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Classical and Quantum Cosmological Solutions in Teleparallel Dark Energy with Anisotropic Background Geometry

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Abstract: We investigate exact and analytic solutions for the field equations in the teleparallel dark energy model, where the physical space is described by the locally rotational symmetric Bianchi I, Bianchi III and Kantowski-Sachs geometries. We make use of the property that a point-like Lagrangian exists for the description of the field equations, and variational symmetries are applied for the construction of invariant functions and conservation laws. The latter are used for the derivation of new analytic solutions for the classical field equations and exact function forms for the wavefunction in the quantum limit.

Keywords: Anisotropic spacetimes; cosmology; teleparallel; exact solutions



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1. Introduction

In teleparallelism, the gravitational Action Integral is defined by the torsion scalar T [1,2], which is constructed by the antisymmetric Weitzenböck connection [3]. The teleparallel theory of gravity is equivalent to General Relativity, and the presence of the cosmological constant plays does not affect the equivalence of the two theories. However, that is not true when other geometric invariants are introduced into the gravitational Action Integral, or scalar fields, which are nonminimally coupled to gravity. There is a plethora of gravitational theories inspired by teleparallelism [4–9], see also the recent review [10] for more details. On the other hand, scalar fields play an important role in the description of the various epochs of the universe. The inflationary era is usually attributed to a scalar field known as inflaton [11], which drives the dynamics to provide expansion of the universe, while the dark energy is assumed to be described by a scalar field. Moreover, various unified dark energy models have also been proposed in the literature [12–14].

In this piece of work, we are interested in a modified teleparallel theory of gravity known as a scalar-torsion theory or teleparallel dark energy model [15–18] in an anisotropic background space. The Action Integral is linear to the torsion scalar T , however, a scalar field is introduced, which interacts to the gravity in the Lagrangian. The theory can be observed as an extension of the scalar-tensor models in teleparallelism. While there are many similarities between the scalar-tensor and scalar-torsion theories, i.e., they are second-order theories with the same degrees of freedom, the theories are totally different. For instance, while the two theories can admit duality transformations, these provide different physical properties [19]. Recently in [20], the cosmological dynamics were studied for the scalar-tensor and scalar-torsion theories in a spatially flat isotropic and homogeneous universe. Various cosmological applications of the scalar-torsion theory can be found, for instance, in [21–23]. Constraints of the scalar-torsion theory with cosmological observations in a isotropic universe are performed in [24]. It was found that the observations favor a nonminimal coupling in scalar-torsion theory.

The purpose of this piece of work is to provide, for the first time, analytic and exact solutions for the classical and quantum limit of the field equations in a scalar-torsion theory

with an anisotropic background space. Specifically, we consider the physical space to be described by the locally rotational spacetimes of Bianchi I, Bianchi III and Kantowski-Sachs spacetimes. These locally rotational spacetimes are homogeneous and anisotropic with two scale factors and four isometries. In the limit of isotropization, the spacetimes are reduced to the spatially flat, closed and open Friedmann–Lemaître–Robertson–Walker geometries. Anisotropic spacetimes are of special interest because they can describe the pre-inflationary [25] era and they can be used as toy models for the description of the small anisotropies in the observed universe [26]. Kasner and Kasner-like are exact solutions of Bianchi I geometry. In teleparallelism, this kind of solution was investigated before in [27,28], while some other studies on exact and analytic solutions of a Bianchi I geometry in teleparallelism can be found, for instance, in [29–31]. In a series of studies, the evolution of the dynamics for the anisotropic parameters investigated in a higher-order modified theory of gravity [32] and references therein. It was found that independently of the initial conditions of the cosmological model on the anisotropy and on the spatially space curvature, the universe evolves to an isotropic and spatially flat geometry.

The study of anisotropic cosmologies is important for a gravitational theory. According to the cosmological principle in large scales, the universe is assumed to be isotropic and homogeneous. Inflation is the mechanism that has been proposed to solve the isotropization of the universe; however, anisotropies may play an important role in the pre-inflation epoch. In his famous paper, R. Wald [33] found that the existence of a positive cosmological constant in the context of General Relativity in anisotropic background geometries leads to isotropic universes in large scales, which lead to the cosmic no-hair conjecture. Additionally, the anisotropic spacetimes studied in this work are related with important initial singularities. Indeed, the dynamical variables of the Bianchi I spacetime describe the behaviour of the Mixmaster universe near the cosmological singularity. On the other hand, Bianchi III and Kantowski-Sachs spacetimes can also describe the dynamics of the physical parameters to the interior of a black hole [34–36]. While exact and analytic solutions for isotropic geometries have been found before for the case of scalar-torsion theory, according to our knowledge, no anisotropic solutions have been derived in the literature. The study of the integrability properties for the field equations and the derivation of solutions is essential, because we can infer that actual solutions exist for the description of the physical parameters in scalar-torsion theory with anisotropic background geometry.

The gravitational field equations for the model of our analysis are nonlinear second-order differential equations. Hence, mathematical techniques which deal with nonlinear dynamical systems should be enrolled in order to determine the exact and analytic solutions. The field equations for this specific cosmological model have the property to be described by the variation of a point-like Lagrangian functions. Hence, in order to construct conservation laws and invariant functions, which are to be used to the derivation of solutions; Noether's theorem for point transformations is considered [37]. The main concept of the Noether symmetry analysis is to constrain the unknown functions and parameters of the gravitational theory, such that the Action Integral is invariant under the application of point transformations, where, according to Noether's second theorem, conservation laws exist [38]. Thus, it is feasible to infer about the integrability properties of the field equations and to determine analytic solutions. Furthermore, the existence of the point-like Lagrangian means that the field equations can be written also by using the Hamiltonian formalism. Thus, under the classical quantization process, the Wheeler-DeWitt equation of quantum cosmology can be written [39]. The Noetherian conservation laws for the classical field equations provide differential invariants for the Wheeler-DeWitt equations, which are necessary in order to determine the closed-form expression for the wavefunction, as described by the Wheeler-DeWitt equation [40]. For a review on the Wheeler-DeWitt equation, we refer the reader to [41]. The structure of this paper is as follows.

In Section 2, we briefly present the basic definitions of teleparallelism and we define the cosmological model of this study, which is that of teleparallel dark energy. Anisotropic background spacetimes are considered and the field equations are derived for the physical

spaces described by the locally rotational Bianchi I, Bianchi III and Kantowski-Sachs geometries. In Section 3, we apply the conditions provided by Noether’s first theorem in order to constrain the unknown functional form of the scalar field potential, where Noether symmetries exist. These results are applied in order to infer the Liouville integrability of the field equations. Classical exact and analytic solutions of the field equations are constructed in Section 4. Furthermore, in Section 5, we write the Wheeler-DeWitt equation, and with the application of the differential invariants provided by the classical conservation laws, we determine the exact solutions for the wavefunction. Finally, in Section 6, we summarize our results.

2. Teleparallel Dark Energy

In teleparallelism, the fundamental geometric objects are the vierbein fields e_i defined by the constraint $g(e_i, e_j) = e_i \cdot e_j = \eta_{ij}$ [1], in which $\eta_{ij} = \text{diag}(-1, 1, 1, 1)$ is the Lorentz metric in canonical form.

In terms of coordinates with a nonholonomic basis $e^i(x^\kappa) = h^i_\mu(x^\kappa)dx^\mu$, the metric tensor $g_{\mu\nu}(x^\kappa)$ is

$$g_{\mu\nu} = \eta_{ij}h^i_\mu h^j_\nu. \tag{1}$$

We define the teleparallel torsion tensor

$$T_{\mu\nu}^\beta = \hat{\Gamma}_{\nu\mu}^\beta - \hat{\Gamma}_{\mu\nu}^\beta = h^\beta_i(\partial_\mu h^i_\nu - \partial_\nu h^i_\mu), \tag{2}$$

which is the antisymmetric part of the affine connection coefficients [3].

In General Relativity, the gravitational Lagrangian is defined by the Ricciscalar of the Levi-Civita. In teleparallelism, the gravitational Lagrangian is defined by the torsion scalar T related with the torsion tensor $T_{\mu\nu}^\beta$.

Specifically, the scalar T is defined as

$$T = S^{\mu\nu} T_{\mu\nu}^\beta \tag{3}$$

in which $S^{\mu\nu}_\beta = \frac{1}{2}(K^{\mu\nu}_\beta + \delta^\mu_\beta T^{\theta\nu} - \delta^\nu_\beta T^{\theta\mu})$ and $K^{\mu\nu}_\beta = -\frac{1}{2}(T^{\mu\nu}_\beta - T^{\nu\mu}_\beta - T^{\mu\nu}_\beta)$ is the contorsion tensor, which equals the difference of the Levi Civita connection in the holonomic and the nonholonomic frame.

The gravitational action integral in the teleparallel equivalence of General Relativity is defined as [1]

$$S = \frac{1}{16\pi G} \int d^4x e(T + L_m), \tag{4}$$

where L_m is the Lagrangian component for the matter source and $e = \sqrt{-g}$.

Similar to the case of General Relativity, scalar fields have been introduced in teleparallelism. Inspired by the Brans-Dicke theory, i.e., the scalar-tensor theories of General Relativity, a Machian teleparallel gravitational theory has been considered. It is called teleparallel dark energy, or scalar-tensor teleparallel gravity and it is defined by the gravitational Action Integral [15–19]

$$S = \frac{1}{16\pi G} \int d^4x e \left[F(\phi) \left(T + \frac{\omega}{2} \phi_{;\mu} \phi^{;\mu} + V(\phi) \right) \right], \tag{5}$$

where $F(\phi)$ is the coupling function of the scalar field with the torsion scalar, ω is a constant nonzero parameter, analogue of the Brans-Dicke parameter, and $V(\phi)$ is the scalar field potential which drives the dynamics.

An equivalent way to write the Action Integral (5) is

$$S = \frac{1}{16\pi G} \int d^4x e \left[\hat{F}(\psi) T + \frac{1}{2} \psi_{;\mu} \psi^{;\mu} + \hat{V}(\psi) \right], \tag{6}$$

where now the new field ψ is related with ϕ as $d\psi = \sqrt{\omega F(\phi)}d\phi$. Note that, when $\hat{F}(\psi) = F_0\psi^2$ or $F(\phi) = F_0e^{2\phi}$, the given gravitational model is the analogue of the Brans-Dicke theory, or of the dilaton equivalent model in teleparallelism.

The teleparallel dilaton model is of special interest because it admits similar properties with the usual dilaton theory. Indeed, in the case of a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background geometry, it has been demonstrated that the field equations admit a discrete symmetry, which is analogue to the duality Gasperini-Veneziano transformation [19].

Anisotropic Spacetimes

In the following analysis, we consider the physical space to be homogeneous and anisotropic, as described by the generic line element

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}\left(e^{2\beta(t)}dx^2 + e^{-\beta(t)}\left(dy^2 + f^2(y)dz^2\right)\right) \tag{7}$$

in which the function $f(y)$ has one of the following functional forms $f_A(y) = 1$, $f_B(y) = \sinh(y)$ or $f_C(y) = \sin(y)$ such that the spacetime is locally rotational; $\alpha(t)$, $\beta(t)$ are the two free scale factors and $N(t)$ is the lapse function. For $f(y) = f_A(y)$, the line element (7) is that of the Bianchi I space, for $f(y) = f_B(y)$ the Bianchi III space is recovered while, when $f(y) = f_C(y)$, the line element (7) is that of the Kantowski-Sachs universe. The parameter $\alpha(t)$ is the expansion rate of the three-dimensional hypersurfaces and $\beta(t)$ is the anisotropic parameter. When $\beta(t) = const.$, the line element (7) describes an isotropic physical space, that is, the flat, open and closed FLRW geometries are recovered.

In order to calculate the torsion scalar, we should define the proper vierbein fields. In order for the limit of General Relativity to be recovered, the vierbein fields should be defined properly. We follow the discussion in [10]; for the Bianchi I spacetime, we assume the vierbein basis

$$e^1 = Ndt, e^2 = e^{\alpha+\beta}dx, e^3 = e^{\alpha-\frac{\beta}{2}}dy, e^4 = e^{\alpha-\frac{\beta}{2}}dz$$

with torsion scalar

$$T_{BI} = \frac{1}{N^2}\left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2\right). \tag{8}$$

For Bianchi III, we consider the basis

$$\begin{aligned} e^1 &= Ndt, \\ e^2 &= i e^{\alpha+\beta} \cos z \sinh y dx + e^{\alpha-\frac{\beta}{2}}(\cosh y \cos z dy - \sinh y \sin z dz), \\ e^3 &= i e^{\alpha+\beta} \sinh y \sin z dx + e^{\alpha-\frac{\beta}{2}}(\cosh y \sin z dy - \sinh y \cos z dz), \\ e^4 &= -e^{\alpha+\beta} \cosh y dx - i e^{\alpha-\frac{\beta}{2}} \sinh y dy, \end{aligned}$$

in which we calculate the torsion scalar

$$T_{BIII} = \frac{1}{N^2}\left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2\right) + 2e^{-2\alpha+\beta}. \tag{9}$$

Finally, for the Kantowski-Sachs spacetime, we assume the vierbein fields

$$\begin{aligned} e^1 &= Ndt, \\ e^2 &= e^{a+\beta} \cos z \sin y dx + e^{a-\frac{\beta}{2}}(\cos y \cos z dy - \sin y \sin z dz), \\ e^3 &= e^{a+\beta} \sin y \sin z dx + e^{a-\frac{\beta}{2}}(\cos y \sin z dy - \sin y \cos z dz), \\ e^4 &= e^{a+\beta} \cos y dx - e^{a-\frac{\beta}{2}} \sin y dy, \end{aligned}$$

from which we calculate the torsion scalar

$$T_{KS} = \frac{1}{N^2} \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 \right) - 2e^{-2\alpha+\beta}. \quad (10)$$

Thus, we define the general torsion scalar

$$T_A(K) = \frac{1}{N^2} \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 \right) + 2Ke^{-2\alpha+\beta},$$

where $K = 0$ is for the Bianchi I space, $K = 1$ is for the Bianchi I space and $K = -1$ corresponds to the torsion scalar of the Kantowski-Sachs space.

By replacing the generic $T_A(K)$ in the Action Integral (5) and assuming that the scalar field inherits the isometries of the background space, that is, $\phi = \phi(t)$, we derive the point-like Lagrangian for the field equations

$$L(N, a, \dot{\alpha}, \beta, \dot{\beta}, \phi, \dot{\phi}) = F(\phi)e^{3\alpha} \left(\frac{1}{N} \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 - \frac{\omega}{2}\dot{\phi}^2 \right) + N \left(2Ke^{-2\alpha+\beta} + V(\phi) \right) \right) \quad (11)$$

while the field equations are

$$0 = F(\phi)e^{3\alpha} \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 - \frac{\omega}{2}\dot{\phi}^2 - V(\phi) - 2Ke^{-2\alpha+\beta} \right), \quad (12)$$

$$0 = \ddot{\alpha} + \frac{3}{2}\dot{\alpha}^2 + \frac{3}{8}\dot{\beta}^2 + \frac{1}{4} \left(\frac{\omega}{2}\dot{\phi}^2 - V(\phi) \right) + \frac{d}{dt}(\ln F(\phi))\dot{\alpha} - \frac{1}{6}Ke^{-2\alpha+\beta}, \quad (13)$$

$$0 = \ddot{\beta} + 3\dot{\alpha}\dot{\beta} + \frac{d}{dt}(\ln F(\phi))\dot{\beta} + \frac{2}{3}Ke^{-2\alpha+\beta}, \quad (14)$$

$$0 = \omega(\ddot{\phi} + 3\dot{\alpha}\dot{\phi})\dot{\phi} + \dot{V} + \frac{d}{dt}(\ln F(\phi)) \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 + V(\phi) + 2Ke^{-2\alpha+\beta} \right), \quad (15)$$

where, without loss of generality, we have selected the lapse function $N(t)$ to be constant, i.e., $N(t) = 1$.

3. Symmetry Analysis

In this section, we proceed with the derivation of variational symmetries for the field equations. Because the field equations for the cosmological model of our consideration follow from a point-like Lagrangian, the variational symmetries are derived with the application of Noether's theorem. In this scenario, variational symmetries are called Noether symmetries; for a recent review on Noether's work, we refer the reader in [38].

Noether's theorem in modified theories of gravity cover a wide range of applications with many important results on the classification for the free functions of the modified theories and the construction of conservation laws, which have been used for the derivation of analytic and exact solutions [37,42–47]. For the convenience of the reader, we discuss the main definitions of Noether's theorems.

3.1. Noether's Theorems

Consider the action integral

$$A = \int L(t, q(t), \dot{q}(t)) dt. \quad (16)$$

where $L(t, q(t), \dot{q}(t))$ is the Lagrangian function, t is the independent variable and $q(t)$ denotes the dependent variable, while $\dot{q}(t) = \frac{dq(t)}{dt}$ is the first-order derivative

Then, under the infinitesimal transformation at any point P in the space of variables $\{t, q\}$

$$\bar{t} = t + \varepsilon\zeta(t, q), \quad \bar{q} = q + \varepsilon\eta(t, q) \quad (17)$$

with the infinitesimal generator, the vector field

$$X = \zeta(t, q)\partial_t + \eta(t, q)\partial_q, \tag{18}$$

the action integral (16) becomes

$$\bar{A} = \int_{\bar{t}_0}^{\bar{t}_1} L(\bar{t}, \bar{q}, \dot{\bar{q}}) d\bar{t}. \tag{19}$$

Thus, according to Noether’s first theorem, the variation of the action integral A remains invariant under the application of the infinitesimal transformation (17), if and only if there exists a function f such that [37]

$$\dot{f} = \zeta \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + (\dot{\eta} - \dot{q}\dot{\zeta}) \frac{\partial L}{\partial \dot{q}} + \zeta L. \tag{20}$$

The function f is a boundary term that has been introduced to allow for the infinitesimal changes in the value of the action integral provided by the infinitesimal transformation.

The novelty of Noether’s work is that there exists a simple formula for the one-to-one correspondence between the symmetry vectors and conservation laws for the equations of motion. Indeed, if X satisfies condition (20) for the Lagrangian function $L(t, q(t), \dot{q}(t))$, then function [37]

$$I(X) = f - \left[\zeta L + (\eta - \dot{q}\zeta) \frac{\partial L}{\partial \dot{q}} \right] \tag{21}$$

is a conservation law for the equations of motion; that is, $\dot{I}(X) = 0$. Formula (21) is the so-called Noether’s second theorem.

3.2. Symmetry Classification

In this study and for the point-like Lagrangian (11) with $F(\phi) = e^{2\phi}$, where without loss of generality, we assume $N(t) = e^{3\alpha+2\phi}$, that is,

$$L(N, a, \dot{a}, \beta, \dot{\beta}, \phi, \dot{\phi}) = \left(6\dot{a}^2 - \frac{3}{2}\dot{\beta}^2 - \frac{\omega}{2}\dot{\phi}^2 \right) + e^{6\alpha+4\phi} \left(2Ke^{-2\alpha+\beta} + V(\phi) \right), \tag{22}$$

we consider the infinitesimal transformation

$$\bar{t} = t + \varepsilon\zeta(t, \alpha, \beta, \phi), \tag{23}$$

$$\bar{\alpha} = \alpha + \varepsilon\eta^\alpha(t, \alpha, \beta, \phi), \tag{24}$$

$$\bar{\beta} = \beta + \varepsilon\eta^\beta(t, \alpha, \beta, \phi), \tag{25}$$

$$\bar{\phi} = \phi + \varepsilon\eta^\phi(t, \alpha, \beta, \phi) \tag{26}$$

with generator the vector field $X = \zeta\partial_t + \eta^\alpha\partial_\alpha + \eta^\beta\partial_\beta + \eta^\phi\partial_\phi$.

Hence, the application of the symmetry condition (20) for the Lagrangian function (22) gives a system of linear partial differential equations, which determine the generator X for various functional forms of the potential $V(\phi)$ and values of the curvature term K .

For the Bianchi I background space, $K = 0$, it follows that, for a zero potential function $V(\phi) = 0$, the admitted Noether symmetries by the field equations are

$$X_1 = \partial_\alpha, X_2 = \phi\partial_\alpha + \frac{12}{\omega}\alpha\partial_\phi, X_3 = \beta\partial_\alpha + 4\alpha\partial_\beta, \tag{27}$$

$$X_4 = \frac{\omega}{3}\phi\partial_\beta - \beta\partial_\phi, X_5 = \partial_\beta, X_6 = \partial_\phi. \tag{28}$$

The symmetry vectors are the isometries of the three-dimensional flat space, $\{X_1, X_5, X_6\}$ are the translation symmetries, while $\{X_2, X_3, X_4\}$ are the three rotations of the three-dimensional minisuperspace.

The corresponding conservation laws are calculated from expression (21) and they are

$$I(X_1) = \dot{\alpha}, I(X_2) = \phi\dot{\alpha} + \alpha\dot{\phi}, I(X_3) = \beta\dot{\alpha} + \alpha\dot{\beta} \tag{29}$$

$$I(X_4) = \phi\dot{\beta} - \beta\dot{\phi}, I(X_6) = \dot{\beta} \text{ and } I(X_6) = \dot{\phi}. \tag{30}$$

On the other hand, for $K = 0$ and $V(\phi) = V_0e^{\lambda\phi}$, the additional admitted Noether symmetries from the Lagrangian of the field equations are

$$X_5, X_7 = \frac{(4 + \lambda)}{2\omega}X_3 + \frac{3}{\omega}X_4, X_8 = (4 + \lambda)X_1 - X_6, \tag{31}$$

for which the resulting conservation laws are

$$I(X_5), I(X_8) = (4 + \lambda)\dot{\alpha} + \omega\dot{\phi}, \tag{32}$$

$$I(X_7) = 2(4 + \lambda)B\dot{\alpha} - (2\alpha(4 + \lambda) + \omega\phi)\dot{\beta} + \omega\beta\dot{\phi}. \tag{33}$$

The admitted Noether symmetries are different in the presence of the curvature term, $K \neq 0$. In this case, for the zero potential $V(\phi) = 0$, the admitted Noether symmetries are

$$Y_1 = X_1 - 4X_5, Y_2 = X_6 - 4X_5, \tag{34}$$

and

$$Y_3 = X_2 - \frac{12}{\omega}X_4, \tag{35}$$

where the resulting conservation laws are

$$I(Y_1) = \dot{\alpha} + \dot{\beta}, I(Y_2) = 12\dot{\beta} - \omega\dot{\phi}, \tag{36}$$

$$I(Y_3) = \phi(\dot{\alpha} + \dot{\beta}) - (\alpha - \beta)\dot{\phi}. \tag{37}$$

Finally, for the exponential potential $V(\phi) = V_0e^{\lambda\phi}$ the field equations admit the unique symmetry vector

$$Y_4 = (\lambda + 4)X_1 - 4(\lambda - 2)X_5 - 6X_6 \tag{38}$$

with conservation law

$$I(Y_4) = (\lambda + 4)\dot{\alpha} + (\lambda - 2)\dot{\beta} - \frac{\omega}{2}\dot{\phi}. \tag{39}$$

From the symmetry analysis, we conclude that the field equations form a Liouville integrable dynamical system according to the Noether point symmetries for the case of Bianchi I for $V(\phi) = 0$ and $V(\phi) = V_0e^{\lambda\phi}$, and for the Bianchi III and Kantowski-Sachs cases $K \neq 0$ only when $V(\phi) = 0$.

4. Classical Solutions

We proceed with the application of the conservation laws for the derivation of analytic solutions for the latter cases.

4.1. Bianchi I Spacetime

For the case of Bianchi I spacetime, for $4 + \lambda \neq 0$, we define the new dependent variable $\Phi = \phi + \frac{6}{4+\lambda}\alpha$, such that the point-like Lagrangian (22) reads

$$L(\alpha, \dot{\alpha}, \beta, \dot{\beta}, \Phi, \dot{\Phi}) = \left(6 - \frac{18\omega}{(4 + \lambda)^2}\right)\dot{\alpha}^3 - \frac{3}{2}\dot{\beta}^2 + \frac{6\omega}{4 + \lambda}\dot{\alpha}\dot{\Phi} - \frac{1}{2}\omega\dot{\Phi}^2 + V_0e^{(4+\lambda)\Phi}. \tag{40}$$

This gives the field equations

$$\left(6 - \frac{18\omega}{(4 + \lambda)^2}\right)\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 + \frac{6\omega}{4 + \lambda}\dot{\alpha}\dot{\Phi} - \frac{1}{2}\omega\dot{\Phi}^2 - V_0e^{(4+\lambda)\Phi} = 0, \tag{41}$$

$$\ddot{\alpha} - \frac{V_0}{2}e^{(4+\lambda)\Phi} = 0, \quad \ddot{\beta} = 0, \tag{42}$$

and

$$\ddot{\Phi} + \frac{V_0((4 + \lambda)^2 - 3\omega)}{(4 + \lambda)\omega}e^{(4+\lambda)\Phi} = 0. \tag{43}$$

Thus, for the anisotropic parameter β , we derive the closed-form expression $\beta(t) = \beta_1 t + \beta_0$.

4.1.1. Arbitrary Parameters

For $\frac{V_0((4+\lambda)^2-3\omega)}{(4+\lambda)\omega} \neq 0$, the analytic solution is

$$\Phi(t) = -\frac{1}{4 + \lambda} \ln\left(\frac{2V_0((4 + \lambda)^2 - 3\omega)}{(4 + \lambda)^2\omega\Phi_1} \cosh^2\left(\frac{4 + \lambda}{2}\sqrt{\Phi_1}(t - t_0)\right)\right) \tag{44}$$

and

$$\alpha(t) = \frac{\omega}{((4 + \lambda)^2 - 3\omega)} \ln\left(\cosh\left(\frac{4 + \lambda}{2}\sqrt{\Phi_1}(t - t_0)\right)\right) + \alpha_1 t + \alpha_0 \tag{45}$$

with constraint equation

$$\frac{12\alpha_1^2((4 + \lambda)^2 - 3\omega)^2 - \omega(\lambda + 4)\Phi_1}{2(\lambda + 4)^4 - (\lambda + 4)^2\omega} - \frac{3}{2}\beta_1^2 = 0. \tag{46}$$

Presently, consider the special solution with $\alpha_1 = 0, a_0 = 0$, and without the loss of generality $t_0 = 0$. This is an anisotropic solution because from the latter constraint, it follows $\beta_1 \neq 0$. That is, we consider the scale factor

$$\alpha(t) = \frac{\omega}{((4 + \lambda)^2 - 3\omega)} \ln\left(\cosh\left(\frac{4 + \lambda}{2}\sqrt{\Phi_1}t\right)\right) \tag{47}$$

and the lapse function

$$N(t) = \left(\frac{\omega(4 + \lambda)^2}{2V_0((4 + \lambda)^2 - 3\omega)}\Phi_1\right)^{\frac{2}{4+\lambda}} \left(\cosh\left(\frac{4 + \lambda}{2}\sqrt{\Phi_1}t\right)\right)^{\frac{3\omega-4(\lambda+4)}{(4+\lambda)^2-3\omega}}. \tag{48}$$

Thus, the expansion rate $\theta(t) = \frac{1}{3N}\dot{\alpha}$ is determined to be

$$(\theta(a))^2 \simeq 1 - a^{-6+\frac{2}{\omega}(4+\lambda)^2}, \quad \alpha = \ln a. \tag{49}$$

The anisotropic parameter $\sigma = \frac{1}{N}\dot{\beta}(t)$ is

$$\sigma(a) \simeq \alpha^{-3+\frac{4}{\omega}(4+\lambda)}, \quad \alpha = \ln a. \tag{50}$$

Thus, for large values of α , the anisotropic parameter vanishes, that is, $\sigma(\alpha) \rightarrow 0$, for $-3 + \frac{4}{\omega}(4 + \lambda) < 0$.

4.1.2. Case $V_0((4 + \lambda)^2 - 3\omega) = 0$

Furthermore, in the case for which $V_0((4 + \lambda)^2 - 3\omega) = 0$ the analytic solution is different. Indeed, for $V_0 = 0$ we derive the closed-form solution

$$\alpha(t) = \alpha_1 t + \alpha_0, \tag{51}$$

$$\Phi(t) = \Phi_1 t + \Phi_0 \tag{52}$$

with constraint equation

$$\left(6 - \frac{18\omega}{(4 + \lambda)^2}\right)\alpha_1^2 - \frac{3}{2}\beta_1^2 + \frac{6\omega}{4 + \lambda}\alpha_1\Phi_1 - \frac{1}{2}\omega\Phi_1^2 = 0. \tag{53}$$

For the latter solution and for $\alpha_0 = 0, \Phi_0$, we derive the expansion rate and the anisotropic parameters to be

$$\theta^2(a) \simeq a^{-\frac{3\lambda}{4+\lambda} + \frac{2\Phi_1}{\alpha_1}}, \sigma^2(a) \simeq a^{-\frac{3\lambda}{4+\lambda} + \frac{2\Phi_1}{\alpha_1}}, \alpha = \ln a \tag{54}$$

from which we infer that the universe becomes isotropic for $-\frac{3\lambda}{4+\lambda} + \frac{2\Phi_1}{\alpha_1} < 0$.

On the other hand, for $((4 + \lambda)^2 - 3\omega) = 0$, the analytic solution is

$$\Phi(t) = \Phi_1 t + \Phi_0, \tag{55}$$

$$\alpha(t) = \frac{e^{(4+\lambda)(\Phi_0+t\Phi_1)}}{2(4 + \lambda)^2\Phi_1^2}V_0 + \alpha_1 t + a_0, \tag{56}$$

with constraint equation $-\frac{1}{6}(4 + \lambda)\Phi_1((4 + \lambda)\Phi_1 - 12\alpha_1) - \frac{3}{2}\beta_1^2 = 0$.

Assume now $\Phi_0 = 0, \alpha_1 = 0$ and $\Phi_1 = 0$. In this case, the expansion rate and the anisotropic parameter are

$$\theta^2(a) \simeq a^{-\frac{3\lambda}{4+\lambda}}(\ln a)^{\frac{2+\lambda}{4+\lambda}}, \alpha = \ln a, \tag{57}$$

and

$$\sigma(a) = a^{\frac{12}{4+\lambda}-3}(\ln a)^{-\frac{2}{4+\lambda}}, \alpha = \ln a. \tag{58}$$

Finally, for $\lambda + 4 = 0$, the closed-form solution for the physical parameters are

$$\alpha(t) = \frac{1}{6} \ln\left(-\frac{6\alpha_1}{V_0 \cosh^2(3\sqrt{\alpha_1}(t - t_0))}\right), \tag{59}$$

$$\beta(t) = \beta_1 t + \beta_0, \tag{60}$$

$$\phi(t) = \phi_1 t + \phi_0, \tag{61}$$

with constraint equation

$$6\alpha_1 - \frac{3}{2}\beta_1^2 - \frac{\omega}{2}\phi_1^2 = 0. \tag{62}$$

4.2. Bianchi III & Kantowski-Sachs Spacetimes

For $K \neq 0$ and zero potential function, i.e., $V(\phi) = 0$, we consider the new variables $\phi = \Phi - \alpha$ and $\beta = B - 4\Phi$, where the point-like Lagrangian (22) becomes

$$4e^B K - 3\dot{\beta}^2 - (\omega - 12)\dot{\alpha}^2 + 2(12\dot{B} + \omega\dot{\alpha})\dot{\Phi} - (\omega + 48)\dot{\Phi}^2 = 0. \tag{63}$$

In the new variables, the field equations read

$$\ddot{\alpha} = \frac{2}{3}Ke^B, \quad \ddot{\Phi} = \frac{2}{3\omega}(\omega - 12)Ke^B, \quad (64)$$

$$\ddot{B} = \frac{2}{3}(\omega - 16)Ke^B \quad (65)$$

with constraint equation

$$4e^B K + 3\dot{\beta}^2 + (\omega - 12)\dot{\alpha}^2 - 2(12\dot{B} + \omega\dot{\alpha})\dot{\Phi} + (\omega + 48)\dot{\Phi}^2 = 0. \quad (66)$$

Thus for $(\omega - 16) \neq 0$, the analytic solution for the field equations is

$$B(t) = \ln\left(\frac{\omega B_1}{4(16 - \omega)} \cosh^{-2}\left(\frac{\sqrt{B_1}}{2}(t - t_0)\right)\right), \quad (67)$$

$$\alpha(t) = -\frac{2\omega}{3(\omega - 16)} \ln\left(\cosh\left(\frac{\sqrt{B_1}}{2}(t - t_0)\right)\right) + \alpha_1 t + \alpha_0, \quad (68)$$

$$\Phi(t) = -\frac{2(\omega - 12)}{3(\omega - 16)} \ln\left(\cosh\left(\frac{\sqrt{B_1}}{2}(t - t_0)\right)\right) + \Phi_1 t + \Phi_0. \quad (69)$$

On the other hand, for $\omega - 16 = 0$, the analytic solution is

$$B(t) = B_1 t + B_0, \quad (70)$$

$$\alpha(t) = \frac{2}{3B_1^2} e^{B_0 + B_1 t} K + \alpha_1 t + \alpha_0, \quad (71)$$

$$\Phi(t) = \frac{1}{6B_1^2} e^{B_0 + B_1 t} K + \Phi_1 t + \alpha_0. \quad (72)$$

Finally, the expansion rate has a similar behaviour with the solution found before.

According to our knowledge, these are the first anisotropic cosmological analytic solutions in the literature in the teleparallel dark energy theory.

5. The Wheeler-DeWitt Equation

Consider the constraint Hamiltonian dynamical system of the form $H = N\mathcal{H}$ with

$$\mathcal{H} = \frac{1}{2}\mathcal{G}^{AB} p_A p_B + \mathcal{U}(\mathbf{q}) \text{ and } \mathcal{H} \equiv 0, \quad (73)$$

where \mathcal{G}^{AB} is the minisuperspace, p_A is the momentum conjugate to the variable q^A and $\mathcal{U}(\mathbf{q})$ is the effective potential.

From the point-like Lagrangian (11), we find

$$p_\alpha = \frac{12}{N}F(\phi)e^{3\alpha}\dot{\alpha}, \quad p_\beta = -\frac{3}{N}F(\phi)e^{3\alpha}\dot{\beta}, \quad p_\phi = -\frac{\omega}{N}F(\phi)e^{3\alpha}\dot{\phi}. \quad (74)$$

Hence, the Hamiltonian is written as

$$H = N \left[\frac{1}{2} \frac{e^{-3\alpha}}{F(\phi)} \left(\frac{1}{12} p_\alpha^2 - \frac{1}{3} p_\beta^2 - \frac{1}{\omega} p_\phi^2 \right) - e^{3\alpha} F(\phi) \left(2Ke^{-2\alpha+\beta} + V(\phi) \right) \right], \quad (75)$$

that is, by using the constraint Equation (12):

$$\mathcal{H} = \frac{1}{2} \frac{e^{-3\alpha}}{F(\phi)} \left(\frac{1}{12} p_\alpha^2 - \frac{1}{3} p_\beta^2 - \frac{1}{\omega} p_\phi^2 \right) - e^{3\alpha} F(\phi) \left(2Ke^{-2\alpha+\beta} + V(\phi) \right) = 0. \quad (76)$$

The Wheeler-DeWitt equation follows under the canonical quantization of the constraint, that is, $\mathcal{H}\Psi(\mathbf{q}) = 0$, where $\Psi(\mathbf{q})$ is the wavefunction of the quantized system.

For the Wheeler-DeWitt equation to remain invariant under conformal transformations, the conformally Laplace operator is applied, such that $\mathcal{H}\Psi(\mathbf{q}) = 0$ to be written as

$$\mathcal{W}(\mathbf{q}, \Psi) \equiv \left(\frac{1}{2}\Delta + \frac{n-2}{8(n-1)}\mathcal{R} - \mathcal{U}(\mathbf{q}) \right) \Psi(\mathbf{q}) = 0, \tag{77}$$

in which \mathcal{R} is the Ricciscalar of the minisuperspace \mathcal{G}_{AB} and $n = \dim \mathcal{G}$.

We consider the case $F(\phi) = e^{2\phi}$ and without a loss of generality, we set $N = e^{3\alpha+2\phi}$, thus, the Wheeler-DeWitt equation becomes

$$\mathcal{W}(\alpha, \beta, \phi, \Psi) \equiv \left(\frac{1}{24}\frac{\partial^2}{\partial \alpha^2} - \frac{1}{6}\frac{\partial^2}{\partial \beta^2} - \frac{1}{2\omega}\frac{\partial^2}{\partial \phi^2} - e^{6\alpha+4\phi}(2Ke^{-2\alpha+\beta} + V(\phi)) \right) \Psi(\alpha, \beta, \phi) = 0. \tag{78}$$

5.1. Quantum Operators from Symmetry Vectors

We briefly discuss the application of the theory of symmetries of differential equations for the construction of quantum operators necessary for the derivation of exact solutions for the Wheeler-DeWitt equation.

Assume now the generic vector field

$$\mathbf{Y} = \xi^{\mathbf{q}}(\mathbf{q}, \Psi)\partial_{\mathbf{q}} + \eta(\mathbf{q}, \Psi)\partial_{\Psi} \tag{79}$$

defined in the jet space $\{\mathbf{q}, \Psi\}$, in which the Wheeler-DeWitt Equation (77) lies.

The vector field \mathbf{Y} is the infinitesimal generator of the point transformation between two points $P(\mathbf{q}, \Psi) \rightarrow P'(\mathbf{q}', \Psi')$ [48,49]:

$$(\mathbf{q}', \Psi') = (\mathbf{q}, \Psi) + \varepsilon(\xi^{\mathbf{q}}(\mathbf{q}, \Psi), \eta(\mathbf{q}, \Psi)), \tag{80}$$

where ε is an infinitesimal parameter such that $\varepsilon^2 \rightarrow 0$.

We say that the Wheeler-DeWitt Equation (77) is an invariant under the point transformation with generator \mathbf{X} if and only if

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{W}'(\mathbf{q}', \Psi') - \mathcal{W}(\mathbf{q}, \Psi)}{\varepsilon} = 0. \tag{81}$$

When the latter condition is true, the vector field \mathbf{Y} is a Lie point symmetry for the differential equation.

In [50], it was found that the generic symmetry vector for Equation (77) is of the form

$$\mathbf{Y} = \xi(\mathbf{q})\partial_{\mathbf{q}} + \left[\frac{(2-n)}{2}\psi(\mathbf{q})\Psi + \mu_0\Psi + \Omega(\mathbf{q}) \right] \partial_{\Psi}, \tag{82}$$

where $\xi(\mathbf{q})$ is a conformal Killing vector field of the minisuperspace \mathcal{G}_{AB} , with the conformal factor $\psi(\mathbf{q})$, which satisfies the condition $\mathcal{L}_{\xi}\mathcal{U}(\mathbf{q}) + 2\psi\mathcal{U}(\mathbf{q}) = 0$, in which \mathcal{L}_{ξ} is the Lie derivative with respect to the vector field ξ . $\Omega(\mathbf{q})$ satisfies the original equation and denotes the infinite number of solutions of the original conformal Laplace equation.

For a given Lie symmetry \mathbf{Y} of the conformal Laplace equation, we can determine in the normal variables the equivalent Lie-Bäcklund vector field $\hat{\mathbf{Y}} = \left(\frac{\partial \Psi}{\partial q^j} - \left(\frac{2-n}{2}\psi + a_0 \right) \Psi \right) \partial_{\Psi}$, which has the property to transform a solution into a solution. That is, $\hat{\mathbf{Y}}\Psi = \mu_1\Psi$, from which it follows that

$$\frac{\partial \Psi}{\partial q^j} - \left(\frac{2-n}{2}\psi + \alpha_0 \right) \Psi = \mu_1\Psi \tag{83}$$

which is nothing else than a quantum operator.

Recall that in the normal variables, the conservation laws of the classical system generated by point symmetries can be written as the linear function of the momentum p_J , that is $I_J = p_J$, from where it follows that under canonical quantization

$$-i \frac{\partial \Psi}{\partial q^J} - I_J \Psi = 0. \quad (84)$$

From the latter, it is clear that Noetherian conservation laws can be used to determine differential invariants for the Wheeler-DeWitt equation, and this is an approach to relate the classical and quantum solutions.

5.2. Exact Solutions

We assume now the case of the Bianchi I universe, $K = 0$, with the exponential potential $V(\phi) = F_0 e^{\lambda\phi}$. Then, with the use of the Noether symmetry vectors, we construct differential operators, which for the general case provide the analytic solution for the Wheeler-DeWitt Equation (78):

$$\Psi(\alpha, \beta, \phi) = \exp\left(c_1\beta - 3 \frac{c_2}{(4+\lambda)^2 - 3\omega} (2c_2\lambda + 8\alpha + \omega\phi)\right) (\Psi_1 J_\kappa(Z) + \Psi_2 Y_\kappa(Z)), \quad (85)$$

where $J_\kappa(\mathbf{q}), Y_\kappa(\mathbf{q})$ are the Bessel functions of the first and second kind, while $\kappa = \frac{2}{3} \sqrt{\frac{\omega(9c_2 - c_1^2((4+\lambda)^2 - 3\omega))}{((4+\lambda)^2 - 3)^2}}$, $Z = 2 \sqrt{\frac{2\omega V_0}{(4+\lambda)^2 - 3\omega}} e^{3\alpha + \frac{1}{2}\phi(4+\lambda)}$, Ψ_1, Ψ_2, c_1 and c_2 are constants, and while c_1 and c_2 are related with the constants of integration for the classical solution.

Similarly, for the case with $K = 0$ and $V(\phi) = 0$, we determine the exact solution

$$\Psi(\alpha, \beta, \phi) = \exp\left(\frac{\omega(c_1(4\phi + \alpha + \beta) - 3c_2\phi) + 48(c_1 - c_2)\alpha - 12c_2\beta}{3(16 - \omega)}\right) (\Psi_1 J_{\bar{\kappa}}(\bar{Z}) + \Psi_2 Y_{\bar{\kappa}}(\bar{Z})), \quad (86)$$

in which $\bar{\kappa} = \frac{1}{3(\omega - 16)} \sqrt{\omega(\omega c_1^2 + 12(2c_1 - c_2)(2c_1 - 3c_2))}$ and $\bar{Z} = -2 \sqrt{\frac{2K\omega}{16 - \omega}} e^{2(\alpha + \phi) + \frac{\beta}{2}}$.

For specific values of the free parameters, the exact solutions have different functional forms. However, they can easily be derived. Indeed, the Wheeler-DeWitt equation with the use of the differential operators is reduced to the Bessel second-order differential equation.

6. Conclusions

Anisotropic spacetimes describe the physical space in the early stages of the universe, and the determination of exact and analytic solutions is of special interest, because the evolution of anisotropies in a given gravitational theory can be studied analytically. In this study, we investigated anisotropic cosmological solutions for the scalar-torsion theory in the context of teleparallelism, where the physical space is described by the locally rotational spaces of the Bianchi I, Bianchi III and Kantowski-Sachs geometries.

For the scalar-torsion theory, we made use of a corresponding vierbein field defined in the proper frame for each background geometry, such that the limit of General Relativity to be recovered when the nonminimally coupled scalar field becomes constant. The resulting field equations have the property to have a point-like Lagrangian description. Furthermore, we assume the existence of a potential function, which drives the dynamics of the scalar field and consequently of all of the physical parameters of the theory.

We applied the Noether symmetry analysis in order to perform a classification of the scalar field potential, by assuming the requirement that there exist nontrivial variational point symmetries for the point-like Lagrangian of the field equations. The latter requirement is equivalent with the existence of the conservation laws for the field equations. That is ensured by Noether's second theorem. From the classification scheme, we are able to identify specific functional forms for the scalar field potentials, where the resulting field equations form a Liouville integrable system. For the potential functions, we solved the

field equations by using the additional conservation laws, and we wrote the resulting analytic solutions in closed-form expressions.

It is known that there exists a unique relation between the Noether symmetries of the classical field equations and of the Lie symmetries of the Wheeler-DeWitt equation of quantum cosmology. Thus, we derived the Wheeler-DeWitt equation and we determined the differential operators, which were used to write the wavefunction in a closed-form expression in terms of the exponential and of the Bessel functions.

In a future study, we plan to further investigate the physical evolution of the field equations for the scalar-torsion theory in an anisotropic background geometry by applying other analytical techniques, such as the analysis of the stationary points for the determination of asymptotic solutions. The results of this analysis as that of the following study will give important information for the initial value problem in scalar-torsion cosmology.

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