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Generalized Arithmetic Staircase Graphs and Their Total Edge Irregularity Strengths

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Abstract: Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a simple undirected graph with finite vertex set V_Γ and edge set E_Γ. A total n-labeling α: $V_\Gamma \cup E_\Gamma \to \{1, 2, \ldots, n\}$ is called a total edge irregular labeling on Γ if for any two different edges xy and $x'y'$ in E_Γ the numbers $\alpha(x) + \alpha(xy) + \alpha(y)$ and $\alpha(x') + \alpha(x'y') + \alpha(y')$ are distinct. The smallest positive integer n such that Γ can be labeled by a total edge irregular labeling is called the total edge irregularity strength of the graph Γ. In this paper, we provide the total edge irregularity strength of some asymmetric graphs and some symmetric graphs, namely generalized arithmetic staircase graphs and generalized double-staircase graphs, as the generalized forms of some existing staircase graphs. Moreover, we give the construction of the corresponding total edge irregular labelings.

Keywords: labeling; total edge irregularity strength; generalized arithmetic staircase graph

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1. Introduction

Graph theory has been developed widely in both theory and application (see [1–7]). According to [8], regarding the application side, graph theory plays a vital role as it is at the foundations, for instance, of the internet, parallel computing, distributed computing, molecular topology, dynamics, energy, electricity, and electronic circuit design. Among many subjects in graph theory, labeling is one that is also continuing to develop for both undirected and directed graphs (see [9–14]). In addition, labeling itself is also playing an important role in many fields, such as coding theory, physics, astronomy, circuit design, and computer science (see [15]). One from various labelings mentioned in [9] is that which is called total edge irregular labeling, introduced by Bača et al. [16]. Given a simple connected undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$ (later written as "a graph" for simplification) with non-empty finite vertex set V_Γ and edge set E_Γ. A labeling on graph Γ is a function from the graph elements into some sets that are usually consisting of numbers. When the codomain of the labeling is the set $\{1, 2, \ldots, n\}$, then the labeling is called n-labeling on Γ. Furthermore, if the domain of the n-labeling is the set V_Γ (E_Γ or $V_\Gamma \cup E_\Gamma$), then Γ is called a vertex (edge or total, respectively) n-labeling. A total n-labeling α: $V_\Gamma \cup E_\Gamma \to \{1, 2, \ldots, n\}$ is called a total edge irregular labeling on Γ if for any two different edges xy and $x'y'$ in E_Γ the numbers $\alpha(x) + \alpha(xy) + \alpha(y)$ and $\alpha(x') + \alpha(x'y') + \alpha(y')$ are distinct. For any edge $xy \in E_\Gamma$, the number $\alpha(x) + \alpha(xy) + \alpha(y)$ is considered as the weight of edge xy under labeling α and denoted by $wt_\alpha(xy)$. The total edge irregularity strength of graph Γ, denoted by $teis(\Gamma)$, is defined as the smallest number n such that we can label Γ by an edge irregular total n-labeling. Bača et al. [16] gave a hint on the lower bound of the total edge irregularity strength of an arbitrary graph, that is, for any graph Γ, $teis(\Gamma)$ is always greater or equal to $max\left\{\left\lceil \frac{|E_\Gamma|+2}{3} \right\rceil, \left\lceil \frac{\Delta_\Gamma+1}{2} \right\rceil\right\}$ where Δ_Γ is the maximum vertex degree of Γ. Thus, to obtain the exact value of the total edge irregularity strength $teis(\Gamma)$ of graph Γ, it is sufficient to show that the upper bound of $teis(\Gamma)$ is equal to the lower bound. This can be done by showing that there exists a total edge irregular n-labeling with $n = max\left\{\left\lceil \frac{|E_\Gamma|+2}{3} \right\rceil, \left\lceil \frac{\Delta_\Gamma+1}{2} \right\rceil\right\}$. In [16], moreover, the authors found the edge irregularity strength of some families of graphs, including path and cycle graphs. For tree, the total edge irregularity strength was given by Ivančo and Jendrol [17], and complete graphs and complete bipartite graphs were given by Jendrol et al. [18]. For some other graphs, the result on the total edge irregularity strength can be found in [19] for generalized Petersen graphs, in [20] for copies of the generalized Petersen graphs, in [21] for the strong product of two paths, in [22] for some large graphs, in [23] for hexagonal girth graphs, in [24] for some series parallel graphs, in [25] for some cartesian products of graphs, in [26] for generalized prism graphs, in [27] for some cactus chain graphs, in [28] for accordion graphs, and in [29] for the disjoint union of sun graphs.

In [30], the author presented the total edge irregularity strength of some staircase graphs. Later, the staircase graphs were modified arithmetically into some odd and even staircase graphs and their total edge irregularity strength were given in [31]. In this paper, we introduce the generalization of the graphs investigated in [30,31] into some generalized arithmetic staircase graphs and generalized arithmetic double-staircase graphs. This generalization is aimed to give a more general setting of staircase graphs containing the existing concept of staircase graphs. Therefore, we obtain a wider scope of the graph class. In this paper, we also give the total edge irregularity strength of these graphs. For each of the graphs, we construct the corresponding total edge irregular labelings.

2. Results

In this section, we introduce the definition of the generalized arithmetic staircase graph, which is a non-symmetric graph, and the generalized arithmetic double-staircase graph, which is a symmetric graph. Moreover, we determine the exact value of the total edge irregularity strength of the graphs by constructing the corresponding total edge irregular n-labelings where n meets the lower bound of the graphs. We also give some examples of the graphs and some labeled ones.

2.1. Generalized Arithmetic Staircase Graph

The first graph we study is the generalized arithmetic staircase graph which is defined as follows.

Definition 1. *Given three arbitrary positive integers $a, b, n \geq 1$. The generalized arithmetic staircase graph $GSC(a, b, n)$ of level n with a initial grids and difference b is a graph with vertex set*

$$V_{GSC(a,b,n)} = \{u_{i,j} | 0 \leq i \leq (a + bj), 0 \leq j \leq (n-1)\} \cup \{u_{i,n} | 0 \leq i \leq (a + bn - b)\}$$

and edge set

$$E_{GSC(a,b,n)} = \{u_{i,j}u_{i,j+1} | 0 \leq i \leq a + bj, 0 \leq j \leq (n-1)\} \cup$$
$$\{u_{i,j}u_{i+1,j} | 0 \leq i \leq (a + bj - 1), 0 \leq j \leq (n-1)\} \cup \{u_{i,n}u_{i+1,n} | 0 \leq i \leq (a + bn - b - 1)\}.$$

Clearly, the $GSC(a, b, n)$ in general is not symmetric. The maximum degree of $GSC(a, b, n)$ is $\Delta_{GSC(a,b,n)} = 2$ or $\Delta_{GSC(a,b,n)} = 3$ for $n = 1$ and is $\Delta_{GSC(a,b,n)} = 4$ for $n \neq 1$. Moreover, it is a routine that $|E_{GSC(a,b,n)}| = bn^2 + (2a + 1)n + a - b$. Therefore,

$$teis(GSC(a, b, n)) \geq max\left\{\left\lceil \frac{|E_{GSC(a,b,n)}| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_{GSC(a,b,n)} + 1}{2} \right\rceil\right\}$$

$$= \left\lceil \frac{|E_{GSC(a,b,n)}| + 2}{3} \right\rceil = \left\lceil \frac{bn^2 + (2a + 1)n + a - b + 2}{3} \right\rceil.$$

The graph $GSC(1, 1, n)$ is isomorphic to the staircase graph SC_n [30]. The graphs $GSC(1, 2, n)$ and $GSC(2, 2, n)$ are isomorphic to the odd staircase graph OSC_n and the even staircase graph ESC_n, respectively [31]. Moreover, from [31], we know that $teis(GSC(a, b, n)) = \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$ for $(a, b) = (1, 1), (1, 2), (2, 2)$ and for any positive number $n \geq 1$.

Now, as an example, we give a visual representation of $GSC(a,b,n)$ for $a = 2, b = 4$ and $n = 3$, as shown in Figure 1.

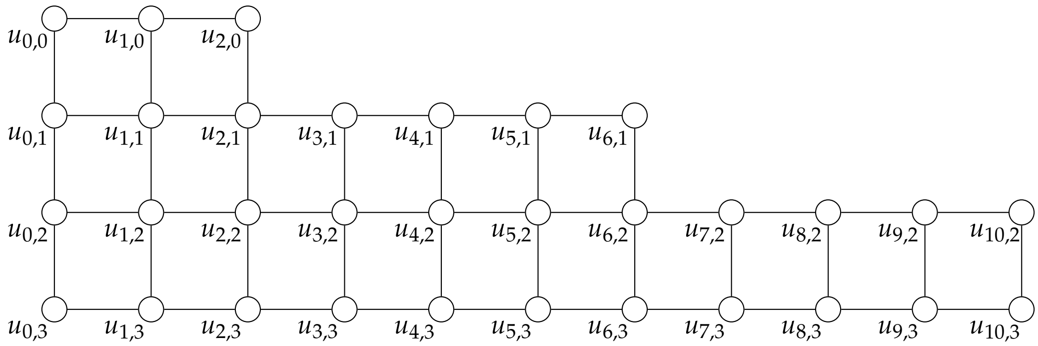

Figure 1. Generalized arithmetic staircase graph $GSC(2,4,3)$.

Before we give the exact value of $teis(GSC(a,b,n))$ for arbitrary positive numbers $a, b, n \geq 1$ such that $(a,b) \neq (1,1),(1,2),(2,2)$, first we give the following lemmas.

Lemma 1. *For any positive integers $b \geq 3$ and $n \geq 1$, we have*

$$teis(GSC(1,b,n)) = \left\lceil \frac{bn^2 + 3n - b + 3}{3} \right\rceil.$$

Proof. It is sufficient to show that there is a total l-labeling with $l = \left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$ that gives different weights for the edges. Before we give the labeling, we determine the biggest positive number s_1 such that

$$\frac{b}{2}s_1^2 + \left(1 - \frac{b}{2}\right)s_1 + 1 \leq \left\lceil \frac{bn^2 + 3n - b + 3}{3} \right\rceil.$$

Now, we give the labeling, namely

$$\beta_{a=1} : V_{GSC(1,b,n)} \cup E_{GSC(1,b,n)} \to \left\{1, 2, \ldots, \left\lceil \frac{bn^2 + 3n - b + 3}{3} \right\rceil \right\}$$

defined in the following way.

(i) Case $s_1 = n$
The vertex and edge labels are given as follows

vertex and edge label	i and j
$\beta_{a=1}(u_{i,j}) = \frac{b}{2}j^2 + (1 - \frac{b}{2})j + 1$	$0 \leq i \leq (bj+1)$ $0 \leq j \leq n-1$
$\beta_{a=1}(u_{i,n}) = \frac{b}{2}n^2 + (1 - \frac{b}{2})n + 1$	$0 \leq i \leq (bn - b + 1)$
$\beta_{a=1}(u_{i,j}u_{i,j+1}) = i + j + 1$	$0 \leq i \leq (bj+1)$ $0 \leq j \leq (n-1)$
$\beta_{a=1}(u_{i,j}u_{i+1,j}) = i + j + 1$	$0 \leq i \leq bj$ $0 \leq j \leq (n-1)$
$\beta_{a=1}(u_{i,n}u_{i+1,n}) = i + j + 1$	$0 \leq i \leq (bn - b)$.

(ii) Case $s_1 < n$
The vertex labels are

vertex label	i and j
$\beta_{a=1}(u_{i,j}) = \frac{b}{2}j^2 + (1-\frac{b}{2})j + 1$	$0 \leq i \leq bj+1$ $0 \leq j \leq s_1$
$\beta_{a=1}(u_{i,j}) = \left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$	$0 \leq i \leq (bj+1)$ $(s_1+1) \leq j \leq (n-1)$
$\beta_{a=1}(u_{i,n}) = \left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$	$0 \leq i \leq (bn-b+1)$

and the edge labels are

edge label
$\beta_{a=1}(u_{i,j}u_{i,j+1}) = i+j+1$ $0 \leq i \leq (bj+1)$, $0 \leq j \leq (s_1-1)$
$\beta_{a=1}(u_{i,j}u_{i,j+1}) = i + \frac{b}{2}s_1^2 + (1+\frac{b}{2}+1)s_1 + 3 - \left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$ $0 \leq i \leq (bs_1+1)$ $j = s_1$
$\beta_{a=1}(u_{i,j}u_{i,j+1}) = i + bs_1^2 + (2br+3)s_1 + br^2 + 3r + 4 - 2\left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$ $0 \leq i \leq (bj+1)$ $j = s_1 + r$ $1 \leq r \leq (n-s_1-1)$
$\beta_{a=1}(u_{i,j}u_{i+1,j}) = i+j+1$ $0 \leq i \leq bj$ $0 \leq j \leq s_1$
$\beta_{a=1}(u_{i,j}u_{i+1,j}) = i + b(s_1+r-1)^2 + (b+3)(s_1+r-1) + 6 - 2\left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$ $0 \leq i \leq bj$ $j = s_1 + r$ $1 \leq r \leq (n-1-s_1)$
$\beta_{a=1}(u_{i,n}u_{i+1,n}) = i + b(n-1)^2 + (b+3)(n-1) + 6 - 2\left\lceil \frac{bn^2+3n-b+3}{3} \right\rceil$ $0 \leq i \leq (bn-b)$.

From the labeling $\beta_{a=1}$, it is easy to check that the weights of $GSC(1,b,n)$'s edges constitute numbers from 3 up to $bn^2+3n-b+3$. Thus, we complete the proof. □

Lemma 2. *For arbitrary positive integers $a \geq 1$ and $n \geq 1$, we have*

$$teis(GSC(a,3,n)) = \left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil.$$

Proof. Just like the previous lemma, we first fix a positive integer s_2 satisfying

$$\frac{3}{2}s_2^2 + \left(a - \frac{3}{2}\right)s_2 + 1 \leq \left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil.$$

And further, to prove the assertion sufficiently, we construct a total labeling

$$\beta_{b=3} : V_{GSC(a,3,n)} \cup E_{GSC(a,3,n)} \to \left\{1,2,\ldots,\left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil\right\}$$

defined in the following way:

(i) Case $s_2 = n$
The vertex and edge labels are

vertex and edge label	i and j
$\beta_{b=3}(u_{i,j}) = \frac{3}{2}j^2 + (a - \frac{3}{2})j + 1$	$0 \le i \le (a + 3j)$ $0 \le j \le n - 1$
$\beta_{b=3}(u_{i,n}) = \frac{3}{2}n^2 + (a - \frac{3}{2})n + 1$	$0 \le i \le (a + 3n - 3)$
$\beta_{b=3}(u_{i,j}u_{i,j+1}) = i + j + 1$	$0 \le i \le (a + 3j)$ $0 \le j \le (n - 1)$
$\beta_{b=3}(u_{i,j}u_{i+1,j}) = i + j + 1$	$0 \le i \le (a + 3j - 1)$ $0 \le j \le (n - 1)$
$\beta_{b=3}(u_{i,n}u_{i+1,n}) = i + j + 1$	$0 \le i \le (a + 3n - 4)$.

(ii) Case $s_2 < n$

We label the vertices as follows

vertex label	i and j
$\beta_{b=3}(u_{i,j}) = \frac{3}{2}j^2 + (a - \frac{3}{2})j + 1$	$0 \le i \le (a + 3j)$ $0 \le j \le s_2$
$\beta_{b=3}(u_{i,j}) = \left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil$	$0 \le i \le (a + 3j)$ $(s_2 + 1) \le j \le (n - 1)$
$\beta_{b=3}(u_{i,n}) = \left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil$	$0 \le i \le (a + 3n - 3)$

and for the edges, we give labels in the following way

edge label
$\beta_{b=3}(u_{i,j}u_{i,j+1}) = i + j + 1$ $0 \le i \le (a + 3j)$, $0 \le j \le (s_2 - 1)$
$\beta_{b=3}(u_{i,j}u_{i,j+1}) = i + \frac{3}{2}s_2^2 + (a + \frac{5}{2})s_2 + a + 2 - \left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil$ $0 \le i \le (a + 3s_2)$ $j = s_2$
$\beta_{b=3}(u_{i,j}u_{i,j+1}) = i + 3s_2^2 + (6r + 2a + 1)s_2 + 3r^2 + 2ar + r + a + 3 - 2k$ $0 \le i \le (a + 3j)$ $j = s_2 + r$ $1 \le r \le (n - s_2 - 1)$
$\beta_{b=3}(u_{i,j}u_{i+1,j}) = i + j + 1$ $0 \le i \le (a + 3j - 1)$ $0 \le j \le s_2$
$\beta_{b=3}(u_{i,j}u_{i+1,j}) = i + 3(s_2 + r - 1)^2 + (2a + 4)(s_2 + r - 1) + 2a + 4 - 2k$ $0 \le i \le (a + 3j - 1)$ $j = s_2 + r$ $1 \le r \le (n - 1 - s_2)$
$\beta_{b=3}(u_{i,n}u_{i+1,n}) = i + 3(n - 1)^2 + (2a + 4)(n - 1) + 2a + 4 - 2k$ $0 \le i \le (a + 3n - 4)$

with $k = \left\lceil \frac{3n^2+(2a+1)n+a-1}{3} \right\rceil$.

Based on the definition of the labeling $\beta_{b=3}$, the edge weights of $GSC(a,3,n)$ vary from 3 up to $3n^2 + (2a + 1)n + a - 1$.

□

From the above two lemmas, then we derive the following theorem on the total edge irregularity strength of any generalized arithmetic staircase graphs $GSC(a,b,n)$.

Theorem 1. *For arbitrary positive integers $a, b, n \ge 1$ and $(a,b) \ne (1,1), (1,2), (2,2)$, it follows that*

$$teis(GSC(a,b,n)) = \left\lceil \frac{bn^2 + (2a+1)n + a - b + 2}{3} \right\rceil.$$

Proof. It is already clear that $teis(GSC(a,b,n)) \geq \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$. Hence, it is sufficient to show that there is a total edge irregularity l-labeling on $GSC(a,b,n)$ with $l = \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$. Prior, it is necessary to determine the largest positive integer s such that

$$\frac{b}{2}s^2 + \left(a - \frac{b}{2}\right)s + 1 \leq \left\lceil \frac{bn^2 + (2a+1)n + a - b + 2}{3} \right\rceil.$$

As an example, we give several integers s for various $a, b,$, and n in Table 1.

Table 1. Largest s for several values of $a, b,$ and n satisfying $\frac{b}{2}s^2 + (a - \frac{b}{2})s + 1 \leq \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$.

a	***b***	***n***	***s***	***a***	***b***	***n***	***s***
1	3	1	1	4	9	200	150
1	3	3	2	7	4	200	163
1	3	100	82	10	11	1225	1000
1	5	120	98	15	15	1300	1061
2	3	130	106	20	20	2450	2000
2	7	135	110	50	23	6125	5001
3	3	2	2	100	100	12250	10,002
3	3	150	122	200	1000	14,600	11,921
4	2	1	1	1000	1400	15,000	12,247
4	2	2	1	10,000	2000	1,000,000	816,470

Then, we construct a function

$$\beta : V_{GSC(a,b,n)} \cup E_{GSC(a,b,n)} \rightarrow \left\{1, 2, \ldots, \left\lceil \frac{bn^2 + (2a+1)n + a - b + 2}{3} \right\rceil \right\}$$

defined in the following manner:

(i) Case $s = n$
The vertex and edge labels are

vertex and edge label	i and j
$\beta(u_{i,j}) = \frac{b}{2}j^2 + (a - \frac{b}{2})j + 1$	$0 \leq i \leq (a + bj)$ $0 \leq j \leq n - 1$
$\beta(u_{i,n}) = \frac{b}{2}n^2 + (a - \frac{b}{2})n + 1$	$0 \leq i \leq (a + bn - b)$
$\beta(u_{i,j}u_{i,j+1}) = i + j + 1$	$0 \leq i \leq (a + bj)$ $0 \leq j \leq (n - 1)$
$\beta(u_{i,j}u_{i+1,j}) = i + j + 1$	$0 \leq i \leq (a + bj - 1)$ $0 \leq j \leq (n - 1)$
$\beta(u_{i,n}u_{i+1,n}) = i + j + 1$	$0 \leq i \leq (a + bn - b - 1)$.

(ii) Case $s < n$
The vertices are labeled in the following way

vertex label	i and j
$\beta(u_{i,j}) = \frac{b}{2}j^2 + (a - \frac{b}{2})j + 1$	$0 \leq i \leq (a + bj)$ $0 \leq j \leq s$
$\beta(u_{i,j}) = \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$	$0 \leq i \leq (a + bj)$ $s + 1 \leq j \leq (n - 1)$
$\beta(u_{i,n}) = \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$	$0 \leq i \leq (a + bn - b)$

and the edge labels are as follows

edge label
$\beta(u_{i,j}u_{i,j+1}) = i + j + 1$ $0 \le i \le (a + bj),$ $0 \le j \le (s - 1)$
$\beta(u_{i,j}u_{i,j+1}) = i + \frac{b}{2}s^2 + (a + \frac{b}{2} + 1)s + a + 2 - l$ $0 \le i \le (a + bs)$ $j = s$
$\beta(u_{i,j}u_{i,j+1}) = i + bs^2 + (2br + 2a + 1)s + br^2 + 2ar + r + a + 3 - 2l$ $0 \le i \le (a + bj)$ $j = s + r$ $1 \le r \le (n - s - 1)$
$\beta(u_{i,j}u_{i+1,j}) = i + j + 1$ $0 \le i \le (a + bj - 1)$ $0 \le j \le s$
$\beta(u_{i,j}u_{i+1,j}) = i + b(s + r - 1)^2 + (2a + b + 1)(s + r - 1) + 2a + 4 - 2l$ $0 \le i \le (a + bj - 1)$ $j = s + r$ $1 \le r \le (n - 1 - s)$
$\beta(u_{i,n}u_{i+1,n}) = i + b(n - 1)^2 + (2a + b + 1)(n - 1) + 2a + 4 - 2l$ $0 \le i \le (a + bn - b - 1)$

with $l = \left\lceil \frac{bn^2 + (2a+1)n + a - b + 2}{3} \right\rceil$.

With respect to the labeling β, the weights of $GSC(a, b, n)$'s edges are varying from 3 up to $bn^2 + (2a + 1)n + a - b + 2$.

□

As an example of the labeling for the graph given in Theorem 1, we give an example of the labeled graph $GSC(2, 4, 3)$, as shown in Figure 2. The vertex labels are given inside the circles while the edge labels are given in blue. For the graph $GSC(2, 4, 3)$, the corresponding value of s is equal to 2.

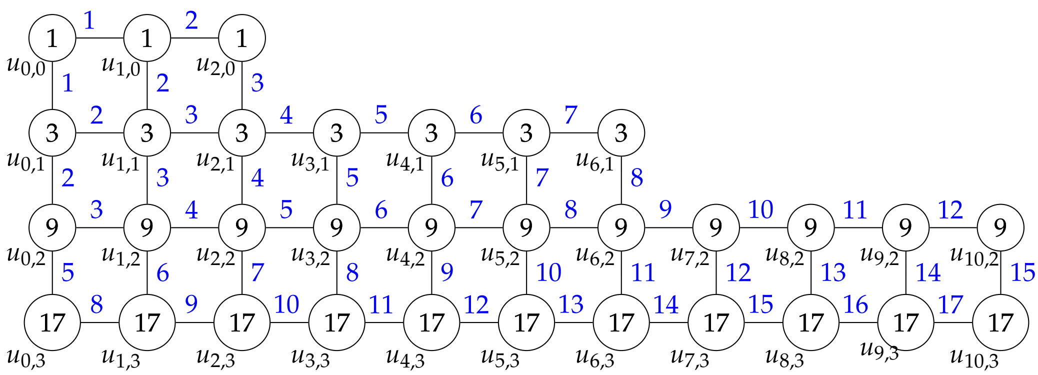

Figure 2. Generalized arithmetic staircase graph $GSC(2, 4, 3)$ with its total edge irregular 17-labeling.

2.2. Generalized Arithmetic Double-Staircase Graph

For the second observation, we give the definition of generalized arithmetic double-staircase graph as the following.

Definition 2. *Given three arbitrary positive integers $a, b, n \ge 1$. The generalized arithmetic double-staircase graph $GDSC(a, b, n)$ of level n with a initial grids and difference b is a graph with vertex set*

$$V_{GDSC(a,b,n)} = \begin{cases} U, & a \text{ odd} \\ U \cup \{u_{0,j} | 0 \le j \le n\}, & a \text{ even} \end{cases} \tag{1}$$

with

$$U = \left\{u_{i,j} | 1 \le |i| \le \left(\left\lceil \frac{a}{2} \right\rceil + bj\right), 0 \le j \le n-1\right\} \cup \left\{u_{i,n} | 1 \le |i| \le \left(\left\lceil \frac{a}{2} \right\rceil + bn - b\right)\right\}$$

and edge set $E_{GDSC(a,b,n)}$ *consisting of edges given as the following*

edge	i and j
$u_{i,j}u_{i,j+1}$	$1 \le \lvert i \rvert \le \left\lceil \frac{a}{2} \right\rceil + bj$ $0 \le j \le (n-1)$
$u_{i,j}u_{i+1,j}$	$-(a+bj) \le i \le -2$ $0 \le j \le (n-1)$
$u_{i,j}u_{i+1,j}$	$1 \le i \le (a+bj-1)$ $0 \le j \le (n-1)$
$u_{-1,j}u_{1,j}$	$0 \le j \le n$
$u_{i,n}u_{i+1,n}$	$-\left(\left\lceil \frac{a}{2} \right\rceil + bn - b\right) \le i \le -2.$
$u_{i,n}u_{i+1,n}$	$1 \le i \le \left(\left\lceil \frac{a}{2} \right\rceil + bn - b - 1\right).$

whenever a is odd and the following edges

edge	i and j
$u_{i,j}u_{i,j+1}$	$\lvert i \rvert \le \left\lceil \frac{a}{2} \right\rceil + bj$ $0 \le j \le (n-1)$
$u_{i,j}u_{i+1,j}$	$-(a+bj) \le i \le (a+bj-1)$ $0 \le j \le (n-1)$
$u_{i,n}u_{i+1,n}$	$-(a+bn-b) \le i \le (a+bn-b-1)$

whenever a is even.

By the definition, it is obvious that the graph $GDSC(a,b,n)$ is a symmetric graph in the sense that it has a symmetric form. As examples, in Figures 3 and 4, we give the graphs $GDSC(a,b,n)$ for $a = 3, b = 2$ $n = 2$ and for $a = 4, b = 2$ $n = 2$, respectively.

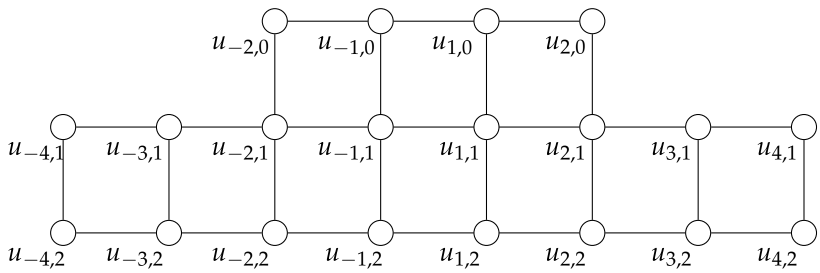

Figure 3. Generalized arithmetic double-staircase graph $GDSC(3,2,2)$.

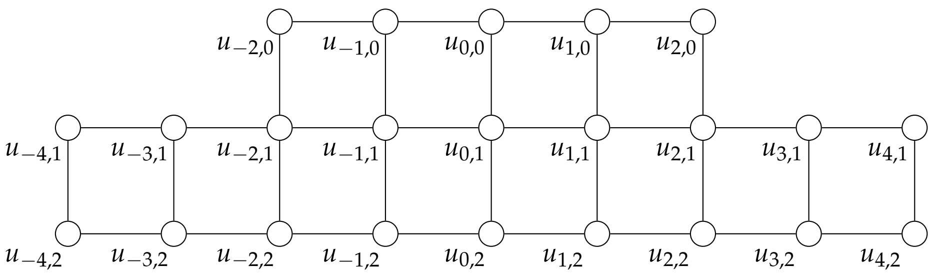

Figure 4. Generalized arithmetic double-staircase graph $GDSC(4,2,2)$.

The maximum degree of $GDSC(a,b,n)$ is obviously $\Delta_{GDSC(a,b,n)} = 2$ or $\Delta_{GDSC(a,b,n)} = 3$ for $n = 1$ and is $\Delta_{GDSC(a,b,n)} = 4$ for $n \neq 1$. Furthermore, it is easy to see that

$$|E_{GDSC(a,b,n)}| = 2|E_{GSC(\frac{a-1}{2},b,n)}| + n + 1 = 2bn^2 + (2a+1)n + a - 2b$$

if a is odd and

$$|E_{GDSC(a,b,n)}| = 2|E_{GSC(\frac{a}{2},b,n)}| - n = 2bn^2 + (2a+1)n + a - 2b$$

if a is even. Hence, clearly the lower bound of $teis(GDSC(a,b,n))$ is given as follows

$$teis(GDSC(a,b,n)) \geq max\left\{ \left\lceil \frac{|E_{GDSC(a,b,n)}| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_{GDSC(a,b,n)} + 1}{2} \right\rceil \right\}$$

$$= \left\lceil \frac{|E_{GDSC(a,b,n)}| + 2}{3} \right\rceil = \left\lceil \frac{2bn^2 + (2a+1)n + a - 2b + 2}{3} \right\rceil.$$

Particularly, the graphs $GDSC(1,1,n)$ and $GDSC(2,1,n)$ are obviously isomorphic to the double-staircase graph DSC_n and the mirror staircase MSC_n, respectively, as presented in [30]. Moreover, the graphs $GDSC(1,2,n)$ and $GDSC(2,2,n)$ are isomorphic to the double odd staircase graph $DOSC_n$ and mirror odd staircase graph $MOSC_n$, respectively, given in [31]. It has been shown in [30,31] that for each graph from those four types of staircase graphs, their total edge irregularity strengths are precisely equal to the lower bound given in [16]. In Theorem 2, we will prove that the same result also holds for arbitrary positive numbers a, b, n with $(a,b) \notin \{1,2\} \times \{1,2\}$.

Before we proof Theorem 2, we give several lemmas below.

Lemma 3. *For arbitrary positive integers $b \geq 3$ and $n \geq 1$, it follows that*

$$teis(GDSC(1,b,n)) = \left\lceil \frac{2bn^2 + 3n - 2b + 3}{3} \right\rceil.$$

Proof. First, we determine the largest positive integer t_1 satisfying

$$bt_1^2 - bt_1 + t_1 + 1 \leq \left\lceil \frac{2bn^2 + 3n - 2b + 3}{3} \right\rceil.$$

We then define a total p-labeling $\gamma_{a=1}$ with $p = \left\lceil \frac{2bn^2+3n-2b+3}{3} \right\rceil$

$$\gamma_{a=1} : V_{GDSC(1,b,n)} \cup E_{GDSC(1,b,n)} \rightarrow \left\{ 1, 2, \ldots, \left\lceil \frac{2bn^2 + 3n - 2b + 3}{3} \right\rceil \right\}$$

in the following manner. The labeling is defined as the following.

(i) Case $t_1 = n$

The labels of the vertices and the edges are

vertex and edge label	i and j
$\gamma_{a=1}(u_{i,j}) = bj^2 - bj + j + 1$	$1 \leq \lvert i \rvert \leq (1+bj)$ $0 \leq j \leq n-1$
$\gamma_{a=1}(u_{i,n}) = bn^2 - bn + n + 1$	$1 \leq i \leq (1 + bn - b)$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + (b+1)j + 2$	$-(1+bj) \leq i \leq -1$ $0 \leq j \leq (n-1)$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + (b+1)j + 1$	$1 \leq i \leq (1+bj)$ $0 \leq j \leq (n-1)$

$\gamma_{a=1}(u_{i,j}u_{i+1,j}) = i + (b+1)j + 2$	$-(1+bj) \leq i \leq -2$ $0 \leq j \leq (n-1)$
$\gamma_{a=1}(u_{-1,j}u_{1,j}) = (b+1)j + 1$	$0 \leq j \leq (n-1)$
$\gamma_{a=1}(u_{i,j}u_{i+1,j}) = i + (b+1)j + 1$	$1 \leq i \leq bj$ $0 \leq j \leq (n-1)$
$\gamma_{a=1}(u_{i,n}u_{i+1,n}) = i + (b+1)n + 2$	$-(1+bn-b) \leq i \leq -2$
$\gamma_{a=1}(u_{-1,n}u_{1,n}) = (b+1)n + 1$	
$\gamma_{a=1}(u_{i,n}u_{i+1,n}) = i + (b+1)n + 1$	$1 \leq i \leq (bn-b)$

(ii) Case $t_1 < n$
The vertex labels are

vertex label	i and j
$\gamma_{a=1}(u_{i,j}) = bj^2 - bj + j + 1$	$1 \leq \lvert i \rvert \leq (1+bj)$ $0 \leq j \leq t_1$
$\gamma_{a=1}(u_{i,j}) = \left\lceil \frac{2bn^2+3n-2b+3}{3} \right\rceil$	$1 \leq \lvert i \rvert \leq (1+bj)$ $(t_1+1) \leq j \leq (n-1)$
$\gamma_{a=1}(u_{i,n}) = \left\lceil \frac{2bn^2+3n-2b+3}{3} \right\rceil$	$1 \leq \lvert i \rvert \leq (bn-b+1)$.

The edge labels are

edge label
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + (b+1)j + 2$ $-(1+bj) \leq i \leq -1$ $0 \leq j \leq (t_1-1)$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + (b+1)j + 1$ $1 \leq i \leq (1+bj)$ $0 \leq j \leq (t_1-1)$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + bt_1^2 + (2b+2)t_1 + 4 - p$ $-(1+bj) \leq i \leq -1$ $j = t_1$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + bt_1^2 + (2b+2)t_1 + 3 - p$ $1 \leq i \leq (1+bj)$ $j = t_1$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + 2b(t_1+r-1)^2 + 5(t_1+r-1) + b(t_1+r) + 3 - 2p$ $-(1+bj) \leq i \leq 1$ $j = t_1 + r$ $1 \leq r \leq (n-t_1-1)$
$\gamma_{a=1}(u_{i,j}u_{i,j+1}) = i + 2b(t_1+r-1)^2 + 5(t_1+r-1) + b(t_1+r) + 2 - 2p$ $1 \leq i \leq (1+bj)$ $j = t_1 + r$ $1 \leq r \leq (n-t_1-1)$
$\gamma_{a=1}(u_{i,j}u_{i+1,j}) = i + (b+1)j + 2$ $-(1+bj) \leq i \leq -2$ $0 \leq j \leq t_1$
$\gamma_{a=1}(u_{-1,j}u_{1,j}) = (b+1)j + 1$ $0 \leq j \leq t_1$
$\gamma_{a=1}(u_{i,j}u_{i+1,j}) = i + (b+1)j + 1$ $1 \leq i \leq bj$ $0 \leq j \leq t_1$
$\gamma_{a=1}(u_{i,j}u_{i+1,j}) = i + 2b(t_1+r-1)^2 + (2b+3)(t_1+r-1) + b(t_1+r) + 7 - 2p$ $-(1+bj) \leq i \leq -2$ $j = t_1 + r$ $1 \leq r \leq (n-1-t_1)$

$\gamma_{a=1}(u_{-1,j}u_{1,j}) = 2b(t_1+r-1)^2 + (2b+3)(t_1+r-1) + b(t_1+r) + 6 - 2p$ $j = t_1 + r$ $1 \le r \le (n-1-t_1)$
$\gamma_{a=1}(u_{i,j}u_{i+1,j}) = i + 2b(t_1+r-1)^2 + (2b+3)(t_1+r-1) + b(t_1+r) + 6 - 2p$ $1 \le i \le bj$ $j = t_1 + r$ $1 \le r \le (n-1-t_1)$
$\gamma_{a=1}(u_{i,n}u_{i+1,n}) = i + 2b(n-1)^2 + (3b+3)(n-1) + 7 - 2p$ $-(1+bn-b) \le i \le -2$
$\gamma_{a=1}(u_{-1,n}u_{1,n}) = 2b(n-1)^2 + (3b+3)(n-1) + 6 - 2p$
$\gamma_{a=1}(u_{i,n}u_{i+1,n}) = i + 2b(n-1)^2 + (3b+3)(n-1) + 6 - 2p$ $1 \le i \le (bn-b)$

with $p = \left\lceil \frac{2bn^2+3n-2b+3}{3} \right\rceil$. With respect to the labeling $\gamma_{a=1}$, it is easy to see that all edges in $GDSC(1,b,n)$ have different weights.

□

Lemma 4. *For arbitrary positive numbers $b \ge 3$ and $n \ge 1$, it follows that*

$$teis(GDSC(a,3,n)) = \left\lceil \frac{6n^2 + (2a+1)n + a - 4}{3} \right\rceil.$$

Proof. We will show that the upper bound of $teis(GDSC(a,3,n))$ meets the lower bound. We first determine the biggest positive integer t_2 satisfying

$$bt_2^2 + (a-3)t_2 + 1 \le \left\lceil \frac{6n^2 + (2a+1)n + a - 4}{3} \right\rceil.$$

Let us construct q-labeling with $q = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$ as follows

$$\gamma_{b=3} : V_{GDSC(a,3,n)} \cup E_{GDSC(a,3,n)} \to \left\{1, 2, \ldots, \left\lceil \frac{6n^2 + (2a+1)n + a - 4}{3} \right\rceil \right\}$$

with labels for the vertices, and the edges are defined in the following way:

(i) Case $t_2 = n$

For a odd, the vertices and the edges are labeled in the following manner:

vertex and edge label	i and j
$\gamma_{b=3}(u_{i,j}) = 3j^2 + (a-3)j + 1$	$1 \le \lvert i \rvert \le (\lceil \frac{a}{2} \rceil + 3j)$ $0 \le j \le n-1$
$\gamma_{b=3}(u_{i,n}) = 3n^2 + (a-3)n + 1$	$1 \le \lvert i \rvert \le (\lceil \frac{a}{2} \rceil + 3n - 3)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + 3j) \le i \le -1$ $0 \le j \le (n-1)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 4j + \lceil \frac{a}{2} \rceil$	$1 \le i \le (\lceil \frac{a}{2} \rceil + 3j)$ $0 \le j \le (n-1)$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + 3j) \le i \le -2$ $0 \le j \le (n-1)$
$\gamma(u_{-1,j}u_{1,j}) = 4j + \lceil \frac{a}{2} \rceil$	$0 \le j \le (n-1)$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 4j + \lceil \frac{a}{2} \rceil$	$1 \le i \le (\lceil \frac{a}{2} \rceil + 3j - 1)$ $0 \le j \le (n-1)$
$\gamma_{b=3}(u_{i,n}u_{i+1,n}) = i + 4n + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + 3n - 3) \le i \le -2$
$\gamma_{b=3}(u_{-1,n}u_{1,n}) = 4n + \lceil \frac{a}{2} \rceil$	
$\gamma_{b=3}(u_{i,n}u_{i+1,n}) = i + 4n + \lceil \frac{a}{2} \rceil$	$1 \le i \le (\lceil \frac{a}{2} \rceil + 3n - 4)$

For a even, the labels of the vertices and the edges are

vertex and edge label	i and j
$\gamma_{b=3}(u_{i,j}) = 3j^2 + (a-3)j + 1$	$\lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $0 \leq j \leq (n-1)$
$\gamma_{b=3}(u_{i,n}) = 3n^2 + (a-3)n + 1$	$\lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3n - 3)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$	$\lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $0 \leq j \leq (n-1)$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq (\lceil \frac{a}{2} \rceil + 3j - 1)$ $0 \leq j \leq (n-1)$
$\gamma_{b=3}(u_{i,n}u_{i+1,n}) = i + 4n + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + 3n - 3) \leq i \leq (\lceil \frac{a}{2} \rceil + 3n - 4)$

(ii) Case $t_2 < n$

For a odd, the vertex labels are

vertex label	i and j
$\gamma_{b=3}(u_{i,j}) = 3j^2 + (a-3)j + 1$	$1 \leq \lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $0 \leq j \leq t_2$
$\gamma_{b=3}(u_{i,j}) = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$	$1 \leq \lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $(t_2+1) \leq j \leq (n-1)$
$\gamma_{b=3}(u_{i,n}) = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$	$1 \leq \lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3n - 3)$

For a even, the vertex labels are

vertex label	i and j
$\gamma_{b=3}(u_{i,j}) = 3j^2 + (a-3)j + 1$	$\lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $0 \leq j \leq t_2$
$\gamma_{b=3}(u_{i,j}) = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$	$\lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $(t_2+1) \leq j \leq (n-1)$
$\gamma_{b=3}(u_{i,n}) = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$	$\lvert i\rvert \leq (\lceil \frac{a}{2} \rceil + 3n - 3)$

For a odd, the edge labels are as follows

edge label
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq -1$ $0 \leq j \leq (t_2 - 1)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 4j + \lceil \frac{a}{2} \rceil$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + 3j)$ $0 \leq j \leq (t_2 - 1)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 3t_2^2 + (a+7)t_2 + \lceil \frac{a}{2} \rceil + a + 2 - q$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq -1$ $j = t_2$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 3t_2^2 + (a+7)t_2 + \lceil \frac{a}{2} \rceil + a + 1 - q$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + 3j)$ $j = t_2$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 6(t_2 + r - 1)^2 + (2a+4)(t_2 + r - 1) + a + \lceil \frac{a}{2} \rceil + 4 - 2q$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq -1$ $j = t_2 + r$ $1 \leq r \leq (n - t_2 - 1)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 6(t_2 + r - 1)^2 + (2a+4)(t_2 + r - 1) + a + \lceil \frac{a}{2} \rceil + 3 - 2q$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + 3j)$ $j = t_2 + r$ $1 \leq r \leq (n - t_2 - 1)$

$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq -2$ $0 \leq j \leq t_2$
$\gamma_{b=3}(u_{-1,j}u_{1,j}) = 4j + \lceil \frac{a}{2} \rceil$ $0 \leq j \leq t_2$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 4j + \lceil \frac{a}{2} \rceil$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + 3j - 1)$ $0 \leq j \leq t_2$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 6(t_2 + r - 1)^2 + (2a + 10)(t_2 + r - 1) + 7 + 2a + \lceil \frac{a}{2} \rceil - 2q$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq -2$ $j = t_2 + r$ $1 \leq j \leq (n - 1 - t_2)$
$\gamma_{b=3}(u_{-1,j}u_{1,j}) = 6(t_2 + r - 1)^2 + (2a + 10)(t_2 + r - 1) + 6 + 2a + \lceil \frac{a}{2} \rceil - 2q$ $j = t_2 + r$ $1 \leq r \leq (n - 1 - t_2)$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 6(t_2 + r - 1)^2 + (2a + 10)(t_2 + r - 1) + 6 + 2a + \lceil \frac{a}{2} \rceil - 2q$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + 3j - 1)$ $j = t_2 + r$ $1 \leq r \leq (n - 1 - t_2)$
$\gamma_{b=3}(u_{i,n}u_{i+1,n}) = i + 6(n - 1)^2 + (2a + 10)(n - 1) + 2a + \lceil \frac{a}{2} \rceil + 4 - 2q$ $-(\lceil \frac{a}{2} \rceil + 3n - 3) \leq i \leq -2$
$\gamma_{b=3}(u_{-1,n}u_{1,n}) = 6(n - 1)^2 + (2a + 10)(n - 1) + 2a + \lceil \frac{a}{2} \rceil + 3 - 2q$
$\gamma_{b=3}(u_{i,n}u_{i+1,n}) = i + 6(n - 1)^2 + (2a + 10)(n - 1) + 2a + \lceil \frac{a}{2} \rceil + 3 - 2q$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + 3n - 4)$

with $q = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$.
For a even, the edge labels are

edge label
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$ $\lvert i \rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $0 \leq j \leq (t_2 - 1)$
$\gamma_{b=3}(u_{i,t_2}u_{i,t_2+1}) = i + 3t_2^2 + (a + 7)t_2 + \lceil \frac{a}{2} \rceil + a + 2 - q$ $\lvert i \rvert \leq (\lceil \frac{a}{2} \rceil + 3t_2)$
$\gamma_{b=3}(u_{i,j}u_{i,j+1}) = i + 6(t_2 + r - 1)^2 + (2a + 4)(t_2 + r - 1) + a + \lceil \frac{a}{2} \rceil + 4 - 2q$ $\lvert i \rvert \leq (\lceil \frac{a}{2} \rceil + 3j)$ $j = t_2 + r$ $1 \leq r \leq (n - t_2 - 1)$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 4j + \lceil \frac{a}{2} \rceil + 1$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq (\lceil \frac{a}{2} \rceil + 3j - 1)$ $0 \leq j \leq t_2$
$\gamma_{b=3}(u_{i,j}u_{i+1,j}) = i + 6(t_2 + r - 1)^2 + (2a + 10)(t_2 + r - 1) + 7 + 2a + \lceil \frac{a}{2} \rceil - 2q$ $-(\lceil \frac{a}{2} \rceil + 3j) \leq i \leq (\lceil \frac{a}{2} \rceil + 3j - 1)$ $j = t_2 + r$ $1 \leq j \leq (n - 1 - t_2)$
$\gamma_{b=3}(u_{i,n}u_{i+1,n}) = i + 6(n - 1)^2 + (2a + 10)(n - 1) + 2a + \lceil \frac{a}{2} \rceil + 4 - 2q$ $-(\lceil \frac{a}{2} \rceil + 3n - 3) \leq i \leq (\lceil \frac{a}{2} \rceil + 3n - 4)$

with $q = \left\lceil \frac{6n^2+(2a+1)n+a-4}{3} \right\rceil$.

It is easy to seet that all edge weights of $GDSC(a, 3, n)$, with respect to $\gamma_{b=3}$, are different numbers. Thus, the proof is complete. □

From the Lemmas 3 and 4, we then have Theorem 2 on total edge irregularity strength of the graph $GDSC(a, b, n)$.

Theorem 2. *For arbitrary $a, b, n \geq 1$ and $(a, b) \notin \{1, 2\} \times \{1, 2\}$, it follows that*

$$teis(GDSC(a,b,n)) = \left\lceil \frac{2bn^2 + (2a+1)n + a - 2b + 2}{3} \right\rceil.$$

Proof. Again, we only need to show that the upper bound of $teis(GDSC(a,b,n))$ meets the lower bound. For this, we will construct h-labeling with $h = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$ so that all edge weights are different and constitute numbers from 3 up to $|E_{GDSC(a,b,n)}| + 2 = 2bn^2 + (2a+1)n + a - 2b + 2$. Now, we first find the largest positive integer t satisfying

$$bt^2 + (a-b)t + 1 \leq \left\lceil \frac{2bn^2 + (2a+1)n + a - 2b + 2}{3} \right\rceil.$$

Several integers t for various $a, b,$, and n are given on Table 2 as an example.

Table 2. Largest t for several values of a, b, and n satisfying $bt^2 + (a-b)t + 1 \leq \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$.

a	***b***	***n***	***t***	***a***	***b***	***n***	***t***
1	3	3	2	20	50	3700	3021
2	5	5	4	20	100	3700	3021
3	4	2	2	50	60	7000	5715
4	2	2	1	60	4	1000	815
4	2	20	16	80	40	500	408
4	9	200	163	100	100	12,250	10,002
7	4	200	163	200	500	14,600	11,921
10	10	135	110	2000	2000	1000	816
10	11	1225	1000	2000	5000	1000	816
15	15	1300	1061	10,000	2000	1,000,000	816,496

We define a total p-labeling γ with $p = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$ as follows

$$\gamma : V_{GDSC(a,b,n)} \cup E_{GDSC(a,b,n)} \rightarrow \left\{1, 2, \ldots, \left\lceil \frac{2bn^2 + (2a+1)n + a - 2b + 2}{3} \right\rceil \right\}.$$

Labels for the vertices and the edges are defined as follows

(i) Case $t = n$
For a odd, the vertex and edge labels are

vertex and edge label	i and j
$\gamma(u_{i,j}) = bj^2 + (a-b)j + 1$	$1 \leq \lvert i \rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,n}) = bn^2 + (a-b)n + 1$	$1 \leq \lvert i \rvert \leq \left(\lceil \frac{a}{2} \rceil + bn - b\right)$
$\gamma(u_{i,j}u_{i,j+1}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$	$-\left(\lceil \frac{a}{2} \rceil + bj\right) \leq i \leq -1$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,j}u_{i,j+1}) = i + (b+1)j + \lceil \frac{a}{2} \rceil$	$1 \leq i \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$	$-\left(\lceil \frac{a}{2} \rceil + bj\right) \leq i \leq -2$ $0 \leq j \leq (n-1)$
$\gamma(u_{-1,j}u_{1,j}) = (b+1)j + \lceil \frac{a}{2} \rceil$	$0 \leq j \leq (n-1)$
$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil$	$1 \leq i \leq \left(\lceil \frac{a}{2} \rceil + bj - 1\right)$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,n}u_{i+1,n}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$	$-\left(\lceil \frac{a}{2} \rceil + bn - b\right) \leq i \leq -2$
$\gamma(u_{-1,n}u_{1,n}) = (b+1)j + \lceil \frac{a}{2} \rceil$	
$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil$	$1 \leq i \leq \left(\lceil \frac{a}{2} \rceil + bn - b - 1\right)$

For a even, the labels of the vertices and edges are

vertex and edge label	i and j
$\gamma(u_{i,j}) = bj^2 + (a-b)j + 1$	$\lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,n}) = bn^2 + (a-b)n + 1$	$\lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bn - b\right)$
$\gamma(u_{i,j}u_{i,j+1}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$	$\lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq \left(\lceil \frac{a}{2} \rceil + bj - 1\right)$ $0 \leq j \leq (n-1)$
$\gamma(u_{i,n}u_{i+1,n}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$	$-(\lceil \frac{a}{2} \rceil + bn - b) \leq i \leq \left(\lceil \frac{a}{2} \rceil + bn - b - 1\right)$

(ii) Case $t < n$

For a odd, the vertices are labeled as the following

vertex label	i and j
$\gamma(u_{i,j}) = bj^2 + (a-b)j + 1$	$1 \leq \lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $0 \leq j \leq t$
$\gamma(u_{i,j}) = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$	$1 \leq \lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $t+1 \leq j \leq (n-1)$
$\gamma(u_{i,n}) = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$	$1 \leq \lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + b(n-1)\right).$

For a even, the vertex labels are given as follows

vertex label	i and j
$\gamma(u_{i,j}) = bj^2 + (a-b)j + 1$	$\lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $0 \leq j \leq t$
$\gamma(u_{i,j}) = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$	$\lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + bj\right)$ $(t+1) \leq j \leq (n-1)$
$\gamma(u_{i,n}) = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$	$\lvert i\rvert \leq \left(\lceil \frac{a}{2} \rceil + b(n-1)\right).$

For a odd, the edge labels are as follows

edge label
$\gamma(u_{i,j}u_{i,j+1}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$ $-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq -1$ $0 \leq j \leq (t-1)$
$\gamma(u_{i,j}u_{i,j+1}) = i + (b+1)j + \lceil \frac{a}{2} \rceil$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + bj)$ $0 \leq j \leq (t-1)$
$\gamma(u_{i,t}u_{i,t+1}) = i + bt^2 + (a+2b+1)t + \lceil \frac{a}{2} \rceil + a + 2 - h$ $-(\lceil \frac{a}{2} \rceil + bt) \leq i \leq -1$
$\gamma(u_{i,t}u_{i,t+1}) = i + bt^2 + (a+2b+1)t + \lceil \frac{a}{2} \rceil + a + 1 - h$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + bt)$
$\gamma(u_{i,j}u_{i,j+1}) = i + 2b(t+r-1)^2 + (2a+b+1)(t+r-1) + a + \lceil \frac{a}{2} \rceil + b + 1 - 2h$ $-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq -1$ $j = t + r$ $1 \leq r \leq (n-t-1)$
$\gamma(u_{i,j}u_{i,j+1}) = i + 2b(t+r-1)^2 + (2a+b+1)(t+r-1) + a + \lceil \frac{a}{2} \rceil + b - 2h$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + bj)$ $j = t + r$ $1 \leq r \leq (n-t-1)$

$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$ $-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq -2$ $0 \leq j \leq t$
$\gamma(u_{-1,j}u_{1,j}) = (b+1)j + \lceil \frac{a}{2} \rceil$ $0 \leq j \leq t$
$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + bj - 1)$ $0 \leq j \leq t$
$\gamma(u_{i,j}u_{i+1,j}) = i + 2b(t+r-1)^2 + (2a+3b+1)(t+r-1) + b + 2a + \lceil \frac{a}{2} \rceil + 4 - 2h$ $-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq -2$ $j = t + r$ $1 \leq r \leq (n-1-t)$
$\gamma(u_{-1,j}u_{1,j}) = 2b(t+r-1)^2 + (2a+3b+1)(t+r-1) + b + 2a + \lceil \frac{a}{2} \rceil + 3 - 2h$ $j = t + r$ $1 \leq r \leq (n-1-t)$
$\gamma(u_{i,j}u_{i+1,j}) = i + 2b(t+r-1)^2 + (2a+3b+1)(t+r-1) + 2a + \lceil \frac{a}{2} \rceil + 3 - 2h$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + bj - 1)$ $j = t + r$ $1 \leq r \leq (n-1-t)$
$\gamma(u_{i,n}u_{i+1,n}) = i + 2b(n-1)^2 + (2a+3b+1)(n-1) + 2a + \lceil \frac{a}{2} \rceil + 4 - 2h$ $-(\lceil \frac{a}{2} \rceil + bn - b) \leq i \leq -2$
$\gamma(u_{-1,n}u_{1,n}) = 2b(n-1)^2 + (2a+3b+1)(n-1) + 2a + \lceil \frac{a}{2} \rceil + 3 - 2h$
$\gamma(u_{i,n}u_{i+1,n}) = i + 2b(n-1)^2 + (2a+3b+1)(n-1) + 2a + \lceil \frac{a}{2} \rceil + 3 - 2h$ $1 \leq i \leq (\lceil \frac{a}{2} \rceil + bn - b - 1)$

with $h = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$.
For a even, the edge labels are

edge label
$\gamma(u_{i,j}u_{i,j+1}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$ $\lvert i \rvert \leq (\lceil \frac{a}{2} \rceil + bj)$ $0 \leq j \leq (t-1)$
$\gamma(u_{i,j}u_{i,j+1}) = i + bt^2 + (a+2b+1)t + \lceil \frac{a}{2} \rceil + a + 2 - h$ $\lvert i \rvert \leq (\lceil \frac{a}{2} \rceil + bj)$ $j = t$
$\gamma(u_{i,j}u_{i,j+1}) = i + 2b(t+r-1)^2 + (2a+b+1)(t+r-1) + a + \lceil \frac{a}{2} \rceil + b + 1 - 2h$ $\lvert i \rvert \leq (\lceil \frac{a}{2} \rceil + bj)$ $j = t + r$ $1 \leq r \leq (n-t-1)$
$\gamma(u_{i,j}u_{i+1,j}) = i + (b+1)j + \lceil \frac{a}{2} \rceil + 1$ $-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq (\lceil \frac{a}{2} \rceil + bj - 1)$ $0 \leq j \leq t$
$\gamma(u_{i,j}u_{i+1,j}) = i + 2b(t+r-1)^2 + (2a+3b+1)(t+r-1) + b + 2a + \lceil \frac{a}{2} \rceil + 4 - 2h$ $-(\lceil \frac{a}{2} \rceil + bj) \leq i \leq (\lceil \frac{a}{2} \rceil + bj - 1)$ $j = t + r$ $1 \leq r \leq n - 1 - t$
$\gamma(u_{i,n}u_{i+1,n}) = i + 2b(n-1)^2 + (2a+3b+1)(n-1) + 2a + \lceil \frac{a}{2} \rceil + 4 - 2h$ $-(\lceil \frac{a}{2} \rceil + bn - b) \leq i \leq (\lceil \frac{a}{2} \rceil + bn - b - 1)$.

with $h = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$.

Under the labeling γ, all edges in $GDSC(a,b,n)$ have different weights and the proof is complete. □

In order to make clear the construction of the labeling given in Theorem 2, we give the following examples of total edge irregular labeling on $GDSC(3,2,2)$ and $GDSC(4,2,2)$, as given in Figures 5 and 6. The labels of the vertices are as shown inside the circles, and the labels of the edges are given in blue. For the graphs $GDSC(3,2,2)$ and $GDSC(4,2,2)$, the corresponding values of t are 2 and 1, respectively.

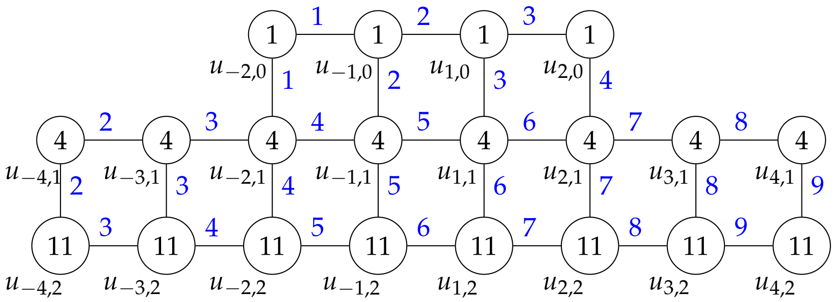

Figure 5. Generalized arithmetic double-staircase graph $GDSC(3,2,2)$ and its total edge irregular 11-labeling.

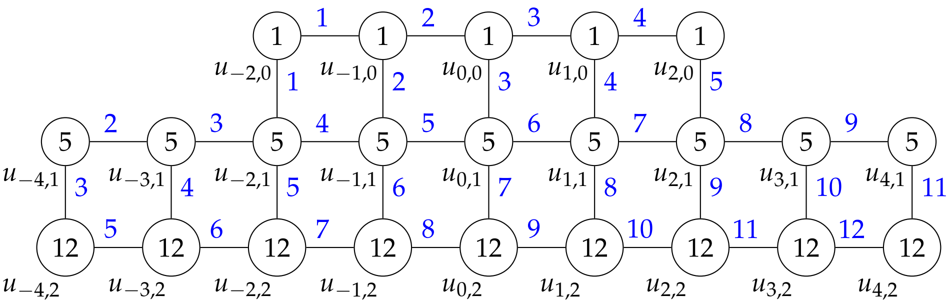

Figure 6. Generalized arithmetic double-staircase graph $GDSC(4,2,2)$ and its total edge irregular 12-labeling.

3. Conclusions

From the previous section, we have that the total edge irregularity strength of the generalized arithmetic staircase graph $GSC(a,b,n)$ for any positive integers $a,b,n \geq 1$ with $(a,b) \neq (1,1),(1,2),(2,2)$ is $teis(GSC(a,b,n)) = \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$. In addition, the total edge irregularity strength of the generalized arithmetic double-staircase graph $GDSC(a,b,n)$ for arbitrary $a,b,n \geq 1$ with $(a,b) \notin \{1,2\} \times \{1,2\}$ is equal to $teis(GDSC(a,b,n)) = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$. From [30,31], we have that $teis(GSC(a,b,n))$ is equal to $\left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$ for $(a,b) = (1,1), (1,2), (2,2)$ and for any positive integer $n \geq 1$ and $teis(GDSC(a,b,n))$ is equal to $\left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$ for any positive numbers a,b with $(a,b) \in \{1,2\} \times \{1,2\}$ and for any positive integer $n \geq 1$. Therefore, we conclude that for all positive integers $a,b,n \geq 1$, it follows that $teis(GSC(a,b,n)) = \left\lceil \frac{bn^2+(2a+1)n+a-b+2}{3} \right\rceil$ and $teis(GDSC(a,b,n)) = \left\lceil \frac{2bn^2+(2a+1)n+a-2b+2}{3} \right\rceil$.

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