Numerical Simulation of Entropy Optimization in Radiative Hybrid Nanofluid Flow in a Variable Features Darcy–Forchheimer Curved Surface

Asif Ullah Hayat 1, Ikram Ullah 2, Hassan Khan 3, Wajaree Weera 4,5* and Ahmed M. Galal 5,6

1 Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan
2 Department of Sciences and Humanities, National University of Computer and Emerging Sciences, Peshawar 25000, Pakistan
3 Department of Mathematics, Near East University TRNC, Mersin 99138, Turkey
4 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
5 Department of Mechanical Engineering, College of Engineering in Wadi Alhaddasir, Prince Sattam bin Abdulaziz University, Alkhair 16273, Saudi Arabia
6 Production Engineering and Mechanical Design Department, Faculty of Engineering, Mansoura University, Mansoura P.O. Box 35516, Egypt
* Correspondence: wajawe@kku.ac.th

Abstract: Studies associated with ethylene glycol (EG) have great significance in various engineering sectors because EG is more useful as a cooling agent in various engines. Furthermore, fluid inspection using two distinct nanoparticles has applications in mechanical systems, electronic devices, medical apparatus, and the diagnosis and treatment of disease. Therefore, present comminution explored the entropy production in magnetized hybrid nanomaterials flowing via Darcy–Forchheimer space with varying permeability. Hybrid nano liquid is synthesized by adding cobalt ferrite and gold nanoparticles to ethylene glycol and water. Effects of thermal radiation, Joule heating, heat sources, and an exponential heat source are considered in the energy expression. The assumed problem is modeled in the form of nonlinear PDEs. Such types of problems have mostly occurred in symmetrical phenomena and are applicable in engineering, physics, and applied mathematics. The obtained system is converted to ODEs using suitable substitution transformations. Resultant ODEs are numerically computed with the help of the NDsolve technique using Mathematica software. Their outcomes are displayed through figures and tables. Obtained results reveal that variable permeability and curvature parameters improve the velocity profile, while an exponential heat source (EHS) enhances the thermal effect. It is also observed that entropy optimization improves with the increment in magnetic parameter.

Keywords: hybrid nanofluid; variable Darcy–Forchheimer law; MHD; thermal radiation; Joule heating; entropy optimization; permeable surface

1. Introduction

In recent years, the analysis of thermal expansion and liquid flow across permeable surfaces has gained the attention of researchers because of its extraordinary range of utilization in the engineering and biomedical sectors, such as enhanced oil recovery, solid matrix heat exchanges, cooling of nuclear reactors, chromatography, geothermal and petroleum resources, thermal insulation drying of porous solids, ceramic processing, filtration processes, etc. Darcy’s law is only applicable for low-porosity and low-velocity systems. As velocities increase and porosity becomes non-uniform, its application in industrial systems and engineering becomes inefficient. Therefore, the Darcy–Forchheimer (DF) model, which includes the inertial aspects and boundaries, can be employed to overcome the limits of Darcy’s law. For these reasons, Forchheimer [1] introduced the squared
velocity term into the expression of motion in 1901. Ikram et al. [2] investigated the Darcy effects in nanofluid (NF) flow in the direction of an infinite rotating and stretchy disc with an exponential heat source and thermal heat source. It has been observed that the exponential space heat source has a greater influence on fluid temperature than the thermal heat source. Alotaibi and Mohamed [3] discussed the DF nanofluid flow across a convectively heated porous expanding surface. The DF flow of Casson material comprising titanium dioxide and graphene oxide in a permeable medium was inspected by Kumar et al. [4]. The DF relation with variable porosity and permeability flow over a stretchable rotating disk in a porous region was scrutinized by Hayat et al. [5]. An extending surface with variable thickness incorporated in a DF space was employed by Gautam et al. [6] to study the Williamson nanofluid flow. It was observed that the rising wall thickness affects the velocity and energy profile. Mallikarjuna et al. [7] evaluated a double-phase (dust phase and fluid phase) DF-hybrid nanofluid flow with dissipation and melting effects. The consequences reveal that a high melting variable decreases the thermal effect in both phases. Saeed et al. [8] studied the flow of a couple of stressed hybrid nanomaterials through DF with variable viscosity, Joule heating, and viscous dissipation. Gowda et al. [9] investigated the incompressible viscous dusty hybrid nanomaterial across a stretching cylinder, with dissipation and DF phenomena. They reported that increasing the curvature parameters increased the temperature and velocity of both liquids. Rasool et al. [10] investigated how the DF medium and radiation affect the Maxwell NF flow when it is subjected to a stretched surface. DF fluid flow with variable permeability and porosity was examined by several researchers [11–14].

Transportation of mass and heat are employed in a broad spectrum of industrial applications, including cars, energy systems, heating and air conditioning, steam-electric power production, electronic devices, and disease detection, but common liquids are unable to fulfill the demand for heat transmission. Consequently, the utilization of nanoparticles in the common fluid is highly suggestive. NF has received much attention in the recent decade, especially in renewable and sustainable energy systems and heat transfer enhancement. NF is utilized in hydropower rotors, ocean power plants, geothermal heat exchangers thermodynamics, wind turbines, and solar collectors [15]. In an experimental context, the dynamic viscosity of MWCNT-MgO/EG hybrid nano liquid was examined by Soltani and Akbari [16]. The results reveal that the dynamic viscosity enhances up to 168% by enhancing the number of nanoparticles from 0.1 to 1%. Waqas et al. [17] analyzed the features of thermal radiation in Sisko nano liquid flow with the mutual effect of heat and mass transmission characteristics using gold nanoparticles over an extending surface. It has been observed that the energy profile of the nanofluid is elevated for both the volume friction of nanoparticles and thermal radiation. Li et al. [18] used gold nanofluid as an optical filter for increasing thermal collection and balancing the energy in the photovoltaic/thermal system. The behavior of blood carrying gold nanoparticles over a curved surface is examined by Khan et al. [19]. Mbambo et al. [20] explore the effectiveness of raising the thermal conductivity of gold nanoparticles in graphene sheets by using an EG-based nanofluid. The results revealed that by changing the structure of nanoparticles, the rate of thermal energy transmission can be boosted. The Buongiorno model is applied by Mishra and Upreti [21] to evaluate the heat and mass transmission processes of FeO-CoFeO$_3$/water-EG hybrid nanomaterial and Ag-MgO/water hybrid nanofluid under the effects of chemical reaction and dissipation. Najiyah et al. [22] numerically evaluated the 3D flow of the magnetic nanofluids (Mn-ZnFe$_2$O$_4$-water, CoFeO$_3$-water, and Fe$_3$O$_4$-water) shrinking surface with radiation phenomena. The influence of the Darcy–Forchheimer and Coriolis forces on nanofluid flow consisting of carbon nanotubes (CNTs) in EG over a revolving frame was explained by Ullah et al. [23]. Entropy is characterized by a growing trend in the temperature ratio and Brinkman number. Ullah et al. [24,25] explored the behavior of melting thermal expansion and activation energy on an unstable Prandtl–Eyring fluid caused by an extended cylinder.
Entropy generation is the quantity of entropy produced by irreversible processes such as diffusion, heat flow via Joule heating, thermal resistance, fluid flow via flow resistance, friction between solid surfaces, the viscosity of a fluid within the system, and so on in a dynamical system [26]. In a reversible reaction, the net entropy of the system remains constant. The entire entropy of a system increases when a nanofluid moves across a substrate by different irreversible mechanisms such as flow resistance, diffusion, thermal resistance, viscosity-induced friction between fluid layers, Joule heating, and so on. Entropy production in a system plays a significant role in decaying the system’s needed energy sources. Thus, researchers focus on improving the performance and effectiveness of various sectors to decay the production of entropy [27,28]. Ikram et al. [29] discuss the entropy considerations associated with the Darcy–Forchheimer flow of hybrid nanomaterial. Their findings reveal that an increase in temperature caused an increment in the Eckert number and the heat source parameter for both nano liquids and hybrid nano liquids. Fares et al. [30] applied a finite element approach to characterize the progressive feature of created entropy in a permeable square enclosure. Aziz et al. [31] discussed entropy optimization in the radiative flow of a nanofluid in a rotating frame with activation energy. Alsallami et al. [32] have quantitatively inspected the NF flow over a heated spinning disk under the effects of nonlinear radiation, Brownian motion, and thermophoresis. The entropy rate and Bejan number are thought to improve as the chemical process, temperature differential factor, and Schmidt number improve. Ullah et al. [33] analyzed the numerical investigation of the entropy characteristics of the hybrid nanoparticles with slip effect.

The goal of the current work is to enhance the energy transference rate for industrial and biomedical applications. It has been observed that none of the previous efforts on the Darcy–Forchheimer hybrid NF flow consisting of gold and cobalt ferrite nanoparticles over an extending curved porous space with variable permeability were given in the literature. Additionally, radiation, Joule heating, and the traditional and exponential heat sources are considered to compute the entropy optimization. To fill the gap, the proposed model has been formulated in the form of a system of PDEs, which is converted into the system of ODEs through transformations. The obtained set of nonlinear differential equations is further processed numerically by the NDsolve technique to investigate the variations in entropy, velocity, temperature, Nusselt number, and skin friction against interesting physical variables. The present model is very useful in distinct areas of applied science, biomedicine, and industries like hybrid power generators, the cooling of nuclear reactors, electronic devices, and heat transportation in milk-tissue pasteurization. The following are major considerations for conducting this study:

- To investigate how hybrid nanofluid is more efficient than nanofluid and base fluid.
- To examine the thermal effects of hybrid nanomaterials with thermal radiation, EHS, and viscous dissipation.
- What are the effects of skin friction and Nusselt number in regard to relevant physical constraints?
- How does the addition of a Darcy–Forchheimer term with variable permeability and porosity features to the momentum equation impact the hybrid nanofluid flow?
- How does the inclusion of cobalt ferrite and gold nano particulates enhance the thermal efficiency of ethylene glycol?

2. Mathematical Modeling

Consider an incompressible two-dimensional hybrid NF flow over curved porous stretching surface with radius \( R \). The Darcy–Forchheimer law is utilized for present flow analysis with variable permeability. For the improvement of the thermal field, Joule heating is added to energy expression. The surface is extended in the \( s \)-direction with velocity \( u_w = b s \), where \( b > 0 \) and \( r \)-direction is Normal to \( s \)-direction (see Figure 1).
Here, \( b = 0 \) indicates the static sheet, \( b < 0 \) demonstrates the shrinking, and \( b > 0 \) specifies the stretching sheet. In this work, we scrutinize the flow across a stretching sheet; therefore, we only consider \( b > 0 \) in this study. Cobalt ferrite and gold nanoparticles were added to ethylene glycol and water to form the hybrid nanofluid. A magnetic field with strength \( B_0 \) is incorporated in the \( r \)-direction. The temperature at the surface is specified as \( T_s \). Heat source and thermal radiation in the energy expression is added to further investigate the temperature variation. Additionally, the second law of thermodynamics is employed to compute entropy production. On the basis of the above-described presumptions, the leading equations can be stated as [34–37]:

\[
\frac{\partial}{\partial r}((r+R)v) + R\frac{\partial u}{\partial s} = 0,
\]

\[
\frac{\partial p}{\partial r} = \frac{\rho_{inf}}{r+R} u^2,
\]

\[
\frac{v\frac{\partial u}{\partial r} + \left( \frac{R}{R+r} \right) u \frac{\partial u}{\partial s} + \frac{v}{R+r} u}{R+u} = \frac{-R}{\rho_{inf}} \frac{\partial p}{\partial s} + v_{inf} \left( \frac{\partial^2 u}{\partial r^2} - \frac{1}{(R+r)^2} u + \frac{1}{(R+r) \frac{\partial u}{\partial r}} \right) - \frac{\sigma_{inf}}{\rho_{inf}} B_0^2 u - \frac{v_{inf} \varepsilon(r)}{k'(r)} u - \frac{C_p \varepsilon^2(r)}{(k'(r))^2} u^2,
\]

\[
\text{Figure 1. Physical sketch of flow over a curved configuration. (a) Flow analysis (b) Preparation of hybrid nanofluid.}
\]
\[
\frac{v \partial T}{\partial r} + u \left( \frac{R}{R + r} \right) \frac{\partial T}{\partial s} = \frac{k_{\text{haf}}}{(\rho C_p)_{\text{haf}}} \left( \frac{\partial^2 T}{\partial r^2} - \left( \frac{1}{R + r} \right) \frac{\partial T}{\partial r} \right) + \frac{16 \sigma^* T^3}{3k(T)_{\text{haf}}} \left( \frac{\partial^2 T}{\partial r^2} - \frac{1}{R + r} \frac{\partial T}{\partial r} \right) \\
+ \frac{\sigma_{\text{haf}} B_0^2 u^2}{(\rho C_p)_{\text{haf}}} (T - T_\infty) + \frac{Q}{(\rho C_p)_{\text{haf}}} (T - T_\infty) \exp \left( - \left( \frac{u_w}{V_f} \right)^{\frac{3}{2}} r \right) + \frac{\mu_{\text{haf}}}{(\rho C_p)_{\text{haf}}} \varepsilon(r) \frac{1}{k^*(r)} u^2 \tag{4}
\]

where

\[
k^*(r) = k_c \left( 1 + d_2 e^{-\gamma r} \right), \quad \varepsilon(r) = \varepsilon_c \left( 1 + d_3 e^{-\gamma r} \right). \tag{5}
\]

where \((u,v)\) demonstrate the velocity component, while \(k^*\) is the porosity term, \(\varepsilon_c\) is the surface porosity, \(k_c\) is surface permeability, \(\gamma\) shows the constant having dimension of length, \(V_{\text{haf}}\) demonstrates the kinematic viscosity, \(d_1\) is the variable permeability and \(d_2\) is the porosity, \(\sigma_{\text{haf}}\) is electrical conductivity, \(\rho_{\text{haf}}\) is density, \(\sigma^*\) is the Stefan Boltzmann coefficient, \(k_{\text{haf}}\) is thermal conductivity, \(C_b\) is the drag coefficient, \(B_0\) is the magnetic field strength, \(Q\) is the heat generation/absorption, \(Qe\) is the exponential heat source, and \((\rho C_p)_{\text{haf}}\) is the volumetric heat capacity [34,35].

The relevant boundary conditions are: [34]

\[
\begin{align*}
&u = u_w = b s, \quad v = 0, \quad T = T_\infty, \quad \text{at} \quad r = 0, \\
&u \to 0, \quad v \to 0, \quad T \to T_\infty, \quad \text{when} \quad r \to \infty.
\end{align*} \tag{6}
\]

Considering the variables:

\[
\begin{align*}
&u = b s F'(\eta), \quad v = -\frac{R \left( b V_f \right)^{\frac{3}{2}}}{R + r} F(\eta), \quad \eta = \frac{u_w}{V_f} r, \\
&T + T_\infty = (T_\infty + T_\infty) \theta(\eta), \quad p = \rho_f (b s)^2 P(\eta).
\end{align*} \tag{7}
\]

Equation (1) is identically satisfied and the system of ODEs form Equations (2)–(4) and are transformed as:

\[
P' = \frac{A_1 F'^2}{K + \eta}; \tag{8}
\]

\[
\frac{2K}{A_1 (\eta + K)} = \frac{1}{A_1 A_2} \left( F'' - \frac{F'}{K + \eta} + \frac{F''}{(K + \eta)} \right) + \frac{K}{(K + \eta)} \left( \frac{F'}{(K + \eta)} - F'^2 + F'' \right) \tag{9}
\]

By putting Equation (9) into Equation (8) we have:
\[
F^{\prime\prime\prime} + \frac{2}{K + \eta} F^{\prime\prime} + \frac{A_1 A_2 K}{(K + \eta)} \left[ (F^{\prime\prime\prime} - F^{\prime\prime}) + \frac{F^{\prime\prime} + FF^{\prime\prime}}{(K + \eta)} - \frac{FF^{\prime}}{(K + \eta)^2} \right] \\
- A_2 A_3 \left( MF^{\prime\prime} + \frac{1}{(K + \eta)} MF^{\prime} - \frac{F^{\prime\prime}}{(K + \eta)^2} + \frac{F^{\prime}}{(K + \eta)^3} \right) \\
- \frac{K}{\alpha A_1 A_2 (K + \eta)^2} \left( (1 + d_2 e^{-\eta}) \right) F^{\prime} - \beta \frac{K \left( 1 + d_2 e^{-\eta} \right)^2}{(K + \eta)^2 \left( 1 + d_1 e^{-\eta} \right)^2} F^{\prime\prime} = 0,
\]

(10)

\[
\theta^{\prime\prime} + \frac{\theta^{\prime}}{\eta + K} + \frac{A_3 P r K}{A_3 A_4 + Ra(\eta + K)} F \theta^{\prime} + \frac{A_4 \times A_3}{A_4 A_4 + Ra} Br MF^{\prime} + \frac{PrS}{A_4 A_4 + Ra} \theta + Q e \left( e^{-\eta} \right) \theta \\
+ \frac{1}{A_4 A_4 + Ra} \left( E_c \left( \frac{F^{\prime\prime} + 1}{\eta + K} F^{\prime} \right) \right) + \frac{Br}{A_4 A_4 \alpha Re_s} \left( \frac{1 + d_2 e^{-\eta}}{1 + d_1 e^{-\eta} F^{\prime\prime}} \right) \\
+ \frac{A_5 \beta \left( 1 + d_2 e^{-\eta} \right)^2}{\left( 1 + d_1 e^{-\eta} \right)^2} F^{\prime\prime} = 0,
\]

(11)

where transformed boundary conditions are:

\[
\begin{align*}
F^{\prime}(\eta) & = 1, \quad F(\eta) = 0, \quad \theta(\eta) = 1, \quad \text{at} \ \eta = 0, \\
F^{\prime}(\eta) & \to 0, \quad F^{\prime\prime}(\eta) \to 0, \quad \theta(\eta) \to 0, \quad \text{as} \ \eta \to \infty.
\end{align*}
\]

(12)

In above expressions

\[
A_1 = (1 - \phi_{Au}) \left\{ (1 - \phi_{CoFeO_4}) + \phi_{CoFeO_4} \frac{\rho_{CoFeO_4}}{\rho_f} \right\} + \phi_{Au} \frac{\rho_{Au}}{\rho_f}, \quad A_2 = (1 - \phi_{CoFeO_4}) \left( 1 - \phi_{CoFeO_4} \right)^{2.5}
\]

(13)

The dimensionless variables are:
\[ K = R \left( \frac{b}{\sqrt{v'}} \right), \quad S = \frac{Q}{b(\rho C_p)_f}, \quad Pr = \left( \frac{\mu C_p}{k_f} \right)_f, \quad \beta = \frac{1}{k_{\infty}^{1/2}} C_p r \varepsilon, \quad Qe = \frac{Q}{u_s (C_p)_f}, \quad \alpha = \frac{k_{\infty}}{\rho f u_w} \varepsilon \mu_f \]

\[ M = \frac{\sigma B_0^2}{b(\rho f)}, \quad Ec = \frac{u_w^2}{(T_w - T_\infty) (C_p)_f}, \quad Ra = 16\sigma^* T_\infty^3, \quad Br = Pr Ec = \frac{\mu_f (bs)^2}{(T_w - T_\infty) k_f}. \]

(14)

In Equation (14), \( K \) represent the dimensionless curvature variable, \( S \) is the heat source, \( Pr \) is the Prandtl number, \( \beta \) is the non-uniform inertia factor, \( Qe \) is the exponential heat source variables, \( \alpha \) is the permeability term, \( M \) is the magnetic variable, \( Ec \) is the Eckert number, \( Ra \) is the radiation parameter, and \( Br \) depict the Brinkman number.

Skin friction is \( C_f \) and Nusselt number are \( Nu_s \) specified as:

\[ C_f = \frac{\tau_{rs}}{\rho_f u_w^2}, \quad Nu_s = \frac{s q_w}{k_f (T_w - T_\infty)}, \]

where wall shear stress \( \tau_{rs} \) and heat flux \( q_w \) are specified as:

\[ \tau_{rs} = \mu_{hf} \left( \frac{\partial u}{\partial r} - \frac{u}{r + R} \right) \bigg|_{r=0}, \quad q_w = -k_{hf} \left( 1 + 16\sigma^* T_\infty^3 \frac{k_f}{3 k_j} \right) \frac{\partial T}{\partial r} \bigg|_{r=0}. \]

(15)

Upon using Equation (7), the above equations become:

\[ (Re_s)^{1/2} C_f = \frac{1}{A_2} \left( f''(0) - \frac{1}{K} f'(0) \right), \quad (Re_s)^{1/2} Nu_s = A_4 \left( 1 + \frac{Ra A_4}{A_3} \right) \theta'(0), \]

(17)

where Reynolds’s number \( Re_s = \frac{b s^2}{V_f} \).

3. Entropy Analysis

Entropy formation for current problem is defined as [34,35]:

\[ S_{gen} = \frac{k_{hf}}{T_\infty^2} \left( \frac{\partial T}{\partial r} \right)^2 + k_{hf} \frac{16\sigma^* T_\infty^3}{3 k_j^2} \left( \frac{\partial T}{\partial r} \right)^2 + \frac{\mu_{hf}}{T_\infty} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{\sigma_{hf} B_0^2}{T_\infty} u^2 + \frac{Q}{T_\infty} (T - T_\infty) \]

\[ + \frac{Q e}{(\rho C_p)_hf} (T - T_\infty) \exp \left( - \frac{u_w}{V_f} \right)^2 r + \frac{\mu_{hf}}{(\rho C_p)_hf} \varepsilon(r) u^2 + \frac{\rho_{hf}}{(\rho C_p)_hf} \varepsilon(r) \frac{\mu_{hf}}{(k^* (r))^2} u^3. \]

(18)

Equation (18) can be revised as follows with the assistance of Equation (7):

\[ N_{EG} = A_4 A_4 \gamma_1 \theta^2 + A_4 A_4 \gamma_1 R a \theta^2 + \frac{Br}{A_2} f''^2 + A_3 A_3 Br^2 \theta' + Pr \theta + \frac{Pr Q e}{A_5} e^{-\eta} \theta + \frac{Br}{A_5 A_4 \alpha} Re_s \]

\[ \left( 1 + d_3 e^{-\eta} \right) F^2 + \frac{A_3 \beta}{A_5} \left( 1 + d_3 e^{-\eta} \right) F^3. \]

(19)

where \( N_{EG} \) denote the rate of entropy generation and \( \gamma_1 \), the temperature ratio parameter, and are specified as:
\[ N_{E_G} = \frac{S_{\text{conv}} V_f T_\infty}{k_j b (T_w - T_x)} \], \quad \gamma_1 = \frac{T_f - T_\infty}{T_\infty}. \] (20)

The Bejan number can be found as:

\[ Be = \frac{\text{Entropy generation on account of heat transfer}}{\text{Entropy generation}}, \]

By employing Equation (19), Be can be expressed as:

\[ Be = \frac{A_A A_A \gamma' f^2 + PrS \theta + \frac{Pr S}{A_S} e^{-\omega \theta}}{A_A A_A \gamma' f^2 + A_A A_A \gamma' Ra f^2 + \frac{Br}{A_2} f^* + A_A A_A Br M f^2 + PrS \theta + \frac{Pr S}{A_S} e^{-\omega \theta}}. \] (21)

4. Numerical Solution

For the results, the system of Equations (10) and (11) with the boundary condition (12) are further solved through the NDSolve technique. This method solves differential systems numerically. The evaluation of data is automatically discretized using this method. The system of ODEs contains a number of equations such as \( h_1, h_2, h_3, \ldots, h_n \), independent variables \( \xi \), dependent variables \( n \), and \( \text{i.e. } N_1, N_2, N_3, \ldots, N_n \), as well as conditions for the boundaries of the domain based on the sequence of the PDEs mechanism. This system can be computed using the NDSolve technique as follows:

\[ \text{NDSolve} \{ \{ h_1, h_2, h_3, \ldots, h_n \}, \{ N_1, N_2, N_3, \ldots, N_n \}, \{ \xi, \xi_{\text{min}}, \xi_{\text{max}} \} \}. \]

This approach achieves remarkable accuracy and is consistently steady. In addition, it yields top performance with minimal CPU usage and short expressions.

5. Discussion or Outcomes

This section analyzes the variation of velocity, skin friction, and temperature gradient against the variation of numerous physical parameters such as curvature parameter \( K \), magnetic field \( M \), variable permeability \( d_x \), variable porosity \( d_y \), inertia coefficient \( \beta \), permeability parameter \( \alpha \), Brinkman number \( Br \), exponential heart source (EHS) parameter \( Q_e \), radiation parameter \( Ra \), heat source parameter \( S \), and Eckert number \( Ec \) for hybrid nanoliquid and nanofluid. For such aims, Figures 2–12 and Tables are declared. In these figures, solid lines manifest the nanofluid, whereas dashed lines show the hybrid nanofluid. Tables 1 and 2 depict the thermophysical relation of nano and hybrid nanomaterials, respectively. Figure 2 elaborates on the role of volume friction on \( F'(\eta) \). The resistance to fluid flow rises as the volume fraction of nanoparticles rises; as a result, the velocity declines. It is also manifest from Figure 2 that hybrid nanofluid has more contribution to the reduction in velocity when compared with nanofluid.

**Table 1.** Thermo-physical features of water + EG and considered \( Au + CoFeO_4 \).

<table>
<thead>
<tr>
<th></th>
<th>( k (\text{W/mK}) )</th>
<th>( \sigma (\Omega \text{m})^{-1} )</th>
<th>( \rho (\text{kg/m}^3) )</th>
<th>( C_p (\text{J/kgK}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water + EG</td>
<td>0.425</td>
<td>0.00509</td>
<td>1056</td>
<td>3288</td>
</tr>
<tr>
<td>( CoFeO_4 )</td>
<td>3.7</td>
<td>5.51 × 10^9</td>
<td>4907</td>
<td>700</td>
</tr>
<tr>
<td>( Au )</td>
<td>318</td>
<td>4.1 × 10^-7</td>
<td>19,300</td>
<td>129</td>
</tr>
</tbody>
</table>
Table 2. Physical relations for hybrid nanofluids ($\phi_1 = \phi_{\text{CoFe}_2\text{O}_4}$, $\phi_2 = \phi_{\text{Au}}$).

<table>
<thead>
<tr>
<th>Properties</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity</td>
<td>$\mu_{\text{inf}} = \frac{1}{(1 - \phi_{\text{CoFe}_2\text{O}<em>4} - \phi</em>{\text{Au}})^2}$</td>
</tr>
<tr>
<td>Density</td>
<td>$\frac{(\rho)<em>{\text{inf}}}{(\rho)</em>{\text{bf}}} = (1 - \phi_{\text{CoFe}<em>2\text{O}<em>4}) \left( 1 - \frac{\rho</em>{\text{Au}}}{\rho</em>{\text{bf}}} \right) \phi_{\text{Au}} + \phi_{\text{CoFe}_2\text{O}<em>4} \left( \frac{\rho</em>{\text{CoFe}_2\text{O}<em>4}}{\rho</em>{\text{bf}}} \right)$</td>
</tr>
<tr>
<td>Thermal Capacity</td>
<td>$\frac{(\rho C_p)<em>{\text{inf}}}{(\rho C_p)</em>{\text{bf}}} = \phi_{\text{CoFe}<em>2\text{O}<em>4} \left( \frac{(\rho C_p)</em>{\text{CoFe}<em>2\text{O}<em>4}}{(\rho C_p)</em>{\text{bf}}} \right) + \phi</em>{\text{Au}} \left( \frac{(\rho C_p)</em>{\text{Au}}}{(\rho C_p)<em>{\text{bf}}} \right) + (1 - \phi</em>{\text{CoFe}_2\text{O}<em>4} - \phi</em>{\text{Au}})$</td>
</tr>
<tr>
<td>Thermal Conductivity</td>
<td>$\frac{k_{\text{inf}}}{k_{\text{bf}}} = \left[ \frac{\phi_{\text{CoFe}<em>2\text{O}<em>4} k</em>{\text{CoFe}<em>2\text{O}<em>4} + \phi</em>{\text{Au}} k</em>{\text{Au}}}{\phi</em>{\text{CoFe}<em>2\text{O}<em>4} + \phi</em>{\text{Au}}} \right] + 2k</em>{\text{bf}} - 2 \left( k_{\text{CoFe}<em>2\text{O}<em>4} \phi</em>{\text{CoFe}<em>2\text{O}<em>4} + k</em>{\text{Au}} \phi</em>{\text{Au}} \right) + \left( \phi</em>{\text{CoFe}<em>2\text{O}<em>4} + \phi</em>{\text{Au}} \right) 2k</em>{\text{bf}}$</td>
</tr>
<tr>
<td>Electrical Conductivity</td>
<td>$\frac{\sigma_{\text{inf}}}{\sigma_{\text{bf}}} = \left[ \frac{\phi_{\text{CoFe}<em>2\text{O}<em>4} \sigma</em>{\text{CoFe}<em>2\text{O}<em>4} + \phi</em>{\text{Au}} \sigma</em>{\text{Au}}}{\phi</em>{\text{Au}} + \phi_{\text{CoFe}<em>2\text{O}<em>4}} \right] + 2\sigma</em>{\text{bf}} - \left( \phi</em>{\text{CoFe}<em>2\text{O}<em>4} \sigma</em>{\text{CoFe}<em>2\text{O}<em>4} + \phi</em>{\text{Au}} \sigma</em>{\text{Au}} \right) + \left( \phi</em>{\text{CoFe}<em>2\text{O}<em>4} + \phi</em>{\text{Au}} \right) \sigma</em>{\text{bf}}$</td>
</tr>
</tbody>
</table>

Figure 2. The impact of volume friction versus velocity $F'(\eta)$. 
Curves for velocity against various values of the curvature parameter and magnetic field is depicted in Figure 3a, b, respectively. An increasing behavior of velocity in Figure 3a is observed for different values of curvature parameter. A high estimation of \( K \) leads to uplift in the surface radius, which improves the velocity. The features of \( M \) on \( F'(\eta) \) is seen in Figure 3b. Here, velocity of the hybrid nanofluid slows down for a higher estimation of \( M \). The magnetic effect induces the Lorentz force, which causes a resistive force to the flow field; consequently, the radial velocity of the flow reduces. A zero value of the magnetic parameter, i.e., \( M = 0 \), corresponds to the hydrodynamic situation. When compared with nanofluid, hybrid nanofluid has a greater contribution toward enhancing the velocity, as shown by Figure 3a, whereas Figure 3b demonstrates the contrary tendency.

Figure 4a, b demonstrate the impact of \( d_1, F'(\eta) \). In this case, velocity \( F'(\eta) \) increases as the variable permeability \( d_1 \) increases (see Figure 4a), while \( d_2 \) experiences the opposing trend (see Figure 4b). Moreover, Figure 4a indicates that hybrid nanofluid contributed more to velocity increase than nanofluid, while the opposite trend
is observed in Figure 4b. The behavior of the inertia coefficient $\beta$ on $F'(\eta)$ is illustrated in Figure 5a. An abatement is observed in velocity for a higher value of $\beta$ because of an increase in internal force. Figure 5b describes the effects of $\alpha$ on velocity $F'(\eta)$. By enlarging the value of $\alpha$ the velocity of the nanomaterial is also augmented. When compared with nanofluid, it is clear that hybrid nanofluid has a greater contribution to the decrease in velocity, as shown by Figure 5a.

![Figure 5](image-url)

**Figure 5.** The variation in velocity $F'(\eta)$ (a) versus inertia coefficient $\beta$ (b) versus permeability parameter $\alpha$.

![Figure 6](image-url)

**Figure 6.** The variation in temperature $\theta(\eta)$. (a) versus volume friction (b) versus EHS parameter $Qe$. 
An escalating behavior is perceived in Figure 6a for different values of volume friction on temperature. As the volume fraction of nanoparticles in the fluid raises, more friction between the particles of the fluid arises, by which the temperature of the hybrid nanomaterial is elevated. As depicted in Figure 6b, $Q_e$ has a significant impact on $\theta(\eta)$. A rise in $Q_e$, causes an enhancement in $\theta(\eta)$. Figure 6a,b show that hybrid nanofluid boosted temperature more than nanofluid. Figure 7a depicts the rising trend of inertia coefficient $\beta$ versus temperature $\theta(\eta)$. The inertia coefficient increases fluid motion resistance, resulting in more heat being created and strengthening the temperature field. Figure 7b captured that $\theta(\eta)$ is an increasing function of $Br$, because $Br$ has a direct relation with the formation of heat through fluid friction, which leads to greater $\theta(\eta)$. Figure 7a,b demonstrate hybrid nanofluid raises temperature more than nanofluid.

Figure 7. The variation in temperature $\theta(\eta)$ (a) versus inertia coefficient $\beta$ (b) versus Brinkman number $Br$.

Figure 8. The variation in temperature $\theta(\eta)$ (a) versus variable permeability $d_1$ (b) versus variable porosity $d_2$.  

The variation in variable permeability $d_1$ and variable porosity $d_2$ against temperature $\theta(\eta)$ is seen in Figure 8a,b. An improvement in $\theta(\eta)$ is reported for a higher value of $d_1$ (see Figure 8a), whereas the reverse tendency is exhibited against a different value of $d_2$ (see Figure 8b). Figure 8a shows that hybrid nanofluid dropped temperature more than nanofluid, while Figure 8b illustrates the inverse.

(a)

(b)

**Figure 9.** The variation in and temperature $\theta(\eta)$ (a) versus radiation parameter $Ra$ (b) versus heat source parameter $S$.

Figure 9a is devoted to analyzing the influence of $Ra$ on temperature. The motion of charged particles in a fluid accelerates as thermal radiation rises, which boosts the temperature. As portrayed in Figure 9b, that increment in heat source parameter $S$ improves the temperature. Greater heat source parameters result in a thicker thermal boundary layer, which raises the temperature.

**Figure 10.** The impact of (EHS) parameter $Qe$ versus $N_G(\eta)$.

As we observed in previous graphs that with a high value of EHS parameter $Qe$, thermal radiation $Ra$, magnetic field $M$, Brickman number $Br$, and the temperature of the fluid increase. Vibrations and internal displacement are two additional phenomena that occur as the fluid temperature rises. Consequently, the fluid entropy improves. As
indicated in Figures 10, 11a,b, and 12a,b, a rise in radiation parameter $\bar{Ra}$, magnetic field $M$, $Br$ Brinkman, and $Ec$ Eckert numbers leads to development in the production of entropy in the fluid. The entropy optimization is demonstrated to be enhanced significantly more by hybrid nanofluid than by nanofluid in Figures 10–12.

![Figure 11](image1.png)

**Figure 11.** The variation in $N_G(\eta)$. (a) versus radiation parameter $\bar{Ra}$  (b) versus magnetic field $M$.

![Figure 12](image2.png)

**Figure 12.** The variation in $N_G(\eta)$. (a) versus Brinkman number $Br$  (b) versus Eckert number $Ec$.

Table 3 illustrates the influence of skin friction coefficient $\sqrt{ReCf}$ against various variables such as curvature parameter $K$, magnetic parameter $M$, variable permeability $d$, variable porosity $d$, inertia coefficient $\beta$, and permeability parameter $\alpha$. The coefficient of skin friction diminishes for higher values of variable permeability $d$. The behavior of the Nusselt number $Nu$, for the different physical parameters is depicted in Table 4. Nusselt number $Nu$, enhances because there is more heat owing to thermal radiation $Ra$, EHS parameter $Qe$, Brinkman number $Br$, variable permeability $d$, and heat source parameter $S$. Table 5 exemplifies the comparison of the current study with the previously published work [37–39] for $m = 1$ and various values of curvature parameters $K$. The analysis reveals a very close match.
Table 3. Computing results of $\sqrt{Re} \cdot Cf_r$ for $\beta$, $M$, $d_1$, $d_2$ and $\alpha$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\sqrt{Re} \cdot Cf_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nanofluid</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$M$</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0</td>
</tr>
<tr>
<td>0.15</td>
<td>0.884051</td>
</tr>
<tr>
<td>1.20</td>
<td>0.980286</td>
</tr>
<tr>
<td>0.10</td>
<td>0.5</td>
</tr>
<tr>
<td>1.0</td>
<td>1.156850</td>
</tr>
<tr>
<td>1.5</td>
<td>1.158347</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0</td>
</tr>
<tr>
<td>6.5</td>
<td>1.182130</td>
</tr>
<tr>
<td>7.2</td>
<td>1.158347</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0</td>
</tr>
<tr>
<td>0.10</td>
<td>0.889332</td>
</tr>
<tr>
<td>0.10</td>
<td>0.903041</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.6</td>
<td>1.31930</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.24639352</td>
</tr>
<tr>
<td>0.3</td>
<td>1.10356053</td>
</tr>
<tr>
<td>0.4</td>
<td>0.86776427</td>
</tr>
<tr>
<td>0.2</td>
<td>0.65271754</td>
</tr>
<tr>
<td>0.5</td>
<td>0.47516205</td>
</tr>
<tr>
<td>0.6</td>
<td>0.42130812</td>
</tr>
<tr>
<td>0.6</td>
<td>0.49321677</td>
</tr>
<tr>
<td>0.2</td>
<td>0.50219742</td>
</tr>
<tr>
<td>0.2</td>
<td>0.42130812</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6219873</td>
</tr>
<tr>
<td>0.2</td>
<td>0.74307507</td>
</tr>
<tr>
<td>0.2</td>
<td>0.69315325</td>
</tr>
<tr>
<td>0.2</td>
<td>0.61025614</td>
</tr>
<tr>
<td>0.2</td>
<td>1.20398502</td>
</tr>
<tr>
<td>0.2</td>
<td>1.63874523</td>
</tr>
<tr>
<td>0.2</td>
<td>0.53866970</td>
</tr>
<tr>
<td>0.2</td>
<td>0.40721617</td>
</tr>
<tr>
<td>0.2</td>
<td>0.21309744</td>
</tr>
</tbody>
</table>

Table 4. Numerical outputs for Nusselt number $\left( \frac{k_{nf}}{k_f} \theta'(0), \frac{k_{inf}}{k_f} \theta'(0) \right)$ for $\beta$, $Br$, $d_1$, $d_2$, $Ra$, and $Qe$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\frac{k_{nf}}{k_f} \theta'(0)$</th>
<th>$\frac{k_{inf}}{k_f} \theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$Br$</td>
<td>$d_1$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>0.3</td>
<td>1.24639352</td>
<td>1.24639463</td>
</tr>
<tr>
<td>0.4</td>
<td>1.10356053</td>
<td>1.10356145</td>
</tr>
<tr>
<td>0.2</td>
<td>0.86776427</td>
<td>0.86776535</td>
</tr>
<tr>
<td>0.5</td>
<td>0.65271754</td>
<td>0.65271869</td>
</tr>
<tr>
<td>0.6</td>
<td>0.47516205</td>
<td>0.47516319</td>
</tr>
<tr>
<td>0.2</td>
<td>0.42130812</td>
<td>0.42130926</td>
</tr>
<tr>
<td>0.5</td>
<td>0.49321677</td>
<td>0.49321789</td>
</tr>
<tr>
<td>0.6</td>
<td>0.50219742</td>
<td>0.50219873</td>
</tr>
<tr>
<td>0.2</td>
<td>0.74307507</td>
<td>0.74307619</td>
</tr>
<tr>
<td>0.2</td>
<td>0.69315325</td>
<td>0.69315406</td>
</tr>
<tr>
<td>0.2</td>
<td>0.61025614</td>
<td>0.61025827</td>
</tr>
<tr>
<td>0.2</td>
<td>1.20398502</td>
<td>1.20398613</td>
</tr>
<tr>
<td>0.6</td>
<td>1.52109772</td>
<td>1.52109891</td>
</tr>
<tr>
<td>0.2</td>
<td>1.63874523</td>
<td>1.63874352</td>
</tr>
<tr>
<td>0.3</td>
<td>0.53866970</td>
<td>0.53867081</td>
</tr>
<tr>
<td>0.2</td>
<td>0.40721617</td>
<td>0.40721708</td>
</tr>
<tr>
<td>0.5</td>
<td>0.21309744</td>
<td>0.21309865</td>
</tr>
</tbody>
</table>
Table 5. The current results for the skin friction coefficient $\sqrt{Re \, Cf_r}$ with the [37–39] for $m = 1$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>Sanni et al. [37]</th>
<th>Sajid et al. [38]</th>
<th>Zaheer et al. [39]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.1576</td>
<td>0.7576</td>
<td>1.1576</td>
<td>1.1584</td>
</tr>
<tr>
<td>10</td>
<td>1.0734</td>
<td>0.8735</td>
<td>1.0735</td>
<td>1.0738</td>
</tr>
<tr>
<td>20</td>
<td>1.0355</td>
<td>0.9356</td>
<td>1.0356</td>
<td>1.0339</td>
</tr>
<tr>
<td>30</td>
<td>1.0235</td>
<td>0.9569</td>
<td>1.0235</td>
<td>1.0240</td>
</tr>
<tr>
<td>40</td>
<td>1.0176</td>
<td>0.9676</td>
<td>1.0176</td>
<td>1.0171</td>
</tr>
<tr>
<td>50</td>
<td>1.0140</td>
<td>0.9741</td>
<td>1.0141</td>
<td>1.0147</td>
</tr>
<tr>
<td>100</td>
<td>1.0070</td>
<td>0.9870</td>
<td>1.0070</td>
<td>1.0083</td>
</tr>
<tr>
<td>200</td>
<td>1.0036</td>
<td>0.9936</td>
<td>1.0036</td>
<td>1.0042</td>
</tr>
<tr>
<td>1000</td>
<td>1.0008</td>
<td>0.9988</td>
<td>1.0008</td>
<td>1.0005</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

6. Conclusions

The objective of this study was to numerically simulate the optimization of entropy in radiative DF hybrid NF flow over a curved stretching surface. Cobalt ferrite and gold nanoparticles are added to ethylene glycol and water to form the hybrid nanofluid. The present work is influenced by the desire to increase the rate of heat transmission for engineering and industrial usage. The key findings are:

- Heat transfer efficiency is escalated by the combination of ethylene glycol and water.
- The inclusion of Au and CoFe$_2$O$_4$ nanoparticles in the base fluid positively affect heat transmission.
- Velocity distribution is enhanced with the higher values of variable permeability $d_1$, while it diminishes with the variable porosity factor and inertia coefficient.
- Temperature $\theta(\eta)$ augmented with the enhancing trend of exponential heat source and the Brinkman number.
- Entropy generation improves with the increment of the radiation and magnetic effect.
- Skin friction shows a decay behavior against the higher value of $d_1$.
- The Nusselt number improves for high values of $Br$ while diminishing for $Ra$ and $Qe$.
- Overall hybrid nanofluid has dominant behavior as compared to nanofluid.
- The present problem may be computed through various techniques as future work [40–45].

**Author Contributions:** Conceptualization, I.U.; methodology, I.U.; software, A.U.H.; validation, A.U.H.; formal analysis, H.K.; resources, W.W.; writing—original draft preparation, A.U.H.; writing—review and editing, A.M.G.; visualization, W.W.; supervision, I.U. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.
Nomenclature

$F'$ Dimensionless velocity
$F$ Stream function
$\eta$ Similarity variable
$\rho$ Density [Kg m$^{-3}$]
$\mu$ Dynamic viscosity [Kg m$^{-1}$ s$^{-1}$]
$\alpha$ Permeability parameter
$T_\infty$ Ambient temperature [K]
$B_r$ Brinkman number
$T$ Temperature of fluid [K]
$q^*$ Kinematic viscosity [m$^{-2}$ s$^{-1}$]
$s_2$ Variable porosity
$k_*$ Variable permeability of porous medium
$Q_e$ Exponential heat source
$\varepsilon_\alpha$ Surface porosity, $C_b$ Drag coefficient
$k$ Thermal conductivity [W m$^{-1}$ K$^{-1}$]
$k_\alpha$ Surface permeability
$u_\infty$ Stretching velocity
$E_c$ Eckert number
$\gamma_1$ Temperature ratio parameter
$\xi$ Porosity of the porous medium
$T_w$ Wall temperature [K]
$P$ Pressure [Kg m$^{-1}$ s$^{-2}$]
$u, v$ Velocity components
$K$ Curvature parameter
$B_0$ Magnetic field strength
$\theta$ Dimensionless temperature
$C_p$ Specific heat capacity [J Kg K$^{-1}$]
$M$ Dimensionless magnetic field
$b < 0$ Shrinking sheet
$Ra$ Radiation variable
$Pr$ Prandtl number
$q_w$ Heat flux
$S$ Dimensionless heat source parameter
$\gamma$ Length dimension,

Subscripts

$Bf$ Base fluid
$Nf$ Nanofluid
$inf$ Hybrid nanofluid
$f$ Fluid

References


44. Stynes, M.; Stynes, D. *Convection-Diffusion Problems*; American Mathematical Society: Providence, RI, USA, 2018; Volume 196.