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E-Connections on the ε-Anti-Kähler Manifolds

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Abstract: The paper undertakes certain special forms of the quarter symmetric metric and non-metric connections on an ε-anti-Kähler manifold. Firstly, we deduce the relation between the Riemannian connection and the special forms of the quarter symmetric metric and non-metric connections. Then, we present some results concerning the torsion tensors of these connections. In addition, we find the forms of the curvature tensor, the Ricci curvature tensor and scalar curvature of such connections and we search the conditions for the ε-anti-Kähler manifold to be an Einstein space with respect to these connections. Finally, we study $U(Ric)-$vector fields with respect to these connections and give some results related to them.

Keywords: ε-anti-Kähler manifold; quarter-symmetric metric; non-metric E-connections; dual connection

MSC: 53C05; 53C15

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1. Introduction

The notion of quarter-symmetric connection on differentiable manifolds was defined and studied by Golab in [1]. A linear connection is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(\xi_1,\xi_2) = u(\xi_2)\phi\xi_1 - u(\xi_1)\phi\xi_2, \tag{1}$$

where u is a $1-$form, ϕ is a tensor of type $(1,1)$, and ξ_i are vector fields $(i = 1,2)$. In particular, if $\phi = id$, then a quarter-symmetric connection reduces to a semi-symmetric connection [2,3]. Thus, the notion of quarter-symmetric connection generalizes the idea of semi-symmetric connection. Note that a quarter-symmetric metric connection is a Hayden connection with torsion tensor of the form (1) [4]. Studies of various types of quarter-symmetric metric connections and their properties include [5–10] among others. In [11], Mishra and Pandey discussed certain special forms of the quarter symmetric metric connection defined by Golab and studied the conditions for Einstein manifolds, Sasakian manifolds and Kähler manifolds equipped with these connections to be flat, projectively flat or conharmonically flat. Furthermore, some of the latest connected studies can be seen in [12–17]. If a quarter-symmetric connection ∇ on a Riemannian manifold (M,g) satisfies the condition

$$(\nabla_{\xi_1}g)(\xi_2,\xi_3) = 0$$

for all vector fields ξ_1,ξ_2,ξ_3 on M, then ∇ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. In this paper, we consider the special forms of the quarter symmetric metric and non-metric connections on an ε-anti-Kähler manifold. These connections have not been defined so far. All forms of curvature tensors of these connections are calculated and some results concerning with their torsion and curvature properties are presented. Finally, we study $U(Ric)-$vector fields with respect to these connections and obtain some interesting results (Theorems 1 and 3). In the future work, we plan to study certain special forms of the quarter symmetric metric and non-metric connections on an ε-anti-Kähler manifold combine with the results and methods in [18–47] to find more new properties.

2. Preliminaries

Let M_n be an $n = 2m-$dimensional differentiable manifold of class C^∞ covered by any system of coordinate neighbourhoods (x^h), where h runs over the range $1, 2, ..., n$.

An almost ε-complex structure E of type $(1,1)$ satisfies the condition $E^2 = \varepsilon I$, where $\varepsilon = \{-1, 1\}$. If $\varepsilon = -1$, it is an almost complex structure and if $\varepsilon = 1$, it is an almost para-complex structure. In this case, the pair (M_n, E) is called an almost ε-complex manifold. In addition, if $N_E = 0$, that is, almost ε-complex structure E is integrable, then the pair (M_n, E) is called an ε-complex manifold, where N_E is the Nijenhuis tensor:

$$N_E(\xi_1, \xi_2) = [E\xi_1, E\xi_2] - E[E\xi_1, \xi_2] - E[\xi_1, E\xi_2] + \varepsilon[\xi_1, \xi_2]$$

for all vector fields ξ_1, ξ_2 on (M_n, E). An ε-anti-Kähler (or ε-Kähler-Norden) manifold (M_n, g, E, ε) is a manifold which consists of an almost ε-complex structure E and a pseudo Riemannian metric g satisfing the following conditions

$$g(E\xi_1, \xi_2) = g(\xi_1, E\xi_2) \tag{2}$$

and

$$\nabla E = 0, \tag{3}$$

where ∇ is the Levi–Civita connection of g.

Examples of ε-Anti-Kähler Manifolds

(1) The (pseudo)-Euclidean space $\mathbb{R}^{2n}$ given by the (pseudo)-Euclidean metric g and the almost ε-complex structure defined by $g = \begin{pmatrix} \varepsilon\delta_{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix}$ and $E = \begin{pmatrix} 0 & \delta^i_j \\ \varepsilon\delta^i_j & 0 \end{pmatrix}$ with respect to the natural basis of $\mathbb{R}^{2n}$ is an ε-anti-Kähler space.

(2) Let TM be the tangent bundle of a Riemannian manifold (M, g) $(dim M = n)$ and let $\pi : TM \longrightarrow M$ be the projection. A point x of the tangent bundle is represented by an ordered pair (z, u), where $z = \pi(x)$ is a point on M and u is a vector on $T_z M$. We refer to [48] for more details and further references on the geometry of tangent bundles. Let ∇ be the Levi–Civita connection on M, and denote by ξ^v and ξ^h the vertical and the horizontal lift, respectively, to the tangent bundle TM of the vector field ξ on M. Consider the following Sasaki-type metric and the almost ε-complex structure on TM, respectively [48,49]:

$$\begin{aligned}
\widetilde{g}(\xi_1^h, \xi_2^h) &= \varepsilon g(\xi_1, \xi_2), \\
\widetilde{g}(\xi_1^h, \xi_2^v) &= \widetilde{g}(\xi_1^v, \xi_2^h) = 0, \\
\widetilde{g}(\xi_1^v, \xi_2^v) &= g(\xi_1, \xi_2),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{E}(\xi^h) &= \varepsilon\xi^v, \\
\widetilde{E}(\xi^v) &= \xi^h.
\end{aligned}$$

It is easy to see that the Sasaki-type metric $\widetilde{g}$ and the almost ε-complex structure $\widetilde{E}$ are related by the equality $\widetilde{g}(\widetilde{E}\widetilde{\xi}_1, \widetilde{\xi}_2) = g(\widetilde{\xi}_1, \widetilde{E}\widetilde{\xi}_2)$ for all vector fields $\widetilde{\xi}_1$, $\widetilde{\xi}_2$ on TM. The corresponding Levi–Civita connection $\widetilde{\nabla}$ of the tangent bundle with the Sasaki-type metric $\widetilde{g}$ satisfies the followings [49]:

$$\begin{cases} \widetilde{\nabla}_{\xi_1^h}\xi_2^h = (\nabla_{\xi_1}\xi_2)^h - \frac{1}{2}(R(\xi_1,\xi_2)u)^v, \\ \widetilde{\nabla}_{\xi_1^h}\xi_2^v = (\nabla_{\xi_1}\xi_2)^v + \frac{1}{2\varepsilon}(R(u,\xi_2)\xi_1)^h, \\ \widetilde{\nabla}_{\xi_1^v}\xi_2^h = \frac{1}{2\varepsilon}(R(u,\xi_1)\xi_2)^h, \\ \widetilde{\nabla}_{\xi_1^v}\xi_2^v = 0, \end{cases}$$

where $R(\xi_1,\xi_2)\xi_3$ is the Riemannian curvature tensor field on (M,g). Firstly,

$$\begin{aligned} \left(\widetilde{\nabla}_{\xi_1^v}\widetilde{E}\right)\xi_2^h &= 0, \\ \left(\widetilde{\nabla}_{\xi_1^v}\widetilde{E}\xi_2^h\right) - \widetilde{E}\left(\widetilde{\nabla}_{\xi_1^v}\xi_2^h\right) &= 0, \\ \varepsilon\widetilde{\nabla}_{\xi_1^v}\xi_2^v - \frac{1}{2\varepsilon}\widetilde{E}\left((R(u,\xi_2)\xi_1)^h\right) &= 0, \\ \frac{1}{2}R(\xi_2,u)\xi_1 &= 0, \end{aligned}$$

from which we have

$$\left(\widetilde{\nabla}_{\xi_1^v}\widetilde{E}\right)\xi_2^h = 0 \Leftrightarrow R(\xi_2,u)\xi_1 = 0,$$

that is, the base manifold (M,g) is flat. In this case, for all vector fields $\widetilde{\xi}_1$, $\widetilde{\xi}_2$ on TM, the conditions $\left(\widetilde{\nabla}_{\widetilde{\xi}_1}\widetilde{E}\right)\widetilde{\xi}_2 = 0$ are provided. Thus, let (M,g) be a Riemannian manifold and let TM be its tangent bundle with the Sasaki-type metric $\widetilde{g}$ and the almost ε-complex structure $\widetilde{E}$. Then, $(TM,\widetilde{g},\widetilde{E})$ is an ε-anti-Kähler manifold if and only if the base manifold (M,g) is flat.

3. Quarter-Symmetric Metric E-Connection on (M_n,g,E,ε)

A linear connection $\overline{\nabla}$ on an ε-anti-Kähler manifold (M_n,g,E,ε) satisfying the relations

$$\overline{\nabla}g = 0 \quad \text{and} \quad \overline{\nabla}E = 0 \tag{4}$$

is called a metric E-connection. Especially, if its torsion tensor is given by

$$\overline{S}(\xi_1,\xi_2) = p(\xi_2)(E\xi_1) - p(\xi_1)(E\xi_2) + p(E\xi_2)(\xi_1) - p(E\xi_1)(\xi_2), \tag{5}$$

where p is called the generator (or 1-form) of the torsion tensor $\overline{S}$, then the connection that satisfies the Equation (4) is as follows:

$$\overline{\nabla}_{\xi_1}\xi_2 = \nabla_{\xi_1}\xi_2 + p(\xi_2)(E\xi_1) - (U)g(E\xi_1,\xi_2) + p(E\xi_2)(\xi_1) - (EU)g(\xi_1,\xi_2), \tag{6}$$

where U is a vector field such that $g(\xi_1,U) = p(\xi_1)$ [4]. From now on, we will call the connection that provides the Equation (6) a quarter-symmetric metric E-connection on the ε-anti-Kähler manifold $(M_n,g,E,\overline{\nabla},\varepsilon)$.

In [50], the author mentioned an operator applied to pure tensor fields K of type (r,s), which is called the Tachibana operator Φ_E, where E is any tensor field of type $(1,1)$. Purity of a (r,s)-tensor field K with respect to E means that

$$\begin{aligned} K(E\xi_1,\xi_2,...,\xi_s,\overset{1}{\zeta},\overset{2}{\zeta},...,\overset{r}{\zeta}) &= K(\xi_1,E\xi_2,...,\xi_s,\overset{1}{\zeta},\overset{2}{\zeta},...,\overset{r}{\zeta}) \\ &\vdots \\ &= K(\xi_1,\xi_2,...,E\xi_s,\overset{1}{\zeta},\overset{2}{\zeta},...,\overset{r}{\zeta}) \\ &= K(\xi_1,\xi_2,...,\xi_s,\mathcal{E}\overset{1}{\zeta},\overset{2}{\zeta},...,\overset{r}{\zeta}) \\ &= K(\xi_1,\xi_2,...,\xi_s,\overset{1}{\zeta},\mathcal{E}\overset{2}{\zeta},...,\overset{r}{\zeta}) \\ &\vdots \\ &= K(\xi_1,\xi_2,...,\xi_s,\overset{1}{\zeta},\overset{2}{\zeta},...,\mathcal{E}\overset{r}{\zeta}) \end{aligned}$$

for vector fields $\xi_1,\xi_2,...,\xi_s$ and covector fields $\overset{1}{\zeta},\overset{2}{\zeta},...,\overset{r}{\zeta}$ on M_n, where $\mathcal{E}$ is the adjoint operator of E defined by

$$(\mathcal{E}\zeta)(\xi) = \zeta(E\xi) = (\zeta \circ E)(\xi).$$

The Tachibana operator applied to a pure (r,s)-tensor field K is as follows [50]:

$$\begin{aligned} &(\Phi_{E\xi}K)\left(\xi_1,...,\xi_s,\overset{1}{\zeta},...,\overset{r}{\zeta}\right) \\ =\ &(E\xi)K\left(\xi_1,...,\xi_s,\overset{1}{\zeta},...,\overset{r}{\zeta}\right) - \xi K\left(E\xi_1,...,\xi_s,\overset{1}{\zeta},...,\overset{r}{\zeta}\right) \\ &+\sum_{\lambda=1}^{s} K\left(\xi_1,...,(L_{\xi_\lambda}E)\xi,...,\xi_s,\overset{1}{\zeta},...,\overset{r}{\zeta}\right) \\ &-\sum_{\mu=1}^{r} K\left(\xi_1,...,\xi_s,\overset{1}{\zeta},...,L_{E\xi}\overset{\mu}{\zeta} - L_{\xi}\left(\overset{\mu}{\zeta}\circ E\right),...,\overset{r}{\zeta}\right). \end{aligned}$$

For $\varepsilon = -1, 1$, if $\Phi_E K = 0$, then K is called a holomorphic and para-holomorphic tensor field according to the ε-complex structure E, respectively. For the sake of simplicity, we will accept the equality of $\Phi_E p = 0$ in the rest of the article, that is, the following equation always holds:

$$(\overline{\nabla}_{E\xi_1}p)(\xi_2) - (\overline{\nabla}_{\xi_1}p)(E\xi_2) = 0. \tag{7}$$

For pure tensors and tensor operators applied to them, we refer to [50,51].

3.1. Properties of the Torsion Tensor $\overline{S}$

In this section, we will examine the properties of the torsion tensor of the quarter-symmetric metric E-connection.

Proposition 1. *The torsion tensor $\overline{S}$ expressed by (5) is a pure tensor according to E, that is,*

$$\overline{S}(E\xi_1,\xi_2) = \overline{S}(\xi_1,E\xi_2) = E\overline{S}(\xi_1,\xi_2).$$

Proof. From the Equation (5), we immediately get

$$\begin{aligned} \overline{S}(E\xi_1,\xi_2) &= \varepsilon[p(\xi_2)\xi_1 - p(\xi_1)\xi_2] - p(E\xi_1)(E\xi_2) + p(E\xi_2)(E\xi_1) \\ &= \overline{S}(\xi_1,E\xi_2) \\ &= E\overline{S}(\xi_1,\xi_2). \end{aligned}$$

□

It is well known that the torsion tensor of the E-connection is pure if and only if the E-connection is pure [50]. Thus, we can easily say that the Connection (6) is pure according to E, that is,

$$\overline{\nabla}_{E\xi_1}\xi_2 = \overline{\nabla}_{\xi_1}(E\xi_2) = E\overline{\nabla}_{\xi_1}\xi_2. \tag{8}$$

Theorem 1. *For* $[(trE)^2 - \varepsilon(4-n)^2] \neq 0$*, the necessary and sufficient condition for the generator* p *to be closed is that*

$$(\overline{\nabla}_{\xi_1}\overline{S})(\xi_2,\xi_3) + (\overline{\nabla}_{\xi_2}\overline{S})(\xi_3,\xi_1) + (\overline{\nabla}_{\xi_3}\overline{S})(\xi_1,\xi_2) = 0,$$

where $\overline{S}$ *is the torsion tensor of the quarter-symmetric metric E-connection* $\overline{\nabla}$ *on* $(M_n, g, E, \nabla, \varepsilon)$ *and* trE *is the trace of the ε-complex structure E.*

Proof. Firstly, let the generator p be closed, that is,

$$\begin{aligned} 2(dp)(\xi_1,\xi_2) &= \xi_1 p(\xi_2) - \xi_2 p(\xi_1) - p([\xi_1,\xi_2]) \\ &= (\nabla_{\xi_1}p)\xi_2 - (\nabla_{\xi_2}p)\xi_1 \\ &= 0. \end{aligned} \tag{9}$$

In addition, from (5), we obtain

$$\begin{aligned} &(\overline{\nabla}_{\xi_1}\overline{S})(\xi_2,\xi_3) + (\overline{\nabla}_{\xi_2}\overline{S})(\xi_3,\xi_1) + (\overline{\nabla}_{\xi_3}\overline{S})(\xi_1,\xi_2) \\ &= [(\overline{\nabla}_{\xi_3}p)\xi_2 - (\overline{\nabla}_{\xi_2}p)\xi_3](E\xi_1) \\ &+[(\overline{\nabla}_{\xi_1}p)\xi_3 - (\overline{\nabla}_{\xi_3}p)\xi_1](E\xi_2) \\ &+[(\overline{\nabla}_{\xi_2}p)\xi_1 - (\overline{\nabla}_{\xi_1}p)\xi_2](E\xi_3) \\ &+[(\overline{\nabla}_{\xi_3}p)(E\xi_2) - (\overline{\nabla}_{\xi_2}p)(E\xi_3)]\xi_1 \\ &+[(\overline{\nabla}_{\xi_1}p)(E\xi_3) - (\overline{\nabla}_{\xi_3}p)(E\xi_1)]\xi_2 \\ &+[(\overline{\nabla}_{\xi_2}p)(E\xi_1) - (\overline{\nabla}_{\xi_1}p)(E\xi_2)]\xi_3. \end{aligned}$$

It is easy to say that $(\overline{\nabla}_{\xi_1}p)\xi_2 - (\overline{\nabla}_{\xi_2}p)\xi_1 = (\nabla_{\xi_1}p)\xi_2 - (\nabla_{\xi_2}p)\xi_1 = 2(dp)(\xi_1,\xi_2)$. From (9) and (8), the last equation becomes

$$\begin{aligned} &(\overline{\nabla}_{\xi_1}\overline{S})(\xi_2,\xi_3) + (\overline{\nabla}_{\xi_2}\overline{S})(\xi_3,\xi_1) + (\overline{\nabla}_{\xi_3}\overline{S})(\xi_1,\xi_2) \\ &= 2[(dp)(\xi_3,\xi_2)(E\xi_1) + (dp)(\xi_1,\xi_3)(E\xi_2) \\ &+(dp)(\xi_2,\xi_1)(E\xi_3) + (dp)(\xi_3,E\xi_2)(\xi_1) \\ &+(dp)(\xi_1,E\xi_3)(\xi_2) + (dp)(\xi_2,E\xi_1)(\xi_3)] \\ &= 0. \end{aligned}$$

Conversely, let $(\overline{\nabla}_{\xi_1}\overline{S})(\xi_2,\xi_3) + (\overline{\nabla}_{\xi_2}\overline{S})(\xi_3,\xi_1) + (\overline{\nabla}_{\xi_3}\overline{S})(\xi_1,\xi_2) = 0$, from which

$$\begin{aligned} 0 = &[(\nabla_{\xi_3}p)\xi_2 - (\nabla_{\xi_2}p)\xi_3](E\xi_1) \\ &+[(\nabla_{\xi_1}p)\xi_3 - (\nabla_{\xi_3}p)\xi_1](E\xi_2) \\ &+[(\nabla_{\xi_2}p)\xi_1 - (\nabla_{\xi_1}p)\xi_2](E\xi_3) \\ &+[(\nabla_{\xi_3}p)(E\xi_2) - (\nabla_{\xi_2}p)(E\xi_3)]\xi_1 \\ &+[(\nabla_{\xi_1}p)(E\xi_3) - (\nabla_{\xi_3}p)(E\xi_1)]\xi_2 \\ &+[(\nabla_{\xi_2}p)(E\xi_1) - (\nabla_{\xi_1}p)(E\xi_2)]\xi_3 \end{aligned}$$

and

$$
\begin{aligned}
0 \;=\; & [(\nabla_{\xi_3} p)\xi_2 - (\nabla_{\xi_2} p)\xi_3] g(E\xi_1, \xi_4) \\
& + [(\nabla_{\xi_1} p)\xi_3 - (\nabla_{\xi_3} p)\xi_1] g(E\xi_2, \xi_4) \\
& + [(\nabla_{\xi_2} p)\xi_1 - (\nabla_{\xi_1} p)\xi_2] g(E\xi_3, \xi_4) \\
& + [(\nabla_{\xi_3} p)(E\xi_2) - (\nabla_{\xi_2} p)(E\xi_3)] g(\xi_1, \xi_4) \\
& + [(\nabla_{\xi_1} p)(E\xi_3) - (\nabla_{\xi_3} p)(E\xi_1)] g(\xi_2, \xi_4) \\
& + [(\nabla_{\xi_2} p)(E\xi_1) - (\nabla_{\xi_1} p)(E\xi_2)] g(\xi_3, \xi_4).
\end{aligned}
$$

If we contract the last equation with respect to ξ_1 and ξ_4, from $(\overline{\nabla}_{E\xi_1} p)(\xi_2) - (\overline{\nabla}_{\xi_1} p)(E\xi_2) = 0$, we get

$$
(4-n)[(\nabla_{\xi_3} p)(E\xi_2) - (\nabla_{E\xi_2} p)\xi_3] + (trE)[(\nabla_{\xi_3} p)\xi_2 - (\nabla_{\xi_2} p)\xi_3] = 0. \tag{10}
$$

In the Equation (10), if we put $\xi_2 = E\xi_2$, then

$$
\varepsilon(4-n)[(\nabla_{\xi_3} p)\xi_2 - (\nabla_{\xi_2} p)\xi_3] + (trE)[(\nabla_{\xi_3} p)(E\xi_2) - (\nabla_{E\xi_2} p)\xi_3] = 0. \tag{11}
$$

Finally, from (10) and (11), we obtain

$$
[(trE)^2 - \varepsilon(4-n)^2][(\nabla_{\xi_3} p)\xi_2 - (\nabla_{\xi_2} p)\xi_3] = 0
$$

and

$$
[(trE)^2 - \varepsilon(4-n)^2](dp)(\xi_1, \xi_2) = 0,
$$

which completes the proof. □

Proposition 2. *On an ε-anti-Kähler manifold* $(M_n, g, E, \overline{\nabla}, \varepsilon)$*, the torsion tensor* $\overline{S}$ *of the quarter-symmetric metric E-connectionsatisfies the following condition:*

$$
\overline{S}(\overline{S}(\xi_1, \xi_2), \xi_3) + \overline{S}(\overline{S}(\xi_3, \xi_1), \xi_2) + \overline{S}(\overline{S}(\xi_2, \xi_3), \xi_1) = 0. \tag{12}
$$

Proof. If the Equation (5) is substituted in the Equation (12), the result is directly obtained. □

It is well known that the curvature tensor of any connection $\overline{\nabla}$ has the form:

$$
\overline{R}(\xi_1, \xi_2, \xi_3) = \overline{\nabla}_{\xi_1} \overline{\nabla}_{\xi_2} \xi_3 - \overline{\nabla}_{\xi_2} \overline{\nabla}_{\xi_1} \xi_3 - \overline{\nabla}_{[\xi_1, \xi_2]} \xi_3.
$$

Especially, if the connection $\overline{\nabla}$ is the quarter-symmetric metric E-connection given by (6), then the curvature tensor of this connection becomes

$$
\begin{aligned}
\overline{R}(\xi_1, \xi_2, \xi_3, \xi_4) \;=\; & R(\xi_1, \xi_2, \xi_3, \xi_4) \\
& + \sigma(\xi_1, \xi_3) g(E\xi_2, \xi_4) - \sigma(\xi_2, \xi_3) g(E\xi_1, \xi_4) \\
& + \sigma(\xi_2, \xi_4) g(E\xi_1, \xi_3) - \sigma(\xi_1, \xi_4) g(E\xi_2, \xi_3) \\
& + \sigma(\xi_1, E\xi_3) g(\xi_2, \xi_4) - \sigma(\xi_2, E\xi_3) g(\xi_1, \xi_4) \\
& + \sigma(\xi_2, E\xi_4) g(\xi_1, \xi_3) - \sigma(\xi_1, E\xi_4) g(\xi_2, \xi_3),
\end{aligned} \tag{13}
$$

where $\overline{R}(\xi_1, \xi_2, \xi_3, \xi_4) = g(\overline{R}(\xi_1, \xi_2, \xi_3), \xi_4)$ and R is the Riemannian curvature tensor of the Levi–Civita connection ∇ and

$$
\begin{aligned}
\sigma(\xi_1, \xi_2) \;=\; & (\nabla_{\xi_1} p)\xi_2 - p(\xi_1)p(E\xi_2) + \frac{1}{2}p(U)g(E\xi_1, \xi_2) \\
& - p(\xi_2)p(E\xi_1) + \frac{1}{2}p(EU)g(\xi_1, \xi_2).
\end{aligned} \tag{14}
$$

It is easy to see that

$$\sigma(\xi_1,\xi_2)-\sigma(\xi_2,\xi_1)=(\nabla_{\xi_1}p)\xi_2-(\nabla_{\xi_2}p)\xi_1=2(dp)(\xi_1,\xi_2)$$

and

$$\sigma(\xi_1,\xi_2)-\sigma(\xi_2,\xi_1)=0 \Longleftrightarrow (dp)(\xi_1,\xi_2)=0.$$

In addition, from (7), we have

$$\sigma(E\xi_1,\xi_2)-\sigma(\xi_2,E\xi_1)=0,$$

that is, the tensor σ is pure according to E. We give the following proposition without proof. Because standard calculations give it. We omit them.

Proposition 3. *The curvature $(0,4)-$tensor $\overline{R}$ of the quarter-symmetric metric E-connection $\overline{\nabla}$ satisfies the following equations:*
(i) $\overline{R}(\xi_1,\xi_2,\xi_3,\xi_4)=-\overline{R}(\xi_2,\xi_1,\xi_3,\xi_4)$,
(ii) $\overline{R}(\xi_1,\xi_2,\xi_3,\xi_4)=-\overline{R}(\xi_1,\xi_2,\xi_4,\xi_3)$,
(iii) $\overline{R}(\xi_1,\xi_2,\xi_3,\xi_4)=\overline{R}(\xi_3,\xi_4,\xi_1,\xi_2) \Longleftrightarrow (dp)(\xi_1,\xi_2)=0$,
(iv) $\overline{R}(\xi_1,\xi_2,\xi_3,\xi_4)+\overline{R}(\xi_3,\xi_1,\xi_2,\xi_4)+\overline{R}(\xi_2,\xi_3,\xi_1,\xi_4)=0 \Longleftrightarrow (dp)(\xi_1,\xi_2)=0.$

Denote by $\overline{R}ic(\xi_1,\xi_2)$ the Ricci tensor of the quarter-symmetric metric E-connection and $Ric(\xi_1,\xi_2)$ the Ricci tensor of the Levi–Civita connection. Then, the Ricci tensor $\overline{R}ic(\xi_1,\xi_2)$ is

$$\begin{aligned}\sum_{i=1}^{n}\overline{R}(e_i,\xi_1,\xi_2,e_i) &= \overline{R}ic(\xi_1,\xi_2) \\ &= Ric(\xi_1,\xi_2)+(4-n)\sigma(\xi_1,E\xi_2)-g(\xi_1,\xi_2)(tr\theta) \\ &\quad -\sigma(\xi_1,\xi_2)(trE)-g(E\xi_1,\xi_2)(tr\sigma),\end{aligned} \tag{15}$$

where $\{e_i\}$, $i=1,...,n$ are orthonormal vector fields on $(M_n,g,E,\overline{\nabla},\varepsilon)$ and

$$\begin{aligned}tr\sigma &= \sum_{i=1}^{n}\sigma(e_i,e_i) \\ &= \sum_{i=1}^{n}(\nabla_{e_i}p)(e_i)+\frac{(n-4)}{2}p(EU)+\frac{(trE)}{2}p(U)\end{aligned}$$

and

$$\begin{aligned}tr\theta &= \sum_{i=1}^{n}(\sigma\circ E)(e_i,e_i)=\sum_{i=1}^{n}\sigma(e_i,Ee_i) \\ &= \sum_{i=1}^{n}(\nabla_{e_i}p)(Ee_i)+\frac{\varepsilon(n-4)}{2}p(U)-\frac{(trE)}{2}p(EU).\end{aligned}$$

It is easy to see that if $(dp)(\xi_1,\xi_2)=0$, then $\overline{R}ic(\xi_1,\xi_2)=\overline{R}ic(\xi_2,\xi_1)$. In fact,

$$\begin{aligned}\overline{R}ic(\xi_1,\xi_2)-\overline{R}ic(\xi_2,\xi_1) &= (4-n)[\sigma(\xi_1,E\xi_2)-\sigma(E\xi_2,\xi_1)] \\ &\quad -(trE)[\sigma(\xi_1,\xi_2)-\sigma(\xi_2,\xi_1)] \\ &= 2(4-n)(dp)(\xi_1,E\xi_2)-2(trE)(dp)(\xi_1,\xi_2).\end{aligned}$$

Moreover, the scalar curvature $\overline{\tau}$ of the quarter-symmetric metric E-connection is given by

$$\begin{aligned}\overline{\tau} &= \sum_{i=1}^{n}\overline{R}ic(e_i,e_i) \\ &= Ric(e_i,e_i)+2(2-n)(tr\theta)-2(tr\sigma)(trE) \\ &= \tau+2(2-n)(tr\theta)-2(tr\sigma)(trE),\end{aligned} \tag{16}$$

where τ is the Riemannian scalar curvature of the Levi–Civita connection of ∇.

An Einstein manifold is a Riemannian manifold that satisfies the condition $Ric(\xi_1, \xi_2) = \lambda g(\xi_1, \xi_2)$, where λ is a scalar function. In addition, for the Ricci tensor $\overline{Ric}(\xi_1, \xi_2)$ of the manifold $(M_n, g, E, \overline{\nabla}, \varepsilon)$, the Einstein case becomes

$$\underset{(\xi_1,\xi_2)}{sym}\ \overline{Ric}(\xi_1, \xi_2) = \frac{1}{2}[\overline{Ric}(\xi_1, \xi_2) + \overline{Ric}(\xi_2, \xi_1)] = \mu g(\xi_1, \xi_2), \tag{17}$$

where μ is a scalar function and the manifold $(M_n, g, E, \overline{\nabla}, \varepsilon, \overline{Ric})$ is called an ε-Einstein–anti-Kähler manifold.

Proposition 4. *Let $(M_n, g, E, \overline{\nabla}, \varepsilon, \overline{Ric})$ be an ε-Einstein–anti-Kähler manifold. Then, the following equation holds:*

$$\mu - \lambda = a_1 \sum_{i=1}^{n} (\nabla_{e_i} p)(Ee_i) + a_2 \sum_{i=1}^{n} (\nabla_{e_i} p)(e_i) + a_3 p(U) + a_4 p(EU),$$

where

$$\begin{aligned} a_1 &= \frac{2(2-n)}{n},\ a_2 = -\frac{2(trE)}{n}, \\ a_3 &= \left(\frac{\varepsilon(2-n)(n-4) - (trE)^2}{n}\right),\ a_4 = \frac{2(trE)}{n}. \end{aligned}$$

Proof. From (15), (17) and $Ric(\xi_1, \xi_2) = \lambda g(\xi_1, \xi_2)$, we get

$$\overline{\tau} = \mu n \quad and \quad \tau = \lambda n.$$

In addition, from the last equation and (16), we obtain

$$\overline{\tau} = \tau + 2(2-n)(tr\theta) - 2(trE)(tr\sigma)$$

and

$$\mu n = \lambda n + 2(2-n)(tr\theta) - 2(trE)(tr\sigma).$$

Finally, substituting $tr\theta$ and $tr\sigma$ in the last equation, we reach the end of the proof. □

3.2. Transposed Connection of the Quarter-Symmetric Metric E-Connection

The transposed connection ${}^t\overline{\nabla}$ of the quarter-symmetric metric E-connection $\overline{\nabla}$ is defined by

$${}^t\overline{\nabla}_{\xi_1}\xi_2 = \overline{\nabla}_{\xi_1}\xi_2 + [\xi_1, \xi_2],$$

for all vector fields ξ_1 and ξ_2 on $(M_n, g, E, \overline{\nabla}, \varepsilon)$ [6,52]. Substituting ${}^t\overline{S}(\xi_1, \xi_2) = {}^t\overline{\nabla}_{\xi_1}\xi_2 - {}^t\overline{\nabla}_{\xi_2}\xi_1 - [\xi_1, \xi_2]$ in the last equation, we get

$${}^t\overline{\nabla}_{\xi_1}\xi_2 = \overline{\nabla}_{\xi_1}\xi_2 - \overline{S}(\xi_1, \xi_2) \tag{18}$$

and

$${}^t\overline{S}(\xi_1, \xi_2) = -\overline{S}(\xi_1, \xi_2).$$

From the Equation (6) and (18), the transposed connection ${}^t\overline{\nabla}$ is as follows:

$${}^t\overline{\nabla}_{\xi_1}\xi_2 = \nabla_{\xi_1}\xi_2 + p(\xi_1)(E\xi_2) - Ug(E\xi_1, \xi_2) + p(E\xi_1)(\xi_2) - (EU)g(\xi_1, \xi_2). \tag{19}$$

It is clear that ${}^t\overline{\nabla}g \neq 0$ and ${}^t\overline{\nabla}E = 0$, that is, the transposed connection ${}^t\overline{\nabla}$ is a non-metric E-connection.

The curvature tensor ${}^t\overline{R}$ of the transposed connection ${}^t\overline{\nabla}$ is as follows:

$$ {}^t\overline{R}(\xi_1,\xi_2,\xi_3) = \overline{R}(\xi_1,\xi_2,\xi_3) - (\nabla_{\xi_1}\overline{S})(\xi_2,\xi_3) + (\nabla_{\xi_2}\overline{S})(\xi_1,\xi_3) - [\overline{S}(\overline{S}(\xi_1,\xi_2),\xi_3) + \overline{S}(\overline{S}(\xi_3,\xi_1),\xi_2) + \overline{S}(\overline{S}(\xi_2,\xi_3),\xi_1)]. $$

From Proposition 2, we obtain

$$ {}^t\overline{R}(\xi_1,\xi_2,\xi_3) = \overline{R}(\xi_1,\xi_2,\xi_3) - (\nabla_{\xi_1}\overline{S})(\xi_2,\xi_3) + (\nabla_{\xi_2}\overline{S})(\xi_1,\xi_3). $$

3.3. $U(\overline{Ric})$-Vector Fields on $(M_n, g, E, \overline{\nabla}, \varepsilon)$

Let φ be a vector field on a Riemannian manifold (M_n, g), which is locally expressed by $\varphi = \varphi^m \frac{\partial}{\partial x^m}$. The vector field φ is a $\varphi(Ric)$-vector field on a Riemannian manifold (M_n, g) such that [53]

$$ \nabla(\overline{\varphi}) = \gamma Ric, \tag{20} $$

where γ is a non-zero scalar function, ∇ is the Levi–Civita connection of g, Ric is the Ricci tensor of (M_n, g) and $g(\varphi,\xi) = \overline{\varphi}(\xi)$. The Equation (20) is locally expressed by

$$ \nabla_j \overline{\varphi}_i = \gamma R_{ij}, $$

where $\overline{\varphi}_i = g_{im}\varphi^m$ and R_{ij} is the Ricci tensor. In the special case, using the generator p we can define a $\varphi(Ric)-$vector field by

$$ (\nabla_{\xi_1} p)\xi_2 = \gamma Ric(\xi_1,\xi_2), \tag{21} $$

where γ is a non-zero scalar function, ∇ is the Levi–Civita connection of g and $g(U,\xi) = p(\xi)$. For this reason, we will call the vector field U a $U(Ric)-$vector field. The Equation (21) implies that $(dp)(\xi_1,\xi_2) = 0$. In fact,

$$ \begin{aligned} 0 &= Ric(\xi_1,\xi_2) - Ric(\xi_2,\xi_1) \\ &= \frac{1}{\gamma}[(\nabla_{\xi_1} p)\xi_2 - (\nabla_{\xi_2} p)\xi_1] \\ &= \frac{1}{\gamma}[\sigma(\xi_1,\xi_2) - \sigma(\xi_2,\xi_1)] \\ &= \frac{2}{\gamma}(dp)(\xi_1,\xi_2). \end{aligned} \tag{22} $$

Proposition 5. *Let $(M_n, g, E, \overline{\nabla}, \varepsilon)$ be an ε-anti-Kähler manifold admitting a $U(Ric)$-vector field. Then, Theorem 1 and Proposition 3 are directly provided.*

On the ε-anti-Kähler manifold $(M_n, g, E, \overline{\nabla}, \varepsilon)$, the $U(\overline{Ric})-$vector field with respect to the quarter-symmetric metric E-connection $\overline{\nabla}$ is defined as follows:

$$ (\overline{\nabla}_{\xi_1} p)\xi_2 = \gamma \overline{Ric}(\xi_1,\xi_2), \tag{23} $$

where γ is a non-zero scalar function on $(M_n, g, E, \overline{\nabla}, \varepsilon)$ and $\overline{Ric}$ is the Ricci tensor of $\overline{\nabla}$. Actually, if $(\nabla_{\xi_1} p)\xi_2 = \gamma Ric(\xi_1,\xi_2)$, then from the Equation (23), we can say that $\overline{Ric}(\xi_1,\xi_2) - \overline{Ric}(\xi_2,\xi_1) = 0$, that is, the Ricci tensor $\overline{Ric}(\xi_1,\xi_2)$ of the quarter-symmetric metric E-connection is symmetric.

Theorem 2. *An anti-Kähler manifold $(M_n, g, E, \overline{\nabla}, \varepsilon = -1)$ with a $U(\overline{Ric})$-vector field of constant length has constant scalar curvature, that is, $\nabla\tau = 0$.*

Proof. From (23), we obtain

$$ \begin{aligned} &(\nabla_{\xi_1} p)\xi_2 - p(\xi_2)p(E\xi_1) - p(E\xi_2)p(\xi_1) + p(U)g(E\xi_1,\xi_2) + p(EU)g(\xi_1,\xi_2) \\ =\ &\gamma[Ric(\xi_1,\xi_2) + (4-n)\sigma(\xi_1,E\xi_2) - g(\xi_1,\xi_2)(tr\theta) - g(E\xi_1,\xi_2)(tr\sigma)] \end{aligned} $$

and
$$-\frac{1}{2\gamma}p(U) = tr\sigma.$$
Substituting $tr\sigma$ in the last equation and from $(\nabla_{\xi_1}p)\xi_2 = \gamma Ric(\xi_1,\xi_2)$, we have
$$\begin{aligned}
-\frac{1}{2\gamma}p(U) &= \sum_{i=1}^{n}(\nabla_{e_i}p)(e_i) + \frac{(n-4)}{2}p(EU) \\
&= \gamma\sum_{i=1}^{n}Ric(e_i,e_i) + \frac{(n-4)}{2}p(EU) \\
&= \gamma\tau + \frac{(n-4)}{2}p(EU)
\end{aligned}$$
and
$$\tau = \frac{(n-4)}{2\gamma}p(EU) - \frac{1}{2\gamma^2}p(U). \tag{24}$$
If we consider that a $U(\overline{R}ic)$-vector field has constant length, we can write
$$||U|| = p(U) = c, \ (c = constant)$$
and from the covariant derivative according to the Levi–Civita connection ∇ of the last equation, we get
$$\nabla_{\xi_1}[p(U)] = (\nabla_{\xi_1}p)U + p(\nabla_{\xi_1}U) = 0. \tag{25}$$
It is clear that $g(\xi_2, U) = p(\xi_2)$ and
$$\begin{aligned}
g(\nabla_{\xi_1}\xi_2, U) + g(\xi_2, \nabla_{\xi_1}U) &= (\nabla_{\xi_1}p)\xi_2 + p(\nabla_{\xi_1}\xi_2) \\
&= (\nabla_{\xi_1}p)\xi_2 + g(\nabla_{\xi_1}\xi_2, U)
\end{aligned}$$
$$g(\xi_2, \nabla_{\xi_1}U) = (\nabla_{\xi_1}p)\xi_2.$$
From the last equation and (25), we obtain
$$(\nabla_{\xi_1}p)U + (\nabla_{\xi_1}p)U = 0$$
and
$$\begin{aligned}
(\nabla_{\xi_1}p)U &= \gamma Ric(\xi_1, U) \\
&= 0.
\end{aligned}$$
In addition, for $p(EU)$ and the last equation, we find
$$\begin{aligned}
\nabla_{\xi_1}[p(EU)] &= (\nabla_{\xi_1}p)(EU) + p(\nabla_{\xi_1}(EU)) \\
&= 2(\nabla_{\xi_1}p)(EU) \\
&= 2\gamma Ric(\xi_1, EU) \\
&= 2\gamma Ric(E\xi_1, U) \\
&= 0.
\end{aligned} \tag{26}$$
Finally, by using (25), (26) and (24), we easily see that $\nabla\tau = 0$. This completes the proof. □

4. Quarter-Symmetric Non-Metric E-Connection on (M_n, g, E, ε)

We construct a linear connection $\widetilde{\nabla}$ on the ε-anti-Kähler manifold (M_n, g, E, ε) whose torsion tensor is in the form (5):

$$\begin{aligned} \widetilde{\nabla}_{\xi_1}\xi_2 &= \nabla_{\xi_1}\xi_2 + (1-\alpha)[p(\xi_2)(E\xi_1) + p(E\xi_2)(\xi_1)] \\ &\quad -\alpha[p(\xi_1)(E\xi_2) + p(E\xi_1)(\xi_2)], \end{aligned} \tag{27}$$

where $\alpha \in \mathbb{R}$. It is provided by standard calculations:

$$\widetilde{\nabla} g \neq 0 \quad \text{and} \quad \widetilde{\nabla} E = 0,$$

that is, the Connection (27) is a quarter-symmetric non-metric E-connection on $(M_n, g, E, \widetilde{\nabla}, \varepsilon)$. The curvature $(0,4)-$tensor of the quarter-symmetric non-metric E-connection can be written in the form:

$$\begin{aligned} \widetilde{R}(\xi_1,\xi_2,\xi_3,\xi_4) &= R(\xi_1,\xi_2,\xi_3,\xi_4) \\ &\quad +\phi(\xi_1,\xi_3)g(E\xi_2,\xi_4) - \phi(\xi_2,\xi_3)g(E\xi_1,\xi_4) \\ &\quad +\phi(\xi_1,E\xi_3)g(\xi_2,\xi_4) - \phi(\xi_2,E\xi_3)g(\xi_1,\xi_4) \\ &\quad +\frac{\alpha}{\alpha-1}[g(\xi_3,\xi_4)(\phi(\xi_1,E\xi_2) - \phi(E\xi_2,\xi_1)) \\ &\quad +g(E\xi_3,\xi_4)(\phi(\xi_1,\xi_2) - \phi(\xi_2,\xi_1))], \end{aligned}$$

where $\alpha \neq 1$ and

$$\phi(\xi_1,\xi_2) = (1-\alpha)(\nabla_{\xi_1}p)\xi_2 - (1-\alpha)^2[p(\xi_1)(E\xi_2) + p(\xi_2)(E\xi_1)].$$

In the last equation, we can say that $\phi(\xi_1,\xi_2) - \phi(\xi_2,\xi_1) = 0 \iff (dp)(\xi_1,\xi_2) = 0$, actually

$$\phi(\xi_1,\xi_2) - \phi(\xi_2,\xi_1) = (1-\alpha)[(\nabla_{\xi_1}p)\xi_2 - (\nabla_{\xi_1}p)\xi_2].$$

In addition, from (7), we can write

$$\phi(E\xi_1,\xi_2) - \phi(\xi_2,E\xi_1) = 0.$$

Thus, the tensor ϕ is pure with respect to E.

It is clear that the curvature $(0,4)-$tensor $\widetilde{R}$ satisfies the following properties:

(i) $\widetilde{R}(\xi_1,\xi_2,\xi_3,\xi_4) = -\widetilde{R}(\xi_2,\xi_1,\xi_3,\xi_4)$,

(ii) $(dp)(X,Y) = 0 \Rightarrow \widetilde{R}(\xi_1,\xi_2,\xi_3,\xi_4) + \widetilde{R}(\xi_3,\xi_1,\xi_2,\xi_4) + \widetilde{R}(\xi_2,\xi_3,\xi_1,\xi_4) = 0.$

The Ricci tensor of the quarter-symmetric non-metric E-connection has the form:

$$\begin{aligned} \widetilde{Ric}(\xi_1,\xi_2) &= Ric(\xi_1,\xi_2) + (2-n)\phi(\xi_1,E\xi_2) - (trE)\phi(\xi_1,\xi_2) \\ &\quad +\frac{2\alpha}{\alpha-1}[\phi(E\xi_2,\xi_1) - \phi(\xi_1,E\xi_2)], \end{aligned} \tag{28}$$

where $Ric(\xi_1, Y)$ is the Ricci tensor of g and $\alpha \neq 1$. It follows that

$$\begin{aligned} \widetilde{Ric}(\xi_1,\xi_2) - \widetilde{Ric}(\xi_2,\xi_1) &= \left(\frac{2(1+\alpha)}{1-\alpha} - n\right)[\phi(\xi_1,E\xi_2) - \phi(E\xi_2,\xi_1)] \\ &\quad -(trE)[\phi(\xi_1,\xi_2) - \phi(\xi_2,\xi_1)]. \end{aligned}$$

If $(dp)(\xi_1,\xi_2) = 0$, then $\widetilde{Ric}(\xi_1,\xi_2) = \widetilde{Ric}(\xi_2,\xi_1)$. Hence, the scalar curvature $\widetilde{\tau}$ of the manifold $(M_n, g, E, \widetilde{\nabla}, \varepsilon)$ becomes

$$\widetilde{\tau} = \tau + (2-n)(tr\psi) - (trE)(tr\phi), \tag{29}$$

where

$$\begin{aligned} tr\phi &= \sum_{i=1}^{n}\phi(e_i,e_i) \\ &= (1-\alpha)\sum_{i=1}^{n}(\nabla_{e_i}p)(e_i) - 2(1-\alpha)^2 p(EU) \end{aligned} \tag{30}$$

and

$$\begin{aligned} tr\psi &= \sum_{i=1}^{n}(\psi\circ E)(e_i,e_i) = \sum_{i=1}^{n}\sigma(e_i,Ee_i) \\ &= (1-\alpha)\sum_{i=1}^{n}(\nabla_{e_i}p)(Ee_i) - 2\varepsilon(1-\alpha)^2 p(U). \end{aligned} \tag{31}$$

The condition for the ε-anti-Kähler manifold $(M_n, g, E, \widetilde{\nabla}, \varepsilon)$ to be Einstein according to the quarter-symmetric non-metric E-connection $\widetilde{\nabla}$ is given by

$$\underset{(\xi_1,\xi_2)}{sym}\ \widetilde{R}ic(\xi_1,\xi_2) = \frac{1}{2}[\widetilde{R}ic(\xi_1,\xi_2) + \widetilde{R}ic(\xi_2,\xi_1)] = \mu g(\xi_1,\xi_2), \tag{32}$$

where μ is a scalar function. In this case, the manifold $(M_n, g, E, \widetilde{\nabla}, \varepsilon, \widetilde{R}ic)$ is called an ε-Einstein–anti-Kähler manifold according to the quarter-symmetric non-metric E-connection $\widetilde{\nabla}$. Then, we get the following proposition.

Proposition 6. *Let $(M_n, g, E, \widetilde{\nabla}, \varepsilon, \widetilde{R}ic)$ be an ε-Einstein–anti-Kähler manifold. Then, the following equation holds:*

$$\mu - \lambda = b_1\sum_{i=1}^{n}(\nabla_{e_i}p)(Ee_i) + b_2 p(U) + b_3,$$

where

$$b_1 = \frac{(n-2)(\alpha-1)}{n},\ b_2 = \frac{2\varepsilon(n-2)(1-\alpha)^2}{n},\ b_3 = -\frac{(trE)(tr\phi)}{n}.$$

Proof. From (28), (32) and $Ric(\xi_1,\xi_2) = \lambda g(\xi_1,\xi_2)$, we have

$$\widetilde{\tau} = \xi n \quad and \quad \tau = \lambda n.$$

In addition, from the last equation and (29), we get

$$\widetilde{\tau} = \tau + (2-n)(tr\psi) - (trE)(tr\phi)$$

and

$$\xi n = \alpha n + (2-n)(tr\psi) - (trE)(tr\phi).$$

Finally, substituting $tr\psi$ and $tr\phi$ in the last equation, the proof is completed. □

4.1. $U(\widetilde{R}ic)$-Vector Fields on $(M_n, g, E, \widetilde{\nabla}, \varepsilon)$

Let (M_n, g, E, ε) be an ε-anti-Kähler manifold with the quarter-symmetric non-metric E-connection $\widetilde{\nabla}$ given by (27). Using the generator p and the quarter-symmetric non-metric E-connection $\widetilde{\nabla}$, we can define a $U(\widetilde{R}ic)$-vector field as follows:

$$(\widetilde{\nabla}_{\xi_1}p)\xi_2 = \eta\widetilde{R}ic(\xi_1,\xi_2),$$

where η is a non-zero scalar function on $(M_n, g, E, \widetilde{\nabla}, \varepsilon)$, $\widetilde{R}ic$ is the Ricci tensor of $\widetilde{\nabla}$ and $g(U,\xi) = p(\xi)$. If $(\nabla_{\xi_1}p)\xi_2 = \eta Ric(\xi_1,\xi_2)$, we can easily see that $\widetilde{R}ic(\xi_1,\xi_2) - \widetilde{R}ic(\xi_2,\xi_1) = 0$.

Theorem 3. *An anti-Kähler manifold* $(M_n, g, E, \widetilde{\nabla}, \varepsilon = -1)$ $(n \geq 3)$ *with a* $U(\widetilde{Ric})$*-vector field of constant length has constant scalar curvature, that is,* $\nabla \tau = 0$.

Proof. As in Theorem 2, we get

$$\tau = \frac{(2-4\alpha)}{\eta^2(1-\alpha)(n-2)} p(U) + \frac{2(1-\alpha)}{\eta} p(EU).$$

If we assume that a $U(\widetilde{Ric})-$vector field has constant length, then $\nabla_{\xi_1}[p(U)] = 0$ and $\nabla_{\xi_1}[p(EU)] = 0$. Thus, we have $\nabla \tau = 0$. □

4.2. Dual Connection of the Quarter-Symmetric Non-Metric E-Connection $\widetilde{\nabla}$

The dual connection ${}^*\widetilde{\nabla}$ of any linear connection $\widetilde{\nabla}$ is given by

$$\xi_1 g(\xi_2, \xi_3) = g(\widetilde{\nabla}_{\xi_1}\xi_2, \xi_3) + g(\xi_2, {}^*\widetilde{\nabla}_{\xi_1}\xi_3) \tag{33}$$

for all vector fields ξ_1, ξ_2 and ξ_3 on M_n [54,55]. Using (33), we find the dual connection ${}^*\widetilde{\nabla}$ of the quarter-symmetric non-metric E-connection on $(M_n, g, E, \widetilde{\nabla}, \varepsilon)$ as follows:

$$\begin{aligned} {}^*\widetilde{\nabla}_{\xi_1}\xi_2 \;=\; & \nabla_{\xi_1}\xi_2 + (\alpha-1)[Ug(E\xi_1, \xi_2) + (EU)g(E\xi_1, \xi_2)] \\ & + \alpha[p(\xi_1)(E\xi_2) + p(E\xi_1)(\xi_2)]. \end{aligned} \tag{34}$$

From (27) and (34), we can write ${}^*\widetilde{S}(\xi_1, \xi_2) = -\alpha\overline{S}(\xi_1, \xi_2)$, where ${}^*\widetilde{S}$ is the torsion tensor of the connection ${}^*\widetilde{\nabla}$. In addition, the connection ${}^*\widetilde{\nabla}$ satisfies the condition ${}^*\widetilde{\nabla} g \neq 0$ and ${}^*\widetilde{\nabla} E = 0$, the connection ${}^*\widetilde{\nabla}$ is a quarter-symmetric non-metric dual E-connection.

In [55], the author has shown that there is a relationship of the form ${}^*\widetilde{R}(\xi_1, \xi_2, \xi_3, \xi_4)$ $= -\widetilde{R}(\xi_1, \xi_2, \xi_4, \xi_3)$ between the curvature tensors of any connection $\widetilde{\nabla}$ and its dual ${}^*\widetilde{\nabla}$. Then, the curvature $(0,4)$-tensor of the dual Connection (34) has the form:

$$\begin{aligned} {}^*\widetilde{R}(\xi_1, \xi_2, \xi_3, \xi_4) \;=\; & R(\xi_1, \xi_2, \xi_3, \xi_4) \\ & + \phi(\xi_2, \xi_4) g(E\xi_1, \xi_3) - \phi(\xi_1, \xi_4) g(E\xi_2, \xi_3) \\ & + \phi(\xi_2, E\xi_4) g(\xi_1, \xi_3) - \phi(\xi_1, E\xi_4) g(\xi_2, \xi_3) \\ & - \frac{\alpha}{\alpha-1}[g(\xi_3, \xi_4)(\phi(\xi_1, E\xi_2) - \phi(E\xi_2, \xi_1)) \\ & + g(E\xi_3, \xi_4)(\phi(\xi_1, \xi_2) - \phi(\xi_2, \xi_1))], \end{aligned}$$

where $\alpha \neq 1$. The Ricci tensor ${}^*\widetilde{Ric}$ and the scalar curvature ${}^*\widetilde{\tau}$ of the dual connection ${}^*\widetilde{\nabla}$ are

$$\begin{aligned} {}^*\widetilde{Ric}(\xi_1, \xi_2) \;=\; & Ric(\xi_1, \xi_2) + 2\phi(\xi_1, E\xi_2) - g(\xi_1, \xi_2)(tr\psi) \\ & - g(E\xi_1, \xi_2)(tr\phi) - \frac{2\alpha}{1-\alpha}[\phi(\xi_1, E\xi_2) - \phi(E\xi_2, \xi_1)] \end{aligned} \tag{35}$$

and

$${}^*\widetilde{\tau} = \tau + (2-n)(tr\psi) - (trE)(tr\phi), \tag{36}$$

respectively. From (35), we obtain

$${}^*\widetilde{Ric}(\xi_1, \xi_2) - {}^*\widetilde{Ric}(\xi_2, \xi_1) = 4(1-3\alpha)(dp)(\xi_1, E\xi_2).$$

Then, we can write the following last proposition.

Proposition 7. *(i)* ${}^*\widetilde{Ric}(\xi_1, \xi_2) - {}^*\widetilde{Ric}(\xi_2, \xi_1) = 0 \Leftrightarrow (dp)(\xi_1, E\xi_2) = 0$

(ii) From (29) and (36), we easily say that the scalar curvature ${}^*\widetilde{\tau}$ *of the connection* ${}^*\widetilde{\nabla}$ *coincides with the scalar curvature* $\widetilde{\tau}$ *of the connection* $\widetilde{\nabla}$*, that is,* ${}^*\widetilde{\tau} = \widetilde{\tau}$.

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