0,1)$, *then, the solution* $\mathfrak{u}$ *of problem (2) and (7) given by (9) satisfies the properties* $\mathfrak{u}(t) \geq 0$ *for all* $t \in [0,1]$ *and* $\mathfrak{u}(t) \geq t^{\gamma-1}\mathfrak{u}(\zeta)$ *for all* $t, \zeta \in [0,1]$.

Proof. By using the properties from Lemma 3, we have $\mathfrak{u}(t) \geq 0$ for all $t \in [0,1]$. In addition, we obtain

$$\begin{aligned}\mathfrak{u}(t) &\geq \int_0^1 t^{\gamma-1}\mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{G}_1(s,\tau)\mathfrak{h}(\tau)\, d\tau\right)ds \\ &\geq t^{\gamma-1}\int_0^1 \mathfrak{G}_2(\zeta,s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{G}_1(s,\tau)\mathfrak{h}(\tau)\, d\tau\right)ds = t^{\gamma-1}\mathfrak{u}(\zeta), \ \forall t,\zeta \in [0,1].\end{aligned}$$

□

3. Existence of Positive Solutions

In this section, we study the existence of positive solutions for our problem (1) and (2). We present the assumptions that we use in this section.

$(I1)$ $\beta \in (1,2]$, $n \in \mathbb{N}$, $n \geq 3$, $\gamma \in (n-1,n]$, $p \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $i = 0,\dots,p$, $0 \leq \alpha_1 < \dots < \alpha_p \leq \alpha_0 < \gamma - 1$, $\alpha_0 \geq 1$, $\mathfrak{d}_1 > 0$, $\mathfrak{d}_2 > 0$, $\mathfrak{H}_i : [0,1] \to \mathbb{R}$, $i = 0,\dots,p$ are nondecreasing functions, $\lambda > 0$, $\mathfrak{q} \in C([0,1],(0,\infty))$, ($\mathfrak{d}_1, \mathfrak{d}_2$ are given by (8)).

$(I2)$ The function $\mathfrak{f} \in C((0,1)\times[0,\infty),\mathbb{R})$ may have a singularity at $t = 0$ and/or $t = 1$, and there exist the functions ξ, $\varphi \in C((0,1),[0,\infty))$, $\psi \in C([0,1]\times[0,\infty),[0,\infty))$ such that

$$-\xi(t) \leq \mathfrak{f}(t,x) \leq \varphi(t)\psi(t,x), \ \ \forall\, t \in (0,1), \ x \in [0,\infty),$$

with $0 < \int_0^1 \xi(s)\,ds < \infty$, $0 < \int_0^1 \varphi(s)\,ds < \infty$.

$(I3)$ There exist θ_1, $\theta_2 \in (0,1)$, $\theta_1 < \theta_2$ such that $\mathfrak{f}_\infty = \lim\limits_{x\to\infty} \min\limits_{t\in[\theta_1,\theta_2]} \dfrac{\mathfrak{f}(t,x)}{x} = \infty$.

$(I4)$ There exist θ_1, $\theta_2 \in (0,1)$, $\theta_1 < \theta_2$ such that $\liminf\limits_{x\to\infty} \min\limits_{t\in[\theta_1,\theta_2]} \mathfrak{f}(t,x) > \Xi_0$, and $\psi_\infty = \lim\limits_{x\to\infty} \max\limits_{t\in[0,1]} \dfrac{\psi(t,x)}{x} = 0$, where

$$\Xi_0 = \frac{2\sigma\omega}{\theta_1^{\gamma-1}}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)\left(\int_{\theta_1}^{\theta_2} \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2} \mathfrak{G}_1(s,\tau)\,d\tau\right)ds\right)^{-1},$$

$\omega = \max_{s\in[0,1]} \mathfrak{J}_1(s)$, and σ, $\mathfrak{G}_1$ $\mathfrak{J}_1$, $\mathfrak{J}_2$ are given in Lemmas 1 and 3.

We consider the fractional differential equation

$$D_{0+}^{\beta}\big(\mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{v}(t)\big) = \lambda(\mathfrak{f}(t,[\mathfrak{v}(t)-\lambda\zeta(t)]^*)+\xi(t)) = 0, \ \ t \in (0,1), \tag{16}$$

with the nonlocal boundary conditions

$$\begin{cases} \mathfrak{v}^{(j)}(0) = 0, \ \ j = 0,\dots,n-2, \ \ D_{0+}^{\gamma}\mathfrak{v}(0) = 0, \\ \mathfrak{q}(1)D_{0+}^{\gamma}\mathfrak{v}(1) = \displaystyle\int_0^1 \mathfrak{q}(t)D_{0+}^{\gamma}\mathfrak{v}(t)\,d\mathfrak{H}_0(t), \ \ D_{0+}^{\alpha_0}\mathfrak{v}(1) = \sum_{i=1}^{p}\int_0^1 D_{0+}^{\alpha_i}\mathfrak{v}(t)\,d\mathfrak{H}_i(t), \end{cases} \tag{17}$$

where $w(t)^* = w(t)$ if $w(t) \geq 0$, and $w(t)^* = 0$ if $w(t) < 0$. Here

$$\zeta(t) = \int_0^1 \mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{G}_1(s,\tau)\xi(\tau)\,d\tau\right)ds, \ \ t \in [0,1]$$

is the solution of problem

$$\begin{cases} D_{0+}^{\beta}\big(\mathfrak{q}(t)D_{0+}^{\gamma}\zeta(t)\big) = \xi(t), \ \ t \in (0,1), \\ \zeta^{(j)}(0) = 0, \ \ j = 0,\dots,n-2, \ \ D_{0+}^{\gamma}\zeta(0) = 0, \\ \mathfrak{q}(1)D_{0+}^{\gamma}\zeta(1) = \displaystyle\int_0^1 \mathfrak{q}(t)D_{0+}^{\gamma}\zeta(t)\,d\mathfrak{H}_0(t), \ \ D_{0+}^{\alpha_0}\zeta(1) = \sum_{i=1}^{p}\int_0^1 D_{0+}^{\alpha_i}\zeta(t)\,d\mathfrak{H}_i(t). \end{cases} \tag{18}$$

Under assumptions $(I1)$, $(I2)$, we obtain $\zeta(t) \geq 0$ for all $t \in [0,1]$. We show that there exists a solution $\mathfrak{v}$ of problem (16) and (17) with $\mathfrak{v}(t) \geq \lambda\zeta(t)$ on $[0,1]$ and $\mathfrak{v}(t) > \lambda\zeta(t)$ on $(0,1]$. In this case, $\mathfrak{u} = \mathfrak{v} - \lambda\zeta$ represents a positive solution of problem (1) and (2). Hence, in what follows, we study problem (16) and (17). By using Lemma 1, $\mathfrak{v}$ is a solution of problem (16) and (17) if and only if $\mathfrak{v}$ is a solution of equation

$$\mathfrak{v}(t) = \lambda\int_0^1 \mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[\mathfrak{v}(\tau)-\lambda\zeta(\tau)]^*)+\xi(\tau))d\tau\right)ds, \ \ t \in [0,1].$$

We introduce the Banach space $X = C[0,1]$ with the supremum norm $\|\cdot\|$, and the cone

$$\mathcal{Q} = \{\mathfrak{v} \in X, \ \mathfrak{v}(t) \geq t^{\gamma-1}\|\mathfrak{v}\|, \ \forall\, t \in [0,1]\}.$$

For $\lambda > 0$ we also define the operator $\mathcal{A} : X \to X$ given by

$$\mathcal{A}\mathfrak{v}(t) = \lambda \int_0^1 \mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)} \int_0^1 \mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[\mathfrak{v}(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))d\tau\right)ds,$$

for $t \in [0,1]$ and $\mathfrak{v} \in X$.

Lemma 5. *We suppose that assumptions $(I1)$ and $(I2)$ hold. Then, $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}$ is a completely continuous operator.*

Proof. Let $\mathfrak{v} \in \mathcal{Q}$ be fixed. By using $(I1)$ and $(I2)$, we find that $\mathcal{A}\mathfrak{v}(t) < \infty$ for all $t \in [0,1]$. In addition, by Lemma 3, we obtain for all $t, t' \in [0,1]$

$$\mathcal{A}\mathfrak{v}(t) \leq \lambda \int_0^1 \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)} \int_0^1 \mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[v(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))d\tau\right)ds,$$

and

$$\begin{aligned}\mathcal{A}\mathfrak{v}(t) &\geq \lambda \int_0^1 t^{\gamma-1}\mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)} \int_0^1 \mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[v(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))d\tau\right)ds \\ &\geq t^{\gamma-1}\mathcal{A}\mathfrak{v}(t').\end{aligned}$$

Hence, $\mathcal{A}\mathfrak{v}(t) \geq t^{\gamma-1}\|\mathcal{A}\mathfrak{v}\|$ for all $t \in [0,1]$. We conclude that $\mathcal{A}\mathfrak{v} \in \mathcal{Q}$, and hence $\mathcal{A}(\mathcal{Q}) \subset \mathcal{Q}$. With a standard approach, we deduce that $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}$ is a completely continuous operator. □

Theorem 1. *We suppose that assumptions $(I1)$, $(I2)$ and $(I3)$ hold. Then, there exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1]$, the problem (1) and (2) has at least one positive solution.*

Proof. We consider a positive number $r_1 > \sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)$, and we define the set $\mathrm{Y}_1 = \{\mathfrak{v} \in X,\ \|\mathfrak{v}\| < r_1\}$. We also define

$$\lambda_1 = \min\left\{1, r_1\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\mathfrak{J}_2(s)\,ds\right)^{-1}\left(\Lambda_1 \int_0^1 \mathfrak{J}_1(\tau)(\varphi(\tau) + \xi(\tau))\,d\tau\right)^{-1}\right\},$$

with $\Lambda_1 = \max\{\max_{t\in[0,1],\,x\in[0,r_1]} \psi(t,x), 1\}$.

Let $\lambda \in (0, \lambda_1]$. Because

$$\begin{aligned}\zeta(t) &\leq \int_0^1 \sigma t^{\gamma-1}\left(\frac{1}{\mathfrak{q}(s)} \int_0^1 \mathfrak{G}_1(s,\tau)\xi(\tau)\,d\tau\right)ds \\ &\leq \sigma t^{\gamma-1} \int_0^1 \frac{1}{\mathfrak{q}(s)}\left(\int_0^1 \mathfrak{J}_1(\tau)\xi(\tau)\,d\tau\right)ds \\ &\leq \sigma\omega t^{\gamma-1}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right),\ \ \forall\, t \in [0,1],\end{aligned}$$

we find for any $\mathfrak{v} \in \mathcal{Q} \cap \partial\mathrm{Y}_1$ and $t \in [0,1]$

$$[\mathfrak{v}(t) - \lambda\zeta(t)]^* \leq \mathfrak{v}(t) \leq \|\mathfrak{v}\| \leq r_1,$$

and

$$\begin{aligned}\mathfrak{v}(t)-\lambda\zeta(t) &\geq t^{\gamma-1}\|\mathfrak{v}\|-\lambda\sigma\omega t^{\gamma-1}\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\\ &= t^{\gamma-1}\left[r_1-\lambda\sigma\omega\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\right]\\ &\geq t^{\gamma-1}\left[r_1-\lambda_1\sigma\omega\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\right]\\ &\geq t^{\gamma-1}\left[r_1-\sigma\omega\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\right]\geq 0.\end{aligned} \tag{19}$$

Therefore, for any $\mathfrak{v}\in\mathcal{Q}\cap\partial\mathrm{Y}_1$ and $t\in[0,1]$, we deduce

$$\begin{aligned}\mathcal{A}\mathfrak{v}(t) &\leq \lambda\int_0^1\mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1\mathfrak{J}_1(\tau)(\varphi(\tau)\psi(\tau,[\mathfrak{v}\tau)-\lambda\zeta(\tau)]^*)+\xi(\tau))\,d\tau\right)ds\\ &\leq \lambda_1\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\mathfrak{J}_2(s)\,ds\right)\left(\Lambda_1\int_0^1\mathfrak{J}_1(\tau)(\varphi(\tau)+\xi(\tau))\,d\tau\right)\leq r_1=\|\mathfrak{v}\|.\end{aligned}$$

Hence, we conclude

$$\|\mathcal{A}\mathfrak{v}\|\leq\|\mathfrak{v}\|,\ \ \forall\,\mathfrak{v}\in\mathcal{Q}\cap\partial\mathrm{Y}_1. \tag{20}$$

Next, for θ_1, $\theta_2\in(0,1)$, $\theta_1<\theta_2$ given in $(I3)$, we choose a constant $\Xi_1>0$ such that

$$\Xi_1\geq\frac{2}{\lambda\theta_1^{\beta+\gamma-2}}\left(\int_{\theta_1}^{\theta_2}\frac{1}{\mathfrak{q}(s)}\mathfrak{J}_2(s)\left(\int_{\theta_1}^{\theta_2}\mathfrak{G}_1(s,\tau)\,d\tau\right)ds\right)^{-1}.$$

By $(I3)$, we deduce that there exists a constant $\Lambda_0>0$ such that

$$\mathfrak{f}(t,x)\geq\Xi_1 x,\ \ \forall\,t\in[\theta_1,\theta_2],\ \ x\geq\Lambda_0. \tag{21}$$

We define now $r_2=\max\left\{2r_1,2\Lambda_0/\theta_1^{\gamma-1}\right\}$ and let $\mathrm{Y}_2=\{\mathfrak{v}\in X,\ \ \|\mathfrak{v}\|<r_2\}$. Then, for any $\mathfrak{v}\in\mathcal{Q}\cap\partial\mathrm{Y}_2$, we obtain as in (19)

$$\begin{aligned}\mathfrak{v}(t)-\lambda\zeta(t) &\geq t^{\gamma-1}\left[r_2-\sigma\omega\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\right]\\ &\geq t^{\gamma-1}\left[r_1-\sigma\omega\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\right]\geq 0,\ \ \forall\,t\in[0,1].\end{aligned}$$

Hence, we find

$$\begin{aligned}&[\mathfrak{v}(t)-\lambda\zeta(t)]^*=\mathfrak{v}(t)-\lambda\zeta(t)\geq t^{\gamma-1}\left[r_2-\sigma\omega\left(\int_0^1\frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1\xi(\tau)\,d\tau\right)\right]\\ &\quad\geq\frac{\theta_1^{\gamma-1}r_2}{2}\geq\Lambda_0,\ \ \forall\,t\in[\theta_1,\theta_2].\end{aligned} \tag{22}$$

Then, for any $\mathfrak{v}\in\mathcal{Q}\cap\partial\mathrm{Y}_2$ and $t\in[\theta_1,\theta_2]$, by (21) and (22), we deduce

$$\begin{aligned}&\mathcal{A}\mathfrak{v}(t)\geq\lambda\int_{\theta_1}^{\theta_2}\mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2}\mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[\mathfrak{v}(\tau)-\lambda\zeta(\tau)]^*)+\xi(\tau))\,d\tau\right)ds\\ &\geq\lambda\int_{\theta_1}^{\theta_2}\mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2}\mathfrak{G}_1(s,\tau)(\Xi_1[\mathfrak{v}(\tau)-\lambda\zeta(\tau)]^*+\xi(\tau))\,d\tau\right)ds\\ &\geq\lambda\int_{\theta_1}^{\theta_2}\mathfrak{G}_2(t,s)\left(\frac{\Xi_1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2}\mathfrak{G}_1(s,\tau)[\mathfrak{v}(\tau)-\lambda\zeta(\tau)]^*\,d\tau\right)ds\\ &\geq\frac{\lambda\theta_1^{\beta+\gamma-2}\Xi_1 r_2}{2}\left(\int_{\theta_1}^{\theta_2}\frac{1}{\mathfrak{q}(s)}\mathfrak{J}_2(s)\left(\int_{\theta_1}^{\theta_2}\mathfrak{G}_1(s,\tau)\,d\tau\right)ds\right)\geq r_2=\|\mathfrak{v}\|.\end{aligned}$$

So,

$$\|\mathcal{A}\mathfrak{v}\|\geq\|\mathfrak{v}\|,\ \ \forall\,\mathfrak{v}\in\mathcal{Q}\cap\partial\mathrm{Y}_2. \tag{23}$$

By (20) and (23) and the Guo–Krasnosel'skii fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $\mathfrak{v}_1 \in \mathcal{Q} \cap (\overline{\mathrm{Y}}_2 \setminus \mathrm{Y}_1)$, that is, $r_1 \le \|\mathfrak{v}_1\| \le r_2$. Since $\|\mathfrak{v}_1\| \ge r_1$, we obtain

$$\begin{aligned}
&\mathfrak{v}_1(t) - \lambda\zeta(t) \ge t^{\gamma-1}\|\mathfrak{v}_1\| - \lambda\sigma\omega t^{\gamma-1}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)\\
&\ge t^{\gamma-1}\left[r_1 - \sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)\right] = \Xi_2 t^{\gamma-1}, \ \ \forall\, t \in [0,1],
\end{aligned}$$

where $\Xi_2 = r_1 - \sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)$, and so $\mathfrak{v}_1(t) \ge \lambda\zeta(t) + \Xi_2 t^{\gamma-1}$ for all $t \in [0,1]$. Let $\mathfrak{u}_1(t) = \mathfrak{v}_1(t) - \lambda\zeta(t)$ for all $t \in [0,1]$. Then $\mathfrak{u}_1$ is a positive solution of problem (1) and (2) with $\mathfrak{u}_1(t) \ge \Xi_2 t^{\gamma-1}$ for all $t \in [0,1]$. $\square$

Theorem 2. *We suppose that assumptions (I1), (I2) and (I4) hold. Then, there exists $\lambda_2 > 0$ such that for any $\lambda \in [\lambda_2, \infty)$, the problem (1) and (2) has at least one positive solution.*

Proof. By $(I4)$ there exists $\Lambda_2 > 0$ such that $\mathfrak{f}(t,x) \ge \Xi_0$, for all $t \in [\theta_1,\theta_2]$ and $x \ge \Lambda_2$. We define $\lambda_2 = \frac{\Lambda_2}{\theta_1^{\gamma-1}\sigma\omega}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)^{-1}\left(\int_0^1 \xi(\tau)\,d\tau\right)^{-1}$. We suppose now $\lambda \ge \lambda_2$. Let $r_3 = 2\lambda\sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)$, and $\mathrm{Y}_3 = \{\mathfrak{v} \in X,\ \|\mathfrak{v}\| < r_3\}$. Then for any $\mathfrak{v} \in \mathcal{Q} \cap \partial\mathrm{Y}_3$, we obtain

$$\begin{aligned}
&\mathfrak{v}(t) - \lambda\zeta(t) \ge t^{\gamma-1}\|\mathfrak{v}\| - \lambda\sigma\omega t^{\gamma-1}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)\\
&= t^{\gamma-1}\left[r_3 - \lambda\sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)\right]\\
&= t^{\gamma-1}\lambda\sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right)\\
&\ge t^{\gamma-1}\lambda_2\sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right) = \frac{\Lambda_2 t^{\gamma-1}}{\theta_1^{\gamma-1}} \ge 0, \ \ \forall\, t \in [0,1].
\end{aligned} \tag{24}$$

Then, for any $\mathfrak{v} \in \mathcal{Q} \cap \partial\mathrm{Y}_3$ and $t \in [\theta_1,\theta_2]$, we find

$$[\mathfrak{v}(t) - \lambda\zeta(t)]^* = \mathfrak{v}(t) - \lambda\zeta(t) \ge \frac{\Lambda_2 t^{\gamma-1}}{\theta_1^{\gamma-1}} \ge \Lambda_2.$$

Hence, for any $\mathfrak{v} \in \mathcal{Q} \cap \partial\mathrm{Y}_3$ and $t \in [\theta_1,\theta_2]$, we deduce

$$\begin{aligned}
&\mathcal{A}\mathfrak{v}(t) \ge \lambda\int_{\theta_1}^{\theta_2} \mathfrak{G}_2(t,s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2} \mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[\mathfrak{v}(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))\,d\tau\right)ds\\
&\ge \lambda\int_{\theta_1}^{\theta_2} t^{\gamma-1}\mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2} \mathfrak{G}_1(s,\tau)(\mathfrak{f}(\tau,[\mathfrak{v}(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))\,d\tau\right)ds\\
&\ge \lambda\int_{\theta_1}^{\theta_2} t^{\gamma-1}\mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2} \mathfrak{G}_1(s,\tau)(\Xi_0 + \xi(\tau))\,d\tau\right)ds\\
&\ge \lambda\Xi_0 t^{\gamma-1}\int_{\theta_1}^{\theta_2} \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_{\theta_1}^{\theta_2} \mathfrak{G}_1(s,\tau)\,d\tau\right)ds = \frac{r_3 t^{\gamma-1}}{\theta_1^{\gamma-1}} \ge r_3.
\end{aligned}$$

Therefore, we conclude

$$\|\mathcal{A}\mathfrak{v}\| \ge \|\mathfrak{v}\|, \ \ \forall\,\mathfrak{v} \in \mathcal{Q} \cap \partial\mathrm{Y}_3. \tag{25}$$

Next, we consider the positive number $\epsilon = \left(2\lambda\int_0^1 \mathfrak{J}_2(s)\frac{1}{\mathfrak{q}(s)}\,ds\right)^{-1}\left(\int_0^1 \mathfrak{J}_1(\tau)\varphi(\tau)\,d\tau\right)^{-1}$. Then by $(I4)$ we obtain that there exists $\Lambda_3 > 0$ such that $\psi(t,x) \le \epsilon x$ for all $t \in [0,1]$

and $x \geq \Lambda_3$. Then, we find $\psi(t,x) \leq \Lambda_4 + \epsilon x$ for all $t \in [0,1]$, $x \geq 0$, where $\Lambda_4 = \max_{t\in[0,1],\, x\in[0,\Lambda_3]} \psi(t,x)$. We define

$$r_4 > \max\left\{r_3, 2\lambda \max\{\Lambda_4, 1\}\left(\int_0^1 \mathfrak{J}_2(s)\frac{1}{\mathfrak{q}(s)}\, ds\right)\left(\int_0^1 \mathfrak{J}_1(\tau)(\varphi(\tau)+\xi(\tau))\, d\tau\right)\right\},$$

and $\Upsilon_4 = \{\mathfrak{v} \in \mathcal{Q},\ \|\mathfrak{v}\| < r_4\}$.

Hence, for any $\mathfrak{v} \in \mathcal{Q} \cap \partial\Upsilon_4$, we deduce as in (24)

$$\begin{aligned}
&\mathfrak{v}(t) - \lambda\zeta(t) \geq t^{\gamma-1}\Big[r_4 - \lambda\sigma\omega\Big(\textstyle\int_0^1 \frac{1}{\mathfrak{q}(s)}\, ds\Big)\Big(\textstyle\int_0^1 \xi(\tau)\, d\tau\Big)\Big] \\
&\geq t^{\gamma-1}\Big[r_3 - \lambda\sigma\omega\Big(\textstyle\int_0^1 \frac{1}{\mathfrak{q}(s)}\, ds\Big)\Big(\textstyle\int_0^1 \xi(\tau)\, d\tau\Big)\Big] \geq \frac{\Lambda_2 t^{\gamma-1}}{\theta_1^{\gamma-1}} \geq 0, \ \ \forall\, t \in [0,1].
\end{aligned} \tag{26}$$

So, for any $\mathfrak{v} \in \mathcal{Q} \cap \partial\Upsilon_4$, we obtain

$$\begin{aligned}
&\mathcal{A}v(t) \leq \lambda \int_0^1 \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{J}_1(\tau)(\mathfrak{f}(\tau, [\mathfrak{v}(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))\, d\tau\right) ds \\
&\leq \lambda \int_0^1 \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{J}_1(\tau)(\varphi(\tau)\psi(\tau, [\mathfrak{v}(\tau) - \lambda\zeta(\tau)]^*) + \xi(\tau))\, d\tau\right) ds \\
&\leq \lambda \int_0^1 \mathfrak{J}_2(s)\left\{\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{J}_1(\tau)[\varphi(\tau)(\Lambda_4 + \epsilon([\mathfrak{v}(\tau) - \lambda\zeta(\tau)]^*)) + \xi(\tau)]\, d\tau\right\} ds \\
&\leq \lambda \max\{\Lambda_4, 1\}\int_0^1 \mathfrak{J}_2(s)\left[\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{J}_1(\tau)(\varphi(\tau) + \xi(\tau))\, d\tau\right] ds \\
&\quad + \lambda\epsilon r_4 \int_0^1 \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_0^1 \mathfrak{J}_1(s)\varphi(\tau)\, d\tau\right) ds \\
&\leq \lambda \max\{\Lambda_4, 1\}\left(\int_0^1 \mathfrak{J}_2(s)\frac{1}{\mathfrak{q}(s)}\, ds\right)\left(\int_0^1 \mathfrak{J}_1(\tau)(\varphi(\tau) + \xi(\tau))\, d\tau\right) \\
&\quad + \lambda\epsilon r_4\left(\int_0^1 \mathfrak{J}_2(s)\frac{1}{\mathfrak{q}(s)}\, ds\right)\left(\int_0^1 \mathfrak{J}_1(\tau)\varphi(\tau)\, d\tau\right) \\
&\leq \tfrac{r_4}{2} + \tfrac{r_4}{2} = r_4 = \|\mathfrak{v}\|, \ \ \forall\, t \in [0,1].
\end{aligned}$$

Hence,

$$\|\mathcal{A}\mathfrak{v}\| \leq \|\mathfrak{v}\|, \ \ \forall\, \mathfrak{v} \in \mathcal{Q} \cap \partial\Upsilon_4. \tag{27}$$

By (25) and (27) and the Guo–Krasnosel'skii fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $\mathfrak{v}_2 \in \mathcal{Q} \cap (\overline{\Upsilon}_4 \setminus \Upsilon_3)$, so $r_3 \leq \|\mathfrak{v}_2\| \leq r_4$. Besides this, similar to (26), we find for all $t \in [0,1]$

$$\begin{aligned}
&\mathfrak{v}_2(t) - \lambda\zeta(t) \geq \mathfrak{v}_2(t) - \lambda\sigma\omega t^{\gamma-1}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\, ds\right)\left(\int_0^1 \xi(\tau)\, d\tau\right) \\
&\geq t^{\gamma-1}\|\mathfrak{v}_2\| - \lambda\sigma\omega t^{\gamma-1}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\, ds\right)\left(\int_0^1 \xi(\tau)\, d\tau\right) \\
&\geq t^{\gamma-1} r_3 - \lambda\sigma\omega t^{\gamma-1}\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\, ds\right)\left(\int_0^1 \xi(\tau)\, d\tau\right) \geq \frac{\Lambda_2 t^{\gamma-1}}{\theta_1^{\gamma-1}}.
\end{aligned}$$

Let $\mathfrak{u}_2(t) = \mathfrak{v}_2(t) - \lambda\zeta(t)$ for all $t \in [0,1]$. Then, $\mathfrak{u}_2(t) \geq \Xi_3 t^{\gamma-1}$ for all $t \in [0,1]$, where $\Xi_3 = \Lambda_2/\theta_1^{\gamma-1}$. Therefore, we deduce that $\mathfrak{u}_2$ is a positive solution of problem (1) and (2). □

With a similar proof to that of Theorem 2, we obtain the next result.

Theorem 3. *We suppose that assumptions* $(I1)$, $(I2)$ *and*

$(\widetilde{I4})$ *There exist* $\theta_1, \theta_2 \in (0,1)$, $\theta_1 < \theta_2$ *such that* $\tilde{\mathfrak{f}}_\infty = \lim_{x\to\infty} \min_{t\in[\theta_1,\theta_2]} \mathfrak{f}(t,x) = \infty$, *and* $\psi_\infty = \lim_{x\to\infty} \max_{t\in[0,1]} \dfrac{\psi(t,x)}{x} = 0$, *hold. Then, there exists $\widetilde{\lambda}_2 > 0$ such that for any $\lambda \in [\widetilde{\lambda}_2, \infty)$, the boundary value problem (1) and (2) has at least one positive solution.*

4. Examples

Let $\beta = 3/2$, $\gamma = 8/3$ $(n = 3)$, $p = 2$, $\alpha_0 = 7/6$, $\alpha_1 = 6/5$, $\alpha_2 = 5/4$, $\mathfrak{q}(t) = \frac{1}{t+2}$ for all $t \in [0,1]$, $\mathfrak{H}_0(t) = \{1,\ t \in [0,1/5);\ 7/3,\ t \in [1/5,1]\}$, $\mathfrak{H}_1(t) = t/4$ for all $t \in [0,1]$, and $\mathfrak{H}_2(t) = \{2,\ t \in [0,1/3);\ 28/9,\ t \in [1/3,1]\}$.

We consider the fractional differential equation

$$D_{0+}^{3/2}\left(\frac{1}{t+2} D_{0+}^{8/3}\mathfrak{u}(t)\right) = \lambda\mathfrak{f}(t,\mathfrak{u}(t)),\ \ t \in (0,1), \tag{28}$$

with the boundary conditions

$$\begin{cases} \mathfrak{u}(0) = \mathfrak{u}'(0) = 0,\ \ D_{0+}^{8/3}\mathfrak{u}(0) = 0,\ \ D_{0+}^{8/3}\mathfrak{u}(1) = \dfrac{20}{11} D_{0^+}^{8/3}\mathfrak{u}\left(\dfrac{1}{5}\right), \\ D_{0+}^{7/6}\mathfrak{u}(1) = \dfrac{1}{4}\displaystyle\int_0^1 D_{0+}^{6/5}\mathfrak{u}(t)\,dt + \dfrac{10}{9} D_{0+}^{5/4}\mathfrak{u}\left(\dfrac{1}{3}\right). \end{cases} \tag{29}$$

We obtain for this problem $\mathfrak{d}_1 \approx 0.40371521 > 0$ and $\mathfrak{d}_2 \approx 0.214979 > 0$, so assumption $(I1)$ is satisfied. In addition, we find

$$\begin{aligned}
&\mathfrak{g}_1(t,s) = \frac{1}{\Gamma(3/2)}\begin{cases} t^{1/2}(1-s)^{1/2} - (t-s)^{1/2},\ \ 0 \le s \le t \le 1, \\ t^{1/2}(1-s)^{1/2},\ \ 0 \le t \le s \le 1, \end{cases} \\
&\mathfrak{g}_2(t,s) = \frac{1}{\Gamma(8/3)}\begin{cases} t^{5/3}(1-s)^{1/2} - (t-s)^{5/3},\ \ 0 \le s \le t \le 1, \\ t^{5/3}(1-s)^{1/2},\ \ 0 \le t \le s \le 1, \end{cases} \\
&\mathfrak{g}_{31}(t,s) = \frac{1}{\Gamma(22/15)}\begin{cases} t^{7/15}(1-s)^{1/2} - (t-s)^{7/15},\ \ 0 \le s \le t \le 1, \\ t^{7/15}(1-s)^{1/2},\ \ 0 \le t \le s \le 1, \end{cases} \\
&\mathfrak{g}_{32}(t,s) = \frac{1}{\Gamma(17/12)}\begin{cases} t^{5/12}(1-s)^{1/2} - (t-s)^{5/12},\ \ 0 \le s \le t \le 1, \\ t^{5/12}(1-s)^{1/2},\ \ 0 \le t \le s \le 1, \end{cases} \\
&\mathfrak{G}_1(t,s) = \mathfrak{g}_1(t,s) + \tfrac{4t^{1/2}}{3\mathfrak{d}_1}\mathfrak{g}_1\left(\tfrac{1}{5},s\right) \\
&= \begin{cases} \frac{1}{\Gamma(3/2)}t^{1/2}(1-s)^{1/2} + \frac{4t^{1/2}}{3\mathfrak{d}_1\Gamma(3/2)}\left[\left(\frac{1}{5}\right)^{1/2}(1-s)^{1/2} - \left(\frac{1}{5}-s\right)^{1/2}\right],\ \ t \le s < \frac{1}{5}, \\ \frac{1}{\Gamma(3/2)}\left[t^{1/2}(1-s)^{1/2} - (t-s)^{1/2}\right] + \frac{4t^{1/2}}{3\mathfrak{d}_1\Gamma(3/2)}\left[\left(\frac{1}{5}\right)^{1/2}(1-s)^{1/2} - \left(\frac{1}{5}-s\right)^{1/2}\right], \\ \quad s < \frac{1}{5},\ s \le t, \\ \frac{1}{\Gamma(3/2)}t^{1/2}(1-s)^{1/2} + \frac{4t^{1/2}}{3\mathfrak{d}_1\Gamma(3/2)}\left(\frac{1}{5}\right)^{1/2}(1-s)^{1/2},\ \ t \le s,\ \frac{1}{5} \le s, \\ \frac{1}{\Gamma(3/2)}\left[t^{1/2}(1-s)^{1/2} - (t-s)^{1/2}\right] + \frac{4t^{1/2}}{3\mathfrak{d}_1\Gamma(3/2)}\left(\frac{1}{5}\right)^{1/2}(1-s)^{1/2},\ \ \frac{1}{5} \le s \le t, \end{cases} \\
&\mathfrak{G}_2(t,s) = \mathfrak{g}_2(t,s) + \frac{t^{5/3}}{\mathfrak{d}_2}\left[\frac{1}{4}\int_0^1 \mathfrak{g}_{31}(\tau,s)\,d\tau + \frac{10}{9}\mathfrak{g}_{32}\left(\frac{1}{3},s\right)\right],\ \ t,s \in [0,1], \\
&\mathfrak{h}_1(s) = \tfrac{1}{\Gamma(3/2)}(1-s)^{1/2},\ \ \mathfrak{h}_2(s) = \tfrac{1}{\Gamma(8/3)}\left((1-s)^{1/2} - (1-s)^{5/3}\right),\ \ s \in [0,1], \\
&\mathfrak{J}_1(s) = \begin{cases} \mathfrak{h}_1(s) + \frac{4}{3\mathfrak{d}_1\Gamma(3/2)}\left[\left(\frac{1}{5}\right)^{1/2}(1-s)^{1/2} - \left(\frac{1}{5}-s\right)^{1/2}\right],\ \ 0 \le s < \frac{1}{5}, \\ \mathfrak{h}_1(s) + \frac{4}{3\mathfrak{d}_1\Gamma(3/2)}\left(\frac{1}{5}\right)^{1/2}(1-s)^{1/2},\ \ \frac{1}{5} \le s \le 1, \end{cases} \\
&\mathfrak{J}_2(s) = \begin{cases} \mathfrak{h}_2(s) + \frac{1}{\mathfrak{d}_2}\Big\{\frac{1}{4\Gamma(37/15)}(1-s)^{1/2} - \frac{1}{4\Gamma(37/15)}(1-s)^{22/15} \\ \quad + \frac{10}{9\Gamma(17/12)}\left[\left(\frac{1}{3}\right)^{5/12}(1-s)^{1/2} - \left(\frac{1}{3}-s\right)^{5/12}\right]\Big\},\ \ 0 \le s < \frac{1}{3}, \\ \mathfrak{h}_2(s) + \frac{1}{\mathfrak{d}_2}\Big[\frac{1}{4\Gamma(37/15)}(1-s)^{1/2} - \frac{1}{4\Gamma(37/15)}(1-s)^{22/15} \\ \quad + \frac{10}{9\Gamma(17/12)}\left(\frac{1}{3}\right)^{5/12}(1-s)^{1/2}\Big],\ \ \frac{1}{3} \le s \le 1, \end{cases} \\
&\omega = \max_{s\in[0,1]} \mathfrak{J}_1(s) \approx 2.49991328,\ \ \sigma \approx 5.24878784.
\end{aligned}$$

Example 1. *We consider the function*

$$\mathfrak{f}(t,x) = \frac{x^2 - x + 1}{3\sqrt[5]{t^2(1-t)^3}} + \ln t, \quad t \in (0,1), \quad x \geq 0. \tag{30}$$

We have $\xi(t) = -\ln t$ *and* $\varphi(t) = \frac{1}{\sqrt[5]{t^2(1-t)^3}}$ *for all* $t \in (0,1)$, $\psi(t,x) = \frac{x^2-x+1}{3}$ *for all* $t \in [0,1]$ *and* $x \geq 0$, *and* $\int_0^1 \xi(t)\,dt = 1$, $\int_0^1 \varphi(t)\,dt = B\left(\frac{3}{5}, \frac{2}{5}\right) \approx 3.303266$. *Then, assumption* $(I2)$ *is satisfied. Besides this, for* θ_1, θ_2 *fixed,* $0 < \theta_1 < \theta_2 < 1$, *we find* $\mathfrak{f}_\infty = \infty$, *so assumption* $(I3)$ *is also satisfied. We also obtain* $\sigma\omega\left(\int_0^1 \frac{1}{\mathfrak{q}(s)}\,ds\right)\left(\int_0^1 \xi(\tau)\,d\tau\right) \approx 32.80378616$, *and then we choose* $r_1 = 33$. *We also have* $\Lambda_1 = \frac{1057}{3}$, $a := \int_0^1 \frac{1}{\mathfrak{q}(s)} \mathfrak{J}_2(s)\,ds \approx 5.14534034$, $b := \int_0^1 \mathfrak{J}_1(\tau)(\varphi(\tau) + \xi(\tau))\,d\tau \approx 5.93232294$, *and then* $\lambda_1 = \min\{1, r_1(ab\Lambda_1)^{-1}\} \approx 0.00306847$. *By Theorem 1, we deduce that problem (28) and (29) with the nonlinearity (30) has at least one positive solution for any* $\lambda \in (0, \lambda_1]$.

Example 2. *We consider the function*

$$\mathfrak{f}(t,x) = \frac{\sqrt{x+1}}{2\sqrt[3]{t^2(1-t)}} - \frac{1}{\sqrt[4]{t}}, \quad t \in (0,1), \quad x \geq 0. \tag{31}$$

Here we have $\xi(t) = \frac{1}{\sqrt[4]{t}}$ *and* $\varphi(t) = \frac{1}{\sqrt[3]{t^2(1-t)}}$ *for all* $t \in (0,1)$, $\psi(t,x) = \frac{\sqrt{x+1}}{2}$ *for all* $t \in [0,1]$ *and* $x \geq 0$. *For* θ_1, θ_2 *fixed,* $0 < \theta_1 < \theta_2 < 1$, *the assumptions* $(I2)$ *and* $(I4)$ *are satisfied,* ($\int_0^1 \xi(t)\,dt = 4/3$, $\int_0^1 \varphi(t)\,dt = B\left(\frac{1}{3}, \frac{2}{3}\right) \approx 3.62759873$, $\lim_{x\to\infty} \min_{t\in[\theta_1,\theta_2]} \mathfrak{f}(t,x) = \infty$ *and* $\psi_\infty = 0$).

For $\theta_1 = 1/4$, $\theta_2 = 3/4$, we obtain

$$\begin{aligned}
A &= \int_{1/4}^{3/4} \mathfrak{J}_2(s)\left(\frac{1}{\mathfrak{q}(s)}\int_{1/4}^{3/4} \mathfrak{G}_1(s,\tau)\,d\tau\right)ds \\
&= \int_{1/4}^{3/4} \mathfrak{J}_2(s)(s+2)\left\{\int_{1/4}^{s}\left\{\frac{1}{\Gamma(3/2)}[s^{1/2}(1-\tau)^{1/2} - (s-\tau)^{1/2}]\right.\right. \\
&\quad \left. + \frac{4s^{1/2}}{3\mathfrak{d}_1\Gamma(3/2)}\left(\frac{1}{5}\right)^{1/2}(1-\tau)^{1/2}\right\}d\tau \\
&\quad \left. + \int_s^{3/4}\left\{\frac{1}{\Gamma(3/2)}s^{1/2}(1-\tau)^{1/2} + \frac{4s^{1/2}}{3\mathfrak{d}_1\Gamma(3/2)}\left(\frac{1}{5}\right)^{1/2}(1-\tau)^{1/2}\right\}d\tau\right\}ds \\
&= \int_{1/4}^{3/4} \mathcal{J}_2(s)(s+2)\left[\frac{1}{\Gamma(5/2)}s^{1/2}\left(\frac{3}{4}\right)^{3/2} - \frac{1}{\Gamma(5/2)}\left(s-\frac{1}{4}\right)^{3/2}\right. \\
&\quad + \frac{4s^{1/2}}{3\mathfrak{d}_1\Gamma(5/2)}\left(\frac{1}{5}\right)^{1/2}\left(\frac{3}{4}\right)^{3/2} - \frac{1}{\Gamma(5/2)}s^{1/2}\left(\frac{1}{4}\right)^{3/2} \\
&\quad \left. - \frac{4s^{1/2}}{3\mathfrak{d}_1\Gamma(5/2)}\left(\frac{1}{5}\right)^{1/2}\left(\frac{1}{4}\right)^{3/2}\right]ds \approx 2.09114717.
\end{aligned}$$

In addition, we find $\Xi_0 = \frac{20}{3}4^{5/3}\sigma\omega A^{-1} \approx 421.63963163$. From the proof of Theorem 2, we deduce $\Lambda_2 \approx 200146.6802$ and $\lambda_2 \approx 46123.1544$. Then, by Theorem 2, we conclude that for any $\lambda \geq \lambda_2$, the problem (28) and (29) with the nonlinearity (31) has at least one positive solution.

5. Conclusions

In this paper, we investigated the existence of positive solutions for the Riemann–Liouville fractional differential Equation (1) with sequential derivatives and a positive parameter, subject to the general nonlocal boundary conditions (2) containing diverse fractional order derivatives and Riemann–Stieltjes integrals. The nonlinearity $\mathfrak{f}$ from (1) can change sign, and it can be singular at $t = 0$ and $t = 1$. In the proof of the main results, we used the Guo–Krasnosel'skii fixed point theorem. We also presented the associated Green functions and their properties, and we gave two examples for the illustration of our results.

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