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Background Independence and Gauge Invariance in General Relativity Part 1—The Classical Theory

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Abstract: The implications of the principles of general and manifest covariance, together with those of the objectivity principle, are considered for the purpose of establishing a DeDonder–Weyl-type Hamiltonian variational formulation for classical general relativity. Based on the analysis of the Einstein–Hilbert variational principle, it is shown that only synchronous variational principles permit the construction of fully 4−tensor Lagrangian and Hamiltonian theories of this type. In addition, the possible validity of an extended Hamiltonian formulation in which Lagrangian variables include also the Ricci tensor is investigated and shown to occur provided the classical cosmological constant is non-vanishing.

Keywords: Einstein–Hilbert variational principle; Hamiltonian theory of GR; ADM Hamiltonian theory; manifest covariance

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1. Introduction

Variational formulations for continuum fields represent a fertile subject of research which pertains to both their classical description as well as the formulation of related quantum theories. The issue becomes particularly intriguing in the case of the gravitational field because of some crucially-important open questions arising both in the framework of the so-called standard formulation of general relativity (SF-GR), i.e., the formulation of GR associated with the Einstein field equations (EFE), customarily named as the metric formulation of GR [1–3], and corresponding theories of quantum gravity (QG). In this regard, it is generally agreed that any physical theory of gravity worthy of this name, either classical (in particular variational formulations of GR) or quantum, should satisfy the following three basic requirements:

- It should fulfill the principle of general covariance (PGC) in arbitrary GR-frames that are related by means of local point transformations (see discussion below in Section 2).
- It should have the goal of determining the structure of space-time, based on the identification of a Riemannian differential manifold associated with it. In particular, this refers to the prescription of its local metric tensor, namely the so-called *background metric field tensor*, to be represented with respect to an arbitrary GR-frame.
- It should determine the Hamiltonian structures, both classical and quantum ones, respectively, realized in the frameworks of SF-GR and QG, both associated with the Einstein field equations (EFE).

The crucial issue is how precisely these requirements can be satisfied and implemented. In this regard, however, further important features must be taken into account that pertain to the physically admissible tensor representations for Lagrangian and equivalent Hamiltonian theories of GR. These include the properties of *objectivity, background independence*and *gauge symmetries*.

This paper is the first part of a two-paper investigation, the present one being devoted to variational theories of classical gravity, while leaving to the second part the treatment of QG. The motivations for splitting the treatment of the two subject areas are several, including that:

- To achieve an admissible candidate for a QG theory worth of this name, the constructions of a, possibly non-unique, Hamiltonian representation for GR, namely of EFE, is required. The crucial characteristics of the same representation that are consistent with the basic principles of GR will be investigated.
- Besides PGC, the consequences of the principle of manifest covariance (PMC) [4] and its relationship with the principle of objectivity must be addressed. The latter principle concerns the determination of the classical background space-time and the prescription of the appropriate variational treatments to be adopted for the gravitational field.
- Furthermore, comparisons with literature are necessary both in GR and QG and must be discussed separately, given the different and peculiar properties of currently available theories.

Regarding specifically GR, a basic conceptual remark in GR refers to the issue of whether and under what conditions (i.e., if any) the validity of PGC in the same context implies or is equivalent to the validity of PMC. Here we wish to point out that if such a conjecture indeed applies, it means that it should always be possible to cast all physical laws, including all the corresponding relevant dynamical variables, in manifestly covariant form, i.e., in an equivalent $4-$tensor form. A related issue then concerns the prescription of the background metric tensor. In fact, two possible choices emerge from the literature. Accordingly, the background metric tensor can be either identified with a classical physical observable (i.e., a uniquely prescribed tensor field) or a dynamical variable. Nevertheless, the construction of the theory should not depend on its particular choice (*background independence*).

Some of the subjects indicated above, which also have much to do with possible alternative representations of the theory of classical and quantum gravity, have already been discussed elsewhere (see, in particular, Refs. [5,6]). For this reason, some of the main outcomes displayed in the same references are adopted as reference background material, particularly those regarding aspects of the Einstein–Hilbert (EH) variational theory of SF-GR, conditions of validity of the ADM Hamiltonian theory (Arnowitt, Deser and Misner [7,8]) and the $4-$tensor manifestly-covariant Hamiltonian theory of classical gravity referred to as CCG-theory [6,9,10].

In order to proceed consistently, the principles of covariance and manifest covariance in GR are first recalled in connection with the tensorial representations of physical laws, the establishment of their objective character and the definition of classical physical observables. The following related questions arise in this connection.

First question: is there a manifestly covariant Hamiltonian theory of GR which can be based on the EH-action principle? The issue concerns the construction of a Hamiltonian theory of GR, for which, historically, a famous starting point is provided by the EH theory for SF-GR, i.e., the EH-action principle for EFE. In particular, the first inquiry that is posed is whether such a theory can be set in manifestly covariant form, i.e., it is expressible in some suitable $4-$tensor form, so that it may identically satisfy the requirement of invariance in form with respect to arbitrary local coordinate (point) transformations. By analyzing in detail the precise nature of the functional setting required for the validity of the EH action principle, we intend to show that the answer to such a question is negative. In other words, in the functional setting required by the EH-action principle, there is no $4-$tensor nor $4-$tensor-density Hamiltonian approach for EFE to be based on the EH-action principle.

Second question: is there a non-trivial manifestly-covariant and $4-$tensor Hamiltonian theory of GR? The issue is whether an alternative variational approach exists which, contrary to the EH case, is capable of leading to a unique 4−tensor Hamiltonian representation of GR and which is also manifestly covariant. The task must be reached consistent with the DeDonder–Weyl formalism of the continuum field dynamics variational formulation, in which Lagrangian coordinates and a corresponding phase-space Hamiltonian state are realized only in terms of 4−tensors [11–17]. Such an approach is shown to exist and to coincide with CCG theory, which satisfies the physical requirements set by the principle of objectivity. The latter demands that it should be possible to identify physical observables with objective observables, namely that such a character must be unique and independent of a particular realization of the GR reference frame where the same observables are evaluated or measured. Necessarily, this notion can only be mathematically formulated after the identification of the metric tensor of the background space-time solution of the Einstein field equations and with respect to which the tensorial properties of a theory and physical observables are established. As shown below, the objectivity principle is therefore strictly related to the principles of general and manifest covariance, both satisfied by CCG theory. It is shown, in fact, that such a theory can be achieved thanks to the adoption of a suitably defined functional setting and the notion of the synchronous variational principle, while within the same approach, a non-trivial Hamiltonian representation of this type can be obtained for GR.

Third question: is there an extended-variable and manifestly covariant 4−tensor Hamiltonian theory of GR? This question, which is closely related to the previous one, concerns the (still unsolved) issue of whether there may also exist independent extended-type variational approaches for EFE, namely based on an extended set of Lagrangian independent variables, to be suitably identified, and in terms of which a corresponding extended-variable manifestly covariant Hamiltonian approach can be achieved.

The first question indicated above may apply, in principle, to the majority of the literature's variational approaches known so far. These include, in particular, the Dirac and ADM approaches, as well as the so-called covariant canonical gauge approach [18]. In this regard, with the exception of CCG theory, almost all previous approaches share a unique feature of the EH-action principle, namely—as clarified below—that of being all based on the adoption of an asynchronous action principle. This means, in other words, that they adopt variational principles in which the relevant space-time volume element is considered a function of the variational space-time metric field tensor (see discussion below in Section 2). The main differences characterizing CCG theory arise, however, because of: (a) the prescription of the functional setting for the variational tensor functions which are adopted, denoted here as $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ and not to be identified with metric tensors; (b) the synchronous type of action variational principle adopted; (c) the further requirement concerning the precise prescription of the gauge invariance property to be adopted in this context. In fact, a natural choice should follow by analogy with the corresponding well-known flat space-time gauge theories available for continuum fields. Nevertheless, it is well-known that such a property is not met by the majority of variational GR approaches to be found in the literature. As a consequence, for the appropriate treatment of the subject posed in this paper, the investigation must necessarily rely on the 4−tensor manifestly covariant Hamiltonian formulation of classical gravity (CCG-theory) established in Refs. [5,9,10], which in turn is at the basis of the corresponding covariant quantum gravity theory (CQG-theory) [19,20].

In detail, the contents of the paper deal with a number of conceptual issues and related goals, which are discussed below according to the following scheme. In Section 2, the possible non-unique realizations of the property of manifest covariance, i.e., invariance in form, implied by the general covariance principle (GCP) is addressed. It is shown that possible realizations of the notion of invariance in form can be in principle obtained either by 4−tensor or 4−tensor-density transformation laws. In particular, the principle of objectivity (PO) is formulated, and the existence of the space-time of the universe (background space-time) is pointed out. Such a space-time is shown to be characterized by prescribed geometric properties, i.e., a background metric tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ and corresponding background Ricci tensor $\widehat{R}_{\mu\nu} \equiv R_{\mu\nu}(\widehat{g}(r))$.

In Section 3, an objective prescription of the variational action principle for GR is achieved. It is shown that validity of PO necessarily requires the introduction of a suitable functional setting and the adoption of an appropriate synchronous variational principle for the fundamental action principle of GR. In such a setting, the variational tensor field $g(r)$ is not treated as a metric tensor by itself, being instead regarded as a tensor function prescribed with respect to the metric field tensor of the background space-time $\widehat{g}(r)$. This property corresponds effectively to a realization of a variational principle of GR in terms of superabundant variables. In particular, it is shown that a unique prescription can be obtained for the invariance properties of the fundamental action functional, while validity of PO warrants that manifest covariance can only be realized in terms of a $4-$scalar variational Lagrangian and $4-$tensor canonical fields.

For this purpose, various different synchronous Lagrangian variational principles are presented. In particular, in this context, the topic will be recalled that concerns the construction of a manifestly covariant Lagrangian dynamical equation for the variational tensor field $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$, which recovers exactly EFE under suitable prescription of the extremal functions. The resulting Lagrangian action principle, in particular, appears significant because it permits the construction of a corresponding manifestly covariant Hamiltonian treatment of SF-GR.

Furthermore, in Section 4, the possible existence of an extended set of independent Lagrangian variables is investigated, which can generalize the customary identification in terms of the tensor $g(r)$ alone, but remain nevertheless associated only with the same EFE. It is proved that this task can be successfully met when the same Lagrangian variables are identified with the set of independent variational coordinates $\{g(r), R(r)\}$, with $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ and $R(r) \equiv \{R_{\mu\nu}(r)\} \equiv \{R^{\mu\nu}(r)\}$ denoting, respectively, the variational tensor field and the variational Ricci tensor field. The corresponding Lagrangian formulation of EFE is referred to as the "metric-Ricci" Lagrangian variational theory. This result permits us to prove in turn in Section 5 that a manifestly covariant formulation of the "metric-Ricci" Hamiltonian theory of GR can be equally achieved. In such a case, the canonical variables are identified with the independent Lagrangian coordinates, represented by the set of variational tensor fields $\{g(r), R(r)\}$ together with corresponding extended conjugate canonical momenta, namely realized in agreement with the DeDonder–Weyl formalism. As a final issue, a fundamental gauge property that must apply for consistency to any variational theory of continuum fields is proved to also hold for the variational treatment of GR only, provided one adopts synchronous variational principles, i.e., consistent with the principle of manifest covariance.

2. The General Covariance Principle

The General Covariance Principle (GCP), which is set at the basis of GR, represents a theoretical cornerstone common to all variational approaches to EFE. In theoretical physics (following Weinberg, 1972 [21]), the notion of general covariance, also known as diffeomorphism covariance, is well-known. It consists of the realization of a suitable condition of invariance in form for the relevant physical laws, or more generally, the condition of (simple) covariance, with respect to suitable coordinate transformations. However, such coordinate transformations are not completely arbitrary since they must be intended as diffeomorphisms of the space-time in itself, i.e., local transformations which preserve the space-time structure. This requires, therefore, prescribing both the said space-time structure and stating explicitly the local nature of the same transformations for such a purpose. Thus, let us assume for definiteness that the space-time is represented by a Riemannian differential manifold of the type $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with $\mathbf{Q}^4$ being the $4-$dimensional real vector space $\mathbb{R}^4$ representing the space-time and $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ being a real and symmetric metric tensor which is parametrized with respect to a coordinate system (or GR-frame) $r \equiv \{r^\mu\} \in \mathbf{Q}^4$. Then, the same coordinate transformations, denoted as local point transformations (or briefly LPT), must preserve the structure of space-time, i.e., they must be realized by local and differentiable bijections of the form

$$r \to r' = r'(r), \tag{1}$$

with inverse

$$r' \to r = r(r'), \tag{2}$$

characterized by a non-singular Jacobian matrix $M \equiv \left\{ M^k_\mu(r) \right\} \equiv \left\{ \frac{\partial r^k(r)}{\partial r'^\mu} \right\}$. Thus, $r \equiv \{r^\mu\}$ and $r' \equiv \{r'^\mu\}$ are arbitrary points belonging to the initial and transformed space-time structures $\{\mathbf{Q}^4, \widehat{g}(r)\}$ and $\{\mathbf{Q}'^4, \widehat{g}'(r')\}$, respectively. By construction, the same space-time structure is preserved under the LPT-group (the group of local point transformations of the type (1)), so that actually $\{\mathbf{Q}^4, \widehat{g}(r)\} \equiv \{\mathbf{Q}'^4, \widehat{g}'(r')\}$, while the metric tensors $\widehat{g}(r)$ and $\widehat{g}'(r')$ transform in each other in accordance with the appropriate tensor (i.e., 4−tensor) transformation laws. For definiteness, denoting $\widehat{g}'(r') \equiv \left\{ \widehat{g}'_{\mu\nu}(r') \right\} \equiv \{\widehat{g}'^{\mu\nu}(r')\}$ as the same symmetric metric 4−tensor expressed in the transformed coordinates r', by construction, it follows that:

(a) The Riemann distance in the two space-times $\{\mathbf{Q}^4, \widehat{g}(r)\}$ and $\{\mathbf{Q}'^4, \widehat{g}'(r')\}$ is the same; namely, it is realized by means of a 4−scalar, so that $ds^2 = \widehat{g}_{\mu\nu}(r) dr^\mu dr^\nu = \widehat{g}'_{\mu\nu}(r') dr'^\mu dr'^\nu$.

(b) The fields $\widehat{g}(r)$ and $\widehat{g}'(r')$ are 4−tensors. Hence, their covariant components $\widehat{g}_{\mu\nu}(r)$ and $\widehat{g}'_{\mu\nu}(r')$ are related via the corresponding covariant 4−tensor transformation laws. In tensor and symbolic form, the direct and inverse transformations $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \to \widehat{g}'(r') \equiv \left\{ \widehat{g}'_{\mu\nu}(r') \right\}$ and $\widehat{g}'(r') \equiv \left\{ \widehat{g}'_{\mu\nu}(r') \right\} \to \widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ read, respectively,

$$\begin{cases} \widehat{g}'_{\alpha\beta}(r') = \widehat{g}_{\mu\nu}(r(r')) \frac{\partial r^\mu}{\partial r'^\alpha} \frac{\partial r^\nu}{\partial r'^\beta} \\ \widehat{g}_{\mu\nu}(r) = \widehat{g}'_{\alpha\beta}(r'(r)) \frac{\partial r'^\alpha}{\partial r^\mu} \frac{\partial r'^\beta}{\partial r^\nu} \end{cases}, \tag{3}$$

$$\begin{cases} \widehat{g}'(r') = M(r(r')) \bullet \widehat{g}(r(r')) \bullet M(r(r')) \\ \widehat{g}(r) = M^{-1}(r'(r)) \bullet \widehat{g}(r'(r)) \bullet M^{-1}(r'(r)) \end{cases}, \tag{4}$$

where $M(r(r')) = \left\{ \frac{\partial r^\mu}{\partial r'^\alpha} \right\}$ and $M^{-1}(r'(r)) = \left\{ \frac{\partial r'^\alpha}{\partial r^\mu} \right\}$ are the direct and inverse Jacobian matrices and, for brevity, "•" denotes here a symbolic matrix product.

(c) The tensor fields $\widehat{g}(r)$ and $\widehat{g}'(r')$ are metric tensors so that they are required to satisfy the orthogonality conditions

$$\widehat{g}_{\mu\nu}(r)\widehat{g}^{\mu\eta}(r) = \delta^\eta_\nu, \tag{5}$$

$$\widehat{g}'_{\mu\nu}(r')\widehat{g}'^{\mu\eta}(r') = \delta^\eta_\nu. \tag{6}$$

(d) Finally, the Ricci and Riemann tensors $R_{\mu\nu}(\widehat{g}(r)), R_{\mu p \nu q}(\widehat{g}(r))$ and $R'_{\mu\nu}(\widehat{g}'(r')), R'_{\mu p \nu q}(\widehat{g}'(r'))$, which are associated, respectively, with the two structures $\{\mathbf{Q}^4, g(r)\}$ and $\{\mathbf{Q}'^4, g'(r')\}$, are transformed in each other in accordance with the covariance 4−tensor transformation laws indicated above by Equations (4) and (8).

Coming now to the specific realization of GCP, all covariant physical laws should hold in arbitrary GR-frames. Of course this does not mean that they should necessarily take the same functional form in all GR-frames (as corresponds to the notion of simple covariance). This means that an arbitrary non-4−tensor smooth real function $F(\widehat{g}(r), r)$ represented with respect to $\{\mathbf{Q}^4, \widehat{g}(r)\}$ will generally transform with respect to the LPT-group (1)–(2) in such a way that its corresponding transformed function $F'(\widehat{g}'(r'), r')$ represented with respect to $\{\mathbf{Q}'^4, \widehat{g}'(r')\}$ takes the form $F'(\widehat{g}'(r'), r') = F'(M(r(r')) \bullet \widehat{g}(r(r')) \bullet M(r(r')), r(r'))$, with $F'(\widehat{g}'(r'), r')$ denoting a real and smooth function of $\widehat{g}'(r')$ and r' generally different from $F(\widehat{g}'(r'), r')$.

2.1. The Principle of Manifest Covariance and Its Extension

In this section we define the precise meaning of manifest covariance and its possible extension. As we intend to show here, the two notions depend actually on the precise representation chosen for the field variables and physical observables in GR, to be all considered here as purely classical ones.

One possible realization of the notion of invariance in form/manifest covariance is provided by the so-called *principle of manifest covariance* (PMC) first introduced in Ref. [4]. According to such a principle, it should always be possible to cast all physical laws in 4−tensor form, i.e., in manifestly covariant form. It is obvious that PMC implies the validity of GCP. The proof is elementary. In fact, let us assume that the space-time is of the form $\{\mathbf{Q}^4, \widehat{g}(r)\}$. Then, according to PMC, it should always be possible to cast all physical laws in 4−tensor form, i.e., to be expressed equivalently, just as the metric 4−tensor (see Equation (3)), in covariant or counter-variant forms. Indeed, all tensor indexes are necessarily raised and lowered by its countervariant and covariant components $\widehat{g}^{\mu\nu}(r)$ and $\widehat{g}_{\mu\nu}(r)$, respectively. Thus, for example, let us consider the case in which a given (arbitrary) set of physical laws is expressed by the covariant 4−tensor equations

$$\begin{cases} \phi(r) = 0, \\ V_\mu(r) = 0, \\ A_{\mu\nu\ldots}(r) = 0. \end{cases} \tag{7}$$

Then, in terms of the local point transformation (1)–(2), it follows that, by assumption, the same observables $\{\phi(r), V_\mu(r), A_{\mu\nu\ldots}(r)\}$ must transform according to the covariant 4−tensor transformation laws

$$\begin{cases} \phi'(r') = \phi(r), \\ V'_{\mu'}(r') = V_\mu(r)\frac{\partial r^\mu}{\partial r'^{\mu'}}, \\ A'_{\mu'\nu'\ldots}(r') = A_{\mu\nu\ldots}(r)\frac{\partial r^\mu}{\partial r'^{\mu'}}\frac{\partial r^\nu}{\partial r'^{\nu'}}\ldots. \end{cases} \tag{8}$$

As a consequence, the transformed 4−tensor equations become

$$\begin{cases} \phi'(r') = 0, \\ V'_\mu(r') = 0, \\ A'_{\mu\nu\ldots}(r') = 0, \end{cases} \tag{9}$$

and are therefore invariant in form, since they take exactly the same form as Equation (7).

We stress, however, that in principle, the covariant 4−tensor transformation laws (8) may not be the only possible realization consistent with the notion of invariance in form, i.e., equation of the type (9). Thus, for example, introducing the determinant of the Jacobi matrix $\left|\frac{\partial r'}{\partial r}\right| = \left|\frac{\partial r}{\partial r'}\right|^{-1}$, another possible realization can be achieved assuming that at least some of the observables $\{\phi(r), V_\mu(r), A_{\mu\nu\ldots}(r)\}$, identify 4−*tensor-densities* of order 0 to n. This requires assuming for them a *covariant* 4−*tensor-density transformation law,* i.e., of the form

$$\begin{cases} \phi'(r') = \phi(r)\left|\frac{\partial r}{\partial r'}\right|^{-1}, \\ V'_{\mu'}(r') = V_\mu(r)\left|\frac{\partial r}{\partial r'}\right|^{-1}\frac{\partial r^\mu}{\partial r'^{\mu'}}, \\ A'_{\mu'\nu'\ldots}(r') = A_{\mu\nu\ldots}(r)\left|\frac{\partial r}{\partial r'}\right|^{-1}\frac{\partial r^\mu}{\partial r'^{\mu'}}\frac{\partial r^\nu}{\partial r'^{\nu'}}\ldots. \end{cases} \tag{10}$$

We also stress that in this case, the transformed equations recover again the form (9) and can therefore also be considered as invariant in form. The conclusion drawn here is, therefore, that the notion of manifest covariance can be conveniently extended to include, besides 4−tensors, 4−tensor-densities as well. This notion will be referred to as extended manifest covariance. Conversely, we shall refer to the extended principle of manifest covariance (EPMC) as the one in which all physical laws are cast either in $4-$tensor or in $4-$tensor-density forms.

2.2. The Principle of Objectivity

From the discussion above, it might appear that, in the case of arbitrary classical physical dynamical variables and/or observables, there is no mandatory physical requirement for the occurrence of either $4-$tensor or $4-$tensor-density transformation properties (i.e., either Equations (8) or (10)). In fact, even if some of them are certainly $4-$tensors, such as the variational metric tensor, it is not obvious nor necessary why all observables should be such.

One such example, in the context of GR, is represented by the structure of space-time (universe) $\left\{\mathbf{Q}^4, \widehat{g}(r)\right\}$, which also lays at the basis of PGC and PMC and is associated with the background metric tensor $\widehat{g}(r) \equiv \left\{\widehat{g}_{\mu\nu}(r)\right\} \equiv \left\{\widehat{g}^{\mu\nu}(r)\right\}$. In this regard, there are two possible viewpoints.

The first one is based on the *principle of objectivity* and is called an *objective viewpoint*. For definiteness, the principle can be stated here as follows: in GR, the classical background metric tensor $\widehat{g}(r)$, which determines the structure of the space-time $\left\{\mathbf{Q}^4, \widehat{g}(r)\right\}$, should be "objective in character". In other words, it should be realized by a classical observable, namely a $4-$tensor field that can actually be measured locally in terms of a classical measurement. Hence, it should always be possible to measure it by setting an arbitrary GR-frame and by means of a classical (ideal) measurement experiment. In particular, within the standard formulation of GR (SF-GR) [6], the same background metric tensor should coincide with a particular solution of EFE (i.e., subject to appropriate boundary conditions). The same $4-$tensor field, which can be locally measured by means of a classical (ideal) measurement experiment, determines the structure of the space-time of the universe (or background space-time).

We notice, however, that an alternate viewpoint is also still possible, i.e., in which the background metric tensor $\widehat{g}(r)$ is considered variational, namely treated as a non-observable tensor field. Thus, in contrast, this will be referred to as a *non-objective viewpoint*. The two viewpoints actually lead to two different possible variational formulations of GR to be reviewed in the next Section. However, what we intend to point out in this paper is that only the first one (i.e., the objective viewpoint) actually permits us to achieve a manifestly covariant Hamiltonian representation for GR.

Before closing, it is worth summarizing the key consequences of the objective viewpoint:

- *Consequence #1*—the validity of the Objectivity Principle implies, therefore, the existence of a classical background space-time, characterized by a prescribed background metric tensor $\widehat{g}(r) \equiv \left\{\widehat{g}_{\mu\nu}(r)\right\} \equiv \left\{\widehat{g}^{\mu\nu}(r)\right\}$.
- *Consequence #2*— all observables depending on $\widehat{g}(r)$ must be regarded as objective observables as well. Equivalently, the geometry of the background space-time $\left\{\mathbf{Q}^4, \widehat{g}(r)\right\}$ depends uniquely on $\widehat{g}(r)$. This implies, in particular, that the Riemann distance and the invariant $4-$volume element of the background space-time should be unique functions of $\widehat{g}(r)$, namely of the form

$$ds^2 = \widehat{g}_{\mu\nu}(r) dr^\mu dr^\nu, \tag{11}$$

$$d\widehat{\Omega} = d^4 r \sqrt{-|\widehat{g}(r)|}. \tag{12}$$

Similarly, the space-time Riemann and Ricci $4-$tensors associated with the same background space-time, i.e.,

$$\widehat{R}_{\mu p \nu q} = R_{\mu p \nu q}(\widehat{g}(r)), \tag{13}$$

$$\widehat{R}_{\mu\nu} = R_{\mu\nu}(\widehat{g}(r)) = \widehat{g}^{pq}(r) R_{\mu p \nu q}(\widehat{g}(r)), \tag{14}$$

should be regarded as objective observables. As a consequence, the covariant derivative of $\widehat{g}(r)$ must vanish, since identically the metric compatibility conditions must hold, namely

$$\widehat{\nabla}_\eta \widehat{g}_{\mu\nu}(r) = \partial_\eta \widehat{g}_{\mu\nu} - \widehat{\Gamma}^p_{\eta\mu}\widehat{g}_{p\nu} - \widehat{\Gamma}^p_{\eta\nu}\widehat{g}_{\mu p} = 0, \tag{15}$$

$$\widehat{\nabla}_\eta \widehat{g}^{\mu\nu}(r) = \partial_\eta \widehat{g}^{\mu\nu} + \widehat{\Gamma}^\mu_{\eta p}\widehat{g}^{p\nu} + \widehat{\Gamma}^\nu_{\eta p}\widehat{g}^{\mu p} = 0, \tag{16}$$

where $\widehat{\Gamma} \equiv \Gamma(\widehat{g}(r)) = \left\{\widehat{\Gamma}^\eta_{\mu\nu}(\widehat{g}(r))\right\}$, with $\widehat{\Gamma}^\eta_{\mu\nu} \equiv \Gamma^\eta_{\mu\nu}(\widehat{g}(r))$, and $\widehat{R}_{\mu\nu}$ as the standard connections (Christoffel symbols) and Ricci tensor expressed in terms of the background metric tensor $\widehat{g}(r)$. All such observables determined in terms of the background metric tensor $\widehat{g}(r)$ therefore coincide with background or extremal tensor fields. In particular, this means that the Riemann and Ricci tensors are necessarily $4-$tensors with respect to the background space-time $\left\{\mathbf{Q}^4, \widehat{g}(r)\right\}$. Hence, the objectivity principle implies the existence of a background space-time, characterized by prescribed geometric properties, i.e., a background metric tensor $\widehat{g}(r) \equiv \left\{\widehat{g}_{\mu\nu}(r)\right\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ and corresponding background Ricci tensor $\widehat{R}_{\mu\nu} \equiv R_{\mu\nu}(\widehat{g}(r))$.

- *Consequence #3*— In the following, we intend to prove that, once the objective viewpoint is taken, then necessarily all relevant dynamical variables and physical observables can always be identified with $4-$tensors. This will be referred to here as the universal $4-$tensor property of classical field variables.

3. Asynchronous and Synchronous Lagrangian Variational Approaches

The comparison between asynchronous and synchronous Lagrangian variational approaches for the Einstein equations has already been discussed at length elsewhere [6]. However, in reference to the issue treated in this paper, i.e., the possible existence of the universal $4-$tensor property, which has been conjectured above, it is important to discuss in detail a closely related issue. As we intend to clarify below, this is represented by the choice of the appropriate functional settings to be adopted in asynchronous and synchronous treatments. In fact, the crucial difference between them lies precisely in the choice of the said functional settings, which can be based either on objective or non-objective viewpoints.

3.1. The Asynchronous and Synchronous Functional Settings

Here, we consider two possible functional settings, to be denoted, respectively, as asynchronous and synchronous ones. Thus, in particular, the asynchronous functional setting is represented by the ensemble $\{g\}_C$ of constrained varied functions $g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r)$ (with $g(r,\theta) \equiv \left\{g_{\mu\nu}(r,\theta)\right\} \equiv \{g^{\mu\nu}(r,\theta)\}$) defined by Equation (A1) in Appendix A, where $g_{extr}(r)$ is a suitable extremal tensor function to be later selected, and such that the variation $\delta g(r) \equiv \frac{d}{d\theta} g(r,\theta)\Big|_{\theta=0}$ on the improper boundary $\mathbf{Q}^4 \equiv \mathbb{R}^4$ is subject to the boundary constraint

$$\delta g(r)|_{\partial \mathbf{Q}^4} = 0, \tag{17}$$

while each variational (or varied) function $g(r,\theta)$ identifies an independent metric tensor for a corresponding (independent) space-time $\left\{\mathbf{Q}^4, g(r)\right\}$, so that its counter and covariant components $\{g^{\mu\nu}\}$ and $\left\{g_{\mu\nu}\right\}$ necessarily raise and lower all tensor indices. Furthermore, in $\{g\}_C$, by assumption, the connections $\Gamma(g(r)) \equiv \left\{\Gamma^\eta_{\mu\nu}\right\}$ and the Ricci tensor $R(g(r)) \equiv \left\{R_{\mu\nu}\right\} \equiv \{R^{\mu\nu}\}$ are considered functions of the variational tensor field $g(r)$. As a consequence, denoting the variation of any smooth function $f(g(r))$ as

$$f(g(r)) = \frac{d}{d\theta} f(g(r,\theta))\Big|_{\theta=0}, \tag{18}$$

it follows that $\delta\Gamma(g(r))$ and $\delta R(g(r))$ are generally non-zero. In addition, the same variation operator δ acts in such a way that it does not preserve the 4−scalar volume element $d\Omega$, since generally

$$\delta d\Omega \equiv d^4 r\delta\sqrt{-|g|} \neq 0. \tag{19}$$

For this reason, the operator δ is referred to here as an *asynchronous variational operator*, and the volume element

$$d\Omega = d^4 r\sqrt{-|g(r)|} \tag{20}$$

is denoted as an *asynchronous* 4−*scalar volume element*. Because of the consequent explicit variational contribution arising due to the non-constant volume element, in analogy with the terminology adopted in classical mechanics, the set $\{g\}_C$ is thus denoted as an *asynchronous functional setting.*

Instead, an alternative functional setting is represented by the ensemble $\{g\}_U$ of unconstrained varied functions $g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r)$ (with $g(r,\theta) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$) defined by Equation (A2) in Appendix A. We notice that in this case, $g(r,\theta)$ are now tensor fields with respect to the background space-time $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with $\widehat{g}(r)$ denoting the metric background, i.e., extremal, tensor field defined above (see Section 2). Furthermore, $g_{extr}(r)$ is an extremal tensor field to be prescribed and no constraint condition on $\delta g(r)|_{\partial\mathbf{Q}^4}$ is set any more. In addition, now the Ricci tensor, the standard connections and the 4−scalar volume element $d\widehat{\Omega}$ are all considered functions of the background tensor field $\widehat{g}(r)$ only, and their tensor indices, when appropriate, are raised and lowered by the same metric field tensor $\widehat{g}(r)$. Thus, since

$$\delta\widehat{g}(r) \equiv \frac{d}{d\theta}\widehat{g}(r)\bigg|_{\theta=0} = 0, \tag{21}$$

it follows that $\delta\Gamma(\widehat{g}(r))$, $\delta R(\widehat{g}(r))$ and $\delta d\widehat{\Omega}$ all vanish identically. The last property justifies the name given to the operator δ in this case, being referred to as a *synchronous variational operator*, with $d\widehat{\Omega}$ being defined by Equation (12) and referred to as a synchronous 4−scalar volume element. This also justifies the label of *synchronous functional setting* given to the set $\{g\}_U$.

Given these premises, we analyze in detail the two cases.

3.2. Asynchronous Lagrangian Action Principle (Non-Objective Viewpoint)

The set $\{g\}_C$ denotes the functional class of "normalized" asynchronous varied tensor functions $g(r)$ represented by the 10 covariant (or counter-variant) independent components of the symmetric tensor field $g(r)$. In this set, the variational tensor field $g(r) \equiv \{g_{\mu\nu}\}$ is also a metric tensor associated with the space-time $\{\mathbf{Q}^4, g\}$ ($\mathbf{Q}^4 \equiv \mathbb{R}^4$). As a consequence, this means that $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ raises and lowers tensor indices, and it satisfies the orthogonality conditions

$$g_{\mu\nu} g^{\mu k} = \delta^k_\nu, \tag{22}$$

which imply in turn the "normalization" condition $g_{\mu\nu}(r)g^{\mu\nu}(r) = 4$. In addition, by assumption in the same functional setting, the tensor $g(r)$ determines both the Christoffel symbols $\Gamma(g(r))$ and the Ricci tensor $R_{\mu\nu}(g)$, so that $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ satisfies the metric compatibility condition, i.e., its covariant derivatives necessarily vanish identically, since:

$$\nabla_\eta g_{\mu\nu}(r) \equiv g_{\mu\nu;\eta}(r) = 0, \tag{23}$$
$$\nabla_\eta g^{\mu\nu}(r) \equiv g^{\mu\nu}_{;\eta}(r) = 0, \tag{24}$$

where ∇_η is the covariant derivative defined with respect to the Christoffel symbols $\Gamma(g(r))$. For the proof we refer to Equations (15) and (16) (see also THM. 3.1.1 of Ref. [22]).

Let us now briefly examine the consequences for the corresponding action variational principles. Consider first the asynchronous-variation action functional which is characteristic of EH theory, namely the Einstein–Hilbert Lagrangian variational approach expressed in terms of asynchronous variations. This is represented by the Einstein-Hilbert (EH) action functional [23]

$$S_{EH}(g(r)) \equiv \int_{\mathbf{Q}^4} d\Omega L_g(g) = \int_{\mathbf{Q}^4} d^4 r \overline{L}(g), \tag{25}$$

where $d\Omega \equiv d^4 r \delta \sqrt{-|g|}$ is the invariant 4−volume element of the Riemann space-time $\{\mathbf{Q}^4, g(r)\}$, $d^4 r \equiv \prod_{i=0,3} dr^i$ its canonical measure, while $L(g)$ and

$$\overline{L}(g) \equiv \sqrt{-|g|} L(g), \tag{26}$$

denote, respectively, the variational Lagrangian and the Lagrangian functions (the latter being defined by Equation (28) below). The following basic features of the asynchronous variational approach based on Definition (25) must be pointed out:

- The first one is that, consistent with the non-objective viewpoint discussed above, the varied functions, i.e., the Lagrangian variables $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ appearing in Equation (25) belong by assumption to $\{g\}_C$.
- The second aspect is that, since the space-time 4−volume element $d\Omega$ by definition depends on the determinant of the variational metric tensor, necessarily its variation $\delta d\Omega$ is non-vanishing, since

$$\delta d\Omega = d^4 r \delta \sqrt{-|g|} \neq 0, \tag{27}$$

  with $\delta \sqrt{-|g|} = \frac{1}{2} \sqrt{-|g|} g^{\mu\nu} \delta g_{\mu\nu}$. As a consequence, the variation of the functional $S_{EH}(g(r))$ does not preserve the space-time volume element (and for this reason is referred to as asynchronous).
- The third feature is that the variational Lagrangian $\overline{L}(g)$, defined by Equation (26), is not a 4−scalar, but rather a 4−scalar-density. This confirms, therefore, that the same 4−tensor-density appears as a consequence of the chosen non-objective viewpoint, implicit in the choice of the functional setting $\{g\}_C$. Indeed, in Equations (25) and (26)

$$L(g, r) = V_{EH}(g) + V_{Fg}(g, r) \tag{28}$$

  identifies a 4−scalar Lagrangian. In particular,

$$V_{EH} = \alpha_L \left[g^{\mu\nu} R_{\mu\nu}(g) - 2\Lambda \right] \tag{29}$$

  denotes the vacuum gravitational contribution, while $V_{Fg}(g, r)$ is a suitably defined non-vacuum contribution (due to possible external fields [5]), and Λ is the cosmological constant. We stress that both $V_{EH}(g)$ and $V_{F_g}(g, r)$ actually contain a common factor of indeterminacy. Thus, they can be defined up to a common factor α_L, to be treated as a suitably defined universal constant [5]. A crucial aspect, which incidentally explains the prescription (26) in terms of a 4−scalar-density, is represented by the functional dependences, which are contained both through $|g|$, the determinant of $g(r)$, and implicitly through the Ricci tensor $R_{\mu\nu}(g)$. The latter, in fact, in $V_{EH}(g)$ is also considered a function of the same variational tensor $g(r)$ through the Christoffel symbols (which are, by assumption, functions of $g(r)$).
- The fourth notable feature lies in the EH variational principle (or EH action principle). This is obtained by requiring that for arbitrary variations $\delta g(r)$ belonging to $\{g\}_C$, it must be

$$\delta S_{EH}(g(r))|_{g=\widehat{g}(r)} = \frac{d}{d\theta} S_{EH}(g_{extr}(r) + \theta \delta g(r)) \Big|_{\theta=0} = 0, \tag{30}$$

with the symbol δ denoting the Frechet derivative [24]. As discussed in Ref. [6], straightforward algebra then delivers

$$\delta S_{EH}(g(r))|_{g=\widehat{g}(r)} = (\delta S_{EH}(g))_{\text{expl}} + (\delta S_{EH}(g))_{\text{impl}}, \tag{31}$$

where

$$(\delta S_{EH}(g))_{\text{impl}} = \alpha_L \int_{\mathbf{Q}^4} d^4r \sqrt{-|g|}\widehat{g}^{\mu\nu}\delta R_{\mu\nu}, \tag{32}$$

while the explicit contributions yield upon identifying $g_{extr}(r) = \widehat{g}(r)$ and upon restoring the correct dimensional units

$$(\delta S_{EH}(g))_{\text{expl}} = \alpha_L \int_{\mathbf{Q}^4} d^4r \sqrt{-|g|}\left[\widehat{R}_{\mu\nu} - \left(\frac{1}{2}\widehat{R} - \Lambda\right)\widehat{g}_{\mu\nu}(r) - \kappa\widehat{T}_{\mu\nu}\right]\delta g^{\mu\nu}. \tag{33}$$

Here, as usual, $\widehat{R}_{\mu\nu} = R_{\mu\nu}(\widehat{g}(r))$ and $\widehat{R} = \widehat{g}^{\mu\nu}(r)R_{\mu\nu}(\widehat{g}(r)) \equiv R((\widehat{g}(r))$ denote, respectively, the background Ricci 4−tensor and Ricci 4−scalar, while $\widehat{T}_{\mu\nu} = T_{\mu\nu}(\widehat{g}(r))$ is the background stress-energy tensor associated with the external source fields described by the external-field Lagrangian density $L_F(g)$. Here, as anticipated, the universal constant α_L which multiplies the rhs (right hand side) of Equation (33), does not affect the EH-action principles, while κ denotes the universal constant

$$\kappa = \frac{8\pi G}{c^4}, \tag{34}$$

where G is the Newtonian constant of gravitation and c is the speed of light in a vacuum.

- Finally, as a fifth notable feature, in order to exactly recover the Einstein field equations, namely

$$\widehat{R}_{\mu\nu} - \left(\frac{1}{2}\widehat{R} - \Lambda\right)\widehat{g}_{\mu\nu} = \kappa\widehat{T}_{\mu\nu}, \tag{35}$$

with κ being defined by Equation (34), it is necessary that the constraint condition

$$(\delta S_{EH}(g))_{\text{impl}} = 0 \tag{36}$$

must be set. As a consequence, as shown in Ref. [6], the EH variational principle in Equation (30) must be considered as a constrained one, the constraint (36) realizing effectively a boundary-constrained EH variational principle. One can show, based on the treatment reported in Ref. [6], that such a constraint is fulfilled identically provided the variations $\delta g(r)$ are subject to the constraint (17). The same constraint condition also implies that its partial derivatives must vanish.

3.3. Synchronous Metric-Lagrangian Action Principle (Objective Viewpoint)

The set $\{g\}_U$ denotes the functional class of "un-normalized" synchronous varied tensor functions $g(r)$ corresponding to the 10 covariant (or counter-variant) components of symmetric tensor field $g(r)$. In this case, the variational tensor field $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ is a tensor field associated with the background space-time $\{\mathbf{Q}^4, \widehat{g}(r)\}$ ($\mathbf{Q}^4 \equiv \mathbb{R}^4$). This means that the tensor indices of $g(r)$, namely $\{g_{\mu\nu}(r)\}$ and $\{g^{\mu\nu}(r)\}$, are lowered and raised by the background metric tensor $\widehat{g}(r)$, i.e.,

$$\begin{aligned} g_{\mu\nu}\widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta} &= g^{\eta\beta}, \\ g^{\eta\beta}\widehat{g}_{\mu\eta}\widehat{g}_{\nu\beta} &= g_{\mu\nu}. \end{aligned} \tag{37}$$

It follows that $g(r)$ is not a metric tensor, since generally $g_{\mu\nu}g^{\mu k} \neq \delta^k_\nu$. In this case, the Ricci tensor $R_{\mu\nu}$ and the Christoffel symbols Γ are background (i.e., extremal) functions of $\widehat{g}(r)$. As a consequence, the covariant derivatives of $\widehat{g}(r)$ vanish identically (see Equations (15) and (16)), namely:

$$\widehat{\nabla}_\eta \widehat{g}_{\mu\nu}(r) \equiv \widehat{g}_{\mu\nu;\eta}(r) = 0, \tag{38}$$
$$\widehat{\nabla}_\eta \widehat{g}^{\mu\nu}(r) \equiv \widehat{g}^{\mu\nu}_{;\eta}(r) = 0, \tag{39}$$

where $\widehat{\nabla}_\eta$ is the covariant derivative defined with respect to the Christoffel symbols $\Gamma(\widehat{g}(r))$. Instead, generally, one has that for the variational tensor:

$$\widehat{\nabla}_\eta g_{\mu\nu}(r) \equiv g_{\mu\nu;\eta}(r) \neq 0, \tag{40}$$
$$\widehat{\nabla}_\eta g^{\mu\nu}(r) \neq 0. \tag{41}$$

Let us now consider the corresponding synchronous action functional, first introduced in the context of CCG-theory (see Ref. [9]). Contrary to the asynchronous case, this is found to be consistent with the choice of the objective viewpoint. Once the Lagrangian variables $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ are required to belong to the functional setting $\{g\}_U$, the following implications follow: (1) the volume element coincides with the extremal volume element (12), so that its variation is synchronous, namely $\delta d\widehat{\Omega} = d^4 r \delta\sqrt{-|\widehat{g}|}$ and it vanishes identically, since $\delta\widehat{g}(r) = 0$; (2) the Ricci tensor is assumed to be a background tensor field, i.e., an extremal function of the type $\widehat{R} = R(\widehat{g}(r))$.

Thus, the variational functional, to be referred to as the *synchronous metric-Lagrangian action functional*, expressed in terms of the Lagrangian variables, is now written as:

$$S_{L_g}(g(r), \widehat{g}(r)) = \int_{\mathbf{Q}^4} d\widehat{\Omega} \overline{L}_g(g(r), \widehat{g}(r)), \tag{42}$$

where $d\widehat{\Omega}$ coincides with the extremal volume element, and $\overline{L}_g(g(r), \widehat{g}(r))$ denotes a $g-$dependent $4-$scalar variational Lagrangian. As a consequence, the variation of the volume element vanishes identically, while the variational Lagrangian is indeed a $4-$scalar. This proves, therefore, that the objective viewpoint discussed above is sufficient to demand that the same variational Lagrangian must be a $4-$scalar. The fact that the variational metric Lagrangian $\overline{L}_g(g(r), \widehat{g}(r))$ is a $4-$scalar follows since

$$\overline{L}_g(g(r), \widehat{g}(r)) \equiv h(g(r), \widehat{g}(r)) L(g(r), \widehat{g}(r)), \tag{43}$$

where

$$h(g(r), \widehat{g}(r)) = 2 - \frac{1}{4} g^{\eta\beta}(r) g^{\mu\nu}(r) \widehat{g}_{\eta\mu}(r) \widehat{g}_{\beta\nu}(r) \tag{44}$$

identifies a $4-$scalar variational weight-factor, while again $L(g(r), \widehat{g}(r))$ coincides with the prescription of the $4-$scalar Lagrangian given above (see Equation (28)). For convenience, here the compact notations

$$\overline{L}_g(g(r), \widehat{g}(r)) = \overline{V}_{og}(g(r), \widehat{g}(r)) + \overline{V}_{Fg}(g(r), \widehat{g}(r)), \tag{45}$$
$$\overline{V}_{og}(g(r), \widehat{g}(r)) = \alpha_L h \left[g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda \right], \tag{46}$$
$$\overline{V}_{F_g}(g(r), \widehat{g}(r)) \equiv -\alpha_L \kappa h L_{F_g}, \tag{47}$$

are introduced, where L_{F_g} is the $4-$scalar Lagrangian associated with external fields, coupled to the gravitational field through the tensors $(g(r), \widehat{g}(r))$. Incidentally, we notice that an alternative equivalent representation of the external-field contribution to the Lagrangian (43) can be adopted in the framework of the synchronous principle [5]. This can be realized by the formal replacement

$$h(g(r), \widehat{g}(r)) \alpha_L \kappa L_{F_g}(g(r), \widehat{g}(r)) \to \alpha_L g^{\mu\nu}(r) \kappa \widehat{T}_{\mu\nu}(r), \tag{48}$$

where $\widehat{T}_{\mu\nu}$ denotes the energy-stress tensor evaluated in terms of the background metric tensor $\widehat{g}(r)$ [23]. Then, the synchronous Lagrangian action principle for EFE follows at once. This is obtained by requiring that the variational equation

$$\delta S_{L_g}(g(r), \widehat{g}(r))\Big|_{g=\widehat{g}(r)} = 0, \tag{49}$$

holds for arbitrary synchronous variations $\delta g(r)$ belonging to the unconstrained set $\{g\}_U$, while noting that by construction, $\delta\widehat{g}(r) \equiv 0$.

Let us now show explicitly that the synchronous Lagrangian action principle (49) yields the Einstein field equations as extremal equations (see also Ref. [6]). The proof is as follows. First, the symbol δ denotes, as usual, the variation operator, i.e., again the Frechet derivative, which in the present case is defined as

$$\delta S_{L_g}(g(r), \widehat{g}(r))\Big|_{g=\widehat{g}(r)} \equiv \frac{d}{d\theta} S_{Lg}(g_{extr}(r) + \theta\delta g(r), \widehat{g}(r))\Big|_{\theta=0}. \tag{50}$$

Recalling the property of the volume element of being extremal under the action of the synchronous operator δ and invoking the Definition (43), we have that

$$\begin{aligned} \delta S_{L_g}(g(r), \widehat{g}(r))\Big|_{g=\widehat{g}(r)} &= \int_{\mathbf{Q}^4} d\widehat{\Omega} \delta \overline{L}_g(g(r), \widehat{g}(r))\Big|_{g=\widehat{g}(r)} \\ &= \int_{\mathbf{Q}^4} d\widehat{\Omega} [\delta h(g(r), \widehat{g}(r))] L(g(r), \widehat{g}(r))\Big|_{g=\widehat{g}(r)} \\ &\quad + \int_{\mathbf{Q}^4} d\widehat{\Omega} h(g(r), \widehat{g}(r)) [\delta L(g(r), \widehat{g}(r))]\Big|_{g=\widehat{g}(r)}. \end{aligned} \tag{51}$$

We treat first the 4−scalar variational factor $h(g(r), \widehat{g}(r))$ defined in Equation (44). By identifying $g_{extr}(r)$ with the background metric tensor field, i.e., letting $g_{extr}(r) = \widehat{g}(r)$, its variation gives

$$\delta h(g(r), \widehat{g}(r))|_{g=\widehat{g}(r)} = -\frac{1}{2}\widehat{g}_{\mu\nu}(r)\delta g^{\mu\nu}, \tag{52}$$

while identically it holds that

$$h(g(r), \widehat{g}(r))|_{g=\widehat{g}(r)} = h(\widehat{g}(r)) = 1. \tag{53}$$

Then, let us consider the 4−scalar Lagrangian function $L(g(r), \widehat{g}(r))$. Using the Definitions (45)–(47), its synchronous variation gives

$$\begin{aligned} \int_{\mathbf{Q}^4} d\widehat{\Omega} h \delta L(g(r), \widehat{g}(r)) &= \int_{\mathbf{Q}^4} d\widehat{\Omega} h \Big\{ \alpha_L \delta \Big[g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda \Big] - \alpha_L \delta L_{F_g} \Big\} \\ &= \int_{\mathbf{Q}^4} d\widehat{\Omega} h \Big\{ \alpha_L \Big[\widehat{R}_{\mu\nu} - \frac{\delta L_{F_g}}{\delta g^{\mu\nu}} \Big] \delta g^{\mu\nu} \Big\}. \end{aligned} \tag{54}$$

On the other hand, the extremal value of $L(g(r), \widehat{g}(r))$ becomes

$$L(g(r), \widehat{g}(r))|_{g=\widehat{g}(r)} \equiv L(\widehat{g}(r)) = \alpha_L \Big[\widehat{g}^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda \Big] - \alpha_L L_{F_g}\Big|_{g=\widehat{g}(r)}. \tag{55}$$

Finally, inserting Equations (52)–(55) into Equation (51), after straightforward algebra, one recovers immediately the Einstein field equations

$$\widehat{R}_{\mu\nu} - \left(\frac{1}{2}\widehat{R} - \Lambda \right) \widehat{g}_{\mu\nu} = \kappa \widehat{T}_{\mu\nu}, \tag{56}$$

where again the variational definition of the tensor $\widehat{T}_{\mu\nu}$ is the customary one that arises in the Einstein–Hilbert theory (see Ref. [23]). The same result can also be obtained if the relation (48) is used in place of (47), since manifestly $\frac{\delta\left(\alpha_L g^{\mu\nu}(r)\kappa\widehat{T}_{\mu\nu}(r)\right)}{\delta g^{\mu\nu}} = \alpha_L\kappa\widehat{T}_{\mu\nu}(r)$.

3.4. Search of a Metric-Hamiltonian Action Principle

The crucial question is whether, based on the two Lagrangian approaches described above (see Sections 3.2 and 3.3), a Hamiltonian variational principle can be achieved that takes either a form consistent either with PMC or EPMC. It is easy to show that the only available route is provided by a synchronous Hamiltonian action principle, i.e., which is consistent with the objectivity principle. The result recovers, therefore, the classical manifestly-covariant theory of GR (CCQ-theory) earlier reported in Refs. [6,9]. In fact, the construction of a Hamiltonian principle requires the introduction of a variational "exchange term", which carries a contribution due to the covariant derivatives of the variational field $g(r)$. A contribution of this type remains excluded for the asynchronous formulation. Indeed, in view of Equations (24) and (23), when the constrained setting $\{g\}_C$ is adopted, the variational derivatives of the corresponding variational field $g(r)$ are always identically vanishing in such a setting. Things change, however, in the context of the unconstrained functional setting $\{g\}_U$, i.e., the manifestly covariant approach based on the adoption of the principle of objectivity stated above. This occurs because only in such a framework the covariant derivatives of the variational tensor field $g(r)$, namely $\widehat{\nabla}_\eta g^{\mu\nu}$ and $\widehat{\nabla}^\eta g_{\mu\nu} \equiv \widehat{g}^{\eta\beta}(r)\widehat{\nabla}_\eta g_{\mu\nu}$, are both non-vanishing, as exemplified by Equations (40) and (41). For this purpose, we introduce here a modified form of the variational Lagrangian $4-$scalar function which, departing from Equation (43) above, is now taken of the form

$$\overline{L}_g(g(r),\widehat{\nabla}g(r),\widehat{g}(r)) = T_g - \sigma\overline{V}_g(g(r),\widehat{g}(r)), \tag{57}$$

$$\overline{V}_g(g(r),\widehat{g}(r)) = \overline{V}_{og}(g(r),\widehat{g}(r)) + \overline{V}_{F_g}(g(r),\widehat{g}(r)), \tag{58}$$

where the symbol $\widehat{\nabla}g(r)$ denotes the covariant derivative, i.e., respectively, $\widehat{\nabla}_\eta g^{\mu\nu}$ or $\widehat{\nabla}^\eta g_{\mu\nu}$. The $4-$scalar function $\overline{L}_g(g(r),\widehat{\nabla}g(r),\widehat{g}(r))$ will be referred to as *metric-Lagrangian*. For completeness, detailed expressions, together with the corresponding synchronous Hamiltonian variational principle, are reported in Appendix B.

4. Extended Lagrangian Variables: The Metric-Ricci Action Principle

Here we address the issue of completeness for the manifestly-covariant Lagrangian description of GR: namely whether there may exist additional independent Lagrangian variables, possibly associated with classical treatment of covariant gravity theory. The complete set of Lagrangian variables determined in this way will be called extended Lagrangian variables. Here, we claim that a possible variational formulation exists in which both the tensor field $g(r) \equiv \{g_{\mu\nu}(r)\} \equiv \{g^{\mu\nu}(r)\}$ (not a metric tensor) and the Ricci tensor $R(r) \equiv \{R_{\mu\nu}(r)\} \equiv \{R^{\mu\nu}(r)\}$ are considered variational and endowed with independent tensor variations $\delta g(r)$ and $\delta R(r)$. A positive answer requires, however, that the Einstein field equations should remain unchanged, i.e., that the extremal values of $g(r)$ and $R(r)$ are mutually related via the standard connections. In order to develop a synchronous variational formulation of the type considered above, this involves adopting the varied tensor fields $\{g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r), R(r,\theta) \equiv R_{extr}(r) + \theta\delta R(r)\}$, hereon referred to briefly as metric and Ricci variational fields, belonging to the *metric-Ricci functional setting* (A4) (see again Appendix A), with the extremal tensor fields $\{g_{extr}(r), R_{extr}(r)\}$ to be properly prescribed. Here, the variations $\delta g(r)|_{\partial\mathbf{Q}^4}$ and $\delta R(r)|_{\partial\mathbf{Q}^4}$ are both required to vanish on the improper boundary of $\mathbf{Q}^4$. Notice that here, the tensor indexes of both variation fields $g(r)$ and $R(r)$ are raised and lowered by the countervariant and covariant components of the background metric tensor $\widehat{g}(r)$, while $\delta g(r)$ and $\delta R(r)$, i.e., the variations of the tensor fields $g(r)$ and $R(r)$, are assumed to be linearly independent. In addition, the standard connections (Christoffel symbols) $\Gamma \equiv \Gamma^\eta_{\mu\nu}$ are again considered functions of the background metric tensor $\widehat{g}(r)$. It follows that the covariant derivatives of $g(r)$ defined in terms of the standard connection are generally non-vanishing (see Equations (40) and (41)). Then, a possible realization of the variational Lagrangian function also applicable in the case of non-vacuum fields is provided by the coupled *metric-Ricci (extended) variational 4-scalar Lagrangian field*, i.e., a linear combination (via a suitable dimensionless and real coupling constant α_1) of two Lagrangians, respectively, of the $g-$dependent Lagrangian defined above, namely $\overline{L}_g(g(r),\widehat{g}(r))$ given by Equation (57), and suitably effective $R-$dependent Lagrangian, $\overline{L}_R\left(R(r),\widehat{g}(r),\widehat{R}(r)\right)$, namely

$$\overline{L}_{g+R}\left(g(r),\widehat{g}(r),R(r),\widehat{R}\right)=\overline{L}_g\left(g(r),\widehat{g}(r),\widehat{R}(r)\right)+\alpha_1\overline{L}_R\left(R(r),\widehat{g}(r),\widehat{R}(r)\right), \tag{59}$$

with α_1 denoting a $4-$scalar coupling coefficient, and with $\overline{L}_R$ being defined here as follows:

$$\overline{L}_R\left(R(r),\widehat{g}(r),\widehat{R}(r)\right)=T_R(\widehat{g}(r),R(r))+\sigma\overline{V}_R\left(R(r),\widehat{g}(r),\widehat{R}(r)\right). \tag{60}$$

Notice that T_R and $\sigma\overline{V}_R$ denote now the effective kinetic and potential energies

$$T_R\left(g(r),\widehat{g}(r),R(r),\widehat{R}\right)=\frac{1}{2\Lambda}\alpha_L\widehat{\nabla}^{\eta}R_{\mu\nu}\widehat{\nabla}_{\eta}R^{\mu\nu}, \tag{61}$$

$$\overline{V}_R=\overline{V}_{oR}\ +\overline{V}_{F_R}. \tag{62}$$

Thus, in particular, $\overline{V}_{oR}$ and $\overline{V}_{F_{R.}}$ identify, respectively,

$$\overline{V}_{oR}\left(R(r),\widehat{g}(r),\widehat{R}(r)\right)=\alpha_L\widehat{g}_{\mu\nu}\left(R^{\mu\nu}-\widehat{R}^{\mu\nu}\right)+\alpha_L\left[\frac{1}{2\Lambda}\left(R^{\mu\nu}R_{\mu\nu}-\widehat{R}^{\mu\nu}\widehat{R}_{\mu\nu}\right)-\frac{1}{4\Lambda}\left(R^2-\widehat{R}^2\right)\right], \tag{63}$$

$$\overline{V}_{F_R}\equiv-\frac{\alpha_L}{\Lambda}\left(R^{\mu\nu}-\widehat{R}^{\mu\nu}\right)\kappa\widehat{T}_{\mu\nu}, \tag{64}$$

while the rest of the notations are standard. In the case of non-vanishing coupling coefficient α_1, the corresponding variational principle

$$\delta S_{L_{g+R}}(g(r),R(r))=0, \tag{65}$$

which is assumed to hold for arbitrary independent tensor variations $(\delta g(r),\delta R(r))$, requires the following Frechet derivatives to vanish identically:

$$\left.\frac{d\overline{L}_g\left(g_{extr}(r)+\theta\delta g,\widehat{g}(r),\widehat{R}(r)\right)}{d\theta}\right|_{\theta=0,g(r)=g_{extr}(r)}=0, \tag{66}$$

$$\left.\frac{d\overline{L}_R\left(R_{extr}(r)+\eta\delta R,\widehat{g}(r),\widehat{R}(r)\right)}{d\eta}\right|_{\eta=0,R(r)=R_{extr}(r)}=0. \tag{67}$$

These equations represent the Euler–Lagrange equations associated with Equation (65). Due to the linear independence of the synchronous variations $\delta g^{\mu\nu}$ and $\delta R^{\mu\nu}$, these can be written in terms of the variational derivatives. Upon letting

$$(g_{extr}(r),R_{extr}(r))=\left(\widehat{g}(r),\widehat{R}(r)\right), \tag{68}$$

then Equations (66) and (67) deliver the metric-Ricci Euler–Lagrange equations

$$\begin{cases} \widehat{\nabla}_\eta \left[\frac{\partial \overline{L}_g(g,\widehat{g},\widehat{R})}{\partial \widehat{\nabla}_\eta g^{\mu\nu}} \right]_{g=\widehat{g}} - \left. \frac{\partial \overline{L}_g(g,\widehat{g},\widehat{R})}{\partial g^{\mu\nu}} \right|_{g=\widehat{g}} = 0, \\ \widehat{\nabla}_\eta \left[\frac{\partial \overline{L}_R(\widehat{g},R,\widehat{R})}{\partial \widehat{\nabla}_\eta R^{\mu\nu}} \right]_{R=\widehat{R}} - \left. \frac{\partial \overline{L}_R(\widehat{g},R,\widehat{R})}{\partial R^{\mu\nu}} \right|_{R=\widehat{R}} = 0. \end{cases} \tag{69}$$

The proof of the statement is an immediate consequence of the extended (synchronous) Lagrangian formalism here pointed out. In fact, from Equation (59), we first notice that the variational field $g^{\mu\nu}$ is only carried by the Lagrangian $\overline{L}_g$, while the variational field $R^{\mu\nu}$ is only contained in $\overline{L}_R$, with the consequence that the two equations in (69) are effectively decoupled. In addition, we have that identically, when evaluated for extremal fields, the new Ricci contribution is vanishing, namely

$$\left. \overline{L}_R\left(\widehat{g}, R, \widehat{R}\right) \right|_{R=\widehat{R}} \equiv \overline{L}_R\left(\widehat{g}, \widehat{R}\right) = 0. \tag{70}$$

The explicit evaluation of the Euler–Lagrange equation for $g^{\mu\nu}$ is then exactly the same as the one outlined in Sections 3.3 and 3.4. On the other hand, one can readily see that the mathematical structure of the two Lagrangians $\overline{L}_g\left(g(r), \widehat{g}(r), \widehat{R}(r)\right)$ and $\overline{L}_R\left(R(r), \widehat{g}(r), \widehat{R}(r)\right)$ is the same for their respective variational fields, so that both contain the same type of quadratic kinetic term and analogous potential contributions. Based on these considerations, thanks to the validity of the customary Lagrangian formalism for continuum fields, one can immediately reach the proof of the formal representation of the Euler–Lagrange equation for $R^{\mu\nu}$ (i.e., the second equation in (69)).

We notice that by construction, $\widehat{\nabla}_\eta \widehat{g}^{\mu\nu} = 0$. If one requires additionally that

$$\widehat{\nabla}_\eta \widehat{R}^{\mu\nu} = 0, \tag{71}$$

should hold identically as well, this means that $\widehat{R}$ must actually depend on $\widehat{g}(r)$, such as, for example, in the case in which the background metric tensor identifies the de Sitter solution. From the Einstein field Equation (35), it follows, therefore, that the extremal stress-energy tensor $\widehat{T}_{\mu\nu}(r)$ must satisfy the constraint condition

$$\widehat{\nabla}_\eta \widehat{T}_{\mu\nu} = 0. \tag{72}$$

Such a requirement is, of course, trivially satisfied in a vacuum. However, subject to the requirement $\widehat{T}_{\mu\nu} = T_{\mu\nu}(\widehat{g}(r))$, it may also be fulfilled in the case of gravitational external sources. In such a case, it follows that the Lagrangian Equation (69) reduces identically to

$$\begin{cases} \left. \frac{\delta \overline{L}_g(g,\widehat{g},\widehat{R})}{\delta g^{\mu\nu}} \right|_{g=\widehat{g}} = \alpha_L \left(-\frac{1}{2}\widehat{g}_{\mu\nu}\widehat{R} + \widehat{R}_{\mu\nu} + \Lambda \widehat{g}_{\mu\nu} \right) - \alpha_L \kappa \widehat{T}_{\mu\nu} = 0, \\ \left. \frac{\delta \overline{L}_R(\widehat{g},R,\widehat{R})}{\delta R^{\mu\nu}} \right|_{R=\widehat{R}} = \alpha_L \left(\frac{1}{\Lambda}\widehat{R}_{\mu\nu} + \widehat{g}_{\mu\nu} - \frac{1}{2\Lambda}\widehat{R}\widehat{g}_{\mu\nu} \right) - \frac{\alpha_L}{\Lambda}\kappa \widehat{T}_{\mu\nu} = 0. \end{cases} \tag{73}$$

Again, the proof of the first equation in (73) is exactly similar to Section 3.3. Instead, the second equation in (73) represents the new theoretical goal of the formalism developed in the present section and deserves a separate proof. The latter can be obtained with straightforward algebra as follows. First, we notice that in validity of the condition (71) on the extremal fields, the dynamical contribution generated by the kinetic term in the Lagrangian $\overline{L}_R\left(R(r), \widehat{g}(r), \widehat{R}(r)\right)$ necessarily vanishes identically. Therefore, in such a setting, only the potential contribution matters. From the second equation in (69), we need only to evaluate the following contribution:

$$\frac{\partial \overline{L}_R\left(\widehat{g}, R, \widehat{R}\right)}{\partial R^{\mu\nu}} = \frac{\partial \overline{V}_R\left(\widehat{g}, R, \widehat{R}\right)}{\partial R^{\mu\nu}} = \frac{\partial \overline{V}_{oR}}{\partial R^{\mu\nu}} + \frac{\partial \overline{V}_{F_R}}{\partial R^{\mu\nu}}. \tag{74}$$

In detail, from Equation (63) and the fact that necessarily $\frac{\partial}{\partial R^{\mu\nu}}\widehat{R}^{\mu\nu} = 0$, one finds

$$\begin{aligned}\frac{\partial \overline{V}_{oR}}{\partial R^{\mu\nu}} &= \alpha_L \widehat{g}_{\mu\nu}\frac{\partial}{\partial R^{\mu\nu}}R^{\mu\nu} + \alpha_L\left[\frac{1}{2\Lambda}\frac{\partial}{\partial R^{\mu\nu}}\left(R^{\mu\nu}R_{\mu\nu}\right) - \frac{1}{4\Lambda}\frac{\partial}{\partial R^{\mu\nu}}R^2\right] \\ &= \alpha_L \widehat{g}_{\mu\nu} + \alpha_L\left[\frac{1}{\Lambda}R_{\mu\nu} - \frac{1}{2\Lambda}R\widehat{g}_{\mu\nu}\right],\end{aligned} \tag{75}$$

$$\frac{\partial \overline{V}_{F_R}}{\partial R^{\mu\nu}} = -\frac{\alpha_L}{\Lambda}\kappa\widehat{T}_{\mu\nu}\frac{\partial}{\partial R^{\mu\nu}}R^{\mu\nu} = -\frac{\alpha_L}{\Lambda}\kappa\widehat{T}_{\mu\nu}. \tag{76}$$

When evaluated for the extremal field $R = \widehat{R}$ these contributions yield the second equation in (73).

The last two equations coincide identically with the non-vacuum Einstein field equations. The first implication from this result is that the original condition on the extremal Ricci tensor introduced in the synchronous approach is eliminated. In other words, the requirement (not a constraint) $R_{\mu\nu} = \widehat{R}_{\mu\nu}$ originally set in the synchronous variational principle (see Equation (49) above) is formally replaced by the new metric-Ricci variational principle and subject to the requirement (68) on the extremal fields $(g_{extr}(r), R_{extr}(r))$. Second, the new classical physical requirement $\Lambda \neq 0$ arises. In the present context, it can be simply explained as a necessary requirement for the validity of the metric-Ricci variational principle. This feature suggests, however, the interesting role taken by the cosmological constant in this approach, which provides a new derivation of the classical Einstein equations based on the variational theory for the Ricci tensor, implying that Λ must be a foundational element of GR theory [25]. In fact, the role of $\Lambda > 0$ is proved to be of high relevance in cosmology and astrophysics, as summarized in Refs. [26,27]. In addition, the request of a positive cosmological constant in the variational principle for the Einstein equations of GR is a condition in agreement with the result pointed out in Ref. [4], where it was shown that $\Lambda > 0$ can be associated with the validity of the orthogonality condition $\widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu$ for the extremal metric tensor.

5. Manifestly Covariant Metric-Ricci Hamiltonian Approach

Given the previous results, we are now in position to attempt formulating a manifestly covariant metric-Ricci Hamiltonian approach in which the canonical variables coincide with the extended Lagrangian variables represented by the set of the metric-Ricci coordinates $(g(r), R(r))$, together with their conjugate momenta, to be suitably defined. Of course, to make sense, it is necessary to assume as before that the covariant derivatives of both the Lagrangian tensor variables $(g(r), R(r))$ are non-vanishing. Again, however, a $4-$tensor extended Hamiltonian approach, i.e., in which both the $4-$tensor fields $(g(r), R(r))$ are variational, cannot be based on the asynchronous EH-action principle (30). The reason for the tensor field $g(r)$ is the same one indicated above, i.e., in the functional set $\{g(r)\}_C$ its covariant derivative is identically vanishing.

However, on the contrary, it is worth stressing that alternate manifestly covariant Hamiltonian approaches are possible. As pointed out in [25], these approaches can be equivalently based on the asynchronous EH-action principle (30) or the synchronous action principle (49). In both cases, the $4-$tensor field $R(r)$ is variational, while the tensor field $g(r)$ (and/or $\widehat{g}(r)$) is considered prescribed. In such a case, in fact, the distinction between the two types of variational principles disappears altogether.

Coming back to the issue of joint variational formulations holding for the whole set of metric-Ricci coordinates $(g(r), R(r))$, no difficulty is expected in case of a synchronous extended Hamiltonian action principle. This can be achieved by generalizing the corresponding extended Lagrangian principle given above (see Equation (65)) to the whole extended set of the metric-Ricci coordinates $(g(r), R(r))$. The key assumption is that of considering all the variational fields as tensor functions with respect to the background metric field tensor $\widehat{g}(r)$. The corresponding metric-Ricci Hamiltonian formulation is straightforward. Thus, following the guidelines adopted in Section 3, upon denoting as $Q(r) = \left\{Q^{\eta}_{\mu\nu}(r)\right\}$ the canonical momenta conjugate to the Ricci tensor $R(r) = \{R_{\mu\nu}(r)\}$, the functional setting is expected to be analogous to (A4), i.e., with the varied fields $\{g(r,\theta), \Pi(r,\theta), R(r,\theta), Q(r,\theta)\}$ required to belong to the synchronous Hamiltonian metric-Ricci setting (A5), while the corresponding extremal fields $\{g_{extr}(r), \Pi_{extr}(r), R_{extr}(r), Q_{extr}(r)\}$ are again to be suitably prescribed. We notice, in particular, that here, all variations $\delta g(r)|_{\partial\mathbf{Q}^4}$, $\delta\Pi(r)|_{\partial\mathbf{Q}^4}$, $\delta R(r)|_{\partial\mathbf{Q}^4}$, $\delta Q(r)|_{\partial\mathbf{Q}^4}$ are required to vanish on the improper boundary of $\mathbf{Q}^4$ (see Appendix A, Item 5). Thus, in analogy with (A10), the relevant *synchronous metric-Ricci Hamiltonian action functional* will be taken of the form

$$S_{H_{g+R}}(g(r), R(r), \Pi(r), Q(r)) = \int_{\mathbf{Q}^4} d\widehat{\Omega} L_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)), \tag{77}$$

$$\begin{aligned} L_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)) \quad = \quad & \Pi^{\eta}_{\mu\nu}(r) g^{\mu\nu}_{;\eta}(r) + Q^{\eta}_{\mu\nu}(r) R^{\mu\nu}_{;\eta}(r) \\ & - H_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)), \end{aligned} \tag{78}$$

where $H_{g+R}(g(r), R(r), \Pi(r), Q(r))$ denotes a $4-$scalar function, to be suitably prescribed consistent with the variational Lagrangian $L_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r))$ defined above (see Equation (78)). Finally, here we notice that the second term in the integral, identifying the new exchange term, is carried by the new canonical momentum $Q^{\eta}_{\mu\nu}(r)$ and the covariant derivative of the countervariant components of the variational Ricci tensor, namely $\widehat{\nabla}_{\eta} R^{\mu\nu}(r) \equiv R^{\mu\nu}_{;\eta}(r)$, with $\widehat{\nabla}_{\eta}$ again being the covariant derivative evaluated with respect to standard connections depending on $\widehat{g}(r)$. From the *metric-Ricci Hamiltonian action principle*

$$\delta S_{H_{g+R}}(g(r), R(r), \Pi(r), Q(r)) = 0, \tag{79}$$

the corresponding Hamilton equations follow:

$$\begin{cases} \widehat{\nabla}_{\eta} g^{\mu\nu} = \frac{\partial H_{g+R}(g(r),R(r),\Pi(r),Q(r))}{\partial \Pi^{\eta}_{\mu\nu}}, \\ \widehat{\nabla}_{\eta} \Pi^{\eta}_{\mu\nu} = -\frac{\partial H_{g+R}(g(r),R(r),\Pi(r),Q(r))}{\partial g^{\mu\nu}}, \\ \widehat{\nabla}_{\eta} R^{\mu\nu} = \frac{\partial H_{g+R}(g(r),R(r),\Pi(r),Q(r))}{\partial Q^{\eta}_{\mu\nu}}, \\ \widehat{\nabla}_{\eta} Q^{\eta}_{\mu\nu} = -\frac{\partial H_{g+R}(g(r),R(r),\Pi(r),Q(r))}{\partial R^{\mu\nu}}. \end{cases} \tag{80}$$

These equations read explicitly

$$\begin{cases} \widehat{\nabla}_{\eta} g^{\mu\nu} = \frac{1}{\alpha_L} \Pi^{\mu\nu}_{\eta}(r), \\ \widehat{\nabla}_{\eta} \Pi^{\eta}_{\mu\nu} = -\sigma\alpha_L \frac{\partial}{\partial g^{\mu\nu}} \left[h\left(g^{\alpha\beta} \widehat{R}_{\alpha\beta} - 2\Lambda \right) \right] + \alpha_L \kappa \widehat{T}_{\mu\nu}, \\ \widehat{\nabla}_{\eta} R^{\mu\nu} = \frac{1}{\alpha_L \Lambda} Q^{\mu\nu}_{\eta}(r), \\ \widehat{\nabla}_{\eta} Q^{\eta}_{\mu\nu} = -\sigma\alpha_L \frac{\partial}{\partial R^{\mu\nu}} \left(\widehat{g}_{\alpha\beta} R^{\alpha\beta} + \frac{1}{2\Lambda} R^{\alpha\beta} R_{\alpha\beta} \right) + \frac{\sigma\alpha_L}{4\Lambda} \left(\frac{\partial}{\partial R^{\mu\nu}} R^2 \right) + \frac{\sigma\alpha_L}{\Lambda} \kappa \widehat{T}_{\mu\nu}, \end{cases} \tag{81}$$

where again it is understood that the equations are evaluated for the appropriate prescribed extremal fields $(g(r), R(r), \Pi(r), Q(r)) = (g_{extr}(r), R_{extr}(r), \Pi_{extr}(r), Q_{extr}(r))$. The proof of the statement can be obtained at once based on straightforward algebra, simply invoking the same calculations developed for the evaluation of the derivatives in the metric-Ricci Euler–Lagrange equations outlined above, recalling that by construction, the Hamiltonian equations for the variables $(g(r), \Pi(r))$ decouple from those holding for the set $(R(r), Q(r))$.

In particular, one has, for example, that $\frac{\partial H_{g+R}(g(r),R(r),\Pi(r),Q(r))}{\partial R^{\mu\nu}} = \frac{\partial \overline{L}_R(\widehat{g},R,\widehat{R})}{\partial R^{\mu\nu}}$, and therefore, the calculation reduces to that reported in Equation (74).

Indeed, the previous equations are equivalent to the corresponding metric-Ricci Euler–Lagrange Equation (69). In particular, the metric-Ricci action principle and the corresponding Hamilton equations imply the validity of the Einstein field equations under prescriptions of the extremal fields $(g_{extr}(r), R_{extr}(r), \Pi_{extr}(r), Q_{extr}(r))$. This happens indeed, as expected, when one sets $(g_{extr}(r), R_{extr}(r), \Pi_{extr}(r), Q_{extr}(r)) \equiv (\widehat{g}(r), \widehat{R}(r), \widehat{\Pi}(r), \widehat{Q}(r))$ and the generalized velocities $\widehat{\nabla}_\eta g^{\mu\nu}$, $\widehat{\nabla}_\eta R^{\mu\nu}$, together with the conjugate momenta $\Pi^{\mu\nu}(r)$ and $Q^{\mu\nu}(r)$, vanish identically (see previous discussion in Section 4). As a result, the second and fourth Hamilton equation recover at once the extremal Euler–Lagrange Equations (73). In particular, the last two equations in (80) become

$$\widehat{\nabla}_\eta \widehat{R}^{\mu\nu} = 0, \tag{82}$$

$$-\sigma\alpha_L \left[\frac{\partial}{\partial R^{\mu\nu}} \left(\widehat{g}^{\alpha\beta} R_{\alpha\beta} + \frac{1}{2\Lambda} R^{\alpha\beta} R_{\alpha\beta} \right) \right]_{R=\widehat{R}(r)} + \frac{\sigma\alpha_L}{4\Lambda} \left[\frac{\partial}{\partial R^{\mu\nu}} \left(\widehat{g}^{\alpha\beta} R_{\alpha\beta} \right)^2 \right]_{R=\widehat{R}(r)} + \frac{\sigma\alpha_L}{\Lambda} \kappa \widehat{T}_{\mu\nu} = 0, \tag{83}$$

where, provided $\Lambda \neq 0$, the second equation recovers identically EFE (35). As stated before, the first equation (83) also implies the validity of the constraint condition (72), a requirement which can be satisfied in the case of vacuum and suitably prescribed external sources. In such cases, Equation (80) provides, therefore, a manifestly covariant Hamiltonian representation for EFE (see Equation (35)) in which all variational canonical fields $g(r), R(r), \Pi(r), Q(r)$ are considered independent $4-$tensor fields.

5.1. Gauge Properties of the metric-Ricci Lagrangian and Hamiltonian Action Principles

An interesting issue concerns the characteristic gauge properties of all the synchronous Lagrangian and Hamiltonian action principles given above, i.e., in particular, the metric-Ricci synchronous Lagrangian and Hamiltonian action principles (A9) and (65), together with its Hamiltonian counterpart (A9). Here, we refer in particular to the additive gauge properties fulfilled by the variational metric-Ricci Lagrangian $\overline{L}_{g+R}\left(g(r), \widehat{g}(r), R(r), \widehat{R}\right)$ (59) and the modified variational metric-Ricci Lagrangian $\overline{L}_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r))$ (78). The first property is achieved noting that a general gauge transformation is provided by

$$\overline{L}_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)) \to \overline{L}_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)) + C(\widehat{g}(r)), \tag{84}$$

which holds for an arbitrary $4-$scalar real field of the form $C = C(\widehat{g}(r))$. Furthermore, if $G^\eta(g(r), \widehat{g}(r), r)$ is an arbitrary smooth $4-$vector field, then it determines a gauge function. In fact, one can show that in the functional settings (A4) and (A5), the map

$$\begin{aligned} &\overline{L}_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)) \to \\ &\overline{L}_{g+R}(g(r), R(r), \Pi(r), Q(r), \widehat{g}(r)) + \widehat{\nabla}_\eta G^\eta(g(r), R(r), \widehat{g}(r), r) \end{aligned} \tag{85}$$

realizes a gauge transformation. This includes the particular cases in which $G^\eta \equiv G^\eta(g(r), R(r), \widehat{g}(r), r)$ and $\widehat{\nabla}_\eta G^\eta(g(r), R(r), \widehat{g}(r), r) = C(\widehat{g}(r))$.

The interesting implication is, therefore, that a necessary condition for the invariance with respect to the previous gauge transformation properties (which are all characteristic of flat-space-time classical field theory) is the variational Lagrangian to be a $4-$scalar. In turn this means that, under the same condition, the adoption of a synchronous variational principle for GR is required. The synchronous variational principle (79), therefore, is characterized by unique characteristic gauge properties.

5.2. Discussion and Comparisons

An interesting comparison can be made with some of the literature approaches which have some apparent similarity with the present treatment. These include, in particular, approaches '*a la Palatini*', based typically on the adoption of an asynchronous variational principle as the original EH-action principle [28]. However, unlike in the asynchronous functional setting $\{g(r)\}_C$ considered above, these involve dropping any a priori relationship between the metric tensor $g(r)$ and the connections $\Gamma(r)$ and considering them as independent geometric quantities [29] while at the same time retaining the relationship between the same connections and the Ricci tensor. In practice, this means that in the resulting action functional, the metric tensor $g(r)$ and the connections now denoted $\Gamma'(r)$ are treated as independent variational quantities, while the Riemann and/or the Ricci tensors are considered prescribed or expressed in terms of the same $\Gamma'(r)$. We stress that an analogous approach is developed in Ref. [18], where the Riemann tensor, and hence the Ricci tensor as well, is assumed prescribed in terms of the same independent connections $\Gamma'(r)$. Such a setting is therefore substantially different from the metric-Ricci one considered in this Section, with the Ricci tensor to be considered variational and independent of the standard connections $\Gamma(r)$. Most importantly, all '*a la Palatini*' approaches violate the principle of manifest covariance because the variational connections $\Gamma'(r)$ are not $4-$tensors. This is the primary motivation of our choice of considering them prescribed in terms of the background metric tensor.

For completeness, however, it is worth pointing out how, despite the feature indicated above (i.e., the intrinsic non-manifestly covariant feature carried by the varied non-tensor fields $\Gamma'(r)$) a manifestly covariant Hamiltonian representation can also be achieved in the contest of '*a la Palatini*' approaches of this type. This information, in fact, can be valuable by itself at least for comparison with the metric-Hamiltonian theory described above in Sections 2 and 3. To begin with, one notices that, as a consequence of such a setting, the covariant derivatives of $g_{\mu\nu}(r)$ and $g^{\mu\nu}(r)$ are now defined in terms of variational connections $\Gamma'(r)$ and therefore take the form

$$\nabla'_\alpha g_{\mu\nu}(r) = \partial_\alpha g_{\mu\nu}(r) - \Gamma'^{d}_{\alpha\mu}(r) g_{d\nu}(r) - \Gamma'^{d}_{\alpha\nu}(r) g_{d\mu}(r), \tag{86}$$

$$\nabla'_\alpha g^{\mu\nu}(r) = \partial_\alpha g^{\mu\nu}(r) + \Gamma'^{\mu}_{\alpha d}(r) g^{d\nu}(r) + \Gamma'^{\nu}_{\alpha d}(r) g^{\mu d}(r), \tag{87}$$

and are generally non-vanishing, while, in addition, the same covariant derivatives identify, up to an arbitrary multiplicative $4-$scalar factor γ, 3rd-order $4-$tensor fields. The conjecture is whether they can actually be associated with canonical momenta of the form $\pi^{\alpha}_{\mu\nu} = \gamma\nabla'^{\alpha} g_{\mu\nu}(r)$. One can show that the prerequisite for this to happen is that the $4-$scalar function $\nabla_\alpha\left[\pi^{\alpha}_{\mu\nu}(r) g^{\mu\nu}(r)\right]$ (where ∇_α still denotes here the standard covariant derivative, i.e., with standard connections defined with respect to $g(r)$) results in a gauge function, namely it is such that the identity

$$\delta \int_{\mathbf{Q}^4} d\Omega \nabla_\alpha\left[\pi^{\alpha}_{\mu\nu}(r) g^{\mu\nu}(r)\right] \equiv \delta \int_{\mathbf{Q}^4} d^4 r \partial_\alpha\left[\sqrt{-|g(r)|}\pi^{\alpha}_{\mu\nu}(r) g^{\mu\nu}(r)\right] = 0 \tag{88}$$

is fulfilled. Such a constraint condition can indeed be satisfied by properly prescribing $\delta g^{\mu\nu}(r)$ and $\delta\pi^{\alpha}_{\mu\nu}(r)$ to vanish on the improper boundary $\partial\mathbf{Q}^4$. In fact, since the rhs of the previous equation does not depend on the choice of the connections (i.e., it holds for arbitrary connections), by introducing the auxiliary $4-$tensor-density [18] $\widetilde{\pi}^{\alpha}_{\mu\nu}(r) = \sqrt{-|g(r)|}\pi^{\alpha}_{\mu\nu}(r)$, then the following identity holds:

$$\delta \int_{\mathbf{Q}^4} d^4 r \widetilde{\pi}^{\alpha}_{\mu\nu}(r) \nabla'_\alpha g^{\mu\nu}(r) + \delta \int_{\mathbf{Q}^4} d^4 r g^{\mu\nu}(r) \nabla'_\alpha \widetilde{\pi}^{\alpha}_{\mu\nu}(r) = 0, \tag{89}$$

in which the contributions carried by the variational connections $\Gamma'(r)$ have been introduced. Let us invoke therefore, in analogy to the prescription for $\{g\}_C$ adopted above (see

Equation (A1)), the *asynchronous Hamiltonian–Palatini functional setting* (A6) for the varied fields $\{g(r,\theta),\pi(r,\theta),\Gamma'(r,\theta)\}$, which are defined with respect to corresponding properly defined extremal fields $\{g_{extr}(r),\pi_{extr}(r),\Gamma'_{extr}(r)\}$.

This implies that if, in such a setting, one considers only variations with respect to the independent tensor functions $\left\{\delta g^{\mu\nu}(r),\delta\widetilde{\pi}^{\alpha}_{\mu\nu}(r)\right\}$ (see Appendix A, Item 6), i.e., performed while also keeping constant the varied connections and at the same time letting $\Gamma'_{extr}(r)\neq\Gamma(r)$, an approach 'a la Palatini' should admit an asynchronous Hamiltonian action principle of the form

$$\delta S_P(g(r),\widetilde{\pi}(r))=\delta\int_{\mathbf{Q}^4}d\Omega\left[\pi^{\alpha}_{\mu\nu}(r)\nabla'_{\alpha}g^{\mu\nu}(r)-H_P(g(r),\widetilde{\pi}(r))\right]=0, \tag{90}$$

with $S_P(g(r),\widetilde{\pi}(r))$ and $H_P(g(r),\widetilde{\pi}(r))$ denoting a $4-$scalar Palatini functional and corresponding $4-$scalar Hamiltonian function. Thus, introducing the corresponding $4-$scalar-density Hamiltonian $\widetilde{H}_P(g(r),\widetilde{\pi}(r))=\sqrt{-|g(r)|}H_P(g(r),\widetilde{\pi}(r))$, the resulting Hamilton equations, defined with respect the modified covariant derivatives (86) and (87), namely

$$\begin{cases}\nabla'_{\alpha}g^{\mu\nu}(r)=\frac{\partial}{\partial\widetilde{\pi}^{\alpha}_{\mu\nu}(r)}\widetilde{H}_P(g(r),\widetilde{\pi}(r)),\\ \nabla'_{\alpha}\widetilde{\pi}^{\alpha}_{\mu\nu}(r)=-\frac{\partial}{\partial g^{\mu\nu}(r)}\widetilde{H}_P(g(r),\widetilde{\pi}(r)),\end{cases} \tag{91}$$

are implied. Notice that, as expected, these equations are cast in $4-$tensor-density form and cannot be cast in an equivalent $4-$tensor form, with the exception of the first equation in (91), which can also be written as

$$\nabla'_{\alpha}g^{\mu\nu}(r)=\frac{\partial}{\partial\pi^{\alpha}_{\mu\nu}(r)}H_P(g(r),\widetilde{\pi}(r)), \tag{92}$$

i.e., in a manifestly covariant $4-$tensor form. However, it must be noted that, once $\Gamma'_{extr}(r)=\Gamma(r)$ is set, the same equation reduces manifestly to

$$\nabla_{\alpha}g^{\mu\nu}(r)=\frac{\partial}{\partial\widetilde{\pi}^{\alpha}_{\mu\nu}(r)}\widetilde{H}_P(g(r),\widetilde{\pi}(r))\equiv 0,$$

so that $\nabla_{\alpha}\widetilde{\pi}^{\alpha}_{\mu\nu}(r)\equiv 0$ is also implied. As a consequence, the second Hamilton equation in (91) is expected to recover exactly EFE (i.e., Equation (35)), which means that $X\equiv\{g_{extr}(r),\pi_{extr}(r),\Gamma'_{extr}(r)\}$ should coincide with $X_{extr}\equiv\left\{\widehat{g}(r),\widehat{\pi}(r)\equiv 0,\widehat{\Gamma}(r)\right\}$. We finally notice that if the variation with respect to $\delta\Gamma'(r)$ is performed, one finds out that the constraint equation

$$\int_{\mathbf{Q}^4}d\Omega\pi^{\alpha}_{\mu\nu}(r)\left[\delta\Gamma'^{\mu}_{\alpha p}g^{p\nu}+\delta\Gamma'^{\nu}_{\alpha p}g^{\mu p}\right]\Big|_{X=X_{extr}}=0 \tag{93}$$

must be fulfilled. This happens if the Ricci tensor is considered a prescribed tensor field which is independent of $\delta\Gamma'$ and if the Palatini Hamiltonian $H_P(g(r),\widetilde{\pi}(r))$ does not depend explicitly on the variational derivatives of the tensor fields $g(r)$ or $\pi(r)$. A first solution is obviously $\delta\Gamma'(r)=0$. Another one is that of requiring identically $\pi_{extr}(r))=0$, which implies in turn that X_{extr} should again coincide with $\left\{\widehat{g}(r),\widehat{\pi}(r)\equiv 0,\widehat{\Gamma}(r)\right\}$.

We stress that Equations (90) and (91) appear formally analogous to the variational principles and Hamilton equations reported in Ref. [18], being also expressed in terms of $4-$tensor-density canonical momenta and a $4-$scalar-density Hamiltonian. In particular, this means that all such variational principles are asynchronous, just as the action principle (90) reported here. Comparison with the $4-$tensor metric-Hamilton Equation (A15) (reported above in Section 3) is also instructive. In contrast with all approaches based on asynchronous variational principles, the latter appears, in fact, as the only possible realization of a manifestly-covariant Hamiltonian formulation of GR which is expressed in

4−tensor form and satisfies the same gauge properties that are characteristic of flat-space-time field theories.

6. Conclusions

In this paper, aspects of the variational theory of general relativity (GR) have been investigated in the context of the general covariance and manifest covariance principles for GR, which concern both the tensor and gauge representations of Lagrangian and Hamiltonian theories for the Einstein field equations.

After recalling the general covariance principle, the possible realization of the conditions of invariance, i.e., the properties of manifest covariance and covariance have been pointed out. In particular, manifest covariance has been shown to correspond either to 4−tensor or 4−tensor-density transformation laws for the relevant tensor fields. The conjecture concerning a possible universal 4−tensor property has been investigated and proved to be realized by the objectivity principle. Its physical implications regarding the notion of background space-time and the identification of objective classical observables have been pointed out. In this reference, an interesting result is that customary non-4-tensor formulations of GR, i.e., represented in non-manifestly covariant form, can, in principle, be reduced to a purely manifestly covariant one, i.e., in which the Lagrangian field variables are represented by 4−tensor fields.

As far as Lagrangian theories of the Einstein field equations are concerned, two possible formulations based either on asynchronous or synchronous variational action principles have been considered. However, their major difference is related to the existence of a possible corresponding manifestly covariant Hamiltonian realization of the Einstein–Hilbert action principle. The notable conclusion which is reached is that the explicit construction of a manifestly covariant Hamiltonian approach, in fact, can only be achieved by adopting a synchronous action principle which is cast in Hamiltonian form.

Subsequently, the issue of completeness for the Lagrangian manifestly covariant description of GR has been addressed, namely whether there may exist additional independent Lagrangian variables, possibly associated with classical treatment of covariant gravity theory. The complete set of Lagrangian variables thus determined, which also includes the variational Ricci tensor, besides the variational tensor $g(s)$, is referred to as extended Lagrangian variables. Such a set is shown to realize an ensemble of independent Lagrangian variational coordinates. For this purpose, a synchronous variational formulation of the Lagrangian action principle based on the extended variables is shown to exist and to recover exactly, under appropriate assumptions, the Einstein field equations. In terms of the same variational principle, the construction of a manifestly covariant extended Hamiltonian approach has been developed, in which the canonical variables coincide with the extended Lagrangian variables represented by the "*metric-Ricci*" set $(g(r), R(r))$ and by their conjugate momenta, to be suitably defined. Additionally, in this case an extended Hamiltonian approach has been achieved by means of a synchronous EH-action principle.

As a final topic, the issue of the prescription of the appropriate gauge properties has been addressed. As a result, it has been shown that for synchronous variational principles, the same gauge properties actually apply, which are characteristic of flat-space-time theories. This property appears notable especially because it is peculiar and unique to the synchronous variational principles considered here. Indeed, on the contrary, it is manifestly violated by all asynchronous variational principles, including the Einstein–Hilbert action variational principle.

The main conclusions of the paper can be summarized as follows:

- The first one is that the asynchronous EH-action principle does not permit the construction of a manifestly covariant Hamiltonian approach for EFE. In fact, such a possibility is simply ruled out by the prescription of the functional setting for the same action principle, the reason being that the "generalized velocities", i.e., the covariant derivatives of the variational metric tensors $g(r)$ vanish.

- On the contrary, a manifestly–covariant $4-$tensor Hamiltonian theory exists, which is based on the adoption of a synchronous action variational principle. As shown here, such a result can be achieved by means of suitably prescribed Lagrangian and Hamiltonian (synchronous) action principles.
- As a third issue, we have shown that based on the same type of synchronous action principles, an extended-variable manifestly-covariant Hamiltonian variational formulation can also be reached, in which Lagrangian coordinates are represented by the set of the metric-Ricci variational fields $\{g(r), R(s)\}$. The resulting Euler–Lagrange equations have been obtained in manifestly covariant $4-$tensor form. In terms of the corresponding metric-Ricci action principle, the Einstein field equations have been recovered as particular solutions.
- Fourth, all the synchronous variational principles considered here have been shown to fulfill the fundamental gauge property characteristic of all classical field theories in flat space-time. Namely, the $4-$scalar variational Lagrangian, which characterizes these theories, is gauge-invariant with respect to an arbitrary additive constant of the same variational Lagrangian.
- Fifth, all synchronous variational principles obtained here are background independent in the sense that they hold for an arbitrary background metric field tensor $\widehat{g}(r)$, namely an arbitrary particular solution of the Einstein field equations.

Finally, further interesting results concern the establishment of alternate possible Hamiltonian formulations for the Einstein field equations.

The most significant one is undoubtedly the first one, which is based on the treatment of the Ricci tensor $R(r)$ as an independent variational tensor field (Section 5) while keeping the tensor field $g(r)$ (together with $\widehat{g}(r)$) as prescribed. This means that generally, two different Hamiltonian treatments can effectively coexist. Indeed, they can actually be coupled by means of a dimensionless and $4-$scalar coupling coefficient (α_1) through their corresponding Lagrangian and Hamiltonian functions in which $g(r)$ and $R(r)$ are treated as independent variations and $R(r)$ and $g(r)$ are treated as constrained quantities. The significant feature in both cases is their unique manifestly covariant character, which affords its joint $4-$tensor representation represented by a linear combination of the corresponding Lagrangian and Hamiltonian functions. This route is interesting in that it represents a possible non-trivial generalization of the manifestly covariant approach represented by CCG-theory (see, in particular, [6,25]).

However, for the sake of reference, a further variational approach is pointed out. This is here achieved in the framework of a 'la Palatini' variational approach, in which—as usual—the connections (denoted as $\Gamma'(r)$) are treated as variational trial functions (Section 5.2). Its characteristic property, as in the original EH-action principle, is due to being based on an asynchronous variational principle. For this reason, the approach appears analogous to the covariant canonical gauge approach earlier developed in Ref. [18]. In fact, both the variational Hamiltonian and the canonical moment are found to be expressed, respectively, by means of a scalar density and a $4-$tensor density. However, this approach (just as all Palatini approaches) is obviously not manifestly covariant since it is not expressible in fully manifestly $4-$tensor form, a conclusion that follows due to the adoption of non-tensor Lagrangian variables, i.e., the same variational connections $\Gamma'(r)$.

These conclusions are interesting because they indicate that, based on the principles of general and manifest covariance, as well as the principle of objectivity, a conceptually sound route to a possible complete Hamiltonian representation of classical GR has actually been achieved, which is still based, as the original CCG-theory, on the adoption of synchronous variational principles. Nevertheless, its actual relevance for the explicit construction of a consistent theory of quantum gravity, i.e., satisfying a suitable set of axioms [19], remains to be ascertained. Fundamental open questions in this regard include, in particular, the possible non-uniqueness of a manifestly covariant theory of quantum gravity achieved on this basis, the notion of background space-time in quantum gravity, the role of manifest covariance and the meaning of background independence, i.e., whether quantum gravity can also self-consistently prescribe its background metric tensor by means of a suitable dynamical equation. These issues will be discussed in the forthcoming Part 2 of the present investigation.

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Appendix A. Functional Settings

1—Asynchronous functional setting. This is the ensemble of 4−tensor functions, which identify metric field tensors, such that:

$$\{g\}_C \equiv \left\{ \begin{array}{c} g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r) \in C^2(\mathbf{Q}^4) \\ \theta \in [-1,1] \\ \delta g(r)|_{\partial\mathbf{Q}^4} = 0 \\ R \equiv R(g(r)), \Gamma = \Gamma(g(r)) \\ d\Omega = d^4r\sqrt{-|g(r)|} \\ g_{\mu\nu}g^{\mu k} = \delta^k_\nu \end{array} \right\}, \tag{A1}$$

where $R \equiv R(g(r)), \Gamma = \Gamma(g(r))$ and $d\Omega$ denote, respectively, the Ricci tensor, the standard connections (Christoffel symbols) and the asynchronous and $g(r)-$dependent 4−scalar volume element.

2—Synchronous functional setting. This is the ensemble of 4−tensor functions $g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r)$ defined with respect to the background metric tensor $\widehat{g}(r)$:

$$\{g\}_U \equiv \left\{ \begin{array}{c} g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r) \in C^2(\mathbf{Q}^4) \\ \theta \in [-1,1] \\ R \equiv R(\widehat{g}(r)), \Gamma = \Gamma(\widehat{g}(r)) \\ d\widehat{\Omega} = d^4r\sqrt{-|\widehat{g}(r)|} \\ g^{\mu\nu} = \widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta}g_{\eta\beta}(r) \\ \widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu \end{array} \right\}. \tag{A2}$$

Here, $\widehat{R} \equiv R(\widehat{g}(r))$ and $\widehat{\Gamma} = \Gamma(\widehat{g}(r))$ denote, respectively, the Ricci tensor and the standard connections (Christoffel symbols) evaluated with respect to the same background metric tensor $\widehat{g}(r)$, while $d\widehat{\Omega}$ is the synchronous $\widehat{g}(r)-$dependent 4−scalar volume element.

3—Synchronous metric-Hamiltonian functional setting. This is the ensemble of 4−tensor functions $\{g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r), \Pi(r,\theta) \equiv \Pi_{extr}(r) + \theta\delta\Pi(r)\}$ defined with respect to the background metric tensor $\widehat{g}(r)$:

$$\{g,\Pi\}_U \equiv \left\{ \begin{array}{c} g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r) \in C^2(\mathbf{Q}^4) \\ \Pi(r,\theta) \equiv \Pi_{extr}(r) + \theta\delta\Pi(r) \in C^2(\mathbf{Q}^4) \\ \theta \in [-1,1] \\ R \equiv R(\widehat{g}(r)), \Gamma = \Gamma(\widehat{g}(r)) \\ d\widehat{\Omega} = d^4r\sqrt{-|\widehat{g}(r)|} \\ g^{\mu\nu} = \widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta}g_{\eta\beta}(r) \\ \widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu \end{array} \right\}. \tag{A3}$$

Here, $\widehat{R} \equiv R(\widehat{g}(r)), \widehat{\Gamma} = \Gamma(\widehat{g}(r))$ and $d\widehat{\Omega}$ have the same meaning of Equation (A2).

4—Synchronous metric-Ricci functional setting. This is the ensemble of extended 4−tensor functions $\{g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r), R(r,\theta) \equiv R_{extr}(r) + \theta\delta R(r)\}$:

$$\{g,R\}_U \equiv \left\{ \begin{array}{c} g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r) \in C^2(\mathbf{Q}^4) \\ R(r,\theta) \equiv R_{extr}(r) + \theta\delta R(r) \in C^2(\mathbf{Q}^4) \\ \theta \in [-1,1] \\ \Gamma = \Gamma(\widehat{g}(r)) \\ d\widehat{\Omega} = d^4r\sqrt{-|\widehat{g}(r)|} \\ \widehat{T} = T(\widehat{g}(r), \widehat{R}(r)) \\ g^{\mu\nu} = \widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta}g_{\eta\beta}(r) \\ R^{\mu\nu} = \widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta}R_{\eta\beta}(r) \\ \widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu \end{array} \right\}. \quad \text{(A4)}$$

Again, here, $\widehat{\Gamma} = \Gamma(\widehat{g}(r))$ denote the standard connections (Christoffel symbols) evaluated with respect to the same background metric tensor $\widehat{g}(r)$, while $d\widehat{\Omega}$ is the synchronous $\widehat{g}(r)$−dependent 4−scalar volume element. In addition, $\widehat{T} = \left\{T_{\mu\nu}\left(\widehat{g}(r), \widehat{R}(r)\right)\right\}$ is the stress-energy tensor expressed in terms of the background fields $(\widehat{g}(r), \widehat{R}(r))$.

5—Synchronous metric-Ricci Hamiltonian functional setting. This is the ensemble of extended Hamiltonian 4−tensor functions $\{g(r,\theta), \Pi(r,\theta), R(r,\theta), Q(r,\theta)\}$:

$$\{g,R\}_U \equiv \left\{ \begin{array}{c} g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r) \in C^2(\mathbf{Q}^4) \\ \Pi(r,\theta) \equiv \Pi_{extr}(r) + \theta\delta\Pi(r) \in C^2(\mathbf{Q}^4) \\ R(r,\theta) \equiv R_{extr}(r) + \theta\delta R(r) \in C^2(\mathbf{Q}^4) \\ Q(r,\theta) \equiv Q_{extr}(r) + \theta\delta Q(r) \in C^2(\mathbf{Q}^4) \\ \theta \in [-1,1] \\ \Gamma = \Gamma(\widehat{g}(r)) \\ d\widehat{\Omega} = d^4r\sqrt{-|\widehat{g}(r)|} \\ \widehat{T} = T(\widehat{g}(r), \widehat{R}(r)) \\ g^{\mu\nu} = \widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta}g_{\eta\beta}(r) \\ R^{\mu\nu} = \widehat{g}^{\mu\eta}\widehat{g}^{\nu\beta}R_{\eta\beta}(r) \\ \widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu \end{array} \right\}. \quad \text{(A5)}$$

As in the previous case, $\widehat{\Gamma} = \Gamma(\widehat{g}(r))$ denote the standard connections (Christoffel symbols) evaluated with respect to the same background metric tensor $\widehat{g}(r)$, while and $d\widehat{\Omega}$ is the synchronous $\widehat{g}(r)$−dependent 4−scalar volume element. In addition, $\widehat{T} = \left\{T_{\mu\nu}(\widehat{g}(r), \widehat{R}(r))\right\}$ is the stress-energy tensor expressed in terms of the background fields $(\widehat{g}(r), \widehat{R}(r))$.

6—Asynchronous Hamiltonian–Palatini functional setting. This is the ensemble of Hamiltonian–Palatini 4−tensor functions $\{g(r,\theta), \pi(r,\theta), \Gamma'(r,\theta)\}$**:**

$$\{g,\pi,\Gamma\}_C \equiv \left\{ \begin{array}{c} g(r,\theta) \equiv g_{extr}(r) + \theta\delta g(r) \in C^2(\mathbf{Q}^4) \\ \pi(r,\theta) \equiv \pi_{extr}(r) + \theta\delta\pi(r) \in C^2(\mathbf{Q}^4) \\ \Gamma'(r,\theta) \equiv \Gamma'_{extr}(r) + \theta\delta\Gamma'(r) \in C^2(\mathbf{Q}^4) \\ \theta \in [-1,1] \\ \delta g(r)|_{\partial\mathbf{Q}^4} = 0 \\ R \equiv R(g(r)), \Gamma = \Gamma(g(r)) \\ d\Omega = d^4r\sqrt{-|g(r)|} \\ g_{\mu\nu}g^{\mu k} = \delta^k_\nu \end{array} \right\}, \quad \text{(A6)}$$

where $g(r,\theta)$ and $\pi(r,\theta)$ are canonically conjugate variables, while $\Gamma'(r,\theta)$ and $\Gamma'_{extr}(r)$ denote, respectively, variational and extremal connections, with $\Gamma'_{extr}(r)$ generally different from the standard connections $\widehat{\Gamma}(r)$.

Appendix B. The Metric-Lagrangian and Hamiltonian Principles

In the definition of the 4−scalar function $\overline{L}_g(g(r),\widehat{\nabla}g(r),\widehat{g}(r))$ (57) given above, $\sigma=\pm 1$ is a signature to be suitably prescribed [5], while T_g denotes the effective kinetic energy, which is defined as

$$T_g = \frac{1}{2}\alpha_L \widehat{\nabla}^{\eta} g_{\mu\nu} \widehat{\nabla}_{\eta} g^{\mu\nu}, \tag{A7}$$

and in the setting, $\{g\}_U$ is non-vanishing. Then, the corresponding Lagrangian variational principle, to be denoted as the *synchronous metric-Lagrangian action principle* (49), requires the evaluation of the Frechet derivative:

$$\delta S_{Lg}(g(r),\widehat{g}(r))\big|_{g=\widehat{g}(r)} \equiv \frac{d}{d\theta} S_{Lg}(g_{extr}(r)+\theta\delta g(r),\widehat{g}(r))\Big|_{\theta=0} = 0. \tag{A8}$$

The synchronous Lagrangian action principle (A8) can readily be cast in Hamiltonian form. This requires, first, properly changing the definition given for $\{g\}_U$ (see Equation (A2)), i.e., requiring that the varied fields $\{g(r,\theta)\equiv g_{extr}(r)+\theta\delta g(r),\Pi(r,\theta)\equiv \Pi_{extr}(r)+\theta\delta\Pi(r)\}$ belong to the synchronous metric-Hamiltonian setting (A3) with the extremal tensor fields $\{g_{extr}(r),\Pi_{extr}(r)\}$ to be properly prescribed and the variations $\delta g(r)|_{\partial \mathbf{Q}^4}$ and $\delta\Pi(r)|_{\partial \mathbf{Q}^4}$ are both required to vanish on the improper boundary of $\mathbf{Q}$. Second, the Hamiltonian variational principle (to be referred to as the synchronous metric-Hamiltonian action principle) is set as

$$\delta S_{H_g}(g(r),\Pi(r),\widehat{g}(r))\Big|_{g=\widehat{g}(r)} = 0. \tag{A9}$$

Here, the functional

$$S_{H_g}(g(r),\Pi(r),\widehat{g}(r)) = \int_{\mathbf{Q}^4} d\widehat{\Omega}\overline{L_g}(g(r),\Pi(r),\widehat{g}(r)) = \int_{\mathbf{Q}^4} d\widehat{\Omega}\left[\Pi^{\eta}_{\mu\nu}\widehat{\nabla}_{\eta}g^{\mu\nu} - \overline{H}_g(g(r),\Pi(r),\widehat{g}(r))\right] \tag{A10}$$

identifies the Hamiltonian action principle in standard notations, with the corresponding modified variational metric Lagrangian

$$\overline{L}_g(g(r),\Pi(r),\widehat{g}(r)) \equiv \Pi^{\eta}_{\mu\nu}\widehat{\nabla}_{\eta}g^{\mu\nu} - \overline{H}_g(g(r),\Pi(r),\widehat{g}(r)), \tag{A11}$$

and $\overline{H}_g(g(r),\Pi(r),\widehat{g}(r))$ being a suitably prescribed 4−scalar variational Hamiltonian density defined by Equation (A13) below. Thus, $\widehat{\nabla}_{\eta}g^{\mu\nu}$ denotes the "generalized velocity" associated with $g^{\mu\nu}(r)$, i.e., its covariant derivative defined with respect to the background metric field tensor $\widehat{g}(r)$, while $\Pi^{\eta}_{\mu\nu}$ is the 4−tensor identifying the conjugate canonical momentum of $g_{\mu\nu}$. Thus, $\Pi^{\eta}_{\mu\nu}(r)$ denotes the 4−tensor canonical momentum, and $\overline{H}_g(g(r),\Pi(r),\widehat{g}(r))$ is the 4−scalar variational Hamiltonian functional, defined as

$$\Pi^{\eta}_{\mu\nu}(r) = \frac{\partial \overline{L}_g(g(r),\widehat{g}(r))}{\partial \widehat{\nabla}_{\eta}g^{\mu\nu}} = \alpha_L \widehat{\nabla}^{\eta} g_{\mu\nu}, \tag{A12}$$

$$\overline{H}_g(g(r),\Pi(r),\widehat{g}(r)) = \frac{1}{2\alpha_L}\Pi^{\eta}_{\mu\nu}(r)\Pi^{\mu\nu}_{\eta}(r) + V_g(g(r),\widehat{g}(r)). \tag{A13}$$

Given these definitions and the validity of the Hamiltonian structure associated with the synchronous Lagrangian theory, it is possible to proceed obtaining the corresponding Hamiltonian equations. Two different but equivalent methods are available. The first one consists of evaluating the variational equations generated by the Lagrangian (A11) for the set of independent variables $(g(r),\Pi(r))$. The explicit derivation follows the steps outlined above, with the additional evaluation of the exchange term $\Pi^{\eta}_{\mu\nu}\widehat{\nabla}_{\eta}g^{\mu\nu}$. An alternative is provided by direct evaluation of the Hamilton equations generated by the Hamiltonian function $\overline{H}_g(g(r),\Pi(r),\widehat{g}(r))$ given by Equation (A13). These are written explicitly in the customary form as

$$\left\{ \begin{array}{l} \widehat{\nabla}_\eta g^{\mu\nu} = \frac{\partial \overline{H}(g(r),\Pi(r),\widehat{g}(r))}{\partial \Pi^\eta_{\mu\nu}}, \\ \widehat{\nabla}_\eta \Pi^\eta_{\mu\nu} = -\frac{\partial \overline{H}_g(g(r),\Pi(r),\widehat{g}(r))}{\partial g^{\mu\nu}}. \end{array} \right. \quad \text{(A14)}$$

The evaluation of the partial derivatives in the previous equations can be established at once with straightforward algebra, being analogous to the variational calculations of Equations (51)–(55). In conclusion, one finally obtains the continuous metric-Hamilton equations, which are expressed as

$$\left\{ \begin{array}{c} \widehat{\nabla}_\eta g^{\mu\nu} = \frac{1}{\alpha_L}\Pi^{\mu\nu}(r), \\ \widehat{\nabla}_\eta \Pi^\eta_{\mu\nu} = -\frac{\partial}{\partial g^{\mu\nu}}\left[\alpha_L h\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda\right] + \overline{V}_{F_g}(g(r),\widehat{g}(r))\right], \end{array} \right. \quad \text{(A15)}$$

where it is understood that the equations are evaluated at $(g(r),\Pi(r)) = (g_{extr}(r),\Pi_{extr}(r))$. Here, we notice that all fields involved, i.e., the canonical variables $(g(r),\Pi(r))$, the variational Hamiltonian function $H_g(g(r),\Pi(r),\widehat{g}(r))$ and the Hamilton equations themselves are $4-$tensor functions with respect to the background metric field tensor $\widehat{g}(r)$. From the previous analysis it follows that the result represented by Equation (A15) is the *only possible Hamiltonian formulation of GR which is expressed in manifestly covariant form by means of* $4-$*tensor equations*. In other words, no analogous Hamiltonian representation can be achieved in terms of the asynchronous functional setting, where it is understood that the equations are evaluated at $(g(r),\Pi(r)) = (g_{extr}(r),\Pi_{extr}(r))$. Here, we notice that all fields involved, i.e., the canonical variables $(g(r),\Pi(r))$, the variational Hamiltonian function $H_g(g(r),\Pi(r),\widehat{g}(r))$ and the Hamilton equations themselves are $4-$tensor functions with respect to the background metric field tensor $\widehat{g}(r)$. From the previous analysis, it follows that the result represented by Equation (A15) is the *only possible Hamiltonian formulation of GR, which is expressed in manifestly covariant form by means of* $4-$*tensor equations*. In other words, no analogous Hamiltonian representation can be achieved in terms of the asynchronous functional setting.

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