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Some Characterizations of Certain Complex Fuzzy Subgroups

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Abstract: The complex fuzzy environment is an innovative tool to handle ambiguous situations in different mathematical problems. In this article, we commence the abstraction of (ρ, η)-complex fuzzy sets, (ρ, η)-complex fuzzy subgroupoid, (ρ, η)-complex fuzzy subgroups and describe important examples of the symmetric group under (ρ, η)-complex fuzzy sets. Additionally, we discuss the conjugacy class of the group with respect to (ρ, η)-complex fuzzy normal subgroups. We define (ρ, η)-complex fuzzy cosets and elaborate upon the certain operation of this analog to group theoretic operation. We prove that factors regarding the (ρ, η)-complex fuzzy normal subgroup form a group and establish an ordinary homomorphism. Moreover, we create the (ρ, η)-complex fuzzy subgroup of the factor group.

Keywords: complex fuzzy set; (ρ, η)-complex fuzzy set; (ρ, η)-complex fuzzy subgroup; (ρ, η)-complex fuzzy normal subgroup

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1. Introduction

Fuzzy logic theory is based on the idea of concerned graded membership influenced by human cognition and perception. Lotfi Zadeh [1] launched his well-known maiden paper on fuzzy logic in 1965. Fuzzy logic can cope with data resulting from computational perceptions that are obscure, imprecise, uncertain and partially accurate. Fuzzy logic deals with unclear human assessments in computing difficulties. It comes with proper tools for contrasting the resolution of various protocols. Fuzzy logic-based new computing procedures can be employed for the designing of intelligence for decision making, pattern recognition, optimizations and time series forecasting. Fuzzy logic is highly fruitful for the people connected with research in areas such as engineers, agricultural, chemical, aerospace, civil, mechanical, geological, industrial and computer software developers. Doubtlessly, the fuzzy logic-based applications were considered vague at a time but are now in use in scientific works. Fuzzy logic-based applications are beneficial in many fields, such as air conditioners, facial identification modes, vacuum cleaners, washing machine mechanisms, transmission systems and weather prediction systems. Several investigators are actively engaged in the development of fuzzy sets and their applications [2–5].

Many differential equations display unreliability and variability in their interpretation [6]. Fuzzy set theory grows more significant in dealing with this type of information. Imai et al. [7] invented the study of BCK/BCI algebras in 1966 as an extrapolated conviction of set-theoretic difference. Rosenfeld [8] built up the erection of fuzzy subgroups on fuzzy sets in 1971. In 1979, fuzzy groups were redefined by Anthony and Sherwood [9]. The concepts of intersection, inclusion, convexity, union, relation, etc., are extended to such sets, and different properties of these convictions in the context of fuzzy sets are demonstrated in [10]. The author of [11] deliberated the abstraction of normal subgroups of fuzzy subgroups in 1984. Liu [12] introduced the concept of invariant fuzzy subgroups. In 1988, Choudhury [13] commenced the notion of fuzzy homomorphisms between two groups and studied its effectiveness on fuzzy subgroups. Mashour et al. pictured many different key properties of fuzzy subgroups in [14]. Filep [15] expanded the structure and construction of fuzzy subgroups in 1992. Many scientists are busily engaged in the development of fuzzy sets in group theory [16]. Gupta and Qi [17] presented some typical T-operators and reviewed T-norm, T-conorm and negation function under T-operators. Malik and Mordeson [18] explored the concept of fuzzy subgroups of Abelian Groups and analyzed their various algebraic characterization. Mishref [19] described the fuzzy normal series of finite groups. Kim [20] introduced the opinion of fuzzy orders of the elements of groups and explored many other algebraic aspects of the fuzzy order of a group. Ray [21] launched the concept of a direct product of fuzzy subgroups in 1999. The idea of a complex fuzzy set was acquainted by Ramot [22,23] et al. in 2002. In 2009, Zhang et al. [24] developed the various algebraic setup of complex fuzzy sets. In 2009, a new structure and construction of Q-fuzzy groups were described by Solairaju [25]. Al-Husban and Salleh [26] defined the notion of complex fuzzy hypergroups based on complex fuzzy spaces. The new applications of complex fuzzy sets in ring theory, metric spaces, graph theory and group theory were introduced in [27–30]. Ma [31] et al. presented some new mathematical operations and laws of complex fuzzy sets, such as absorption law, involution law, symmetrical difference formula, simple difference and disjunctive sum. Moreover, they developed a new algorithm using complex fuzzy sets for application signals. Trevijano and Elorza [32] determined the new opinion of annihilators of fuzzy subgroups and examined their various algebraic properties. Ilieva [33] commenced a novel technique for fuzzy forecasting of time series with supervised learning and k-order fuzzy relationships. The recent applications of complex fuzzy sets in ring theory and BCk/BCI algebras may be viewed in [34,35]. Imtiaz [36] et al. explored the new structure of ξ-complex fuzzy sets and ξ-complex fuzzy subgroups. Many authors [37–43] initiated several interesting techniques and approaches to solve complicated systems in fuzzy group theory, fuzzy ring theory and fuzzy fractional calculus, whereas the computational effects are very vague and straightforward. Gulzar [44] et al. presented the novel concept of complex fuzzy subfields. Verma [45] et al. discussed a systematic review on the development in the inquiry of fuzzy variational problems. Masmali [46] et al. introduced the concept of μ-fuzzy subgroups and discussed many algebraic properties of fact. Recently, Razzaque and Razaq [47] initiated the concept of q-rung orthopair fuzzy subgroups and level subgroups of q-rung orthopair fuzzy subgroups.

Considering the above literature and importance of complex fuzzy set and group theory, this paper discloses the concept of (ρ, η)-complex fuzzy subgroups. Additionally, (ρ, η)-complex fuzzy sets have the ability to play an effective role in solving symmetric groups. This article describes element structures of symmetric groups under the environment of (ρ, η)-complex fuzzy subgroups. The basic objective and fundamental contribution of this paper are to;

1. Define different algebraic properties of (ρ, η)-complex fuzzy subgroups.
2. Describe the abstraction (ρ, η)-complex fuzzy normal subgroups and (ρ, η)-complex fuzzy cosets along with the related fundamental theorems.
3. Discuss the factor group regarding (ρ, η)-complex fuzzy normal subgroup forming a group and establishing an ordinary homomorphism.

2. Preliminaries

In this section, we specify essential definitions and algebraic features of complex fuzzy sets and complex fuzzy subgroups that play important roles in our next exploration.

Definition 1 ([1])**.** *A fuzzy subset is just like a function from the universe, set to a unit interval [0, 1].*

Definition 2 ([22])**.** *A complex fuzzy set λ of universe of discourse X is mapping from a non-empty set to a unit disk and is described by the rule $\alpha_\lambda : X \to \{x \in Ç : |x| \leq 1\}$, where is a set of complex numbers. The $\alpha_\lambda(x) = \xi_\lambda(x)e^{i\varphi_\lambda(x)}$ is a complex membership function of complex fuzzy set λ, where $i = \sqrt{-1}$.*

Definition 3 ([29]). *Let R be a universe of discourse and $\lambda = \{(x, \lambda(x)) : x \in R\}$ be a fuzzy subset. Then, π-fuzzy subset of R is defined as*

$$\lambda_\pi = \{(x, \lambda_\pi(x)) : \lambda_\pi(x) = 2\pi\lambda x), x \in R\}.$$

Definition 4 ([29]). *Let Z be a group. A π-fuzzy set λ_π of Z is a π-fuzzy subgroup of Z if*

1. $\lambda_\pi(xy) \geq min\{\lambda_\pi(x), \lambda_\pi(y)\}$, *for all* $x, y \in Z$,
2. $\lambda_\pi(x^{-1}) \geq \lambda_\pi(x)$, *for all* $x \in Z$.

Definition 5 ([29]). *Let λ and γ be two complex fuzzy subsets of Z. Then,*

1. *A complex fuzzy set λ is a homogeneous complex fuzzy subset if, for all $x, y \in Z$ we have $\xi_\lambda(x) \leq \xi_\lambda(y)$ if and only if $\varphi_\lambda(x) \leq \varphi_\lambda(y)$.*
2. *A complex fuzzy set λ is a homogeneous complex fuzzy set with γ if, for all $x, y \in Z$ we have $\xi_\lambda(x) \leq \xi_\gamma(y)$ if and only if $\varphi_\lambda(x) \leq \varphi_\gamma(y)$.*

Note: All complex fuzzy sets used in this paper are homogeneous complex fuzzy sets.

Definition 6 ([24]). *Let $\lambda = \{(x, \xi_\lambda(x)e^{i\varphi_\lambda(x)}) : x \in M\}$ and $\gamma = \{(x, \xi_\gamma(x)e^{i\varphi_\gamma(x)}) : x \in Z\}$ be a complex fuzzy set of set M. Then, the set theoretic operations are defined as:*

1. $(\lambda \cap \gamma)(x) = \xi_{(\lambda\cap\gamma)}(x)e^{i\varphi_{(\lambda\cap\gamma)}(x)} = min\{\xi_\lambda(x)e^{i\varphi_\lambda(x)}, \xi_\gamma(x)e^{i\varphi_\gamma(x)}\}, \forall\, x \in M$,
2. $(\lambda \cup \gamma)(x) = \xi_{(\lambda\cap\gamma)}(x)e^{i\varphi_{(\lambda\cup\gamma)}(x)} = max\{\xi_\lambda(x)e^{i\varphi_\lambda(x)}, \xi_\gamma(x)e^{i\varphi_\gamma(x)}\}, \forall\, x \in M$.

Definition 7 ([29]). *Let Z be a group and $\lambda = \{(x, \xi_\lambda(x)e^{i\varphi_\lambda(x)}) : x \in Z\}$ be a complex fuzzy set of Z. Then, λ is said to be a complex fuzzy subgroup of Z if the below axioms are satisfied.*

1. $\xi_\lambda(xy)e^{i\varphi_\lambda(xy)} \geq \min\{\xi_\lambda(x)e^{i\varphi_\lambda(x)}, \xi_\lambda(y)e^{i\varphi_\lambda(y)}\}$,
2. $\xi_\lambda(x^{-1})e^{i\varphi_\lambda(x^{-1})} \geq \xi_\lambda(x)e^{i\varphi_\lambda(x)}$, *for all* $x, y \in Z$.

Example 1. *Let $Z_4 = \{0, 1, 2, 3\}$ be a group of order four. We define a complex fuzzy set $\lambda = \{(0, 0.5e^{i\frac{\pi}{2}}), (1, 0.4e^{i\frac{\pi}{3}}), (2, 0.3e^{i\frac{\pi}{4}}), (3, 0.4e^{i\frac{\pi}{3}})\}$ as complex fuzzy subgroup of Z. We can see that both axioms of complex fuzzy subgroup are satisfied on λ. Hence, λ is a complex fuzzy subgroup.*

Definition 8. *Let Z be a group. A complex fuzzy set $\lambda = \{(x, \xi_\lambda(x)e^{i\varphi_\lambda(x)}) : x \in Z\}$ of Z is called a complex fuzzy normal subgroup of Z if : $\xi_\lambda(xy)e^{i\varphi_\lambda(xy)} = \xi_\lambda(yx)e^{i\varphi_\lambda(yx)}$, for all $x, y \in Z$.*

3. Algebraic Properties of (ρ,η)-Complex Fuzzy Subgroups

In this section, we commence the abstraction of the (ρ, η)-complex fuzzy set. Moreover, we initiate the new algebraic structure of (ρ, η)-complex fuzzy subgroups and explore their fundamental algebraic properties.

Definition 9. *Let $\lambda = \{(x, \xi_\lambda x)e^{i\varphi_\lambda(x)}); x \in Z\}$ be a complex fuzzy set of group Z. For any $\rho \in [0, 1]$ and $\eta \in [0, 2\pi]$, then the complex fuzzy set $\lambda_{(\rho,\eta)}$ is a (ρ, η)-complex fuzzy set of Z in regards to a complex fuzzy set λ and is defined as: $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_\lambda x)e^{i\varphi_\lambda(x)} \bullet \rho e^{i\eta} = max\{0, \xi_\lambda(x) + \rho - 1\}e^{imax\{0, \phi_\lambda(x) + \eta - 2\pi\}}$.*

Definition 10. *Let $\lambda_{(\rho,\eta)}$ and $\gamma_{(\rho,\eta)}$ be two (ρ, η)-complex fuzzy subsets of Z. Then,*

1. *A (ρ, η)-complex fuzzy set $\lambda_{(\rho,\eta)}$ is a homogeneous (ρ, η)-complex fuzzy subset if, for all $x, y \in Z$ we have $\xi_{\lambda_\rho}(x) \leq \xi_{\lambda_\rho}(y)$ if and only if $\varphi_{\lambda_\eta}(x) \leq \varphi_{\lambda_\eta}(y)$.*
2. *A (ρ, η)-complex fuzzy set $\lambda_{(\rho,\eta)}$ is homogeneous $(\rho, \eta) - CFS$ with $\gamma_{(\rho,\eta)}$ if, for all $x, y \in Z$ we have $\xi_{\lambda_\rho}(x) \leq \xi_{\gamma_\rho}(y)$ if and only if $\varphi_{\lambda_\eta}(x) \leq \varphi_{\gamma_\eta}(y)$.*

Note: All (ρ, η)-complex fuzzy sets used in this paper are homogeneous.

Remark 1. *If $\rho = 1$ and $\eta = 2\pi$ in the above definition, it becomes a classical complex fuzzy set.*

Remark 2. *Let $\lambda_{(\rho,\eta)}$ and $\gamma_{(\rho,\eta)}$ be two (ρ,η)-complex fuzzy subsets of Z. Then, $(\lambda \cap \gamma)_{(\rho,\eta)} = \lambda_{(\rho,\eta)} \cap \gamma_{(\rho,\eta)}$.*

Definition 11. *Let $\lambda_{(\rho,\eta)}$ be a (ρ,η)-complex fuzzy subset of group Z, for $\rho \in [0,1]$ and $\eta \in [0,2\pi]$. A (ρ,η)-complex fuzzy subgroupoid of Z is a fuzzy algebraic structure that satisfies the following condition: $\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} \geq \min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}$.*

Definition 12. *Let $\lambda_{(\rho,\eta)}$ be a (ρ,η)-complex fuzzy subset of group Z for $\rho \in [0,1]$ and $\eta \in [0,2\pi]$. A (ρ,η)-complex fuzzy subgroup of group Z is a fuzzy algebraic structure that satisfies the following axioms:*

1. $\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} \geq \min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}$,
2. $\xi_{\lambda_\rho}(x^{-1})e^{i\varphi_{\lambda_\eta}(x^{-1})} \geq \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}$, *for all $x, y \in Z$.*

In the following example, we explain a (ρ,η)-complex fuzzy subset and a (ρ,η)-complex fuzzy subgroup.

Example 2. *Consider a group $Z = \{(1), (1234), (1432), (13)(24), (14)(23), (12)(34), (24), (13)\}$ as a group of order 8. We describe a complex fuzzy subset of Z as follows:*

$$\xi_\lambda(x) = \begin{cases} 0.6e^{\frac{\pi}{2}} & if\, x \in \{(1), (13)(24)\} \\ 0.5e^{\frac{\pi}{4}} & if\, x \in \{(14)(23), (12)(34)\} \\ 0.3e^{\frac{\pi}{6}} & otherwise \end{cases}$$

In view of Definition 9, we have
$\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_\lambda x)e^{i\varphi_\lambda(x)} \bullet \rho e^{i\eta} = max\{0, \xi_\lambda(x) + \rho - 1\}e^{imax\{0, \phi_\lambda(x) + \eta - 2\pi\}}$.
Describe (ρ,η)-complex fuzzy subgroup of Z, for $\rho = 0.9$ and $\eta = 5\pi/2$, then

$$\xi_{\lambda_{(0.9,\frac{5\pi}{2})}}(x) = \begin{cases} 0.5e^{\pi} & if\, x \in \{(1), (13)(24)\} \\ 0.4e^{\frac{3\pi}{4}} & if\, x \in \{(14)(23), (12)(34)\} \\ 0.2e^{\frac{2\pi}{3}} & otherwise \end{cases}$$

Now we take $x = (14)(23), y = (24), and\ xy = (1234)$, we have
$\xi_{\lambda_{(0.9,\frac{5\pi}{2})}}(xy) = 0.2e^{\frac{2\pi}{3}}$,
$min\{\xi_{\lambda_{(0.9,\frac{5\pi}{2})}}(x), \xi_{\lambda_{(0.9,\frac{5\pi}{2})}}(y)\} = min\{0.4e^{\frac{3\pi}{4}}, 0.2e^{\frac{2\pi}{3}}\} = 0.2e^{\frac{2\pi}{3}}$,
Moreover, $\xi_{\lambda_{(0.9,\frac{5\pi}{2})}}(x^{-1}) = \xi_{\lambda_{(0.9,\frac{5\pi}{2})}}(x)$.
Similarly, both conditions of Definition 12 are satisfied for all elements of group Z. Hence λ is a $(0.9, \frac{5\pi}{2})$-complex fuzzy subgroup of Z.

Remark 3. *If $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy set of group Z for $\rho \in [0,1]$. Then,*
$\xi_{\lambda_\rho}(x^{-1}y)e^{i\varphi_{\lambda_\eta}(x^{-1}y)} \geq min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}$.

Theorem 1. *Let $\lambda_{(\rho,\eta)}$ be a (ρ,η)-complex fuzzy subgroup of group $Z, \forall\ x, y \in Z$. Then,*

1. $\xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)} \leq \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$
2. $\xi_{\lambda_\rho}(xy^{-1})e^{i\varphi_{\lambda_\eta}(xy^{-1})} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$
   This means that $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}$

Proof. Obviously. □

Theorem 2. *Let Z be a finite group. If $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroupoid of Z, then $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroup of Z.*

Proof. Assume that $x \in Z$ is a member of Z; thus, $|x| = n$. Here, n is a finite number such that $x^n = e$, where e is the identity element of group Z, we know that $x^{-1} = x^{n-1}$. We use the application of Definition 12. Then, we have

$$\begin{aligned}\xi_{\lambda_\rho}(x^{-1})e^{i\varphi_{\lambda_\eta}(x^{-1})} &= \xi_{\lambda_\rho}(x^{n-1})^{i\varphi_{\lambda_\eta}(x^{n-1})} \\ &= \xi_{\lambda_\rho}(x^{n-2}x)^{i\varphi_{\lambda_\eta}(x^{n-2}x)} \\ &\geq \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}\end{aligned}$$

□

Theorem 3. *Let Z be a group. If $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroup of Z, for some $x \in Z$, $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$ and then $\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} = \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}$ for all $y \in Z$.*

Proof. Given that $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. Then, from Theorem 1, we have

$$\xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)} \leq \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \forall\, y \in Z$$

Consider

$$\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} \geq min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}$$

$$\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} \geq \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)} \text{ from Theorem 1.} \tag{1}$$

Now, assume that

$$\begin{aligned}\xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)} &= \xi_{\lambda}(x^{-1}xy)^{i\varphi_{\lambda}(x^{-1}xy)} \\ &\geq min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)}\}\end{aligned}$$

Again, from Theorem 1, we have

$$min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)}\} = \xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)}$$

Therefore, we obtain

$$\xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)} \geq \xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)}, \forall\, y \in Z \tag{2}$$

From Equations (1) and (2), we have

$$\xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)} = \xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)}, \forall\, y \in Z.$$

□

Theorem 4. *Let Z be a group. If λ is a complex fuzzy subgroup of Z, then $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroup of Z.*

Proof. Assume that λ is a complex fuzzy subgroup of Z, for all $x, y \in Z$.

Consider

$$\begin{aligned}
\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} &= \xi_\lambda xy)e^{i\varphi_\lambda(xy)} \bullet \rho e^{i\eta} \\
&\geq min\{\xi_\lambda x)e^{i\varphi_\lambda(x)}, \xi_\lambda y)e^{i\varphi_\lambda(y)}\} \bullet \rho e^{i\eta} \\
&= min\{\xi_\lambda x)e^{i\varphi_\lambda(x)} \bullet \rho e^{i\eta}, \xi_\lambda y)e^{i\varphi_\lambda(y)} \bullet \rho e^{i\eta}\} \\
&= min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}.
\end{aligned}$$

Further,

$$\begin{aligned}
\xi_{\lambda_\rho}(x^{-1})e^{i\varphi_{\lambda_\eta}(x^{-1})} &= \xi_\lambda x^{-1})^{i\varphi_\lambda(x^{-1})} \bullet \rho e^{i\eta} \\
&\geq \xi_\lambda x)e^{i\varphi_\lambda(x)} \bullet \rho e^{i\eta} \\
&= \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}.
\end{aligned}$$

□

Remark 4. *The below example proves that the converse of Theorem 4 is generally false.*

Example 3. *Let $Z = \{e, a, b, ab\}$ be a Klein four group. We can see that $\lambda = \{< e, 0.2e^{i\frac{\pi}{12}} >, < b, 0.4e^{i\frac{\pi}{4}} >, < a, 0.4e^{i\frac{\pi}{4}} >, < ab, 0.3e^{i\frac{\pi}{7}} >\}$ is not a complex fuzzy subgroup of Z. Take $\rho = 0$ and $\eta = 0$ then we have $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} \geq 0e^{i0}$, for all $x \in Z$. Then we get $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \rho e^{i\eta}, \forall\, x \in Z$. Therefore, $\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} \geq min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}$ for all $x, y \in Z$. Moreover, $\lambda^{-1} = a, b^{-1} = b, (a\gamma)^{-1} = ab$. Thus, $\xi_{\lambda_\rho}(x^{-1})e^{i\varphi_{\lambda_\eta}(x^{-1})} \geq \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}$. Hence, $\lambda_{(\rho,\eta)}$ is a (ρ, η)-complex fuzzy subgroup.*

Theorem 5. *Let $\lambda_{(\rho,\eta)}$ and $\gamma_{(\rho,\eta)}$ be a two (ρ, η)-complex fuzzy subgroups of group Z, then, $\lambda_{(\rho,\eta)} \cap \gamma_{(\rho,\eta)}$ is also (ρ, η)-complex fuzzy subgroup of Z.*

Proof. Given that $\lambda_{(\rho,\eta)}$ and $\gamma_{(\rho,\eta)}$ be two (ρ, η)-complex fuzzy subgroups of Z, for any $x, y \in Z$.

Consider,

$$\begin{aligned}
\xi_{(\lambda\cap\gamma)_\rho}(xy)e^{\varphi((\lambda\cap\gamma))_\eta(xy)} &= \xi_{\lambda_\rho\cap\gamma_\rho}(xy)e^{i\varphi_{\lambda_\eta\cap\gamma_\eta}(xy)} \\
&= min\{\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_{x\eta}}(xy)}, \xi_{\gamma_\rho}(xy)e^{i\varphi_{\gamma_\eta}(xy)}\} \\
&\geq min\begin{cases} min\{\, \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}\}, \\ min\{\xi_{\gamma_\rho}(x)e^{i\varphi_{\gamma_\eta}(x)}, \xi_{\gamma_\rho}(y)e^{i\varphi_{\gamma_\eta}(y)}\}. \end{cases} \\
&= min\begin{cases} min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\gamma_\rho}(x)e^{i\varphi_{\gamma_\eta}(x)}\}, \\ min\{\xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(y)}, \xi_{\gamma_\rho}(y)e^{i\varphi_{\gamma_\eta}(y)}\}. \end{cases} \\
&= min\{\xi_{\lambda_\rho\cap\gamma_\rho}(x)e^{i\varphi_{\gamma_\eta\cap\lambda_\eta}(x)}, \xi_{\gamma_\rho\cap\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta\cap\gamma_\eta}(y)}\} \\
&= min\{\xi_{(\lambda\cap\gamma)_\rho}(x)e^{\varphi_{(\lambda\cap\gamma)_\eta}(x)}, \xi_{(\lambda\cap\gamma)_\rho}(y)e^{\varphi_{(\lambda\cap\gamma)_\eta}(y)}\}
\end{aligned}$$

Further,

$$\begin{aligned}
\xi_{(\lambda\cap\gamma)_\rho}(x^{-1})e^{\varphi_{(\lambda\cap\gamma)_\eta}(x^{-1})} &= \xi_{\lambda_\rho\cap\gamma_\rho}(x^{-1})e^{i\varphi_{\gamma_\eta\cap\lambda_\eta}(x^{-1})} \\
&= min\{\xi_{\lambda_\rho}(x^{-1})e^{i\varphi_{\lambda_\eta}(x^{-1})}, \xi_{\gamma_\rho}(x^{-1})e^{i\varphi_{\gamma_\eta}(x^{-1})}\} \\
&\geq min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\gamma_\rho}(x)e^{i\varphi_{\gamma_\eta}(x)}\} \\
&= \xi_{(\lambda\cap\gamma)_\rho}(x)e^{\varphi_{(\lambda\cap\gamma)_\eta}(x)}
\end{aligned}$$

Hence, $\lambda_{(\rho,\eta)} \cap \gamma_{(\rho,\eta)}$ is (ρ, η) of Z. □

Remark 5. *The converse of Theorem 5 may not be true.*

Example 4. *Consider a permutation group S_4. Considering two (ρ, η)-complex fuzzy subgroups $\lambda_{(0.7,2\pi)}$ and $\gamma_{(0.7,2\pi)}$ of S_4, we take $\rho e^{i\eta} = 0.7e^{2\pi}$, described as:*

$$\lambda_{(0.7,2\pi)}(x) = \begin{cases} 0.6e^{\frac{\pi}{3}} & if\ x \in <(13)> \\ 0.5e^{\frac{\pi}{5}} & otherwise \end{cases}$$

and

$$\gamma_{(0.7,2\pi)}(x) = \begin{cases} 0.7e^{2\pi} & if\ x \in <(1324)> \\ 0.4e^{\frac{\pi}{6}} & otherwise \end{cases}$$

This implies that

$$(\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(x) = \begin{cases} 0.7e^{2\pi} & if x \in <(1234)> \\ 0.6e^{\frac{\pi}{3}} & if x \in <(13)> - <(1324)> \\ 0.5e^{\frac{\pi}{5}} & otherwise \end{cases}$$

Take $x = (12)(34), y = (13)$ and $xy = (1324)$. Therefore, $(\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(x) = 0.7e^{2\pi}$. $(\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(y) = 0.6e^{\frac{\pi}{3}}$ and $(\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(xy) = 0.5e^{\frac{\pi}{5}}$. $(\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(xy) \ngeq min\{(\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(x), (\lambda_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(y)\}$. Hence, this proves the claim.

Definition 13. *Let $\lambda_{(\rho,\eta)}$ be a (ρ, η)-complex fuzzy subgroup of group Z. Then (ρ, η)-complex fuzzy subgroup $x\lambda_{(\rho,\eta)}(p) = (p, \xi_{x\lambda_\rho}(p)e^{i\varphi_{x\lambda_\eta}(p)})$, of Z is said to be a (ρ, η)-complex fuzzy left coset of Z established by $\lambda_{(\rho,\eta)}$ and x is defined as:*

$$\xi_{x\lambda_\rho}(p)e^{i\varphi_{x\lambda_\eta}(p)} = \xi_{\lambda_\rho}(x^{-1}(p))e^{i\varphi_{\lambda_\eta}(x^{-1}(p))} = \xi_\lambda x^{-1}(p))e^{i\varphi_\lambda(x^{-1}(p))} \bullet \rho e^{i\eta}\ \forall\ p, x \in Z.$$

In the same way, we describe the (ρ, η)-complex fuzzy right coset $\lambda_{(\rho,\eta)}x(p) = (p, \xi_{\lambda_\rho x}(p)e^{i\varphi_{\lambda_\eta x}(p)}$, $x \in Z$ and x is described as:

$$\xi_{\lambda_\rho x}(p)e^{i\varphi_{\lambda_\eta x}(p)} = \xi_{\lambda_\rho}(p^{-1}(x))e^{i\varphi_{\lambda_\eta}(p^{-1}(x))} = \xi_\lambda p^{-1}(x))e^{i\varphi_\lambda(p^{-1}(x))} \bullet \rho e^{i\eta}\ \forall\ p, x \in Z.$$

Example 5. *Take $G = \{(1), (1234), (1432), (13)(24), (14)(23), (12)(34), (24), (13)\}$ as a group of order 8. Describe (ρ, η)-complex fuzzy subgroup of Z, $\rho = 1$ and $\eta = \pi/2$, then*

$$\lambda_{(1,\frac{\pi}{2})}(p) == \begin{cases} 0.6e^{\frac{\pi}{2}} & if x \in \{(1), (13)(24)\} \\ 0.5e^{\frac{\pi}{4}} & if x \in \{(14)(23), (12)(34)\} \\ 0.4e^{\frac{\pi}{6}} & otherwise \end{cases}$$

From Definition 13, we have $\xi_{x\lambda_{(1,\frac{\pi}{2})}}(p)e^{i\varphi_{x\lambda_{(1,\frac{\pi}{2})}}(p)} = \xi_{\lambda_{(1,\frac{\pi}{2})}}(x^{-1}(p))e^{i\varphi_{\lambda_{(1,\frac{\pi}{2})}}(x^{-1}(p))}$. *Hence, the* $(1, \pi/2)$*-complex fuzzy left coset of* $\lambda_{(1,\frac{\pi}{2})}(p)$ *in* G *for* $x = (24)$ *is defined as:*

$$x\lambda_{(1,\frac{\pi}{2})}(p)\begin{cases} 0.6e^{\frac{\pi}{2}} & if x \in \{(13)(24)\} \\ 0.5e^{\frac{\pi}{4}} & x \in \{(1432),(1234)\} \\ 0.4e^{\frac{\pi}{6}} & otherwise \end{cases}$$

In the same way, we can define the $(1, \pi/2)$*-complex fuzzy right coset of* $\lambda_{(1,\frac{\pi}{2})}(p)$*, for all* $x \in Z$.

Definition 14. *Let* Z *be a group and* $\lambda_{(\rho,\eta)}$ *be a* (ρ, η)*-complex fuzzy subgroup of* Z*. Then,* $\lambda_{(\rho,\eta)}$ *is said to be a* (ρ, η)*-complex fuzzy normal subgroup if* $\lambda_{(\rho,\eta)}(xy) = \lambda_{(\rho,\eta)}(yx)$*. Equivalently,* (ρ, η)*-complex fuzzy subgroup* $\lambda_{(\rho,\eta)}$ *is a* (ρ, η)*-complex fuzzy normal subgroup of* Z *if* $\lambda_{(\rho,\eta)}x(y) = x\lambda_{(\rho,\eta)}(y)$*, for all* $x, y \in Z$.

Remark 6. *Let* $\lambda_{(\rho,\eta)}$ *be a* (ρ, η)*-complex fuzzy subgroup of group* Z*. Then* $\lambda_{(\rho,\eta)}(y^{-1}xy) = \lambda_{(\rho,\eta)}(x)$ *for all* $x, y \in G$.

Theorem 6. *Every complex fuzzy normal subgroup of group* Z *is a* (ρ, η)*-complex fuzzy normal subgroup of* Z.

Proof. Suppose that z, x are elements of Z. Then

$$\xi_\lambda x^{-1}z)^{i\varphi_\lambda(x^{-1}z)} = \xi_\lambda z^{-1}x)^{i\varphi_\lambda(z^{-1}x)}$$

This implies that

$$\xi_\lambda x^{-1}z)^{i\varphi_\lambda(x^{-1}z)} \bullet \rho e^{i\eta} = min\xi_\lambda z^{-1}x)^{i\varphi_\lambda(z^{-1}x)} \bullet \rho e^{i\eta}$$

which implies that

$$\xi_{x\lambda_\rho}(z)e^{i\varphi_{x\lambda_\eta}(z)} = \xi_{\lambda_\rho x}(z)e^{i\varphi_{x\lambda_\eta}(z)}.$$

This implies that $x\lambda_{(\rho,\eta)}(z) = \lambda_{(\rho,\eta)}x(z)$. Consequently, $\lambda_{(\rho,\eta)}$ is a (ρ, η)-complex fuzzy normal subgroup of Z. □

Theorem 7. *Let* $\lambda_{(\rho,\eta)}$ *be a* (ρ, η)*-complex fuzzy subgroup of group* Z*. Then,* $\lambda_{(\rho,\eta)}$ *remains the same in the conjugacy class of* Z *if and only if* $\lambda_{(\rho,\eta)}$ *is a* (ρ, η)*-complex fuzzy normal subgroup.*

Proof. Suppose that $\lambda_{(\rho,\eta)}$ is a (ρ, η)-complex fuzzy normal subgroup of group Z. Then,

$$\begin{aligned} \xi_{\lambda_\rho}(y^{-1}xy)e^{i\varphi_{\lambda_\eta}(y^{-1}xy)} &= \xi_{\lambda_\rho}(xyy^{-1})^{i\varphi_{\lambda_\eta}(xyy^{-1})} \\ &= \xi_{\lambda_\rho}(x)^{i\varphi_{\lambda_\eta}(x)}, \forall\, x, y \in Z \end{aligned}$$

Conversely, assume that $\lambda_{(\rho,\eta)}$ is remains same in every conjugate class of Z. Then,

$$\begin{aligned} \xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} &= \xi_{\lambda_\rho}(xyxx^{-1})^{i\varphi_{\lambda_\eta}(xyxx^{-1})} \\ &= \xi_{\lambda_\rho}(x(yx)x^{-1})^{i\varphi_{\lambda_\eta}(x(yx)x^{-1})} \\ &= \xi_{\lambda_\rho}(yx)^{i\varphi_{\lambda_theta}(yx)}, \forall\, x, y \in Z. \end{aligned}$$

□

Theorem 8. *Let Z be a group and $\lambda_{(\rho,\eta)}$ be a (ρ,η)-complex fuzzy subgroup of Z, Then $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy normal subgroup if and only if $\xi_{\lambda_\rho}([x,y])e^{i\varphi_{\lambda_\eta}([x,y])} \geq \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}$, $\forall\, x,y \in Z$.*

Proof. Take $\lambda_{(\rho,\eta)}$ as a (ρ,η)-complex fuzzy normal subgroup of Z. Let $x,y \in Z$ be a member of the group. Assume that

$$\begin{aligned} \xi_{\lambda_\rho}(x^{-1}y^{-1}xy)e^{i\varphi_{\lambda_\eta}(x^{-1}y^{-1}xy)} &\geq min\{\xi_{\lambda_\rho}(y^{-1}xy)e^{i\varphi_{\lambda_\eta}(y^{-1}xy)}, \xi_{\lambda_\rho}(x^{-1})e^{i\varphi_{\lambda_\eta}(x^{-1})}\} \\ &= min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}\} \\ \xi_{\lambda_\rho}([x,y])e^{i\varphi_{\lambda_\eta}([x,y])} &\geq \xi_\lambda x)^{i\varphi_{\lambda_\eta}(x)} \end{aligned}$$

Conversely, suppose that $\xi_{\lambda_\rho}([x,y])e^{i\varphi_{\lambda_\eta}([x,y])} \geq \xi_\lambda(x)e^{i\varphi_{\lambda_\eta}(x)}$. Let $x,r \in Z$ be an element

$$\begin{aligned} \text{Consider } \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)} &= \xi_{\lambda_\rho}(rr^{-1}x^{-1}rx)e^{i\varphi_{\lambda_\eta}(rr^{-1}x^{-1}rx)} \\ &\geq min\{\xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)}, \xi_{\lambda_\rho}([r,p])e^{i\varphi_{\lambda_\eta}([r,x])}\} \\ &= \xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)} \qquad (3) \\ \text{Thus, } \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)} &\geq \xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)}\forall r,x \in Z. \qquad (4) \\ \text{Now, } \xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)} &= \xi_{\lambda_\rho}(xx^{-1}rxx^{-1})e^{i\varphi_{\lambda_\eta}(xx^{-1}rxx^{-1})} \\ &\geq min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)}\}. \qquad (5) \end{aligned}$$

We describe two cases.
Case 1
If $min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)}\} = \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}$

$$\text{Then, we obtain } \xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)} \geq \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \forall\, r,x \in Z.$$

$\lambda_{(\rho,\eta)}$ is a constant portray and the result is satisfied trivially.
Case 2
If $min\{\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}, \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)}\} = \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)}$.
Then, from Equation (5), we have

$$\xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)} \geq \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)} \tag{6}$$

From Equations (4) and (6), we get

$$\xi_{\lambda_\rho}(r)e^{i\varphi_{\lambda_\eta}(r)} = \xi_{\lambda_\rho}(x^{-1}rx)e^{i\varphi_{\lambda_\eta}(x^{-1}rx)}.$$

Hence $\lambda_{(\rho,\eta)}$ is a constant. □

Theorem 9. *If $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy normal subgroup of Z, Then, the set $\lambda'_{(\rho,\eta)} = \{p \in Z : \lambda_{(\rho,\eta)}(p^{-1}) = \lambda_{(\rho,\eta)}(e)\}$ is a normal subgroup of Z.*

Proof. We have $\lambda'_{(\rho,\eta)} \neq \varphi$ because $e \in Z$. Let $p,q \in \lambda'_{(\rho,\eta)}$ be any members. Consider $\xi_{\lambda_\rho}(pq)e^{i\varphi_{\lambda_\eta}(pq)} \geq min\{\xi_{\lambda_\rho}(p)e^{i\varphi_{\lambda_\eta}(p)}, \xi_{\lambda_\rho}(q)e^{i\varphi_{\lambda_\eta}(q)}\} = min\{\xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}, \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}\}$. Which means that $\xi_{\lambda_{\rho(pq)}}e^{i\varphi_{\lambda_\eta}(pq)} \geq \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. However, $\xi_{\lambda_\rho}(pq)e^{i\varphi_{\lambda_\eta}(pq)} \leq \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. Therefore, $\xi_{\lambda_\rho}(pq)e^{i\varphi_{\lambda_\eta}(pq)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. It means that $\lambda_{(\rho,\eta)}(p^{-1}) = \lambda_{(\rho,\eta)}(e)$, it means that $pq \in \lambda'_{(\rho,\eta)}$. Further, $\xi_{\lambda_\rho}(q^{-1})e^{i\varphi_{\lambda_\eta}(q^{-1})} \geq \xi_{\lambda_\rho}(q)e^{i\varphi_{\lambda_\eta}(q)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. However,

$\xi_{\lambda_\rho}(q)e^{i\varphi_{\lambda_\eta}(q)} \leq \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. Thus, $\lambda'_{(\rho,\eta)}$ is a subgroup of Z. Let $p \in \lambda'_{(\rho,\eta)}$ and $q \in Z$. We know, $\xi_{\lambda_\rho}(q^{-1}pq)e^{i\varphi_{\lambda_\eta}(q^{-1}pq)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$. This implies that $q^{-1}pq \in \lambda'_{(\rho,\eta)}$. Therefore, $\lambda'_{(\rho,\eta)}$ is a normal subgroup. □

Example 6. *Consider $G = \{(1),(123),(132),(12),(13),(23)\}$ as a symmetric group. Let the $\lambda_(\rho,\eta)$-complex fuzzy normal subgroup of G be defined as*

$$\lambda_{(0.8,\frac{5\pi}{3})}(x) = \begin{cases} 0.4e^{\frac{2\pi}{3}} & if\ x \in \{(1),(123),(132)\} \\ 0.2e^{\frac{\pi}{6}} & otherwise \end{cases}$$

From Theorem 3.24, we have $\lambda'_{(\rho,\eta)} = \{(1),(123),(132)\}$
We can see that λ' is a normal subgroup of index 2.

Theorem 10. *Let $\lambda_{(\rho,\eta)}$ be a (ρ,η)-complex fuzzy normal subgroup of Z. Then,*

(i) $x\lambda_{(\rho,\eta)} = y\lambda_{(\rho,\eta)}$ *if and only if* $x^{-1}y \in \lambda'_{(\rho,\eta)}$
(ii) $\lambda_{(\rho,\eta)}x = \lambda_{(\rho,\eta)}y$ *if and only if* $xy^{-1} \in \lambda'_{(\rho,\eta)}$

Proof. (i) For any $x, y \in G$ we have $x\lambda_{(\rho,\eta)} = y\lambda_{(\rho,\eta)}$. Consider,

$$\begin{aligned} \xi_{\lambda_\rho}(x^{-1}y)e^{i\varphi_{\lambda_\eta}(x^{-1}y)} &= \xi_\lambda x^{-1}y)e^{i\varphi_\lambda(x^{-1}y)} * \rho e^{i\eta} \\ &= \xi_{x\lambda_\rho}(y)e^{i\varphi_{x\lambda_\rho}(y)} \\ &= \xi_{y\lambda_\rho}(y)e^{i\varphi_{y\lambda_\rho}(y)} \\ &= \xi_{\lambda_\rho}(y^{-1}y)e^{i\varphi_{\lambda_\eta}(y^{-1}y)} \\ &= \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}. \end{aligned}$$

Therefore, $x^{-1}y \in \lambda'_{(\rho,\eta)}$. Conversely, let $x^{-1}y \in \lambda'_{(\rho,\eta)}$ imply that $\xi_{\lambda_\rho}(x^{-1}y)e^{i\varphi_{\lambda_\eta}(x^{-1}y)} = \xi_\lambda e)e^{i\varphi_\lambda(e)} * \rho e^{i\eta}$. Consider,

$$\begin{aligned} \xi_{x\lambda_\rho}(a)e^{i\varphi_{x\lambda_\rho}(a)} &= \xi_\lambda x^{-1}a)e^{i\varphi_\lambda(x^{-1}a)} \bullet \rho e^{i\eta} \\ &= \xi_{\lambda_\rho}(x^{-1}a)e^{i\varphi_{\lambda_\eta}(x^{-1}a)} \\ &= \xi_{\lambda_\rho}(x^{-1}y)(y^{-1}a)e^{i\varphi_{\lambda_\eta}(x^{-1}a)} \\ &\geq min\{\xi_{\lambda_\rho}(x^{-1}y)e^{i\varphi_{\lambda_\eta}(x^{-1}y)}, \xi_{\lambda_\rho}(y^{-1}a)e^{i\varphi_{\lambda_\eta}(y^{-1}a)}\} \\ &= min\{\xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}, \xi_{\lambda_\rho}(y^{-1}a)e^{i\varphi_{\lambda_\eta}(y^{-1}a)}\} \\ &= \xi_{\lambda_\rho}(y^{-1}a)e^{i\varphi_{\lambda_\eta}(y^{-1}a)}\} \\ &= \xi_{y\lambda_\rho}(a)e^{i\varphi_{y\lambda_\rho}(a)}. \end{aligned}$$

Exchange the function of x and y, and we get $\xi_{y\lambda_\rho}(a)e^{i\varphi_{y\lambda_\eta}(a)} \geq \xi_{x\lambda_\rho}(a)e^{i\varphi_{x\lambda_\eta}(a)}$. Therefore, $\xi_{x\lambda_\rho}(a)e^{i\varphi_{x\lambda_\eta}(a)} = \xi_{y\lambda_\rho}(a)e^{i\varphi_{y\lambda_\eta}(a)}$.

(ii) In the same way, we can show that as part (i).
□

Theorem 11. *Let Z be a group and $\lambda_{(\rho,\eta)}$ be a (ρ,η)-complex fuzzy normal subgroup of Z and c, d, u, and v be any members in Z. If $c\lambda_{(\rho,\eta)} = u\lambda_{(\rho,\eta)}$ and $d\lambda_{(\rho,\eta)} = v\lambda_{(\rho,\eta)}$. Then, $cd\lambda_{(\rho,\eta)} = uv\lambda_{(\rho,\eta)}$.*

Proof. We have $c\lambda_{(\rho,\eta)} = u\lambda_{(\rho,\eta)}$ and $d\lambda_{(\rho,\eta)} = v\lambda_{(\rho,\eta)}$. This means that $c^{-1}u, d^{-1}v \in \lambda^e_{(\rho,\eta)}$ Consider, $(cd)^{-1}(uv) = y^{-1}(c^{-1}u)v = d^{-1}(c^{-1}u)(dd^{-1})v = [d^{-1}(c^{-1}u)(d)](d^{-1}v)$. As $\lambda'_{(\rho,\eta)}$ is a normal subgroup of Z. Thus, $(cd)^{-1}(uv) \in \lambda^e_{(\rho,\eta)}$. As result, $cd\lambda_{(\rho,\eta)} = uv\lambda_{(\rho,\eta)}$. □

Theorem 12. *Assume that $Z/\lambda_{(\rho,\eta)} = \{x\lambda_{(\rho,\eta)}; x \in Z\}$ is the assemblage of all (ρ,η)-complex fuzzy co-sets of $\lambda_{(\rho,\eta)}$ and is a (ρ,η)-complex fuzzy normal subgroup of group Z. Therefore, a well-defined binary operation $\star$ of $Z/\lambda_{(\rho,\eta)}$ is described as $x\lambda_{(\rho,\eta)} \star y\lambda_{(\rho,\eta)} = xy\lambda_{(\rho,\eta)} \forall\, x, y \in Z$.*

Proof. We have $x\lambda_{(\rho,\eta)} = y\lambda_{(\rho,\eta)}$ and $f\lambda_{(\rho,\eta)} = g\lambda_{(\rho,\eta)}$ for any $f, g, x, y \in Z$. Let $c \in Z$ be any element. Then, $[x\lambda_{(\rho,\eta)} \star f\lambda_{(\rho,\eta)}](c)$=$(xf\lambda_{(\rho,\eta)}(c))$= $(c, \xi_{xf\lambda_\rho}(c)e^{i\varphi_{y\lambda_\rho}(c)})$. Consider,

$$\begin{aligned}
\xi_{xf\lambda_\rho}(c)e^{i\varphi_{x\lambda_\rho}(c)} &= \xi_{xf\lambda_\rho}(c)e^{i\varphi_{x\lambda_\rho}(c)} \bullet \rho e^{i\eta} \\
&= \xi_{\lambda_\rho}((xf)^{-1}c)e^{i\varphi_{\lambda_\eta}(xf)^{-1}c)} \\
&= \xi_{\lambda_\rho}(f^{-1}x^{-1})c)e^{i\varphi_{\lambda_\eta}(f^{-1}x^{-1}c)} \\
&= \xi_{\lambda_\rho}(f^{-1}(x^{-1}c)e^{i\varphi_{\lambda_\eta}(f^{-1}(x^{-1}c)} \\
&= \xi_{f\lambda_\rho}(x^1c)e^{i\varphi_{x\lambda_\rho}(x^{-1}c)} \\
&= \xi_{g\lambda_\rho}(x^1c)e^{i\varphi_{x\lambda_\rho}(x^{-1}c)} \\
&= \xi_\lambda g^{-1}(x^{-1}c)e^{i\varphi_\lambda(g^{-1}(x^{-1}c)} \\
&= \xi_\lambda x^{-1}(cg^{-1}))e^{i\varphi_\lambda(x^{-1}(cg^{-1})} \\
&= \xi_{x\lambda_\rho}(cg^{-1})e^{i\varphi_{x\lambda_\rho}(cg^{-1})} \\
&= \xi_{y\lambda_\rho}(cg^{-1})e^{i\varphi_{y\lambda_\rho}(cg^{-1})} \\
&= \xi_{\lambda_\rho}(y^{-1}(cg^{-1}))e^{i\varphi_{\lambda_\eta}(y^{-1}(cg^{-1})} \\
&= \xi_{\lambda_\rho}((y^{-1}c)g^{-1})e^{i\varphi_{\lambda_\eta}(y^{-1}c)g^{-1})} \\
&= \xi_{\lambda_\rho}((y^{-1}g^{-1})c)e^{i\varphi_{\lambda_\eta}((y^{-1}g^{-1})c)} \\
&= \xi_{\lambda_\rho}((yg)^{-1}c)e^{i\varphi_{\lambda_\eta}((yg)^{-1}c)} \\
&= \xi_{yg\lambda_\rho}(c)e^{i\varphi_{yg\lambda_\rho}(c)}
\end{aligned}$$

Therefore, $\star$ is a well-defined operation and satisfies the associative property on $Z/\lambda_{(\rho,\eta)}$. Further, $\xi_{\lambda_\rho}e^{i\varphi_{\lambda_\eta}} \star \xi_{x\lambda_\rho}e^{i\varphi_{x\lambda_\eta}} = \xi_{e\lambda_\rho}e^{i\varphi_{e\lambda_\eta}} \star \xi_{x\lambda_\rho}e^{i\varphi_{x\lambda_\eta}} = \xi_{x\lambda_\rho}e^{i\varphi_{x\lambda_\eta}} = \xi_{x\lambda_\rho}e^{i\varphi_{x\lambda_\eta}} \longrightarrow \xi_{\lambda_\rho}e^{i\varphi_\lambda}$ is an identity element of $Z/\lambda_{(\rho,\eta)}$. Obviously, the inverse of each member of $Z/\lambda_{(\rho,\eta)}$ exists if $\xi_{x\lambda_\rho}e^{i\varphi_{x\lambda_\eta}} \in Z/\lambda_{(\rho,\eta)}$, and then there exists a member, $\xi_{x^{-1}\lambda_\rho}e^{i\varphi_{x^{-1}\lambda_\eta}} \in Z/\lambda_{(\rho,\eta)}$ such that $\xi_{x^{-1}x\lambda_\rho}e^{i\varphi_{x^{-1}x\lambda_\eta}} = \xi_{\lambda_\rho}e^{i\varphi_{\lambda_\eta}}$. Consequently $Z/\lambda_{(\rho,\eta)}$ is a group. □

Lemma 1. *Let Z be a group and $L : Z \to Z/\lambda_{(\rho,\eta)}$ be a classical homomorphism from Z onto $Z/\lambda_{(\rho,\eta)}$ and is described by $L(x) = x\lambda_{(\rho,\eta)}$ with the kernel $L = \lambda^e_{(\rho,\eta)}$.*

Proof. Let x, y be arbitrary members of Z, and $L(xy) = xy\lambda_{(\rho,\eta)} = \xi_{xy\lambda_\rho}e^i\varphi_{xy\lambda_\eta} = \xi_{x\lambda_\rho}e^i\varphi_{x\lambda_\eta} \star \xi_{y\lambda_\rho}e^i\varphi_{y\lambda_\eta} = x\lambda_{(\rho,\eta)} \star y\lambda_{(\rho,\eta)} = h(x) \star h(y)$. Therefore, L is a homomorphism. Moreover, L is as follows

$$\begin{aligned}\text{Now, } KernelL &= \{x \in Z : L(x) = e\lambda_{(\rho,\eta)}\}\\ &= \{x \in Z : x\lambda_{(\rho,\eta)} = e\lambda_{(\rho,\eta)}\}\\ &= \{x \in Z : xe^{-1} \in \lambda'_{(\rho,\eta)}\}\\ &= \{x \in Z : x \in \lambda^{e}_{(\rho,\eta)}\}\\ &= \lambda^{e}_{(\rho,\eta)}.\end{aligned}$$

□

Theorem 13. *Let $\lambda^{e}_{(\rho,\eta)}$ be a normal subgroup of Z. If $\lambda_{(\rho,\eta)} = \{(x, \xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)}) : x \in Z\}$ is a (ρ,η)-complex fuzzy subgroup, then the (ρ,η)-complex fuzzy set $\bar{\lambda}_{(\rho,\eta)} = \{(x\lambda^{e}_{(\rho,\eta)}, \bar{\bar{\xi}}_{\lambda_\rho}(x\lambda^{e}_{(\rho,\eta)})e^{i\bar{\varphi}_{\lambda_\eta}(x\lambda^{e}_{(\rho,\eta)})}) : x \in Z\}$ of $Z/\lambda^{e}_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroup of $Z/\lambda^{e}_{(\rho,\eta)}$, where $\bar{\bar{\xi}}_{\lambda_\rho}(x\lambda^{e}_{(\rho,\eta)}e^{i\bar{\varphi}_{\lambda_\eta}(x\lambda^{e}_{(\rho,\eta)})}) = max\{\xi_{\lambda_\rho}(xa)e^{i\varphi_{\lambda_\eta}(xa)} : a \in \lambda^{e}_{(\rho,\eta)}\}$.*

Remark 7. *If $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroup of Z, and $x \in Z$ and $\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} = \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(x)}$ for all $y \in Z$ then $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$.*

Remark 8. *If $\lambda_{(\rho,\eta)}$ is a (ρ,η)-complex fuzzy subgroup of Z, and $x \in Z$ and $\xi_{\lambda_\rho}(xy)e^{i\varphi_{\lambda_\eta}(xy)} = \xi_{\lambda_\rho}(y)e^{i\varphi_{\lambda_\eta}(x)}$ for all $y \in Z$ then $\xi_{\lambda_\rho}(x)e^{i\varphi_{\lambda_\eta}(x)} = \xi_{\lambda_\rho}(e)e^{i\varphi_{\lambda_\eta}(e)}$.*

4. Conclusions

We have commenced the abstraction of a (ρ,η)-complex fuzzy subset and a (ρ,η)-complex fuzzy subgroup. We have proved that every (ρ,η)-complex fuzzy subgroup is a complex fuzzy subgroup, but the converse is not true. We have proved that the factor group regarding a (ρ,η)-complex fuzzy normal subgroup forms a group and establishes an ordinary homomorphism from the group to its factor group regard. In the future, we will extend this approach to the advanced structure of group theory, such as Sylow theorems and fundamental theorems of isomorphism. The Sylow theorems form a fundamental part of finite group theory and play an important role in the classification of finite simple groups.

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References

1. Zadeh, L.A. Fuzzy Sets. *Inf. Control.***1965**, *8*, 338–353. [CrossRef]
2. Ilieva, G. Fuzzy Group Full Consistency Method for Weight Determination. *Cybern. Inf. Technol.* **2020**, *20*, 50–58. [CrossRef]
3. Ilieva, G.; Yankova, T.; Radeva, I.; Popchev, I. Blockchain Software Selection as a Fuzzy Multi-Criteria Problem. *Computers* **2021**, *10*, 120. [CrossRef]
4. Sharma, H.K.; Singh, A.; Yadav, D.; Kar, S. Criteria selection and decision making of hotels using Dominance Based Rough Set Theory. *Oper. Res. Eng. Sci. Theory Appl.* **2022**, *5*, 41–55. [CrossRef]

5. Garg, H.; Vimala, J.; Rajareega, S.; Preethi, D.; Dominguez, L.P. Complex Intuitionistic Fuzzy Soft SWARA-COPRAS Approach: An Application of ERP Software Selection. *AIMS Math.* **2022**, *7*, 5895–5909. [CrossRef]
6. Alaoui, M.K.; Alharbi, F.M.; Zaland, S. Novel Analysis of Fuzzy Physical Models by Generalized Fractional fuzzy operators. *J. Funct. Spaces* **2022**, *2022*, 2504031. [CrossRef]
7. Imai, Y.; Iseki, K. On axiom system of propositional calculi, XIV. *Proc. Jpn. Acad.***1966**, *42*, 19–22. [CrossRef]
8. Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl.* **1971**, *35*, 512–517. [CrossRef]
9. Anthony, J.M.; Sherwood, H. Fuzzy groups redefined. *J. Math. Anal. Appl.* **1979**, *69*, 124–130. [CrossRef]
10. Mizumoto, M. Fuzzy sets and their operations. *Inf. Control* **1981**, *48*, 30–48. [CrossRef]
11. Mukherjee, N.P.; Bhattacharya, P. Fuzzy normal subgroups and fuzzy cosets. *Inf. Sci.* **1984**, *34*, 225–239. [CrossRef]
12. Liu, W.J. Fuzzy invariant subgroups and fuzzy ideals. *Fuzzy Sets Syst.* **1982**, *8*, 133–139. [CrossRef]
13. Choudhury, F.P.; Chakraborty, A.B.; Khare, S.S. A note on fuzzy subgroups and fuzzy homomorphism. *J. Math. Anal. Appl.* **1988**, *131*, 537–553. [CrossRef]
14. Mashour, A.S.; Ghanim, H.; Sidky, F. I. Normal fuzzy subgroups. *Ser. Math.* **1990**, *20*, 53–59.
15. Filep, L. Structure and construction of fuzzy subgroups of a group. *Fuzzy Set Syst.* **1992**, *51*, 105–109. [CrossRef]
16. Kumar, I.J.; Saxena, P.K.; Yadav, P. Fuzzy normal subgroups and fuzzy quotients. *Fuzzy Sets Syst.* **1992**, *46*, 121–132. [CrossRef]
17. Gupta, M.M.; Qi, J. Theory of T-norm and fuzzy inference methods. *Fuzzy Sets Syst.* **1991**, *40*, 431–450. [CrossRef]
18. Malik, D.S.; Mordeson, J.N.; Nair, P.S. Fuzzy normal subgroups in fuzzy subgroups. *J. Korean Math. Soc.* **1992**, *29*, 1–8.
19. Mishref, M.A. Normal fuzzy subgroups and fuzzy normal series of finite groups. *Fuzzy Sets Syst.* **1995**, *72*, 379–383. [CrossRef]
20. Kim, J.G. Fuzzy order relative to fuzzy subgroups. *Inf. Sci.* **1994**, *80*, 341–348. [CrossRef]
21. Ray, A.K. On product of fuzzy subgroups. *Fuzzy Sets Syst.* **1999**, *105*, 181–183. [CrossRef]
22. Ramot, D.; Milo, R.; Friedman M.; Kandel A. Complex fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 450–461. [CrossRef]
23. Ramot, D.; Friedman, M.; Langholz, G.; Kandel, A. Complex fuzzy logic. *IEEE Trans. Fuzzy Syst.* **2003**, *11*, 171–186. [CrossRef]
24. Zhang, G.; Dillon, T.S.; Cai, K.Y.; Ma, J.; Lu, J. Operation properties and λ-equalities of complex fuzzy sets. *Int. J. Approx. Reason.* **2009**, *50*, 1227–1249. [CrossRef]
25. Solairaju, A.; Nagarajan. R. A new structure and construction of Q-fuzzy subgroup. *Adv. Fuzzy Math.* **2009**, *4*, 23–29.
26. Al-Husban, A.; Salleh, A.R. Complex fuzzy group based on complex fuzzy space. *Glob. J. Pure. Appl. Math.* **2016**, *12*, 1433–1450.
27. Singh, D.; Joshi, V.; Imdad, M.; Kumam, P. A novel framework of complex valued fuzzy metric spaces and fixed point theorems. *J. Intell. Fuzzy Syst.* **2016**, *164*, 3227–3238. [CrossRef]
28. Thirunavukarasu, P.; Suresh, R.; Viswanathan, K.K. Energy of complex fuzzy graph. *Int. J. Math. Sci. Eng. Appl.* **2016**, *10*, 243–248.
29. Alsarahead, M.O.; Ahmad, A.G. Complex fuzzy subgroups. *Appl. Math. Sci.* **2017**, *11*, 2011–2021. [CrossRef]
30. Alsarahead, M.O.; Ahmad, A.G. Complex fuzzy subrings. *Int. J. Pure Appl. Math.* **2017**, *117*, 563–577.
31. Ma, X.; Zhan, J.; Khan, M.; Zeeshan, M.; Anis, S.; Awan, A.S. Complex fuzzy sets with Applications in signals. *Comput. Appl. Math.* **2019**, *38*, 150. [CrossRef]
32. Trevijano, S.A.; Chasco, M.J.; Elorza, J. The annihilator of fuzzy subgroups. *Fuzzy Sets Syst.* **2019**, *369*, 122–131. [CrossRef]
33. Ilieva, G. Fuzzy Supervised Multi-Period Time Series Forecasting. *Cybern. Inf. Technol.* **2019**, *19*, 74–85. [CrossRef]
34. Gulistan, M.; Yaqoob, N.; Nawaz, S.; Azhar, S. A study of $(\alpha,\ \beta)$-complex fuzzy hyperideals in non-associative hyperrings. *J. Intell. Fuzzy Syst.* **2019**, *36*, 1–17. [CrossRef]
35. Yun, Y.B.; Xin, X.L. Complex fuzzy sets with application in BCK/BCI-Algebras. *Bull. Sect. Log.* **2019**, *48*, 173–185.
36. Imtiaz, A.; Shuaib, U.; Alolaiyan, H.; Razaq, A.; Gulistan, M. On structural Properties of ξ-complex fuzzy sets and their application. *Complexity* **2020**, *2020*, 2038724. [CrossRef]
37. Mirvakili, S.; Naraghi, H.; Shirvani, M.; Ghiasvand, P. Some Fuzzy Multigroups Obtained from Fuzzy Subgroups. *J. Math. Ext.* **2021**, *16*, 1–25.
38. Gulzar, M.; Abbas, G.; Dilawar, F. Algebraic properties of ω-Q-fuzzy subgroup. *Int. J. Math. Comput. Sci.* **2020**, *15*, 265–274.
39. Gulzar, M.; Alghazzawi, D.; Mateen, M.H.; Kausar, N. A Certain Class of t-Intuitionistic Fuzzy Subgroups. *IEEE Access* **2020**, *8*, 163260–163268. [CrossRef]
40. Gulzar, M.; Mateen, M.H.; Alghazzawi, D.; Kausar, N. A Novel Applications of Complex Intuitionistic Fuzzy Sets in Group Theory. *IEEE Access* **2020**, *8*, 196075–196085. [CrossRef]
41. Gulzar, M.; Alghazzawi, D.; Mateen, M.H.; Premkumar, M. On some characterization of Q complex fuzzy subrings. *J. Math. Comput. Sci.* **2021**, *22*, 295–305. [CrossRef]
42. Ali, U.; Alyousef, H.A.; Ahmed, K.; Ali, S. Solving Nonlinear Fractional differential equations for Contractive and weakly compatible Mapping in neutrosophic metric spaces. *J. Funct. Spaces* **2022**, *2022*, 1491683. [CrossRef]
43. Aresh, M.; El-Tantawy, S.A.; Alotaibi, B.M.; Zaland, S. Study of fuzzy fractional third-order dispersive KdV equation in a plasma under Atangana-Bleanu Derivative. *J. Funct. Spaces* **2022**, *2022*, 7922001.
44. Gulzar, M.; Alghazzawi, D.; Dilawar, F.; Mateen, M.H. A note on complex fuzzy subfields. *Indones. J. Electr. Eng. Comput. Sci.* **2021**, *21*, 1048–1056. [CrossRef]
45. Verma, M.; Chen, Y.C.; Kilicman, A.; Hasim, R.M. A systematic Review on the advancement in the study of fuzzy variational problems. *J. Funct. Spaces* **2022**, *2022*, 8037562. [CrossRef]

46. Masmali, I.; Shuaib, U.; Razaq, A.; Fatima, A.; Alhamzi, G. On Fundamental Algebraic Characterizations of μ-Fuzzy Normal Subgroups. *J. Funct. Spaces* **2022**, *2022*, 2703489. [CrossRef]
47. Razzaque, A.; Razaq, A. On q-Rung Orthopair Fuzzy Subgroups. *J. Funct. Spaces* **2022**, *2022*, 8196638. [CrossRef]