Approximating Common Solution of Minimization Problems Involving Asymptotically Quasi-Nonexpansive Multivalued Mappings

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Abstract: In this paper, an iterative scheme for finding common solutions of the set of fixed points for a pair of asymptotically quasi-nonexpansive mapping and the set of minimizers for the minimization problem is constructed. Using the idea of the jointly demicloseness principle, strong convergence results are achieved without imposing any compactness condition on the space or the operator. Our results improve, extend and generalize many important results in the literature.

Keywords: strong convergence; variational inequality; asymptotically quasi-nonexpansive multivalued mapping; modified proximinal point algorithm; common fixed point; hilbert space

1. Introduction

Over the years, fixed point theory has been revealed to be a powerful and all-encompassing tool in the study of nonlinear phenomena. In this direction, several researchers have obtained different results relating to practical consequences of this field of study in areas such as tomography, quantize design, signal enhancement, signal and image reconstruction, signal and filter synthesis, telecommunications, interpolation, extrapolation, and several others. An instance of the above applications can be seen in [1]. It is a known fact that many interesting problems that dwell in physical systems could be recast as fixed-point problems. For instance, let \( \varpi \) be a signal of interest and let \( \varrho \) be its accompanying distorted version. Suppose, in addition, that \( \varpi \) and \( \varrho \) are related by the operator equation

\[
\varrho = \varpi + u(\varrho - \varpi),
\]

where we have assumed that \( \varrho \) is the signal measured at the reception point of the transmission system \( \varpi \) and \( \varrho \) is the transmitted signal. The problem of interest is how to approximate \( \varrho \) given \( \varrho \) and the model \( \varpi \) of the distortion that \( \varrho \) underwent. Assume \( \varrho \) satisfies the constraint equation

\[
\varrho = \varpi.
\]

Then, the identity

\[
\varrho = \varrho + u(\varrho - \varpi(\varrho))
\]

holds. Under the seemingly general condition, the solution to the equation will give the unknown signal \( \varrho \).

Recently, the concept of symmetry, a congenial characteristic of a Banach space, which is closely connected to fixed point problems [2], has drawn the attention of renowned
mathematicians worldwide. This unwavering interest has been known to stem from the practical application of this subject to different fields. Recall that a symmetry is a mapping of the object \( X \), considered to be structured onto itself such that the structure is preserved. Saleem et al. [3] and Saint [4], illustratively gave different ways this mapping could occur. Neugebauer [2], using the concept of symmetry, obtained several applications of a layered compression-expansion fixed-point theorem in the existence of solutions of a second-order difference equation with Dirichlet boundary conditions. Let \( Y \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and induced norm \( \| \cdot \| \). In this paper, \( \Theta \) denotes the nonempty, close and convex subset of \( Y \) and by \( \mathbb{N} \), the set of natural numbers. Additionally, if \( \{ \omega_n \} \) is any sequence in \( Y \), then we shall adopt the following notations:

1. \( \omega_n \to \omega^* \) means that \( \omega_n \) converges strongly to \( \omega^* \);
2. \( \omega_n \rightharpoonup \omega^* \) means that \( \omega_n \) converges weakly to \( \omega^* \).

Let \( \Delta : \Theta \to \Theta \) be a nonlinear mapping with domain \( D(\Delta) \) and range \( R(\Delta) \). The set of fixed points of \( \Delta \) will be denoted by \( F(\Delta) \), while the set of common fixed points of the mappings \( \Delta, \Gamma : \Theta \to \Theta \) will be denoted by \( F(\Delta) \cap F(\Gamma) \).

**Definition 1.** The mapping \( \Delta : \Theta \to \Theta \) is:

1. **Lipschitzian** if there exists a constant \( L \), such that
   \[
   \| \Delta \omega - \Delta q \| \leq L \| \omega - q \|, \text{ for all } \omega, q \in D(\Delta),
   \]
   where \( L \) is the Lipschitzian constant of \( \Delta \). It is worth noting that if \( L \in (0, 1) \) in (1), then \( \Delta \) is a contraction, while \( \Delta \) is called nonexpansive if \( L = 1 \) in (1).

2. **Asymptotically nonexpansive** (see [5]) if for all \( \omega, q \in D(\Delta) \), there exists a sequence \( \{k_n\} \subseteq [1, +\infty) \) with \( \lim_{n \to +\infty} k_n = 1 \) such that
   \[
   \| \Delta^n \omega - \Delta^n q \| \leq k_n \| \omega - q \|, \text{ for all } n \in \mathbb{N}.
   \]

3. **Uniformly Lipschitzian** if there exists a constant \( L \) such that
   \[
   \| \Delta^n \omega - \Delta^n q \| \leq L \| \omega - q \|, \text{ for all } \omega, q \in D(\Delta),
   \]

4. **Asymptotically quasi-nonexpansive** if \( F(\Delta) \neq \emptyset \) and (3) is satisfied; that is, if \( F(\Gamma) \neq \emptyset \) and for all \( (\omega \times q) \in D(\Delta) \times F(\Delta) \), there exists a sequence \( \{k_n\} \subseteq [1, +\infty) \) with \( \lim_{n \to +\infty} k_n = 1 \) such that
   \[
   \| \Delta^n \omega - \Delta^n q \| \leq k_n \| \omega - q \|, \text{ for all } n \in \mathbb{N}.
   \]

**Remark 1.** A uniformly Lipschitzian mapping is a superclass of the classes of nonexpansive mappings (which is a subclass of the class of asymptotically nonexpansive mapping) and asymptotically nonexpansive mappings, while the class of asymptotically quasi-nonexpansive mappings is a superclass of the classes of asymptotically nonexpansive mappings and quasi-nonexpansive mappings (recall that a nonlinear mapping \( \Delta : \Theta \to \Theta \) is called quasi-nonexpansive if \( F(\Delta) \neq \emptyset \) and for all \( (\omega \times q) \in D(\Delta) \times F(\Delta) \), we have \( \| \Delta \omega - \Delta q \| \leq \beta \| \omega - q \| \) (see examples 4.1, 4.3 and 4.9 in [6] for more details).

**Definition 2.** A nonlinear mapping \( \Delta : Y \to Y \) with domain \( D(\Delta) \subseteq Y \) and range \( R(\Delta) \subseteq Y \) is called:

1. **A strongly positive bounded linear operator** if there exists a constant \( \alpha > 0 \), such that inequality
   \[
   \langle \Delta \omega, \omega \rangle_Y \geq \alpha \| \omega \|^2, \text{ for all } \omega \in Y
   \]
   holds.

2. **Monotone** if
   \[
   \langle \omega - q, \Delta \omega - \Delta q \rangle \geq 0, \text{ for all } \omega, q \in D(\Delta).
   \]
\((g)\) \(\alpha\)-strongly monotone if there exists \(\alpha > 0\), such that
\[
\langle \omega - q, \Delta \omega - \Delta q \rangle \geq \alpha \| \omega - q \|^2, \text{ for all } \omega, q \in D(\Delta).
\] (7)

\((h)\) Let \(V : Y \to Y\), then \(\delta\)-inverse strongly monotone (for short \(\delta\)-ism), such that
\[
\langle \omega - q, \Delta \omega - \Delta q \rangle \geq \alpha \| V \omega - V q \|^2, \text{ for all } \omega, q \in D(\Delta).
\] (8)

**Remark 2.** The map \(I - \Xi\) is monotone if \(\Delta\) is nonexpansive. The projection operator \(P_\Xi\) is inverse strongly monotone. It is a common knowledge that inverse strongly monotone operators are indispensable in solving practical problems of traffic assignment problems (see [7,8] for more details).

Approximating a fixed point for single-valued mappings is an active area of research and has attracted many established mathematicians worldwide. Since exact solutions for practical problems in different fields of human endeavor are difficult to attain, approximation via iteration scheme has become a vital tool for solving fixed-point problems. Let \(f\) be a contraction map on \(\Upsilon\). Starting from an initial point \(\omega_1 \in \Upsilon\), define the sequence \(\{\omega_n\}\) iteratively, as follows
\[
\omega_{n+1} = \tau_n f(\omega_n) + (1 - \tau_n) \Delta \omega_n, n \geq 0,
\] (9)

where \(\{\tau_n\}\) is a sequence in \((0,1)\). The iteration sequence (9) was first introduced by Moudafi [9] and has been positively used to approximate fixed points of different nonlinear mappings in recent times (see [10,11] and the reference therein for further study).

The iteration scheme for the fixed point of non-expansive mappings has been extensively investigated, mainly because of the intimate connection between non-expansive mappings and monotonicity methods. In this direction, Marino and Xu [12] and Xu [11] discovered that the iterative method for non-expansive mappings could be used to solve the convex minimization problem. More precisely, it was shown in [11] that a typical minimization problem of a quadratic function of the form:
\[
\min_{x \in F(\Xi)} \frac{1}{2} \langle \Delta b, x \rangle - \langle b, x \rangle
\] (10)

over the set of fixed points for nonexpansive mappings in a real Hilbert space could be solved using the iteration scheme
\[
\omega_{n+1} = \alpha_n b + (1 - \alpha_n \Lambda) \Delta \omega_n, n \geq 0,
\]
where \(\Delta\) is a nonexpansive mapping and \(\Lambda\) is a strongly positive bounded linear operator (Recall \((e)\) and \((g)\) from Definition 2 that a strongly bounded linear operator is a \(\| V \|\)-Lipschitzian and \(\alpha\)-strongly monotone operator).

Motivated by the results in [9], Xu [10] generalized (9) as follows: let \(f\) be a contraction on \(\Upsilon\) and \(V : \Upsilon \to \Upsilon\) be a strongly positive bounded linear operator. Let \(\{s_n\}\) be the sequence generated from an arbitrary point \(s_0 \in \Upsilon\), such that
\[
s_{n+1} = \alpha_n \gamma f(s_n) + (1 - \alpha_n \Lambda) \Delta s_n, n \geq 0,
\] (11)

where \(\alpha_n\) is a sequence in \([0,1]\). He showed that (11) converges strongly to the fixed point of \(\Delta\), which at the same time serves as the solution to the variational inequality problem below:
\[
\langle Vs^* - \gamma f(\omega^*), \omega^* - q \rangle \leq 0, \text{ for all } q \in F(\Delta).
\] (12)

Despite the practical value of some crucial fixed-point results obtained for the case of single-valued mappings, there have been a concentrated efforts in evaluating fixed points for nonlinear multivalued mappings. This special interest is believed to have come from the various practical applications of multivalued mappings. For instance, a monotonic
operator in optimization theory is the multivalued mapping of the subdifferential of the function $g, \partial g : D(g) \subseteq Y \rightarrow 2^Y$, and is defined by
\[
\partial g(t) = \{ s \in Y : \langle t - s, s \rangle \leq g(t) - g(s), \text{for all } s \in \Theta \},
\]
and $0 \in \partial g(t)$ satisfies the condition
\[
\langle t - s, 0 \rangle = 0 \leq g(t) - g(s), \text{for all } s \in \Theta.
\]

In particular, if $g : \Theta \rightarrow \mathbb{R}$ is a convex, continuously differentiable function, then $\Lambda = \nabla g$, the gradient is a subdifferential, which is a single-valued mapping, and the condition $\nabla g(t) = 0$ is an operator equation and $(\nabla g(t), t - 3) \geq 0$ is variational inequalities and both conditions are closely related to optimality conditions. Hence, finding fixed points or common fixed points for multivalued mapping is an important area in applications. However, we have noticed (with concern) that fewer iteration schemes exist, especially in the direction of asymptotically nonlinear multivalued mappings.

Let $(E, \rho)$ be a metric space, $D$ a nonempty subset of $E$ and $\Delta : D \rightarrow 2^D$ a multivalued mapping. A point $\omega \in D$ is said to be a fixed point of $\Delta$ if $\omega \in \Delta \omega$. The fixed-point set of $\Delta$ is denoted by $F(\Delta) = \{ \omega \in D : \omega \in \Delta \omega \}$. Let $CB(E), KC(E)$ and $P(E)$ represent the family of closed and bounded subset of $E$, the family of nonempty compact and convex subset of $E$ and the family of proximinal subset of $E$, respectively. A subset $D$ of $E$ is called proximinal if, for each $\omega \in D$, there exists a point $k \in D$ for which (13) holds.

\[
\rho(\omega, k) = \inf\{ \| \omega - q \| : q \in D \} = \rho(\omega, D),
\]
where $\rho(\omega, q) = \| \omega - q \|$, for all $\omega, q \in E$. It is known that every nonempty closed and convex subset of a real Hilbert is proximinal.

Let $\Lambda, B \in CB(D)$. The Pompeiu Hausdorff metric $Y$ induced by the metric $\rho$ is defined as
\[
Y(\Lambda, B) = \max_{\omega \in \Lambda} \{ \sup_{q \in B} \rho(\omega, B), \sup_{q \in B} \rho(q, \Lambda) \}
\]

**Definition 3.** A multivalued mapping $\Delta : D(\Delta) \subseteq E \rightarrow CB(E)$ is called:

1. $\beta$-Lipschitzian if there exists $\beta > 0$ such that
\[
Y(\Delta \omega, \Delta \varphi) \leq \beta \| \omega - \varphi \|, \text{for all } \omega, \varphi \in D(\Delta).
\]
   Note that in (14), $\Delta$ is a contraction if $\beta \in (0, 1)$ and nonexpansive if $\beta = 1$.

2. Uniformly Lipschitzian if there exists $\beta > 0$, such that
\[
Y(\Delta^n \omega, \Delta^n \varphi) \leq \beta \| \omega - \varphi \|, \text{for all } \omega, \varphi \in D(\Delta), \text{for all } n \geq 1.
\]

3. asymptotically nonexpansive if, for all $\omega, \varphi \in D(\Delta)$, there exists a sequence $\{ \mu_n \} \subset [1, +\infty)$ with $\lim_{n \rightarrow +\infty} \mu_n = 1$, such that
\[
Y(\Delta^n \omega, \Delta^n \varphi) \leq \mu_n \| \omega - \varphi \|, \text{for all } n \geq 1.
\]

4. asymptotically quasi-nonexpansive mapping in $\Delta$ if $F(\Delta) \neq \emptyset$ and (3) holds; that is, if $F(\Delta) \neq \emptyset$ and for all $(\omega \times \varphi) \in D(\Delta) \times F(\Delta)$, there exists a sequence $\{ \mu_n \} \subset [1, +\infty)$ with $\lim_{n \rightarrow +\infty} \mu_n = 1$ such that
\[
Y(\Delta^n \omega, \Delta^n \varphi) \leq \mu_n \| \omega - \varphi \|, \text{for all } n \geq 1.
\]

**Remark 3.** The class of asymptotically quasi-nonexpansive multivalued mapping is a superclass of the classes of asymptotically nonexpansive multivalued mappings and quasi-nonexpansive multivalued mappings (where a multivalued mapping $\Delta : D(\Delta) \subseteq E \rightarrow CB(E)$ is called quasi-nonexpansive (a superclass of the class of nonspreading-type multivalued mapping) if $F(V) \neq \emptyset$ and
for all $\omega \times q \in D(V) \times F(\Delta)$, we have $Y(\Delta \omega, \Delta q) \leq \|\omega - q\|$. Also, $V$ is called nonspread-type if the inequality $2Y(\Delta \omega, \Delta q)^2 \leq \rho(\omega, \Delta q)^2 + \rho(q, \Delta \omega)^2$ holds for all $\omega, q \in D(\Delta)$. Every nonspread-type multivalued mapping with a nonempty fixed point set is quasi-nonexpansive.

The minimization problem, an invaluable problem in application, especially in the area of nonlinear analysis and optimization theory, is defined as follows: find $t \in Y$, such that

$$\dot{g}(t) = \min_{\dot{g} \in Y} g(\dot{z}),$$

where $g : Y \to (-\infty, +\infty)$ is a proper convex and lower semicontinuous function. Note that problem (18) is consistent if it has a solution. The set of all solutions (minimizers) of $g$ on $Y$ is defined as $\text{argmin}_{\dot{z} \in Y} g(\dot{z})$.

Let $g : Y \to (-\infty, +\infty)$ be a proper convex and lower semicontinuous function. Starting from an arbitrary point $\omega_1 \in Y$, define the iteration scheme $\{\omega_n\}$ as follows:

$$\begin{align*}
\omega_1 & \in Y, \\
\omega_{n+1} &= \text{argmin}_{u \in Y} \left[ g(u) + \frac{1}{2\lambda_n} \|u - \omega_n\|^2 \right], n \in \mathbb{N}.
\end{align*}$$

(19)

The fixed-point algorithm (19) for solving problem (18) was first introduced in 1970 by Martinet [14], and was called a proximal point algorithm (for short, PPA). In recent times, many researchers have studied and generalized (19), and many interesting results have been obtained for different classes of nonlinear single-valued and multivalued mappings: Rockfeller [15] solved problem (18) using (19); Marino and Xu [12], and subsequently Phuengrattan and Lerkchaiyaphum [16], obtained weak and strong convergence to the common solution of the minimization problem and fixed-point problem using the modified version of (19) in the setting of real Hilbert spaces.

More recently, Chang et al. [17] used the scheme

$$\begin{align*}
\omega_1 & \in Y, \\
\omega_n & = \text{argmin}_{u \in Y} \left[ f(u) + \frac{1}{2\lambda_n} \|u - \omega_n\|^2 \right]; \\
\lambda_n & \in (1 - \beta_n)\omega_n + \beta_n w_n, w_n \in \Xi \omega_n, \\
\omega_{n+1} & = (1 - \alpha_n)\omega_n + \beta_n v_n, v_n \in T\omega_n, n \in \mathbb{N},
\end{align*}$$

(20)

where $\{\alpha_n\}, \{\beta_n\} \in [0, 1], \Theta$ is a closed and convex subset of a real Hilbert space $Y$ and $g : \Theta \to (-\infty, +\infty)$ is a proper convex and lower semicontinuous function, in order to prove that (20) converges weakly and strongly to the common solutions of the minimization problem of (18) and fixed point problem of nonspread-type multivalued mapping $\Xi$ in the framework of a real Hilbert space $Y$.

Most recently, El-Yekheir et al. [18] introduced and studied the following modified PPA: Let $\Theta$ be a closed and convex subset of a real Hilbert space $Y$, $g : \Theta \to (-\infty, +\infty)$ a proper convex and lower semicontinuous function and $\Xi : \Theta \to \text{CK}(\Theta)$ a multivalued quasi-nonexpansive mapping. The PPA-Ishikawa iteration method is defined as follows:

$$\begin{align*}
\omega_1 & \in Y, \\
\lambda_n & = \text{argmin}_{u \in Y} \left[ f(u) + \frac{1}{2\lambda_n} \|u - \omega_n\|^2 \right]; \\
\lambda_n & \in (1 - \beta_n)\lambda_n + \beta_n w_n, w_n \in \Xi \lambda_n, \\
\omega_{n+1} & = P_{\Theta}(\alpha_n g(\omega_n) + (1 - \eta_n \lambda_n) \lambda_n), n \in \mathbb{N},
\end{align*}$$

(21)

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Using (21), they proved strong convergence results under mild conditions on the control sequences. More precisely, they proved the following theorem:
Theorem 1. Let $\Theta$ be a closed and convex subset of a real Hilbert space $Y$ and $g : \Theta \rightarrow \langle -\infty, +\infty \rangle$ a proper convex and lower semicontinuous function. Let $f : \Theta \rightarrow Y$ be a $b$-Lipschitzian mapping and $\Xi : \Theta \rightarrow \text{CB}(\Theta)$ be a multivalued quasi-nonexpansive mapping, such that $\text{Fix} = f(\Xi) \cap \text{argmin}_{\theta \in \Theta} f(\theta) \neq \emptyset$ and $T \Xi = \{p\}$, for all $p \in F(\Xi)$. Let $A$ be a $k$-strongly monotone and $L$-Lipschitzian operator. Assume that $0 < \eta < \frac{2k}{L^2}, 0 < \gamma b < \tau$, where $\tau = \eta (k - \frac{L^2 \eta}{2})$ and $I - \Xi$ is demiclosed at the origin. Let $\{\omega_n\}$ be as defined by (21), where $\{a_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n \to +\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = +\infty$, $\lim inf_{n \to \infty} a_n (1 - \beta_n) > 0$ and $\lambda_n$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 0$ and some $\lambda$. Then, the sequence $\{\omega_n\}$ generated by (21) converges strongly to $\omega^* \in F$, the unique solution of the variational inequality problem $\langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - p \rangle \leq 0$, for all $p \in F$.

The demiclosedness principle, first studied by Opial [19], is one of the essential tools for proving weak and strong convergence theorems for both single-valued and multivalued nonlinear mappings. It is on record that the theory of fixed points with the associated mappings satisfying demiclosedness principle due to Opial [20] has been extensively studied for the past 40 years or so, and much more intensively recently (see, e.g., [19–23] and the references therein). Although some invaluable results have been obtained, it is worth mentioning that, in some cases, the mapping $\Xi$ of the class of nonexpansive mappings defined in the setting of a real Hilbert space $Y$ does not necessarily satisfies the demiclosedness principle due to Opial [20] (see example 2.1 in reference [19] for more details). Consequently, it is natural to ask:

Question 1. Is there any way one can obtain strong convergence theorems of Halpern’s type for such mappings that fail to satisfy the original demiclosedness principle due to Opial in the setting of Banach spaces?

Naraghrad [19] answered the above question in the affirmative, satisfying the jointly demiclosedness principle (if $C$ is a nonempty subset of a Banach space $E$, then a pair $S, \Xi : C \rightarrow C$ satisfies jointly demiclosedness principle if $\omega_n \subset C$ converges weakly to a point $z \in C$ and $\lim_{n \to +\infty} \|S\omega_n - \Xi \omega_n\| = 0$, then $Sz = z$ and $Tz = z$; that is, $S - \Xi$ is jointly demiclosed at zero) which they introduced. Later, Agwu [24] extended the ideas of the jointly demiclosedness principle to a more general class of multivalued mappings and gave the following definition.

Definition 4 ([24]). Let $D$ be a nonempty closed convex subset of a Banach space $E$. A pair $(S, \Xi)$ of multivalued mappings $S, \Xi : D \rightarrow \text{CB}(D)$ satisfies the jointly demiclosedness principle, in the sense of Naraghrad [19], if for any sequence $\{\omega_n\}_{n \geq 1} \subset D$ converging weakly to a point $q \in D$ and there exist $u_n \in S \omega_n$ and $v_n \in \Xi \omega_n$ with $\|u_n - v_n\| = d(S \omega_n, \Xi \omega_n)$ such that $\|u_n - v_n\| \to 0$ as $n \to +\infty$, then $Sz = q$ and $\Xi q = q$; that is, $S - \Xi$ is jointly demiclosed at zero. In particular, if $S = I$, where $I$ is the identity mapping on $E$, then $I - \Xi$ is demiclosed at zero.

Inspired and motivated by the results in [12,18,24], it is natural to ask the following question:

Question 2. Can we construct a modified proximinal point algorithm that is independent of (21)? If so, can the proposed modified PPA be used to achieve convergence results for a larger class of asymptotically quasi-nonexpansive multivalued mappings that fail to satisfy the original demiclosedness principle due to Opial in the setting of real Hilbert spaces?
It is our purpose in this paper to give an affirmative answer to Question 2. Let $\Theta$ be a closed and convex subset of a real Hilbert space $Y, g : \Theta \to (-\infty, +\infty)$ a proper convex and lower semicontinuous function and $\Delta : \Theta \to CB(\Theta)$ be an asymptotically quasi-nonexpansive multivalued mappings. Then, the modified PPA iteration scheme generated by $\{\alpha_n\}$ for the above mentioned mappings is as follows:

$$\begin{aligned}
\omega_0 &\in \Theta; \\
s_n &= f_{\lambda}^I(\omega) = \arg\min_{\omega \in \Theta} \left\{g(\omega) + \frac{1}{2\lambda} \|v - \omega\|^2\right\}; \\
t_n &= (1 - \beta_n)u_n + \beta_n \xi_n \in \Delta^n s_n, u_n \in \Gamma^n \omega_n; \\
\omega_{n+1} &= P_\Theta(\alpha_n g(\omega_n) + \gamma_n t_n + ((1 - \gamma_n))n - \eta_n \lambda_n) \omega_n),
\end{aligned}$$

(22)

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$

Remark 4. Observe that if:

1. $\alpha_n = 0$ in (22), we get

$$\begin{aligned}
\omega_0 &\in \Theta; \\
s_n &= f_{\lambda}^I(\omega) = \arg\min_{\omega \in \Theta} \left\{g(\omega) + \frac{1}{2\lambda} \|v - \omega\|^2\right\}; \\
t_n &= (1 - \beta_n)u_n + \beta_n \xi_n \in \Delta^n s_n, u_n \in \Gamma^n \omega_n; \\
\omega_{n+1} &= \gamma_n t_n + (1 - \gamma_n) \omega_n
\end{aligned}$$

(23)

Note that (23) generalizes (19), (20) and many other iteration schemes in this direction.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results:

**Assumption V**

Assume that:

(a) $Y$ is a real Hilbert space, $\emptyset \neq \Theta \subset Y$ is closed and convex and $P_\Theta : Y \to \Theta$ is a projection operator;

(b) $S : D(S) \subset Y \to 2^Y, g : Y \to (-\infty, +\infty)$ and $\Xi : \Theta \to \Theta(C)$ are multivalued mapping and proper, convex and lower semicontinuous function and multivalued nonsparing mapping, respectively;

(c) $F(f^I_\lambda)$ and $\arg\min_{\omega \in Y}$ are set of fixed points of the resolvent of $g$ and set of minimizers of $g$ and

(d) $\lambda$ is a set of real numbers and $\Lambda : \Theta \to Y$ be a $k$-strongly monotone and $L$-Lipschitzian operator with $k > 0, L > 0$.

Let $Y, \Theta$ and $P_\Theta$ be as described in Assumption V. Then, $P_\Theta$, assigns to each $t \in Y$ the unique point of $\Theta$, $P_\Theta t$, such that

$$\|t - P_\Theta t\| \leq \|t - s\|, \text{ for all } s \in \Theta$$

It has been established that for every $t \in Y$,

$$\langle t - P_\Theta t, s - P_\Theta t \rangle \leq 0, \text{ for all } s \in \Theta.$$  

(24)

**Definition 5.** Let $Y$ and $S$ be as described in Assumption V. Then, the mapping $I - S$ is known to be demiclosed at the origin if for any sequence $\{t_n\} \subset D(S)$, such that $t_n \to q$ and $d(t_n, St_n) \to 0$; then $q \in S q$, where $I$ is the identity map of $Y$. 

Lemma 1. Let \( g, F(f_\lambda^g), \arg\min_{q \in \Omega} \), and \( \Omega \) be as described in Assumption V. Given any \( \lambda > 0 \), define the Moreau–Yosida resolvent of \( f \) in \( \Omega \) as

\[
f_\lambda^g(\omega) = \arg\min_{q \in \Omega} \left[ g(q) + \frac{1}{2\lambda} \| q - \omega \|^2 \right], \text{ for all } \omega \in \Omega.
\]

Then,

1. \( F(f_\lambda^g) \) coincides with \( \arg\min_{q \in \Omega} \) (see [25]), and for any \( \lambda > 0 \), the resolvent \( f_\lambda^g \) of \( g \) is firmly nonexpansive mapping, and hence is nonexpansive in [26].

2. Since \( f_\lambda^g \) is firmly nonexpansive mapping, if \( F(f_\lambda^g) \neq \emptyset \), then we have

\[
\| f_\lambda^g(\omega) - q \|^2 \leq \| \omega - q \|^2 - \| f_\lambda^g(\omega) - \omega \|^2, \text{ for all } \omega \in \Omega, q \in F(f_\lambda^g).
\]

Lemma 2 ([27]). Let \( Y, \Theta \) and \( \Xi \) be as described in Assumption V. Then, \( I - \Xi \) is demiclosed at zero.

Lemma 3 ([28]). Let \( Y \) be as in Assumption V. Then, for every \( \omega \in \Omega \) and for every \( \lambda \in [0,1] \), the following inequalities hold

(i) \( \| \omega - q \|^2 \leq \| \omega \|^2 + 2\langle q, \omega + q \rangle \)

(ii) \( \| \lambda \omega + (1 - \lambda)q \|^2 \leq \| \omega \|^2 + (1 - \lambda)\| q \|^2 - \lambda(1 - \lambda)\| \omega - q \|^2. \)

Lemma 4 ([11]). Let \( \{a_n\}_{n \geq 0} \in R^+ \cup \{0\} \) with \( a_{n+1} = (1 - \alpha_n)a_n + b_n, n \geq 0 \), where \( \{\alpha_n\}_{n \geq 0} \in (0, 1) \) and \( \{b_n\}_{n \geq 0} \in R \) such that \( \sum_{n=0}^{+\infty} = +\infty \) and \( \limsup_{n \to +\infty} \frac{b_n}{\alpha_n} \leq 0 \). Then, \( a_n \to 0 \) as \( n \to +\infty \).

Lemma 5 ([29]). Let \( Y, \Theta \) and \( \Lambda \) be as described in Assumption V. Assume that \( 0 < \eta < \frac{2k}{1^2} \) and \( \tau = \eta \left( k - \frac{1^2}{1} \right) \). Then, we have

\[
\| (1 - s\eta \Lambda) \omega - (1 - s\eta \Lambda)q \| \leq (1 - t\tau)\| \omega - q \|, \text{ for all } \omega, q \in \Theta.
\]

Lemma 6 ([30]). \( Y, \Theta \) and \( g \) be as described in Assumption V. Given \( r > 0 \) and \( \mu > 0 \), the identity below holds:

\[
f_\lambda^g(\omega) = f_\mu^g\left( \frac{\mu}{r} \omega + (1 - \frac{\mu}{r})f_\lambda^g(\omega) \right)
\]

Lemma 7 ([31]). \( Y, \Theta \) and \( g \) be as described in Assumption V. Then, for every \( \omega, q \in \Omega \) and \( \lambda > 0 \), the following subdifferential inequality holds

\[
\frac{1}{\lambda} \| f_\lambda^g(\omega) - q \|^2 - \frac{1}{\lambda} \| \omega - q \|^2 + \frac{1}{\lambda} \| \omega - f_\lambda^g(\omega) \|^2 + g(f_\lambda^g(\omega)) \leq f(q).
\]

Lemma 8 ([32]). Let \( \{s_n\}_{n \geq 0} \in R \) and which does not increase at infinity in the sense that there exists a subsequence \( s_{n_k} \) of \( \{s_n\} \), such that \( s_{n_k} \leq s_{n_k} + 1 \) for \( k \geq 0 \). For \( n \in N \) (sufficiently large), let \( \tau(n) \) be the sequence of integers defined as follows

\[
\tau(n) = \max\{ j \leq n : s_j \leq s_j + 1 \}.
\]
Then, $\tau(n) \to +\infty$ as $n \to +\infty$ and
\[
\max \{s_{\tau(n)}, s_n\} \leq s_{\tau(n)} + 1.
\] (28)

Agwu and Igbokwe [33] worked on a two-step extragradient-viscosity algorithm for the common fixed-point problem of two asymptotically nonexpansive and variational inequality problems in Banach spaces. Khatoon et al. [34,35] carried out great work on a modified proximal point algorithm for nearly asymptotically quasi-nonexpansive mappings. Khunpanuk et al. [36] worked on a proximal point algorithm for solving common fixed-point problem of two asymptotically nonexpansive mappings and fixed-point problems for multivalued type-one demicontractive mappings. Ezeora et al. [37] established a strong convergence of an inertial-type algorithm to a common solution of a minimization problem and a fixed-point problem.

3. Main Results

Assumption Z

Assume that:

(I) $Y$ is a real Hilbert space and $\emptyset \neq \Theta \subset Y$ is closed and convex;

(II) $R$ is a set of real numbers, $g : \Theta \rightarrow R$ and $f : \Theta \rightarrow Y$ are proper convex and lower semicontinuous function and $\rho$-Lipschitzian mappings, respectively. For a given $\lambda > 0$, we define the Moreau–Yosida resolvent of $g$ in $\Theta$ as
\[
s_n = f_{\lambda}^\Theta(\omega) = \arg\min_{v \in \Theta} \left\{g(v) + \frac{1}{2\lambda} \|v - \omega\|^2\right\}.
\] (29)

(III) $\Delta, \Gamma : \Theta \rightarrow \Theta$ are two asymptotically quasi-nonexpansive multivalued mappings with $\sum_{n=1}^{+\infty} (k^{(1)}_n - 1) < +\infty$ and $\sum_{n=1}^{+\infty} (k^{(2)}_n - 1) < +\infty$, such that $\Delta - \Gamma$ are jointly demiclosed at the origin.

(IV) Let $\Lambda : \Theta \rightarrow Y$ be an $\alpha$–strongly monotone and $L$-Lipschitzian operator.

Algorithm Q

Let $\Theta, \Delta, \Gamma, \Lambda$ and $s_n$ be as described in Assumption Z. Then, $\{\omega_n\}^{+\infty}_{n=0}$ is a sequence generated as follows:

\[
\begin{align*}
\omega_0 & \in \Theta; \\
s_n = f_{\lambda}^\Theta(\omega) = \arg\min_{v \in \Theta} \left\{g(v) + \frac{1}{2\lambda} \|v - \omega\|^2\right\}; \\
l_n &= (1 - \beta_n)u_n + \beta_n\lambda \in \Delta^n s_n, u_n \in \Gamma^n \omega_n; \\
\omega_{n+1} &= P_\Theta(\omega_n + \gamma_n l_n + (\eta_n + (1 - \gamma_n)I - \eta_n \lambda - \Lambda)\omega_n),
\end{align*}
\] (30)

Further, let $Y$ and $\Theta$ retain their usual meaning. A multivalued mapping $\Xi : \Theta \rightarrow CB(\Theta)$ is called asymptotically $\beta$-nonsparing if there exists $\beta > 0$ such that
\[
Y(\Xi^n \omega, \Xi^n \varphi)^2 \leq \beta (d(\Xi^n \omega, \varphi)^2 + d(\omega, \Xi^n \varphi)^2), \text{ for all } \omega, \varphi \in \Theta.
\]

Note that a multivalued mapping $\Xi$ is called asymptotically nonsparing-type if $\beta = \frac{1}{2}$; that is,
\[
2Y(\Xi^n \omega, \Xi^n \varphi)^2 \leq d(\Xi^n \omega, \varphi)^2 + d(\omega, \Xi^n \varphi)^2, \text{ for all } \omega, \varphi \in \Theta.
\]
Again, if $\Xi$ is an asymptotically nonspreading-type and $F(\Xi) \neq \emptyset$, then $\Xi$ is asymptotically quasi-nonexpansive mapping. Indeed, for all $\omega \in \Theta$ and $q \in F(\Xi)$, we have

$$2Y(\Xi^n \omega, \Xi^n q)^2 \leq d(\Xi^n \omega, q)^2 + d(\omega, \Xi^n q)^2 \leq Y(\Xi^n \omega, \Xi^n q)^2 + \|\omega - q\|^2.$$ 

Thus, it follows that

$$Y(\Xi^n \omega, \Xi^n q) \leq \|\omega - q\|.$$ 

**Theorem 2.** Let $\Theta, \Delta, F, \omega, f$ and $s_n$ be as described in Assumption Z. Suppose $F = F(\Delta) \cap F(\Gamma) \cap \arg\min_{x \in GB_\delta(v)} \| \gamma x + (1 - \gamma)I - \eta \lambda x \| \neq \emptyset$; $\Gamma q = \{q\}$; $\Delta q = q$ for all $q \in F(\Delta) \cap F(\Gamma)$; $0 < \gamma_n < \kappa = \left(1 - \frac{\gamma(1 + \rho)}{2\tau}\right)$, $0 < \eta < \frac{2\alpha}{L^2}$; $0 < \gamma \rho < \tau$, where $\tau = \min\left\{\alpha - \frac{L^2 \eta}{2}, \|\omega - q\| \right\}$ and $\Delta - \Gamma$ is jointly demiclosed at the origin. Let $\{\omega_n\}$ be a sequence generated by Algorithm Q, such that $\{\alpha_n\}_{n \geq 1}$, $\{\beta_n\}_{n \geq 1}$ satisfy the conditions:

1. $\lim_{n \to +\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = +\infty$ and $\lim_{n \to +\infty} \frac{\Delta - \Gamma}{\alpha_n} = 0$;
2. $0 < \liminf \beta_n (1 - \beta_n) \leq \limsup \beta_n (1 - \beta_n) < 1$, for each $i = 1, 2, \ldots, m$;
3. $\{\lambda_n\}$ is such that $\lambda_n \geq \lambda > 0$, for all $n \geq 1$ and for some $\lambda$.

Then, $\{\omega_n\}_{n=0}^\infty$ strongly converges to $\omega^* \in F$, which simultaneously serves as a unique solution to the problem:

$$\langle \eta \Lambda \omega - \gamma f(\omega^*), \omega^* - q \rangle, q \in F.$$ (31)

**Proof.** Firstly, we prove that the solution to problem (31) is unique. To achieve this, we assume for contradiction that there exist two points $\omega^*, q^* \in F$ which are solutions of (31) and $\omega^* \neq q^*$. Then, using the same argument as in the proof of the uniqueness of (31) in [18], we have $\omega^* = q^*$ as required.

Again, we note that the operator $P_{\Theta}[\gamma I + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]$ is a contraction. Now, for any two fixed numbers $\alpha, \gamma \in R$ such that $\alpha, \gamma$ are in $\left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$, and for all $\omega, \varphi \in Y$, we get, using Lemma 5, $X = [\gamma + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]\omega$ and $Y = [\gamma + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]\varphi$, that

$$\|P_{\Theta} X - P_{\Theta} Y\| \leq \|\gamma \omega + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]\omega - \gamma \varphi + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]\varphi\| \leq \alpha \gamma \|f(\omega) - f(\varphi)\| + \gamma \|\omega - \varphi\| + \|((1 - \gamma)I - \eta \lambda I)\omega - \varphi\| \leq \alpha \gamma \rho \|\omega - \varphi\| + \gamma \|\omega - \varphi\| + \rho \gamma \|\omega - \varphi\| \leq 0 \leq 0.$$ 

According to the Banach contraction principle, the mapping $P_{\Theta}[\gamma I + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]$ has a fixed point, say $\omega' = P_{\Theta}[\gamma I + (\alpha \gamma f + ((1 - \gamma)I - \eta \lambda I)]\omega'$. This is equivalent to the problem:

$$\langle \eta \Lambda \omega - \gamma f(\omega), \omega - q \rangle, q \in F.$$ 

The rest of the proof of Theorem 2 will be presented in stages:

Stage 1: We show that the sequence $\{\omega_n\}$ and $\{\psi_n\}$ are bounded. Let $q \in F$ and $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$. Then, $\Delta q = \{q\}$, $I q = \{q\}$ and $g(q) \leq g(u)$, for all $u \in \Theta$. Hence, $f_n^q q = q$ for all $n \geq 1$.

Since $f_n^q$ is firmly nonexpansive (and hence nonexpansive), we have

$$\|s_n - q\| = \|f_n^q \omega_n - q\| \leq \|\omega_n - q\|.$$ (32)
Also, from (30), we have
\[
\|t_n - q\| = \|(1 - \beta_n)u_n + \beta_n s_n\| \\
\leq (1 - \beta_n)\|u_n - q\| + \beta_n\|s_n - q\| \\
\leq (1 - \beta_n)\|\Delta^n \omega_n, \Delta^n q\| + \beta_n\|\Gamma^n s_n, \Gamma^n q\| \\
\leq (1 - \beta_n)\|\omega_n - q\| + \beta_n\|s_n - q\| \\
\leq k_n\|\omega_n - q\|. 
\]
(33)

According to condition (i), there exists a constant \( \epsilon \) with \( 0 < \epsilon < 1 - \delta \) and \( \gamma_n(k_n - 1) < \epsilon \alpha_n \). Moreover, from (30), (32), (33) and Lemma 5, we get
\[
\|\omega_{n+1} - q\| = \|P_\Omega(\alpha_n f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n)I - \eta \alpha_n \Lambda)\omega_n) - P_\Omega q\| \\
\leq \|\alpha_n \gamma f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n)I - \eta \alpha_n \Lambda)\omega_n - q\| \\
\leq \|\alpha_n (\gamma f(\omega_n) - \eta \Lambda q) + \gamma_n (t_n - q) + ((1 - \gamma_n)I - \eta \alpha_n \Lambda)(\omega_n - q)\| \\
\leq \alpha_n \|f(\omega_n) - \eta \Lambda q\| + \gamma_n \|t_n - q\| + ((1 - \gamma_n)I - \alpha_n \tau)\|\omega_n - q\| \\
\leq \alpha_n \|f(\omega_n) - f(q)\| + \alpha_n \|\gamma f(q) - \eta \Lambda q\| + \gamma_n\|t_n - q\| \\
+ ((1 - \gamma_n)I - \alpha_n \tau)\|\omega_n - q\| \\
\leq \alpha_n \|f(\omega_n) - f(q)\| + \alpha_n \|f(q) - \eta \Lambda q\| + \gamma_n k_n\|\omega_n - q\| \\
+ ((1 - \gamma_n)I - \alpha_n \tau)\|\omega_n - q\| \\
\leq \alpha_n \|f(\omega_n) - q\| + \alpha_n \|f(q) - \eta \Lambda q\| + (1 - \gamma_n \tau + \gamma_n (k_n - 1))\|\omega_n - q\| \\
\leq \max\left\{\|\omega_n - q\|, \frac{\|\gamma f(q) - \eta \Lambda q\|}{\tau - \epsilon - \gamma \rho}\right\}, n \geq 1.
\]

By induction, it is easy to see that
\[
\|\omega_n - q\| \leq \max\left\{\|\omega_0 - q\|, \frac{\|\gamma f(q) - \eta \Lambda q\|}{\tau - \epsilon - \gamma \rho}\right\}, n \geq 1.
\]

Hence, \( \{\omega_n\} \) is bounded, and so are the sequences \( \{\phi_n\}, \{f(\omega_n)\} \) and \( \{\lambda \omega_n\} \).

Stage 2: We prove that the sequence \( \{\omega_n\} \) converges strongly to \( \omega^* \).

From (30) and Lemma 3 (ii), we obtain
\[
\|t_n - q\|^2 = \|(1 - \beta_n)u_n + \beta_n s_n\|^2 \\
\leq (1 - \beta_n)\|u_n - q\|^2 + \beta_n\|s_n - q\|^2 - \beta_n(1 - \beta_n)\|u_n - s_n\|^2 \\
\leq (1 - \beta_n)\|\Delta^n \omega_n, \Delta^n q\|^2 + \beta_n Y(\Gamma^n s_n, \Gamma^n q)^2 - \beta_n(1 - \beta_n)\|u_n - s_n\|^2 \\
\leq (1 - \beta_n)(k_n^2)^2\|\omega_n - q\|^2 + \beta_n(k_n^2)^2\|s_n - q\|^2 - \beta_n(1 - \beta_n)\|u_n - s_n\|^2 \\
\leq (1 - \beta_n)k_n^2\|\omega_n - q\|^2 + \beta_n k_n^2\|s_n - q\|^2 - \beta_n(1 - \beta_n)\|u_n - s_n\|^2 \\
\leq k_n^2\|\omega_n - q\|^2 - \beta_n(1 - \beta_n)\|u_n - s_n\|^2 
\]
(34)

Set \( h_n = \alpha_n \|f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n)I - \eta \alpha_n \Lambda)\omega_n \). Then, from (30), (31), (35), Lemmas 3(i) and 5, we get
\[ \|\omega_{n+1} - q\|^2 \leq \|\alpha_n\gamma f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n)I - \eta\alpha_n\Lambda)\omega_n - q\|^2 
= \|\alpha_n(\gamma f(\omega_n) - \eta\Lambda q) + \gamma_n(t_n - q) + ((1 - \gamma_n)I - \eta\alpha_n\Lambda)(\omega_n - q)\|^2 
\leq \|((1 - \gamma_n)I - \eta\alpha_n\Lambda)(\omega_n - q) + \gamma_n(t_n - q)\|^2 + 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
= \|((1 - \gamma_n)I - \eta\alpha_n\Lambda)(\omega_n - q)\|^2 + \gamma_n^2\|t_n - q\|^2 
+ 2\gamma_n((1 - \gamma_n)I - \eta\Lambda q, h_n - q) + 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
\leq \|((1 - \gamma_n)I - \alpha_n\tau)^2\|\omega_n - q\|^2 + \gamma_n^2\|t_n - q\|^2 
+ 2\gamma_n((1 - \gamma_n)I - \alpha_n\tau)\|\omega_n - q\|\|t_n - q\| + 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
= [1 - 2\gamma_n + \gamma_n^2 - \alpha_n(1 - \gamma_n)\tau + \alpha_n^2\tau^2]\|\omega_n - q\|^2 + \gamma_n^2\|t_n - q\|^2 
+ \gamma_n(1 - \alpha_n\tau)\|\omega_n - q\|^2 + \gamma_n((1 - \gamma_n)I - \alpha_n\tau)\|t_n - q\|^2 
+ 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
\leq [1 - \gamma_n + \gamma_n^2 + \alpha_n\tau\|\omega_n - q\|^2 + \gamma_n^2\|t_n - q\|^2 - \gamma_n\alpha_n\tau\|\omega_n - q\|^2 
- \gamma_n^2\|\omega_n - q\|^2 + \gamma_n(1 - \alpha_n\tau)\|t_n - q\|^2 - \gamma_n^2\|t_n - q\|^2 
+ 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
\leq [1 - \gamma_n(1 - \alpha_n\tau) + \alpha_n^2\tau^2]\|\omega_n - q\|^2 + \gamma_n(1 - \alpha_n\tau)\|k_n - 1\|\|\omega_n - q\|^2 - \beta_n(1 - \beta_n)\|u_n - j_n\|^2 
+ 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
\leq [1 + \epsilon\alpha_n + \alpha_n^2\tau^2]\|\omega_n - q\|^2 - \beta_n(1 - \beta_n)\|u_n - j_n\|^2 + 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\| 
- \eta\Lambda q, h_n - q]. 
\]

Set \(D_n = \beta_n(1 - \beta_n)\|u_n - j_n\|^2\) so that the last inequality becomes:

\[D_n \leq \|\omega_n - q\|^2 - \|\omega_{n+1} - q\|^2 + \epsilon\alpha_n + \alpha_n^2\tau^2\|\omega_n - q\|^2 
+ 2\alpha_n\|\gamma f(\omega_n) - \eta\Lambda q, h_n - q\|. \tag{36}\]

To show that \(\{\omega_n\}\) is convergent, we consider the following two cases:

**Case 1**: Assume that the sequence \(\{\|\omega_n - q\|\}\) is monotonically decreasing. Then, \(\{\|\omega_n - q\|\}\) is convergent. Indeed, we have

\[\lim_{n \to +\infty} [\|\omega_n - q\| - \|\omega_{n+1} - q\|] = 0. \tag{37}\]

Thus, by (36), condition (i) and (ii) and the fact that \(\lim_{n \to +\infty} k_n = 1\), we have

\[\lim_{n \to +\infty} D_n = \lim_{n \to +\infty} \beta_n(1 - \beta_n)\|u_n - j_n\|^2 = 0. \tag{38}\]

Since, \(\beta_n \in [a, b] \subset (0, 1)\), for \(a \geq 0\) and \(b \leq 1\) we get

\[\lim_{n \to +\infty} \|u_n - j_n\| = 0. \tag{39}\]

Additionally, using (30), we get

\[\|\omega_{n+1} - t_n\| \leq \|\alpha_n\gamma f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n)I - \eta\alpha_n\Lambda)\omega_n - \varphi_u\| 
= \|\alpha_n(\gamma f(\omega_n) - \omega_n) - (1 - \gamma_n)(\omega_n - t_n)\| 
\leq \alpha_n\|\gamma f(\omega_n) - \eta\alpha_n\Lambda\| + (1 - \gamma_n)\|\omega_n - t_n\|. \tag{40}\]
and condition (i) imply that
\[
\lim_{n \to +\infty} (\|\omega_{n+1} - l_n\| - \|\omega_n - l_n\|) = 0. \tag{41}
\]
It follows from (41) that
\[
\lim_{n \to +\infty} \|\omega_{n+1} - \omega_n\| = 0. \tag{42}
\]
Again, for any \(q \in F\), using Lemma 7 and the fact that \(g(q) \leq g(s_n)\), we obtain
\[
\|\omega_n - s_n\|^2 \leq \|\omega_n - q\|^2 - \|s_n - q\|^2. \tag{43}
\]
Furthermore, from (30), (34), (42), (43) and Lemma 3(i), we obtain
\[
\begin{align*}
\|\omega_{n+1} - q\|^2 &\leq \|a_n \gamma f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n) I - \eta \Delta A) \omega_n - q\|^2 \\
&= \|a_n (\gamma f(\omega_n) - \eta \Delta A) - \gamma_n (t_n - l_n) + ((1 - \gamma_n) I - \eta \Delta A) (\omega_n - q)\|^2 \\
&\leq \|((1 - \gamma_n) I - \eta \Delta A) (\omega_n - q) - \gamma_n (t_n - l_n)\|^2 + 2a_n \gamma f(\omega_n) - \eta \Delta A \omega_{n+1} - q\|^2 \\
&\leq \|((1 - \gamma_n) I - \eta \Delta A) (\omega_n - q)\|^2 + \gamma_n^2 \|t_n - l_n\|^2 + 2a_n \gamma f(\omega_n) - f(q), \omega_n - q\|^2 \\
&+ 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q) \\
&\leq ((1 - \gamma_n) I - \gamma_n \tau)^2 \|\omega_n - q\|^2 + \gamma_n^2 ((1 - \beta_n) \kappa_n^2 \|\omega_n - q\|^2 + \beta_n k_n^2 \|s_n - q\|^2 \\
&- \beta_n (1 - \beta_n) \|t_n - s_n\|^2) + 2a_n \gamma f(\omega_n) - f(q) \|\omega_{n+1} - q\| \\
&+ 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q) \\
&\leq [1 - 2\gamma_n + \gamma_n^2 (1 - \gamma_n) \tau + \alpha_n^2 \tau^2] \|\omega_n - q\|^2 + \gamma_n^2 (k_n^2 \omega_n - q) \|\omega_n - q\|^2 \\
&- \beta_n k_n^2 \|\omega_n - s_n\|^2 + 2a_n \gamma f(\omega_n) - f(q) \|\omega_{n+1} - q\| \\
&+ 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q) \\
&\leq [1 - 2\gamma_n + \gamma_n^2 (\alpha_n^2 \tau^2) \|\omega_n - q\|^2 + \gamma_n k_n^2 \|\omega_n - q\|^2 - \beta_n \gamma_n k_n^2 \|\omega_n - s_n\|^2 \\
&+ 2a_n \gamma f(\omega_n - q) \|\omega_{n+1} - q\| + 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q) \\
&\leq \|\omega_{n+1} - \omega_n\|^2 + \|\omega_n - q\|^2 + \alpha_n^2 \tau^2 \|\omega_n - q\|^2 + \gamma_n (k_n^2 - 1) \|\omega_n - q\|^2 \\
&- \beta_n \gamma_n k_n^2 \|\omega_n - s_n\|^2 + 2a_n \gamma f(\omega_n - q) \|\omega_{n+1} - q\| + 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q) \\
&\leq \|\omega_{n+1} - s_n\|^2 \|\omega_n - q\|^2 + \alpha_n^2 \tau^2 \|\omega_n - q\|^2 + \gamma_n (k_n^2 - 1) \|\omega_n - q\|^2 \\
&- \beta_n \gamma_n k_n^2 \|\omega_n - s_n\|^2 + 2a_n \gamma f(\omega_n - q) \|\omega_{n+1} - q\| \\
&+ 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q). \tag{44}
\end{align*}
\]
(44) implies that
\[
\beta_n \gamma_n^2 k_n^2 \|\omega_n - s_n\|^2 \leq \|\omega_n - \omega_{n+1}\|^2 + (\alpha_n^2 \tau^2 + \epsilon) \|\omega_n - q\|^2 \\
+ 2a_n \gamma f(\omega_n - q) \|\omega_{n+1} - q\| \\
+ 2a_n \gamma f(q - \eta \Delta A \omega_n, \omega_{n+1} - q). \tag{45}
\]
The last inequality and condition (i) yields
\[
\lim_{n \to +\infty} \|\omega_n - s_n\| = 0.
\]
Since
\[ \| \omega_{n+1} - s_n \| \leq \| \omega_{n+1} - \omega_n \| + \| \omega_n - s_n \|, \]
it follows from (43) and (45) that
\[ \lim_{n \to +\infty} \| \omega_{n+1} - s_n \| = 0. \] (46)

In addition, using (30) and Lemma 6, we get
\[
|\omega_n - f^g_{\lambda_n} \omega_n| = |s_n - f^g_{\lambda_n} \omega_n + s_n - \omega_n| \\
\leq |s_n - f^g_{\lambda_n} \omega_n| + |s_n - \omega_n| \\
\leq |f^g_{\lambda_n} \omega_n - f^g_{\lambda_n} \omega_n| + |s_n - \omega_n| \\
\leq |s_n - \omega_n| + \| \frac{\lambda}{\lambda_n} \omega_n + \frac{\lambda_n - \lambda}{\lambda_n} f^g_{\lambda_n} \omega_n \| \\
\leq |s_n - \omega_n| + \left( \frac{\lambda}{\lambda_n} \right) \| s_n - \omega_n \| \\
\leq \left( 2 - \frac{\lambda}{\lambda_n} \right) \| s_n - \omega_n \|. \] (47)

(45) and (47) imply that
\[ \lim_{n \to +\infty} |\omega_n - f^g_{\lambda_n} \omega_n| = 0. \] (48)

Additionally, by the boundedness of the sequence \( \{ \omega_n \} \), there exists a subsequence \( \{ \omega_{n_i} \}_{i \in \mathbb{N}} \) of \( \{ \omega_n \}_{n \in \mathbb{N}} \), such that \( \omega_{n_i} \to q_1 \in \Theta \) as \( i \to +\infty \). Again, from (39), we get
\[ \lim_{n \to +\infty} \| u_{n_i} - s_{n_i} \| = 0. \] (49)

Moreover, since by assumption, the pair \( (\Delta, \Gamma) \) jointly satisfy the demiclosedness principle, it follows from (49) that \( q_1^* \in F(\Delta) \cap F(\Gamma) \). Assume that another subsequence \( \{ \omega_{n_j} \}_{j \in \mathbb{N}} \) of \( \{ \omega_n \}_{n \in \mathbb{N}} \) exists, such that \( \omega_{n_j} \to q_2 \) as \( j \to +\infty \), where \( q_2^* \in F \). Then, following the same argument as in [24] with \( p = q_1^* \) and \( q = q_2^* \), we get that \( q_1^* = q_2^* \).

Next, we show that \( \limsup_{n \to +\infty} \langle \eta \Lambda \omega_n^* - \gamma f(\omega_n^*), \omega_n^* - \omega_n \rangle \leq 0 \). Now, since \( Y \) is a Hilbert space and the sequence \( \{ \omega_n \} \) is bounded, it follows that there exists a subsequence \( \{ \omega_{n_i} \} \) of \( \{ \omega_n \} \) that converges weakly to \( \omega \in \Theta \) and
\[ \lim_{n \to +\infty} \langle \eta \Lambda \omega_{n_i}^* - \gamma f(\omega_{n_i}^*), \omega_{n_i}^* - \omega_{n_i} \rangle = \limsup_{j \to +\infty} \langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - \omega_n \rangle. \]

Following the information above, with \( q_1^* = \omega \), we have \( \omega \in F(\Delta) \cap F(\Gamma) \). Additionally, since \( f^g_{\lambda_n} \) is single-valued and nonexpansive, using (48), we get \( \omega \in F(f^g_{\lambda_n}) = \arg \min_{\mu \in \Theta} g(\mu) \). Therefore, \( \omega \in F \). Thus,
\[
\lim_{n \to +\infty} \langle \eta \Lambda \omega_n^* - \gamma f(\omega_n^*), \omega_n^* - \omega_n \rangle = \limsup_{j \to +\infty} \langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - \omega_n \rangle \\
\leq \langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - \omega \rangle \\
\leq 0. \] (50)

Lastly, we prove that \( \omega_n \to \omega^* \) as \( n \to +\infty \). From (30) and (35) and Lemma 5, we have
\[
\begin{align*}
\|\omega_{n+1} - \omega^*\|^2 & \leq \|\alpha_n f(\omega_n) + \gamma_n t_n + ((1 - \gamma_n)I - \eta \Lambda \alpha)|\omega_n\| - \omega^*\|^2 \\
& = \|\alpha_n (f(\omega_n) - \eta \Lambda \alpha) - \gamma_n (\omega^* - t_n) + ((1 - \gamma_n)I - \eta \Lambda \alpha)(\omega_n - \omega^*)\|^2 \\
& \leq \|((1 - \gamma_n)I - \eta \Lambda \alpha)(\omega_n - \omega^*) - \gamma_n (\omega^* - t_n)\|^2 \\
& + 2\alpha_n \langle \gamma f(\omega_n) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^* \rangle \\
& \leq \|((1 - \gamma_n)I - \eta \Lambda \alpha)(\omega_n - \omega^*)\|^2 + \gamma^2_n \|\omega^* - t_n\|^2 \\
& + 2\alpha_n \gamma \langle f(\omega_n) - f(\omega^*), \omega_{n+1} - \omega^* \rangle + 2\alpha_n \gamma \langle f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^* \rangle \\
& \leq ((1 - \gamma_n)I - \alpha_n t)^2 \|\omega_n - \omega^*\|^2 + \gamma^2_n \|\omega^* - t_n\|^2 \\
& + 2\alpha_n \gamma \|f(\omega_n) - f(\omega^*)\| \|\omega_{n+1} - \omega^*\| + 2\alpha_n \gamma \|f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^*\| \\
& \leq \left[1 - 2\gamma_n + \gamma^2_n - 2\alpha_n (1 - \gamma_n) \tau + \alpha^2_n \gamma^2 \|\omega_n - \omega^*\|^2 + \gamma^2_n \|k_n^2\|\omega_n - q\|^2 - \beta_n (1 - \beta_n) \|u_n - \beta_n\|^2 \right] \\
& + 2\alpha_n \gamma \rho \|\omega_n - \omega^*\| \|\omega_{n+1} - \omega^*\| + 2\alpha_n \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^*\| \\
& \leq \left[1 - 2\gamma_n + \gamma^2_n - 2\alpha_n (1 - \gamma_n) \tau + \alpha^2_n \gamma^2 \|\omega_n - \omega^*\|^2 + \gamma^2_n \|k_n^2\|\omega_n - q\|^2 - \gamma^2_n \beta_n (1 - \beta_n) \|u_n - \beta_n\|^2 \right] \\
& + \alpha_n \gamma \rho \|\omega_n - \omega^*\| + 2\alpha_n \gamma \rho \|\omega_{n+1} - \omega^*\| + 2\alpha_n \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^*\| \\
& \leq \left[1 - 2(1 - \gamma_n) \tau - \gamma \rho \|\omega_n - \omega^*\|^2 + \alpha^2_n \gamma^2 \|\omega_n - \omega^*\|^2 + \gamma^2_n \|k_n^2\|\omega_n - q\|^2 + \gamma^2_n \beta^2_n (1 - \beta_n) \|u_n - \beta_n\|^2 \right] \\
& + \alpha_n \gamma \rho \|\omega_n - \omega^*\| + 2\alpha_n \gamma \rho \|\omega_{n+1} - \omega^*\| + 2\alpha_n \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^*\| \\
& \leq \left[1 - 2(1 - \gamma_n) \tau - \gamma \rho \|\omega_n - \omega^*\|^2 + \alpha^2_n \gamma^2 \|\omega_n - \omega^*\|^2 + \gamma^2_n \|k_n^2\|\omega_n - q\|^2 + \gamma^2_n \beta^2_n (1 - \beta_n) \|u_n - \beta_n\|^2 \right] \\
& + \alpha_n \gamma \rho \|\omega_n - \omega^*\| + 2\alpha_n \gamma \rho \|\omega_{n+1} - \omega^*\| + 2\alpha_n \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^*\|. \\
\end{align*}
\]

From (51), we obtain

\[
\begin{align*}
(1 - \alpha_n \gamma \rho) \|\omega_{n+1} - \omega^*\|^2 & \leq \left[1 - 2(1 - \gamma_n) \tau - \gamma \rho \|\omega_n - \omega^*\|^2 \right] \\
& + \alpha_n \left\{ \frac{\alpha_n^2 \gamma^2}{\alpha_n} + \frac{\gamma^2_n \|k_n^2 - 1\|}{\alpha_n} \right\} M \\
& + 2\langle \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^* \rangle. \\
\end{align*}
\]

where \(M = \sup_{n \geq 1} \|\omega_n - \omega^*\|^2\).

Put

\[
\begin{align*}
c_n &= \frac{(2(1 - \gamma_n) \tau - \gamma \rho)\alpha_n}{1 - \alpha_n \gamma \rho}, \\
\delta_n &= \frac{\alpha_n}{1 - \alpha_n \gamma \rho} \left\{ \frac{\alpha_n^2 \gamma^2}{\alpha_n} + \frac{\gamma^2_n \|k_n^2 - 1\|}{\alpha_n} \right\} M \\
& + 2\langle \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^* \rangle \\
\end{align*}
\]

and

\[
\begin{align*}
\tau_n &= \frac{\delta_n}{c_n} \\
& = \frac{1}{2(1 - \gamma_n) \tau - \gamma \rho} \left\{ \frac{\alpha_n^2 \gamma^2}{\alpha_n} + \frac{\gamma^2_n \|k_n^2 - 1\|}{\alpha_n} \right\} M \\
& + 2\langle \gamma f(\omega^*) - \eta \Lambda \alpha^*, \omega_{n+1} - \omega^* \rangle. \\
\end{align*}
\]
Then, from condition (i), we get

\[ \sigma_n \to 0 \quad \text{as} \quad n \to +\infty, \quad \sum_{n=1}^{+\infty} \sigma_n = +\infty \quad \text{and} \quad \limsup_{n \to +\infty} \tau_n \leq 0. \]

Thus, using Lemma 4 and (52), the result follows as required (i.e., \( \omega_n \to \omega^* \) as \( n \to +\infty \)).

Case 2:

Suppose \( \{\|\omega_n - \omega^*\|\} \) is a monotonically increasing sequence. Set \( B_n = \|\omega_n - \omega^*\| \) and the mapping \( \tau : \mathbb{N} \to \mathbb{N} \), for all \( n \geq n_0 \) (for some \( n_0 \) large enough), by \( \tau_n = \max \{ k \in \mathbb{N} : k \leq n, B_k \leq B_{k+1} \} \). Then, \( \tau \) is a nondecreasing sequence, such that \( \tau_n \to +\infty \) as \( n \to +\infty \) and \( B_{\tau(n)} \leq B_{\tau(n)+1} \) for \( n \geq n_0 \). From (36), we have

\[ D_{\tau_n} \leq \|\omega_{\tau_n} - \omega^*\|^2 - \|\omega_{\tau_n+1} - \omega^*\|^2 + (\epsilon \alpha_n + \alpha_n^2 e^2) M \]

\[ + 2\alpha_{\tau_n}(\gamma f(\omega_n) - \eta \Lambda \omega^*, h_{\tau_n} - \omega^*) \to 0 \quad \text{as} \quad n \to +\infty, \]

where \( D_{\tau_n} = \beta_{\tau_n}(1 - \beta_{\tau_n})\|u_{\tau_n} - \lambda_{\tau_n}\|^2 \). Moreover, we have

\[ \lim_{n \to +\infty} \|u_{\tau_n} - \lambda_{\tau_n}\|^2 = 0. \tag{53} \]

Following a similar argument to the one above in Case 1, we conclude that

\[ \limsup_{n \to +\infty} \langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - \omega_{\tau(n)+1} \rangle \leq 0. \]

Hence, for all \( n \geq n_0 \), we have

\[ 0 \leq \|\omega_{\tau(n)+1} - \omega^*\|^2 - \|\omega_{\tau(n)} - \omega^*\|^2 \leq \alpha_{\tau(n)} \left[ \frac{(1 - \gamma_n)\tau - \gamma - \gamma \rho}{1 - \alpha_{\tau(n)} \gamma \rho} \right] \|\omega_{\tau(n)} - \omega^*\|^2 \]

\[ + \frac{\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)} \gamma \rho} \left\{ |\alpha_{\tau(n)}^2 + \frac{\gamma_{\tau_n}(k_{\tau(n)}^2 - 1)}{\alpha_{\tau(n)}}| M \right\} \]

\[ + 2(\gamma f(\omega^*) - \eta \Lambda \omega^*, \omega_{n+1} - \omega^*) \],

from which we get

\[ \|\omega_{\tau(n)} - \omega^*\|^2 \leq \frac{1}{2(1 - \gamma_n)\tau - \gamma - \gamma \rho} \left\{ |\alpha_{\tau(n)}^2 + \frac{\gamma_{\tau_n}(k_{\tau(n)}^2 - 1)}{\alpha_{\tau(n)}}| M \right\} \]

\[ + 2(\gamma f(\omega^*) - \eta \Lambda \omega^*, \omega_{n+1} - \omega^*) \].

Consequently, we have

\[ \lim_{n \to +\infty} \|\omega_{\tau(n)} - \omega^*\| = 0. \]

and

\[ \lim_{n \to +\infty} B_{\tau(n)} = \lim_{n \to +\infty} B_{\tau(n)+1} = 0. \tag{54} \]

Therefore, we conclude, using Lemma 8, that

\[ 0 \leq B_{\tau(n)} \leq \max\{B_{\tau(n)}, B_{\tau(n)+1}\} = B_{\tau(n)+1}. \tag{55} \]

Hence, \( B_n \to 0 \) as \( n \to +\infty \); that is, \( \omega_n \to \omega^* \) as \( n \to +\infty \). This ends the proof Theorem 2. \( \square \)
Now, in application, we employ Theorem 2 to estimate a point \( q \in F = F(\Delta) \cap F(\Gamma) \cap \text{argmin}_{v \in \Theta} \xi(v) \neq \emptyset \) without any compactness assumptions on either the space or the operators, where \( \Delta, \Gamma \) is a pair of asymptotically nonspreading-type mappings.

**Theorem 3.** Let \( \Theta, \Lambda, g, f \) and \( s_n \) be as described in Assumption Z. Let \( \Lambda, \Gamma : \Theta \rightarrow C(\Theta) \) be two asymptotically nonspreading-type multivalued mappings, such that \( F = F(\Delta) \cap F(\Gamma) \cap \text{argmin}_{v \in \Theta} \xi(v) \neq \emptyset ; \Gamma q = \{ q \} ; \Delta q = q, \) for all \( q \in F(\Delta) \cap F(\Gamma) \). Assume that \( 0 < \gamma_n < \kappa = \left( 1 - \frac{\gamma(1 + \rho)}{2\tau} \right), 0 < \eta < 2\kappa \), \( 0 < \gamma \rho < \tau \), where \( \tau = \eta \left( \kappa - \frac{L^2\eta}{2} \right) \) and \( \Delta - \Gamma \) is jointly demiclosed at the origin. Let \( \{ \omega_n \}_{n \geq 1} \) be a sequence generated by Algorithm Q, such that \( \{ \alpha_n \}_{n \geq 1}, \{ \beta_n \}_{n \geq 1} \in [0,1] \) satisfy conditions (i), (ii) and (iii) of Theorem 2. Then, \( \{ \omega_n \}_{n=0}^{+\infty} \) strongly converges to \( \omega^* \in F \), which, at the same time, serves as a unique solution to the problem:

\[
\langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - q \rangle, q \in F.
\] (56)

**Proof.** Since every asymptotically nonspreading-type mapping with a nonempty fixed point set is asymptotically quasi nonexpansive multivalued mapping, the proof of Theorem 3 immediately follows from Lemma 2 and Theorem 2. \( \square \)

Again, if \( \Delta \) and \( \Gamma \) are asymptotically quasi-nonexpansive single-valued mapping and \( \Lambda \) is a strongly positive bounded linear operator, then the following theorem can be obtained from Theorem 3.

**Theorem 4.** Let \( \Theta, \Lambda, g, f \) and \( s_n \) be as described in Assumption Z. Let \( \Lambda, \Gamma : \Theta \rightarrow \Theta \) be two asymptotically quasi-nonexpansive multivalued mappings, such that \( F = F(\Delta) \cap F(\Gamma) \cap \text{argmin}_{v \in \Theta} \xi(v) \neq \emptyset ; \Gamma q = \{ q \} ; \Delta q = q, \) for all \( q \in F(\Delta) \cap F(\Gamma) \). Assume that \( 0 < \gamma_n < \kappa = \left( 1 - \frac{\gamma(1 + \rho)}{2\tau} \right), 0 < \eta < 2\kappa \), \( 0 < \gamma \rho < \tau \), where \( \tau = \eta \left( \kappa - \frac{L^2\eta}{2} \right) \) and \( \Delta - \Gamma \) is jointly demiclosed at the origin. Let \( \{ \omega_n \}_{n \geq 1} \) be a sequence generated by Algorithm Q, such that \( \{ \alpha_n \}_{n \geq 1}, \{ \beta_n \}_{n \geq 1} \in [0,1] \) satisfy conditions (i), (ii) and (iii) of Theorem 3.1. Then, \( \{ \omega_n \}_{n=0}^{+\infty} \) strongly converges to \( \omega^* \in F \), which satisfies the optimality condition of the minimization problem

\[
\min_{\omega \in Y} \frac{\eta}{2} \langle \Lambda \omega, \omega \rangle - h(\omega),
\] (57)

where \( h \) is a potential function for \( \gamma f \) (i.e., \( h'(\omega) = \gamma f(\omega) \) on \( Y \)).

4. Conclusions

In this paper, a modified proximinal point algorithm used to approximate a common element of the set of solutions of a fixed-point problem for a pair of asymptotically quasi-nonexpansive multivalued mappings and a minimization problem of (18) is introduced and studied. Under appropriate conditions on the control sequences, strong convergence results were obtained using the algorithm so introduced, providing an affirmative answer to Question 1.2 raised in the paper. Since asymptotically quasi-nonexpansive multivalued mapping is much more general than asymptotically nonexpansive multivalued mapping, asymptotically nonspreading-type multivalued mapping, quasi-nonexpansive multivalued mapping, nonexpansive multivalued mapping, nonspreading-type multivalued mapping, the problem studied in our paper is quite general and includes problems with optimization, variational inequality and fixed point as its special cases. Thus, it follows that Theorem 2 in our paper improves, extends and generalizes the results obtained in [10,12,14–18,33], and many more others currently existing in the literature.

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