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Jensen-Mercer Type Inequalities in the Setting of Fractional Calculus with Applications

Bandar Bin-Mohsin [1], Muhammad Zakria Javed [2], Muhammad Uzair Awan [2,*], Marcela V. Mihai [3], Hüseyin Budak [4], Awais Gul Khan [2] and Muhammad Aslam Noor [5]

1 Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia
2 Department of Mathematics, Government College University, Faisalabad 38000, Pakistan
3 Department Scientific-Methodical Sessions, Romanian Mathematical Society-Branch Bucharest, Academy Street no. 14, RO-010014 Bucharest, Romania
4 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey
5 Department of Mathematics, COMSATS University Islamabad, Islamabad 4000, Pakistan
* Correspondence: awan.uzair@gmail.com

Abstract: The main objective of this paper is to establish some new variants of the Jensen–Mercer inequality via harmonically strongly convex function. We also propose some new fractional analogues of Hermite–Hadamard–Jensen–Mercer-like inequalities using $\mathcal{AB}$ fractional integrals. In order to obtain some of our main results, we also derive new fractional integral identities. To demonstrate the significance of our main results, we present some interesting applications to special means and to error bounds as well.

Keywords: Hermite–Hadamard–Mercer inequality; Hölder's inequality; Riemann–Liouville fractional integrals; harmonic convex function; special means; error estimation

MSC: 26A33; 26A51; 26D07; 26D10; 26D15; 26D20

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Citation: Bin-Mohsin, B.; Javed, M.Z.; Awan, M.U.; Mihai, M.V.; Budak, H.; Khan, A.G.; Noor, M.A. Jensen-Mercer Type Inequalities in the Setting of Fractional Calculus with Applications. *Symmetry* **2022**, *14*, 2187. https://doi.org/10.3390/sym14102187

Academic Editor: Fan Yang

Received: 13 September 2022
Accepted: 10 October 2022
Published: 18 October 2022

1. Introduction and Preliminaries

Fractional calculus is one of the leading interdisciplinary areas due to its richness of applications in real-world problems. In recent years, it has become a very interesting field of research. We use different versions of definitions of derivatives and integrals for the generalizations of different phenomena. These definitions also interact with the theory of inequalities. One of the fascinating ways of refining and generalizing the fundamental inequalities is by using fractional calculus concepts. The most investigated fractional integral is the R–L (Riemann–Liouville) integral and derivative with a power function as a kernel [15]. Several modifications of the indicated integral have been introduced by using some special functions such as [16] who introduced a new fractional integral associated with a new extension of the Mittag–Leffler function which is known as Raina's farctional integral. In [17] the authors used the exponential function as a non-singular kernel. Inspired by the ongoing research in [18], Caputo and Fabrizio introduced a new fraction integral in terms of a normalization function where functions belong to Hilbert spaces $H^1(\bar{\ltimes}_s, \bar{\ltimes}_e)$, which is stated as:

Definition 1 ([18])**.** *Let $\tilde{\mathfrak{F}} \in H^1(\bar{\ltimes}_s, \bar{\ltimes}_e)\bar{\ltimes}_e > \bar{\ltimes}_s, \upsilon \in [0,1]$. Then*

$$_{CF}D^{\upsilon}\tilde{\mathfrak{F}}(\bar{\ltimes}_e) = \frac{B(\upsilon)}{1-\upsilon}\int_{\bar{\ltimes}_s}^{b} f'(\bar{\delta})\exp\left[-\frac{\upsilon}{1-\upsilon}(\bar{\ltimes}_e - \bar{\delta})\right]\mathrm{d}\bar{\delta}$$

where $B(\upsilon)$ is the normalization function.

On the basis of the derivative defined in [18], the following integral operators are defined:

Definition 2 ([19])**.** *Let* $\tilde{\mathfrak{F}} \in H^1(\bar{\ltimes}_s, \bar{\ltimes}_e)\bar{\ltimes}_e > \bar{\ltimes}_s, v \in [0,1]$*. Then, left and right Caputo–Fabrizio fractional integrals are*

$${}^{CF}_{\bar{\ltimes}_s}I^{v}\tilde{\mathfrak{F}}(\bar{\ltimes}_e) = \frac{1-v}{B(v)}\tilde{\mathfrak{F}}(\bar{\ltimes}_e) + \frac{v}{B(v)}\int_{\bar{\ltimes}_s}^{b} f(\bar{\delta})\mathrm{d}\bar{\delta},$$

and

$${}^{CF}I^{v}_{\bar{\ltimes}_e}\tilde{\mathfrak{F}}(\bar{\ltimes}_s) = \frac{1-v}{B(v)}\tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \frac{v}{B(v)}\int_{\bar{\ltimes}_s}^{b} f(\bar{\delta})\mathrm{d}\bar{\delta},$$

where $B(v)$ *is the normalization function.*

Due to the limitation of the above discussed operators at $v = 1$, the following new operator was introduced and investigated by Atangana and Baleanu in both a Caputo and Fabrizio and an R–L sense using the Mittag–Leffler function as a kernel, which is stated as:

Definition 3 ([20])**.** *Let* $\tilde{\mathfrak{F}} \in H^1(\bar{\ltimes}_s, \bar{\ltimes}_e)\bar{\ltimes}_e > \bar{\ltimes}_s, v \in [0,1]$*. Then*

$${}^{ABC}D^{v}_{\bar{\ltimes}_s}\tilde{\mathfrak{F}}(\bar{\ltimes}_e) = \frac{B(v)}{1-v}\int_{\bar{\ltimes}_s}^{b} f'(\bar{\delta})E_v\left[-\frac{v}{1-v}(\bar{\ltimes}_e - \bar{\delta})^v\right]\mathrm{d}\bar{\delta}$$

where $B(v)$ *is the normalization function.*

Definition 4 ([20])**.** *Let* $\tilde{\mathfrak{F}} \in H^1(\bar{\ltimes}_s, \bar{\ltimes}_e)\bar{\ltimes}_e > \bar{\ltimes}_s, v \in [0,1]$*. Then*

$${}^{ABC}D^{v}_{\bar{\ltimes}_s}\tilde{\mathfrak{F}}(\bar{\ltimes}_e) = \frac{B(v)}{1-v}\frac{d}{\mathrm{d}\bar{\delta}}\int_{\bar{\ltimes}_s}^{b} f(\bar{\delta})E_v\left[-\frac{v}{1-v}(\bar{\ltimes}_e - \bar{\delta})^v\right]\mathrm{d}\bar{\delta}$$

where $B(v)$ *is the normalization function.*

The associated integral operators of the above described derivatives is stated as

Definition 5 ([20])**.** *The fractional integral related to the new nonlocal kernel of a mapping* $\tilde{\mathfrak{F}} \in L^1(\bar{\ltimes}_s, \bar{\ltimes}_e)$ *is defined as follows:*

$${}^{\mathcal{AB}}_{\bar{\ltimes}_s}I^{v}_{\bar{\delta}}\tilde{\mathfrak{F}}(\bar{\delta}) = \frac{1-v}{B(v)}\tilde{\mathfrak{F}}(\bar{\delta}) + \frac{v}{B(v)\Gamma(v)}\int_{\bar{\ltimes}_s}^{\bar{\delta}} \tilde{\mathfrak{F}}(\mathfrak{y}_1)(\bar{\delta} - \mathfrak{y}_1)^{v-1}\mathrm{d}\mathfrak{y}_1,$$

The right-hand side of the integral operator as follows:

$${}^{\mathcal{AB}}I^{v}_{\bar{\ltimes}_e}\tilde{\mathfrak{F}}(\bar{\delta}) = \frac{1-v}{B(v)}\tilde{\mathfrak{F}}(\bar{\delta}) + \frac{v}{B(v)\Gamma(v)}\int_{\bar{\delta}}^{\bar{\ltimes}_e} \tilde{\mathfrak{F}}(\mathfrak{y}_1)(\mathfrak{y}_1 - \bar{\delta})^{v-1}\mathrm{d}\mathfrak{y}_1,$$

where $\bar{\ltimes}_e > \bar{\ltimes}_s$*,* $v \in [0,1]$*. Here,* $\Gamma(v)$ *is the gamma function.* $B(v) > 0$ *is called the normalization function.*

We now discuss some preliminary concepts and results pertaining to the theory of convexity.

Definition 6. *Let* $\mathcal{C} \subseteq \mathbb{R}$ *be a convex set. A function* $\tilde{\mathfrak{F}} : \mathcal{C} \to \mathbb{R}$ *is said to be convex, if*

$$\tilde{\mathfrak{F}}\big((1-\bar{\delta})\mathfrak{y}_1 + \bar{\delta}\mathfrak{y}_2\big) \leq (1-\bar{\delta})\tilde{\mathfrak{F}}(\mathfrak{y}_1) + \bar{\delta}\tilde{\mathfrak{F}}(\mathfrak{y}_2), \quad \forall \mathfrak{y}_1, \mathfrak{y}_2 \in \mathcal{C},\ \bar{\delta} \in [0,1].$$

Analysis of convex sets, convex functions, and their properties is one of the most investigated areas of mathematical analysis. It covers both applied and pure mathematics due to its a great many applications. It has a great impact on topological vector spaces, separation theorem, and normed spaces as well. On the other hand, it has huge applications in economics and engineering and has a genetic relationship with the theory of optimization. In [1], the author discussed an interesting property of convexity with respect to a differential equation. For example, let $\Pi : \mathcal{I} \times \Upsilon \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, where interval $\mathcal{I} \subset \mathbb{R}$ and $\Upsilon \subset \mathbb{R}^n$ are open sets. Consider the second-order differential equation

$$\mathfrak{x}'' = \Pi(\mathfrak{t}, \mathfrak{x}, \mathfrak{x}'). \tag{1}$$

Definition 7 ([1]). *A set $\mathcal{C} \subset \Upsilon$ is said to be convex, with respect to the differential equation (1), if for arbitrary points $\mathfrak{x}$ and $\mathfrak{y}$ in $\mathcal{C}$ and the solution χ of (1) passing through these points the segment of χ joining them is contained in $\mathcal{C}$.*

Remark 1 ([1]). *Note that the set $\mathcal{C}$ is convex in the classical sense if and only if it is convex with respect to the differential equation $\mathfrak{x}'' = 0$.*

Convexity also has a close relation with the concept of symmetry. There are many interesting properties of symmetric convex sets. A beneficial point of relation between convexity and symmetry is that we work on one and apply it to the other. For details, see [2].

In addition, the study of inequalities in association with convexity is a major and active area of research. Many well-known inequalities are the direct consequences of the applications of convex functions. One of them is Jensen's inequality, which is stated as:

Theorem 1 ([3]). *Let $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ be a convex function and $\mathfrak{y}_{1i} \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$ and $\sum_{i=1} \bar{\delta}_i = 1$, then*

$$\tilde{\mathfrak{F}}\left(\sum_{i=1} \bar{\delta}_i \mathfrak{y}_{1i}\right) \leq \sum_{i=1} \bar{\delta}_i f(\mathfrak{y}_{1i}).$$

It is one of the most investigated and studied inequalities due to its generic property. A number of well known inequalities are obtained from Jensen's result. In [4], Mercer proved the discrete version of Jensen's inequality for convex functions, which is described as:

Theorem 2. *Let $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ be a convex function, then*

$$\tilde{\mathfrak{F}}\left(\bar{\ltimes}_s + \bar{\ltimes}_e - \sum_{i=1} \bar{\delta}_i \mathfrak{y}_{1i}\right) \leq \tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \sum_{i=1} \bar{\delta}_i f(\mathfrak{y}_{1i}),$$

for all $\mathfrak{y}_{1i} \in [\bar{\ltimes}_s, \bar{\ltimes}_e], \bar{\delta} \in [0,1]$ with $\sum_{i=1} \bar{\delta}_i = 1$.

We now recall the definition of harmonically convex sets.

Definition 8 ([5]). *A set $\mathcal{H} \subset (0, \infty)$ is said to be harmonic convex set, if*

$$\frac{\mathfrak{y}_1 \mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1 + (1-\bar{\delta})\mathfrak{y}_2} \in \mathcal{H}, \quad \forall \mathfrak{y}_1, \mathfrak{y}_2 \in \mathcal{H}, \ \bar{\delta} \in [0,1].$$

The class of harmonically convex functions is defined as:

Definition 9 ([5]). *A function* $\tilde{\mathfrak{F}} : \mathcal{H} \to \mathbb{R}$ *is said to be harmonic convex, if*

$$\tilde{\mathfrak{F}}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1 + (1-\bar{\delta})\mathfrak{y}_2}\right) \leq (1-\bar{\delta})\tilde{\mathfrak{F}}(\mathfrak{y}_1) + \bar{\delta}\tilde{\mathfrak{F}}(\mathfrak{y}_2), \quad \forall \mathfrak{y}_1, \mathfrak{y}_2 \in \mathcal{H},\ \bar{\delta} \in [0,1]. \tag{2}$$

Harmonic mean is useful in electric circuit theory and different fields of engineering sciences. It is well-known that the total resistance of a set of parallel resistors can be calculated by adding the reciprocals of each individual resistance value and then taking the reciprocal of the total resistance. For example, if $\bar{\ltimes}_s$ and $\bar{\ltimes}_e$ are the resistance of two parallel resistors, then the total resistance is

$$T = \frac{1}{2}H(\bar{\ltimes}_s, \bar{\ltimes}_e),$$

where $H(\bar{\ltimes}_s, \bar{\ltimes}_e) = \frac{2\bar{\ltimes}_s\bar{\ltimes}_e}{\bar{\ltimes}_s + \bar{\ltimes}_e}$ is the harmonic mean. For details, see [6].
For finding the solution of non-linear problems, parallel-algorithm can be established with the aid of harmonic mean, for details, see [7].

Remark 2. *We consider the function* $\bar{\mathfrak{G}} : \left[\frac{1}{\bar{\ltimes}_s}, \frac{1}{\bar{\ltimes}_e}\right] \to \mathbb{R}$ *and is defined as* $\bar{\mathfrak{G}}(\mathfrak{y}_1) = \frac{1}{\mathfrak{y}_1}$*, then* $\tilde{\mathfrak{F}}$ *is harmonic convex function if and only if* $\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}$ *is convex function. It is not hard to check that (2) is equivalent to*

$$\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1-\bar{\delta}}{\mathfrak{y}_1} + \frac{\bar{\delta}}{\mathfrak{y}_2}\right) \leq (1-\bar{\delta})\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right) + \bar{\delta}\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right).$$

In [8], Polyak introduced the notion of strongly convex functions, as follows:

Definition 10. *A function* $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ *is a strongly convex function with modulus* $\mu > 0$*, if*

$$\tilde{\mathfrak{F}}(\bar{\delta}\bar{\ltimes}_s + (1-\bar{\delta})\bar{\ltimes}_e) \leq \bar{\delta}\tilde{\mathfrak{F}}(\bar{\ltimes}_s) + (1-\bar{\delta})\tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \mu\bar{\delta}(1-\bar{\delta})(\bar{\ltimes}_e - \bar{\ltimes}_s)^2,$$

$\bar{\delta} \in [0,1]$.

Noor et al. [9] introduced the concept of a strongly harmonic convex function as:

Definition 11. *A function* $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ *is a strongly convex function with modulus* $\mu > 0$ *if*

$$\tilde{\mathfrak{F}}\left(\frac{\bar{\ltimes}_s\bar{\ltimes}_e}{\bar{\delta}\bar{\ltimes}_s + (1-\bar{\delta})\bar{\ltimes}_e}\right) \leq (1-\bar{\delta})\tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \bar{\delta}\tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \mu\bar{\delta}(1-\bar{\delta})\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2$$

$\bar{\delta} \in [0,1]$.

Remark 3. *We consider the function* $\bar{\mathfrak{G}} : \left[\frac{1}{\bar{\ltimes}_s}, \frac{1}{\bar{\ltimes}_e}\right] \to \mathbb{R}$ *and is defined as* $\bar{\mathfrak{G}}(\mathfrak{y}_1) = \frac{1}{\mathfrak{y}_1}$*, then* $\tilde{\mathfrak{F}}$ *is harmonically strongly convex function if and only if* $\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}$ *is strongly convex function.*

Definition 11 can be easily verified using Remark 3. For further investigation about harmonically convex functions, see [10–12].
In [13], Baloch et al. investigated the Jensen–Mercer inequality in the context of harmonic convex functions.

Theorem 3. *A function* $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ *be a harmonic convex function, then we have*

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \sum_{i=1}\frac{\bar{\delta}_i}{\mathfrak{y}_{1i}}}\right) \leq \tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \sum_{i=1}\bar{\delta}_i f(\mathfrak{y}_{1i}),$$

for all $\mathfrak{y}_{1i} \in [\bar{\ltimes}_s, \bar{\ltimes}_e], \bar{\delta} \in [0,1]$ *with* $\sum_{i=1} \bar{\delta}_i = 1$.

In [14], Moradi et al. established the the Jensen–Mercer Inequality for strongly convex functions, which is given as:

Theorem 4. *Let* $\tilde{\mathfrak{F}} : I = [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ *be a strongly convex function with modulus* $\mu > 0$

$$\tilde{\mathfrak{F}}\left(\bar{\ltimes}_s + \bar{\ltimes}_e - \sum_{i=1} \bar{\delta}_i \mathfrak{y}_{1i}\right) \leq \tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \sum_{i=1} \bar{\delta}_i f(\mathfrak{y}_{1i}) - \mu\left(2\sum_{i=1} \bar{\delta}_i \lambda_i (1-\lambda_i)(\bar{\ltimes}_e - \bar{\ltimes}_s)^2 + \sum_{i=1} \bar{\delta}_i \left(\mathfrak{y}_{1i} - \sum_{i=1} \bar{\delta}_i \mathfrak{y}_{1i}\right)^2\right),$$

where for all $\mathfrak{y}_{1i} \in I$, $\min_{1 \leq i \leq \phi} \mathfrak{y}_{1i} = \bar{\ltimes}_s$, $\max_{1 \leq i \leq \phi} \mathfrak{y}_{1i} = \bar{\ltimes}_e$, $\sum_{i=1} \bar{\delta}_i = 1$ *and* $\lambda_i \in [0,1]$.

The association of theory of inequality and fractional calculus has produced an ample amount of literature. Firstly in 2013, Sarikaya et al. [21] used the fractional concepts to obtain the refinements of H.H.I (Hermite–Hadamard's inequality). After this idea, generalization of inequalities through fractional calculus became a powerful tool to obtain new improvements of classical inequalities. It is impossible to list all the articles related to fractional inequalities, but we list some remarkable papers regarding the H.H.J.M inequality. In [22], the authors discussed the fractional variants of H.H.J.M-like inequalities. In 2020 Zho et al. [23], derived the fractional H.H.J.M type inequalities involving Caputo derivatives. In the following sequel, Butt et al. [24,25] discussed the following inequality using k-fractional and ψ factional integral,s respectively. Recently, Cortez et al. [26] proved some new generalizations of H.H.J.M type inequalities. In [27], Butt et al. concluded some more estimates of H.H.J.M-type inequalities for harmonic convexity. Iscan et al. [28] proved the fractional analogues of H.H.J.M inequality for GA-convex. In [29], the authors discussed some new variational inequalities involving harmonic convexity. In [30], Noor et al. used the concept of harmonic convexity to obtain some integral inequalities in the fractal domain. In 2022, Noor. et al. [31] established some variational inclusions involving three operators. Awan et al. [32] studied the exponential convexity in the perspective of inequalities. In 2022, Faisal et al. [33] used the concept of majorized tuples to obtain some new refinements of H.H.J.M-type inequalities for convex functions. For more details see [34–39].

The main motivation of our study is to establish some strong versions of H.H.J.M-type inequalities via fractional calculus. For this purpose, we prove a new generalization of H.H.J.M inequality for strongly convex functions. With the aid of this result, we conclude some H.H.J.M-type inequalities. In order to obtain some estimates of H.H.J.M inequality, we prove two new fractional identities. For the validity of our main outcomes, we discuss some applications to means and error bounds as well. We hope that the ideas and techniques of the paper will inspire the interested readers.

We have organized our study into the following sections. In the first section, we give some preliminaries and basic facts of convexity and fractional calculus. In the second section, we prove fractional H.H.J.M inequalities involving $\mathcal{AB}$ fractional integrals. In the proceeding section, we derive two equalities for differentiable functions. Then, in the next section, we drive some bounds of H.H.J.M inequality with the help of lemmas. At the end, we present some applications.

2. H.H.J.M Inequalities

We now discuss our main results. We give a stronger version of the well-known Jensen–Mercer inequality for strongly harmonic convexity and H.H.J.M-type inequalities.

We first give the following Jensen–Mercer inequality for strongly harmonic convex functions.

Theorem 5. *Let $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ be a strongly harmonic convex function with modulus $\mu > 0$ where $0 < \bar{\ltimes}_s < \bar{\ltimes}_e$. Let $\mathfrak{y}_{1i} \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$ be -points and let $\mu_i, \lambda_i \in [0,1]$, be such that $\sum_{i=1} \mu_i = 1$. Then we have the following inequality:*

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \sum_{i=1} \frac{\mu_i}{\mathfrak{y}_{1i}}}\right) \leq \tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \sum_{i=1} \mu_i \tilde{\mathfrak{F}}(\mathfrak{y}_{1i})$$
$$- \mu\left(2\sum_{i=1} \mu_i\lambda_i(1-\lambda_i)\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2 + \sum_{i=1} \mu_i\left(\frac{1}{\mathfrak{y}_{1i}} - \sum_{i=1}\frac{\mu_i}{\mathfrak{y}_{1i}}\right)^2\right)$$

Proof. From Remark 3, we can write as

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \sum_{i=1} \frac{\mu_i}{\mathfrak{y}_{1i}}}\right)$$
$$+ \mu\left(2\sum_{i=1} \mu_i\lambda_i(1-\lambda_i)\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2 + \sum_{i=1} \mu_i\left(\frac{1}{\mathfrak{y}_{1i}} - \sum_{i=1}\frac{\mu_i}{\mathfrak{y}_{1i}}\right)^2\right)$$
$$= \tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \sum_{i=1}\frac{\mu_i}{\mathfrak{y}_{1i}}\right) + \mu\left(2\sum_{i=1} \mu_i\lambda_i(1-\lambda_i)\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2 + \sum_{i=1} \mu_i\left(\frac{1}{\mathfrak{y}_{1i}} - \sum_{i=1}\frac{\mu_i}{\mathfrak{y}_{1i}}\right)^2\right).$$

Since $\tilde{\mathfrak{F}}$ is harmonically strongly convex function, then by Remark 3, $\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}$ is strongly convex function, and we have

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \sum_{i=1} \frac{\mu_i}{\mathfrak{y}_{1i}}}\right) + \mu\left(2\sum_{i=1} \mu_i\lambda_i(1-\lambda_i)\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2 + \sum_{i=1} \mu_i\left(\frac{1}{\mathfrak{y}_{1i}} - \sum_{i=1}\frac{\mu_i}{\mathfrak{y}_{1i}}\right)^2\right)$$
$$\leq \tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}\right) + \tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_e}\right) - \sum_{i=1} \mu_i f \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_{1i}}\right).$$

Again using the definition of $\bar{\mathfrak{G}}$, we obtain our required result. □

Now we establish Hermite–Hadamard–Mercer inequality for strongly harmonic convex functions involving the Atangana–Baleanu fractional integral.

Theorem 6. *Let $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ be a strongly harmonic convex function on $[\bar{\ltimes}_s, \bar{\ltimes}_e]$ with $\bar{\ltimes}_s < \bar{\ltimes}_e$, then*

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right) - \frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2 - \mathfrak{y}_1}\right)^v \left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right) + \frac{1-v}{B(v)}\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$\leq [\tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \tilde{\mathfrak{F}}(\bar{\ltimes}_e)] - \frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2 - \mathfrak{y}_1}\right)^v \left[{}^{\mathcal{AB}}_{\frac{1}{\mathfrak{y}_2}}I^v_{\frac{1}{\mathfrak{y}_1}}\,\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right) + {}^{\mathcal{AB}}I^v_{\frac{1}{\mathfrak{y}_1}}\,\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$- \mu\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2 - \frac{\mu}{4}\frac{v^2 - v + 2}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2$$
$$\leq \tilde{\mathfrak{F}}(\bar{\ltimes}_s) + \tilde{\mathfrak{F}}(\bar{\ltimes}_e) - \tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1 + \mathfrak{y}_2}\right) - \frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2 - \mathfrak{y}_1}\right)^v \left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right) + \frac{1-v}{B(v)}\tilde{\mathfrak{F}} \circ \bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$- \mu\left[(\lambda_1 - \lambda_1^2) + (\lambda_2 - \lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s} - \frac{1}{\bar{\ltimes}_e}\right)^2 - \frac{\mu}{2}\frac{v^2 - v + 2}{(v+2)(v+1)}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2,$$

where $\mathfrak{y}_1, \mathfrak{y}_2 \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$ *and* $B(v)$ *is normalization with* $v, \bar{\delta} \in [0,1]$.

Proof. Let us consider $\mathfrak{y}_{11} = \frac{\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(1-\bar{\delta})\mathfrak{y}_2}$ and $\mathfrak{y}_{12} = \frac{\mathfrak{y}_1\mathfrak{y}_2}{(1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}$ in Theorem 5; we obtain

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right) \le \tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)-\frac{1}{2}\left[\tilde{\mathfrak{F}}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(1-\bar{\delta})\mathfrak{y}_2}\right)+\tilde{\mathfrak{F}}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{(1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}\right)\right]$$
$$-\mu\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{4}(2\bar{\delta}-1)^2\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Multiplying both sides by $\bar{\delta}^{v-1}$ and integrating with respect to $\bar{\delta}$ on $[0,1]$, then we have

$$\frac{1}{v}\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right) \le \frac{\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)}{v}-\frac{1}{2}\int_0^1\bar{\delta}^{v-1}\left[\tilde{\mathfrak{F}}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(1-\bar{\delta})\mathfrak{y}_2}\right)+\int_0^1\tilde{\mathfrak{F}}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{(1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}\right)\right]$$
$$-\frac{\mu}{v}\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\mathrm{d}\bar{\delta}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{4}\int_0^1\bar{\delta}^{v-1}(2\bar{\delta}-1)^2\mathrm{d}\bar{\delta}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Multiplying both sides of the above inequality by $\frac{2v}{B(v)\Gamma(v)}\left(\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2}\right)^v$ and then subtracting $\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)$ from both sides, we have

$$\frac{2}{B(v)\Gamma(v)}\left(\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2}\right)^v\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right)-\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$\le\frac{2}{B(v)\Gamma(v)}\left(\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2}\right)^v[\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)]-\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\frac{v}{B(v)\Gamma(v)}\left[\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}\left(\frac{1}{\mathfrak{y}_1}-u\right)^{v-1}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}(u)\mathrm{d}u+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}\left(u-\frac{1}{\mathfrak{y}_2}\right)^{v-1}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}(u)\mathrm{d}u\right]$$
$$-\mu\frac{2}{B(v)\Gamma(v)}\left(\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2}\right)^v\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2$$
$$-\frac{\mu}{2B(v)\Gamma(v)}\left(\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2}\right)^v\frac{v^2-v+2}{(v+2)(v+1)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Now, by comparing the above inequality with the Atangana–Baleanu fractional integral, we have the

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right)-\frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$\le[\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)]-\frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[{}^{\mathcal{AB}}_{\frac{1}{\mathfrak{y}_2}}I^v_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}I^v_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\mu\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{4}\frac{v^2-v+2}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2. \quad (3)$$

This completes the proof of our first inequality. To prove our second inequality, we use the notion of strongly harmonic convexity.

$$\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)=\tilde{\mathfrak{F}}\left(\frac{2}{\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}+\frac{1-\bar{\delta}}{\mathfrak{y}_1}+\frac{\bar{\delta}}{\mathfrak{y}_2}}\right)$$
$$\le\frac{1}{2}\left[\tilde{\mathfrak{F}}\left(\frac{2}{(1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}\right)+\tilde{\mathfrak{F}}\left(\frac{2}{\bar{\delta}\mathfrak{y}_1+(1-\bar{\delta})\mathfrak{y}_2}\right)\right]-\frac{\mu}{4}(2\bar{\delta}-1)^2\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Now, multiplying both sides by $\bar{\delta}^{v-1}$ and integrating over $[0,1]$, we have

$$\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right) \le \frac{v}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}\left(\frac{1}{\mathfrak{y}_1}-u\right)^{v-1}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}(u)\mathrm{d}u+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}\left(u-\frac{1}{\mathfrak{y}_2}\right)^{v-1}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}(u)\mathrm{d}u\right]$$
$$-\frac{\mu}{4}\frac{v^2-v+2}{(v+2)(v+1)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Adding both sides $\frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$ of the above inequality, we have

$$\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)+\frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$\le \frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[{}^{\mathcal{AB}}_{\frac{1}{\mathfrak{y}_2}}I^{v}_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}I^{v}_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]-\frac{\mu}{4}\frac{v^2-v+2}{(v+2)(v+1)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

After simplification, we obtain the following result

$$\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)-\frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[{}^{\mathcal{AB}}_{\frac{1}{\mathfrak{y}_2}}I^{v}_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}I^{v}_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\mu\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{4}\frac{v^2-v+2}{(v+2)(v+1)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2$$
$$\le \tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)-\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)-\frac{B(v)\Gamma(v)}{2}\left(\frac{\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\mu\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{2}\frac{v^2-v+2}{(v+2)(v+1)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2. \quad (4)$$

From (3) and (4), we obtain the required result. □

Theorem 7. *Let $\tilde{\mathfrak{F}}:[\bar{\ltimes}_s,\bar{\ltimes}_e]\to\mathbb{R}$ be a strongly harmonic convex function on $[a,\bar{\ltimes}_e]$ with $\bar{\ltimes}_s<\bar{\ltimes}_e$, then*

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right)-\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$\le [\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)]-\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[{}^{\mathcal{AB}}_{\frac{\mathfrak{y}_1+ny}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}I^{v}_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}I^{v}_{\frac{nx+\mathfrak{y}_2}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\mu\left[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2$$
$$-\frac{\mu}{4(+1)^2}\frac{4(v^2+v)+(+1)^2(v^2+3v+2)-2(+1)(v^2+2v)}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2$$
$$\le \tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)-\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)-\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^{v}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\mu\left[(\lambda_1-\lambda_1^{)}+(\lambda_2-\lambda_2^2)\right]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2$$
$$-\frac{\mu}{4(+1)^2}\frac{2(v^2+v)+(+1)^2(v^2+3v+2)-2(+1)(v^2+2v)}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2$$

where $\mathfrak{y}_1,\mathfrak{y}_2\in[\bar{\ltimes}_s,\bar{\ltimes}_e]$ with $\mathfrak{y}_1<\mathfrak{y}_2$ and $B(v)$ is normalization with $v,\bar{\delta}\in[0,1]$.

Proof. Let us consider $\mathfrak{y}_{11}=\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(+1-\bar{\delta})\mathfrak{y}_2}$ and $\mathfrak{y}_{12}=\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{(+1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}$ in Theorem 5; we have

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right)\leq\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)-\frac{1}{2}\left[\tilde{\mathfrak{F}}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(+1-\bar{\delta})\mathfrak{y}_2}\right)+\tilde{\mathfrak{F}}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{(+1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}\right)\right]$$
$$-\mu\Big[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\Big]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{4(+1)^2}(2\bar{\delta}-+1)^2\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Multiplying both sides by $\bar{\delta}^{v-1}$ and integrating with respect to $\bar{\delta}$ on $[0,1]$, then we have

$$\frac{1}{v}\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right)\leq\frac{\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)}{v}-\frac{1}{2}\int_0^1\bar{\delta}^{v-1}\left[\tilde{\mathfrak{F}}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(+1-\bar{\delta})\mathfrak{y}_2}\right)\right.$$
$$\left.+\tilde{\mathfrak{F}}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{(+1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}\right)\right]\mathrm{d}\bar{\delta}$$
$$-\frac{\mu}{v}\Big[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\Big]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2-\frac{\mu}{4(+1)^2}\int_0^1\bar{\delta}^{v-1}(2\bar{\delta}-+1)^2\mathrm{d}\bar{\delta}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Subtracting $\frac{B(v)\gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$ from both sides of the above inequality, and comparing with Atangana–Baleanu fractional integrals, we have

$$\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{\mathfrak{y}_1+\mathfrak{y}_2}{\mathfrak{y}_1\mathfrak{y}_2}}\right)-\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$\leq[\tilde{\mathfrak{F}}(\bar{\ltimes}_s)+\tilde{\mathfrak{F}}(\bar{\ltimes}_e)]-\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[{}^{\mathcal{AB}}_{\frac{\mathfrak{y}_1+ny}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}I^v_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}I^v_{\frac{nx+\mathfrak{y}_2}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$-\mu\Big[(\lambda_1-\lambda_1^2)+(\lambda_2-\lambda_2^2)\Big]\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2$$
$$-\frac{\mu}{4(+1)^2}\frac{4(v^2+v)+(+1)^2(v^2+3v+2)-2(+1)(v^2+2v)}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2.$$

This completes the proof of our first inequality. To prove our second inequality, we use the notion of strongly harmonic convexity, we have

$$\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)=\tilde{\mathfrak{F}}\left(\frac{2}{\frac{\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_2}+\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{\bar{\delta}}{(+1)\mathfrak{y}_2}}\right)$$
$$\leq\frac{1}{2}\left[\tilde{\mathfrak{F}}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{(+1-\bar{\delta})\mathfrak{y}_1+\bar{\delta}\mathfrak{y}_2}\right)+\tilde{\mathfrak{F}}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\bar{\delta}\mathfrak{y}_1+(+1-\bar{\delta})\mathfrak{y}_2}\right)\right]-\frac{\mu}{4(+1)^2}(2\bar{\delta}-+1)^2\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.$$

Now, multiplying both sides by $\bar{\delta}^{v-1}$ and integrating over $[0,1]$, we have

$$\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)$$
$$\leq\frac{v}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[\int_{\frac{\mathfrak{y}_1+ny}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}^{\frac{1}{\mathfrak{y}_1}}\left(\frac{1}{\mathfrak{y}_1}-u\right)^{v-1}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}(u)\mathrm{d}u+\int_{\frac{1}{\mathfrak{y}_2}}^{\frac{nx+\mathfrak{y}_2}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\left(u-\frac{1}{\mathfrak{y}_2}\right)^{v-1}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}(u)\mathrm{d}u\right]$$
$$-.\frac{\mu}{4(+1)^2}\frac{4(v^2+v)+(+1)^2(v^2+3v+2)-2(+1)(v^2+2v)}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2.$$

Adding $\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)]$ to both sides of the above inequality, we have

$$\begin{aligned}
&\tilde{\mathfrak{F}}\left(\frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1+\mathfrak{y}_2}\right)+\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1))\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]\\
&\leq\frac{B(v)\Gamma(v)}{2}\left(\frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_2-\mathfrak{y}_1}\right)^v\left[{}^{\mathcal{AB}}_{\frac{\mathfrak{y}_1+ny}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}I^v_{\frac{1}{\mathfrak{y}_1}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}I^v_{\frac{nx+\mathfrak{y}_2}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\mathfrak{y}_2}\right)\right]\\
&\quad-\frac{\mu}{4(+1)^2}\frac{4(v^2+v)+(+1)^2(v^2+3v+2)-2(+1)(v^2+2v)}{(v+2)(v+1)}\left(\frac{1}{\bar{\ltimes}_s}-\frac{1}{\bar{\ltimes}_e}\right)^2.
\end{aligned}$$

After simplification, we obtain the required inequality. □

3. Key Lemmas

We now derive some new generalized fractional identities involving $\mathcal{AB}$-fractional integrals, which will serve as an auxiliary result in proving our main result in successive sections. It is worth mentioning that theses fractional equalities are quite different from the lemmas obtained with the help of Riemann–Liouville fractional integrals.

Lemma 1. *Let $\tilde{\mathfrak{F}}'\in L[\bar{\ltimes}_s,\bar{\ltimes}_e]$. Suppose $\tilde{\mathfrak{F}}:[\bar{\ltimes}_s,\bar{\ltimes}_e]\to\mathbb{R}$ be a differentiable function on $[\bar{\ltimes}_s,\bar{\ltimes}_e]$ with $\bar{\ltimes}_s<\bar{\ltimes}_e$, then*

$$\begin{aligned}
\Omega(\bar{\ltimes}_s,\bar{\ltimes}_e,v,\tilde{\mathfrak{F}})=&\frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[\int_0^1\frac{[(1-\bar{\delta})^v-\bar{\delta}^v]}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)}\right)\mathrm{d}\bar{\delta}\right.\\
&\left.-2v\mu\int_0^1[\bar{\delta}^v-\bar{\delta}^{v+1}]\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)\right],
\end{aligned}\tag{5}$$

where

$$\begin{aligned}
\Omega(\bar{\ltimes}_s,\bar{\ltimes}_e,v,\tilde{\mathfrak{F}})=&\frac{\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}}\right)+\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}}\right)}{2}-\frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2\\
&+\frac{(\mathfrak{y}_1\mathfrak{y}_2)^v(1-v)\Gamma(v)}{2(\mathfrak{y}_2-\mathfrak{y}_1)^v}\left[\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)+\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)\right]\\
&-\frac{(\mathfrak{y}_1\mathfrak{y}_2)^vB(v)\Gamma(v)}{2(\mathfrak{y}_2-\mathfrak{y}_1)^v}\left[{}^{\mathcal{AB}}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)\right.\\
&\left.+{}^{\mathcal{AB}}_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)\right]
\end{aligned}$$

Here, $B(v)$ is normalizing function with $v>0,\mathfrak{y}_1,\mathfrak{y}_2\in[\bar{\ltimes}_s,\bar{\ltimes}_e]$ and $\bar{\delta}\in[0,1]$.

Proof. Consider the right-hand side of (5)

$$\begin{aligned}
I=&\frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[\int_0^1\frac{[(1-\bar{\delta})^v-\bar{\delta}^v]}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)}\right)\mathrm{d}\bar{\delta}-2v\mu\int_0^1[\bar{\delta}^v-\bar{\delta}^{v+1}]\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)\right]\\
=&\frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}[I_1-I_2-2v\mu\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)I_3],
\end{aligned}\tag{6}$$

where

$$I_1 = \int_0^1 \frac{(1-\bar{\delta})^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2} \tilde{\mathfrak{F}}' \left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)} \right) \mathrm{d}\bar{\delta}$$
$$= \frac{(\mathfrak{y}_1\mathfrak{y}_2)\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}}\right)}{(\mathfrak{y}_2-\mathfrak{y}_1)} + \frac{(\mathfrak{y}_1\mathfrak{y}_2)^{v+1}B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^{v+1}}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)\right]$$
$$- \frac{(\mathfrak{y}_1\mathfrak{y}_2)^{v+1}B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^{v+1}} {}^{\mathcal{AB}}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right), \tag{7}$$

$$I_2 = \int_0^1 \frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2} \tilde{\mathfrak{F}}' \left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)} \right) \mathrm{d}\bar{\delta}$$
$$= -\frac{(\mathfrak{y}_1\mathfrak{y}_2)\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}}\right)}{(\mathfrak{y}_2-\mathfrak{y}_1)} - \frac{(\mathfrak{y}_1\mathfrak{y}_2)^{v+1}B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^{v+1}}\left[\frac{1-v}{B(v)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)\right]$$
$$+ \frac{(\mathfrak{y}_1\mathfrak{y}_2)^{v+1}B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^{v+1}} {}^{\mathcal{AB}}_{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right), \tag{8}$$

and

$$I_3 = \int_0^1 [\bar{\delta}^v - \bar{\delta}^{v+1}]\mathrm{d}\bar{\delta} = \frac{1}{(v+1)(v+2)}. \tag{9}$$

Using the values of (7), (8), and (9) in (2), yields the required result. □

Next, we have another identity for differentiable functions, which provides us some stronger versions of midpoint H.H.J.M-type inequalities.

Lemma 2. *Let $\tilde{\mathfrak{F}}' \in L[\bar{\ltimes}_s, \bar{\ltimes}_e]$. Suppose $\tilde{\mathfrak{F}} : [\bar{\ltimes}_s, \bar{\ltimes}_e] \to \mathbb{R}$ be a differentiable function on $[\bar{\ltimes}_s, \bar{\ltimes}_e]$ with $\bar{\ltimes}_s < \bar{\ltimes}_e$, then*

$$\omega(\bar{\ltimes}_s, \bar{\ltimes}_e, v, \tilde{\mathfrak{F}}) = \frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(\ +1)^2}\left[\int_0^1 \frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\mathrm{d}\bar{\delta}\right.$$
$$-\int_0^1 \frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-1\left(\frac{\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{+1-\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\mathrm{d}\bar{\delta}$$
$$\left.-(\ +1)\mu\int_0^1[\bar{\delta}^v-\bar{\delta}^{v+1}]\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)\mathrm{d}\bar{\delta}\right],$$

where

$$
\begin{aligned}
&\omega(\bar{\ltimes}_s, \bar{\ltimes}_e, v, \tilde{\mathfrak{F}}) \\
&\frac{\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(\mathfrak{y}_1+ny)}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\right)+\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(nx+\mathfrak{y}_2)}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\right)}{+1}+\frac{(+1)^{v-1}(\mathfrak{y}_1\mathfrak{y}_2)^v(1-v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^v} \\
&\times\left[\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)+\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)\right]-\frac{(+1)^{v-1}(\mathfrak{y}_1\mathfrak{y}_2)^v B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^v} \\
&\times\left[{}^{\mathcal{AB}}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(\mathfrak{y}_1+ny)}{\mathfrak{y}_1\mathfrak{y}_2(+1)}\right)}fog\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)+{}^{\mathcal{AB}}_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(nx+\mathfrak{y}_2)}{\mathfrak{y}_1\mathfrak{y}_2(+1)}\right)}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}fog\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)\right] \\
&-\frac{\mu}{(+1)(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.
\end{aligned}
$$

Here, $B(v)$ is a normalizing function and $\forall\mathfrak{y}_1,\mathfrak{y}_2\in[\bar{\ltimes}_s,\bar{\ltimes}_e]$ and $\in\mathbb{N}$.

Proof. Consider

$$
\begin{aligned}
J=&\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+1)^2}\left[\int_0^1\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\mathrm{d}\bar{\delta}\right. \\
&-\int_0^1\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-1\left(\frac{\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{+1-\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\mathrm{d}\bar{\delta} \\
&\left.-(+1)\mu\int_0^1[\bar{\delta}^v-\bar{\delta}^{v+1}]\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)\mathrm{d}\bar{\delta}\right] \\
=&\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+1)^2}[J_1-J_2-J_3(+1)\mu\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)],
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
J_1=&\int_0^1\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\mathrm{d}\bar{\delta} \\
=&\frac{(+1)(\mathfrak{y}_1\mathfrak{y}_2)}{(\mathfrak{y}_2-\mathfrak{y}_1)}\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(\mathfrak{y}_1+ny)}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\right)+\frac{(+1)^{v+1}(\mathfrak{y}_1\mathfrak{y}_2)^{v+1}(1-v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^v}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right) \\
&-\frac{(+1)^{v-1}(\mathfrak{y}_1\mathfrak{y}_2)^v B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^v}{}^{\mathcal{AB}}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(\mathfrak{y}_1+ny)}{\mathfrak{y}_1\mathfrak{y}_2(+1)}\right)}fog\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_2=&\int_0^1\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{+1-\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\mathrm{d}\bar{\delta} \\
=&-\frac{(+1)(\mathfrak{y}_1\mathfrak{y}_2)}{(\mathfrak{y}_2-\mathfrak{y}_1)}\tilde{\mathfrak{F}}\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(nx+\mathfrak{y}_2)}{(+1)\mathfrak{y}_1\mathfrak{y}_2}}\right)-\frac{(+1)^{v+1}(\mathfrak{y}_1\mathfrak{y}_2)^{v+1}(1-v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^v}\tilde{\mathfrak{F}}\circ\bar{\mathfrak{G}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right) \\
&+\frac{(+1)^{v-1}(\mathfrak{y}_1\mathfrak{y}_2)^v B(v)\Gamma(v)}{(\mathfrak{y}_2-\mathfrak{y}_1)^v}{}^{\mathcal{AB}}_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(nx+\mathfrak{y}_2)}{\mathfrak{y}_1\mathfrak{y}_2(+1)}\right)}I_{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}fog\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right),
\end{aligned}
$$

and

$$J_3 = (\,+1)\mu \int_0^1 [\bar{\delta}^v - \bar{\delta}^{v+1}]\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)\mathrm{d}\bar{\delta} = \frac{(\,+1)\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right).$$

Substituting the values of J_1, J_2 and J_3 in J , then we obtain our required result. □

4. Estimates of H.H.J.M-Type Inequalities Involving Strongly Harmonic Convex Functions

In this portion, we consider the strongly harmonic convex property of the functions to obtain some new fractional bounds of H.H.J.M-type inequalities using some fundamental inequalities such as Hölder's, Power mean inequalities.

Theorem 8. *All the assumptions of Lemma 5 are satisfied, and let $|\tilde{\mathfrak{F}}|$ be an harmonic convex function, then*

$$|\Omega(v, \bar{\ltimes}_s, \bar{\ltimes}_e, \tilde{\mathfrak{F}})| \leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[T_1(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)| + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|) - T_2|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)| - T_3|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|\right] + \frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2,$$

where

$$\begin{aligned}
T_1 &= \frac{B_2^{-2}}{v+1} {}_2F_1\left(2, v+1, v+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}\right)}\right) + \frac{B_2^{-2}}{(v+1)} {}_2F_1\left(2, 1, v+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}\right)}\right) \\
&\quad + \frac{M^{-2}}{(v+1)} {}_2F_1\left(2, v+1, v+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2}\right)}\right). \\
T_2 &= \frac{B_2^{-2}}{v+2} {}_2F_1\left(2, v+1, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}\right)}\right) + \frac{B_2^{-2}}{(v+1)(v+2)} {}_2F_1\left(2, 2, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}\right)}\right) \\
&\quad + \frac{M^{-2}}{(v+1)(v+2)} {}_2F_1\left(2, v+1, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2 M}\right)}\right). \\
T_3 &= \frac{B_2^{-2}}{(v+1)(v+2)} {}_2F_1\left(2, v+1, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}\right)}\right) + \frac{B_2^{-2}}{(v+2)} {}_2F_1\left(2, 1, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}\right)}\right) \\
&\quad + \frac{M^{-2}}{(v+1)(v+2)} {}_2F_1\left(2, v+1, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} + \frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2 M}\right)}\right) \\
&\quad + \frac{M^{-2}}{(v+1)} {}_2F_1\left(2, v+1, v+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2 M}\right)}\right),
\end{aligned}$$

where $B_1 = \frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_1}$, $B_2 = \frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}$ $M = \frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2}$, and $\forall \mathfrak{y}_1, \mathfrak{y}_2 \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$ with $\mathfrak{y}_1 < \mathfrak{y}_2$.

Proof. From Lemma 5, the modulus property and harmonic convexity of $|\tilde{\mathfrak{F}}'|$, we have

$$
\begin{aligned}
&|\Omega(v, \bar{\ltimes}_s, \bar{\ltimes}_e, \tilde{\mathfrak{F}})| \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[\int_0^1 \left|\frac{[(1-\bar{\delta})^{\frac{v}{k}} - \bar{\delta}^{\frac{v}{k}}]}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\right| \left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)}\right)\right| \mathrm{d}\bar{\delta} \right.\\
&\quad \left. -2v\mu \int_0^1 [\bar{\delta}^v - \bar{\delta}^{v+1}]\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)\right] \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[\int_0^1 \left|\frac{[(1-\bar{\delta})^{v} - \bar{\delta}^{v}]}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\right| \left(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)| + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)| - \bar{\delta}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|\right.\right. \\
&\quad \left.\left. -(1-\bar{\delta})|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|\right)\mathrm{d}\bar{\delta}\right] + \frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2 \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[\int_0^1 \frac{[(1-\bar{\delta})^{v} - \bar{\delta}^{v}]}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2} \left(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)| + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)| - \bar{\delta}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|\right.\right. \\
&\quad -(1-\bar{\delta})|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|\big)\mathrm{d}\bar{\delta} + 2\int_0^{\frac{1}{2}} \frac{(1-2\bar{\delta})^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1} + \frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\left(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)| + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)| - \bar{\delta}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|\right. \\
&\left.\left. -(1-\bar{\delta})|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|\right)\mathrm{d}\bar{\delta}\right] + \frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2.
\end{aligned}
$$

After simple calculations, we obtain the required result. □

Theorem 9. *All the assumptions of Lemma 5 are satisfied, and let $|\tilde{\mathfrak{F}}'|^q$ be an harmonic convex function, then we have*

$$
\begin{aligned}
&|\Omega(v, \bar{\ltimes}_s, \bar{\ltimes}_e, \tilde{\mathfrak{F}})| \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2} T_4^{\frac{1}{p}}\left[\left(\frac{1}{vq+1}(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q) - \frac{1}{(vq+2)(vq+1)}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q\right.\right. \\
&\quad \left. - \frac{1}{vq+2}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|^q\right)^{\frac{1}{q}} + \left(\frac{1}{vq+1}(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q) - \frac{1}{(vq+2)(vq+1)}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|^q\right. \\
&\quad \left.\left. - \frac{1}{vq+2}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q\right)^{\frac{1}{q}}\right] + \frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2,
\end{aligned}
$$

where

$$
T_4 = \frac{\mathfrak{y}_1\mathfrak{y}_2[B_1^{1-2p} - B_2^{1-2p}]}{(\mathfrak{y}_2 - \mathfrak{y}_1)(1-2p)},
$$

B_1 and B_2 are already defined, $\forall \mathfrak{y}_1, \mathfrak{y}_2 \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$ with $\mathfrak{y}_1 < \mathfrak{y}_2$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 5, the modulus property, Hölder's inequality and harmonic convexity of $|\tilde{\mathfrak{F}}'|^q$, we have

$$
\begin{aligned}
&|\Omega(v,\bar{\ltimes}_s,\bar{\ltimes}_e,\tilde{\mathfrak{F}},\bar{\mathfrak{G}})| \\
&\leq \frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left[\int_0^1 \frac{(1-\bar{\delta})^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)}\right)\right|\mathrm{d}\bar{\delta}\right. \\
&\left.+\int_0^1 \frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^2}\left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)}\right)\right|\mathrm{d}\bar{\delta}\right]+\frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2 \\
&\leq \frac{\mathfrak{y}_2-\mathfrak{y}_1}{2\mathfrak{y}_1\mathfrak{y}_2}\left(\int_0^1\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1}+\frac{1-\bar{\delta}}{\mathfrak{y}_2}\right)\right)^{-2p}\mathrm{d}\bar{\delta}\right)^{\frac{1}{p}}\left[\left(\int_0^1(1-\bar{\delta})^{vq}(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q+|\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q)\right.\right. \\
&-\bar{\delta}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q-(1-\bar{\delta})|\tilde{\mathfrak{F}}'(\mathfrak{y}_2|^q)\Big)^{\frac{1}{q}}+\left(\int_0^1\bar{\delta}^{vq}(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q+|\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q)\right. \\
&\left.-\bar{\delta}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q-(1-\bar{\delta})|\tilde{\mathfrak{F}}'(\mathfrak{y}_2|^q)\Big)^{\frac{1}{q}}\right]+\frac{v\mu}{(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.
\end{aligned}
$$

Simply integrating the above inequality, we establish the desired result. □

Theorem 10. *All the assumptions of Lemma 2 are satisfied, and let $|\tilde{\mathfrak{F}}|$ be an harmonic convex function, then*

$$
\begin{aligned}
|\omega(\bar{\ltimes}_s,\bar{\ltimes}_e,v,)| \leq{}& \frac{\mathfrak{y}_2-\mathfrak{y}_1}{(+1)^2\mathfrak{y}_1\mathfrak{y}_2}\left[C_1(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|+|\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|)-(C_2|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|-C_3|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|)\right] \\
&+\frac{\mu}{(+1)(v+1)(v+2)}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 ={}& \frac{B_1^{-2}}{v+1}{}_2F_1\left(2,v+1,v+2,\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)}\right) \\
&+\frac{B_2^{-2}}{v+1}{}_2F_1\left(2,v+1,v+2,\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}\right). \\
C_2 ={}& \frac{B_1^{-2}}{v+1}{}_2F_1\left(2,v+1,v+2,\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)}\right) \\
&-\frac{B_1^{-2}}{(+1)(v+2)}{}_2F_1\left(2,v+2,v+3,\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)}\right) \\
&+\frac{B_2^{-2}}{(+1)(v+2)}{}_2F_1\left(2,v+2,v+3,\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)}\right).
\end{aligned}
$$

$$C_3 = \frac{B_2^{-2}}{v+1}{}_2F_1\left(2, v+1, v+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{\mathfrak{y}_2}\right)}\right)$$
$$- \frac{B_2^{-2}}{(+1)(v+2)}{}_2F_1\left(2, v+2, v+3, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{\mathfrak{y}_2}\right)}\right)$$
$$+ \frac{B_1^{-2}}{(+1)(v+2)}{}_2F_1\left(2, v+2, v+3, \frac{(\mathfrak{y}_2 - \mathfrak{y}_1)}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{\mathfrak{y}_1}\right)}\right),$$

and $B_1 = \frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_1}$ *and* $B_2 = \frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \frac{1}{\mathfrak{y}_2}$, $\forall u, v \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$ *and* $\in \mathbb{N}$.

Proof. Using Lemma 2, the modulus property and harmonic convexity, we have

$$|\omega(\bar{\ltimes}_s, \bar{\ltimes}_e, v,)|$$
$$\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+1)^2}\left[\int_0^1 \left|\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1} + \frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\right|\left[|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)| + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)| - \frac{+1-\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|\right.\right.$$
$$\left. - \frac{\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|\right]\mathrm{d}\bar{\delta} + \int_0^1 \left|\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1} + \frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\right|$$
$$\left[|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)| + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)| - \frac{+1-\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)| - \frac{\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|\right]\mathrm{d}\bar{\delta}$$
$$\left. + (+1)\mu\int_0^1 [|\bar{\delta}^v - \bar{\delta}^{v+1}|]\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)\mathrm{d}\bar{\delta}\right].$$

After simple calculation, we obtain the required result. □

Theorem 11. *All the assumptions of Lemma 2 are satisfied, and let $|\tilde{\mathfrak{F}}|^q$ be an harmonic convex function, then we have*

$$|\omega(\bar{\ltimes}_s, \bar{\ltimes}_e, v,)| \leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(+1)^2\mathfrak{y}_1\mathfrak{y}_2}\left[C_4^{\frac{1}{p}}\left(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q - \frac{2+1}{2(+1)}|\tilde{\mathfrak{F}}(\mathfrak{y}_1)|^q - \frac{1}{2(+1)}|\tilde{\mathfrak{F}}(\mathfrak{y}_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + C_5^{\frac{1}{p}}\left(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q - \frac{1}{2(+1)}|\tilde{\mathfrak{F}}(\mathfrak{y}_1)|^q - \frac{2+1}{2(+1)}|\tilde{\mathfrak{F}}(\mathfrak{y}_2)|^q\right)^{\frac{1}{q}}\right]$$
$$+ \frac{\mu}{+1}\left(\frac{1}{vp+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+q}\right)^{\frac{1}{q}}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2,$$

where

$$C_4 = \frac{B_1^{-2p}}{(vp+1)}{}_2F_1\left(2, vp+1, vp+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{\mathfrak{y}_1}\right)}\right).$$
$$C_5 = \frac{B_2^{-2p}}{(vp+1)}{}_2F_1\left(2, vp+1, vp+2, \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(+1)\mathfrak{y}_1\mathfrak{y}_2\left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{\mathfrak{y}_2}\right)}\right).$$

Moreover, B_1 and B_2 are defined in an earlier Theorem, $\forall u, v \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$, $\in \mathbb{N}$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2, the modulus property, Hölder's inequality and harmonic convexity of $|\tilde{\mathfrak{F}}'|^q$, we have

$$
\begin{aligned}
&|\omega(\bar{\ltimes}_s, \bar{\ltimes}_e, v,)| \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+ 1)^2}\left[\left(\int_0^1 \left|\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1} + \frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\right|^q \mathrm{d}\bar{\delta}\right)^{\frac{1}{q}}\right. \\
&\quad \left(\int_0^1 \left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{+1-\bar{\delta}}{\mathfrak{y}_1(+1)} + \frac{\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\right|^q \mathrm{d}\bar{\delta}\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left|\frac{\bar{\delta}^v}{\left(\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{(+1)\mathfrak{y}_1} + \frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\right|^p \mathrm{d}\bar{\delta}\right)^{\frac{1}{p}} \\
&\quad \left.\left(\int_0^1 \left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s} + \frac{1}{\bar{\ltimes}_e} - \left(\frac{\bar{\delta}}{\mathfrak{y}_1(+1)} + \frac{+1-\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\right|^q \mathrm{d}\bar{\delta}\right)^{\frac{1}{q}}\right] \\
&\quad + \frac{\mu}{+1}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2 \left(\int_0^1 \bar{\delta}^{vp}\mathrm{d}\bar{\delta}\right)^{\frac{1}{p}} \left(\int_0^1 (1-\bar{\delta})^q \mathrm{d}\bar{\delta}\right)^{\frac{1}{q}} \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+ 1)^2}\left[C_4^{\frac{1}{p}}\left(\int_0^1 \left[|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q - \frac{+1-\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q - \frac{\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|^q\right]\mathrm{d}\bar{\delta}\right)^{\frac{1}{q}}\right. \\
&\quad \left. + C_5^{\frac{1}{p}}\left(\int_0^1 \left[|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q - \frac{\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q - \frac{+1-\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|^q\right]\mathrm{d}\bar{\delta}\right)^{\frac{1}{q}}\right] \\
&\quad + \frac{\mu}{+1}\left(\frac{1}{vp+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+q}\right)^{\frac{1}{q}}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2.
\end{aligned}
$$

After simple calculation, we obtain the required result. □

Theorem 12. *All the assumptions of Lemma 2 are satisfied, and let $|\tilde{\mathfrak{F}}|^q$ be an harmonic convex function, then we have*

$$
\begin{aligned}
&|\omega(\bar{\ltimes}_s, \bar{\ltimes}_e, v,)| \\
&\leq \frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(+ 1)^2\mathfrak{y}_1\mathfrak{y}_2}\left[C_6^{\frac{1}{p}}\left(\frac{1}{vq+1}(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q) - \frac{(vq+2)+1}{(+1)(vq+1)(vq+2)}|\tilde{\mathfrak{F}}(\mathfrak{y}_1)|^q - \right.\right. \\
&\quad \left.\frac{1}{(+1)(vq+2)}|\tilde{\mathfrak{F}}(\mathfrak{y}_2)|^q\right)^{\frac{1}{q}} + C_7^{\frac{1}{p}}\left(\frac{1}{vq+1}(|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q + |\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q)\right. \\
&\quad \left.\left. - \frac{(vq+2)+1}{(+1)(vq+1)(vq+2)}|\tilde{\mathfrak{F}}(\mathfrak{y}_1)|^q - \frac{1}{(+1)(vq+2)}|\tilde{\mathfrak{F}}(\mathfrak{y}_2)|^q\right)^{\frac{1}{q}}\right] \\
&\quad + \frac{\mu}{+1}\left(\frac{1}{vp+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+q}\right)^{\frac{1}{q}}\left(\frac{1}{\mathfrak{y}_1} - \frac{1}{\mathfrak{y}_2}\right)^2,
\end{aligned}
$$

where

$$C_6 = \frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{(\mathfrak{y}_2-\mathfrak{y}_1)(1-2p)}\left[\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_1}\right)^{1-2p}-\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1(\mathfrak{y}_1+ny)}{(+1)\mathfrak{y}_1\mathfrak{y}_2}\right)^{1-2p}\right].$$

$$C_7 = \frac{(+1)\mathfrak{y}_1\mathfrak{y}_2}{(\mathfrak{y}_2-\mathfrak{y}_1)(1-2p)}\left[\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(nx+\mathfrak{y}_2)}{(+1)\mathfrak{y}_1\mathfrak{y}_2}\right)^{1-2p}-\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{\mathfrak{y}_2}\right)^{1-2p}\right].$$

$\forall u,v \in [\bar{\ltimes}_s, \bar{\ltimes}_e]$, $\in \mathbb{N}$ *and* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. .

Proof. Using Lemma 2, the modulus property, Hölder's inequality and harmonic convexity of $|\tilde{\mathfrak{F}}'|^q$, we have

$$\begin{aligned}
&|\omega(\bar{\ltimes}_s,\bar{\ltimes}_e,v,)| \\
&\leq \frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+1)^2}\left[\left(\int_0^1\left|\frac{1}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\right|^p d\bar{\delta}\right)^{\frac{1}{p}}\right. \\
&\quad\left(\int_0^1 \bar{\delta}^{vq}\left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{+1-\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\right|^q d\bar{\delta}\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\left|\frac{1}{\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{(+1)\mathfrak{y}_1}+\frac{+1-\bar{\delta}}{(+1)\mathfrak{y}_2}\right)\right)^2}\right|^p d\bar{\delta}\right)^{\frac{1}{p}} \\
&\quad\left.\left(\int_0^1 \bar{\delta}^{vq}\left|\tilde{\mathfrak{F}}'\left(\frac{1}{\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\left(\frac{\bar{\delta}}{\mathfrak{y}_1(+1)}+\frac{+1-\bar{\delta}}{\mathfrak{y}_2(+1)}\right)}\right)\right|^q d\bar{\delta}\right)^{\frac{1}{q}}\right] \\
&\quad+\frac{\mu}{+1}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2\left(\int_0^1\bar{\delta}^{vp}d\bar{\delta}\right)^{\frac{1}{p}}\left(\int_0^1(1-\bar{\delta})^q d\bar{\delta}\right)^{\frac{1}{q}} \\
&\leq \frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_1\mathfrak{y}_2(+1)^2}\left[C_6^{\frac{1}{p}}\left(\int_0^1\bar{\delta}^{vq}\left[|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q+|\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q-\frac{+1-\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q-\frac{\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|^q\right]d\bar{\delta}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+C_7^{\frac{1}{p}}\left(\int_0^1\bar{\delta}^{vq}\left[|\tilde{\mathfrak{F}}'(\bar{\ltimes}_s)|^q+|\tilde{\mathfrak{F}}'(\bar{\ltimes}_e)|^q-\frac{\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_1)|^q-\frac{+1-\bar{\delta}}{+1}|\tilde{\mathfrak{F}}'(\mathfrak{y}_2)|^q\right]d\bar{\delta}\right)^{\frac{1}{q}}\right] \\
&\quad+\frac{\mu}{+1}\left(\frac{1}{vp+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+q}\right)^{\frac{1}{q}}\left(\frac{1}{\mathfrak{y}_1}-\frac{1}{\mathfrak{y}_2}\right)^2.
\end{aligned}$$

After simple calculation, we obtain the required result. □

5. Applications

Finally, we give some applications to special means between two positive real numbers. Now, we rewrite some well-known means.

1. The arithmetic mean:

$$A(\bar{\ltimes}_s,\bar{\ltimes}_e)=\frac{\bar{\ltimes}_s+\bar{\ltimes}_e}{2},$$

2. The generalized log-mean:

$$L_r(\bar{\ltimes}_s,\bar{\ltimes}_e)=\left[\frac{\bar{\ltimes}_e^{\,r+1}-\bar{\ltimes}_s^{\,r+1}}{(r+1)(\bar{\ltimes}_e-\bar{\ltimes}_s)}\right]^{\frac{1}{r}};\ \ r\in\Re\setminus\{-1,0\}.$$

Proposition 1. *Under the assumptions of Theorem 8, we have*

$$
\begin{aligned}
(I)\quad & \Big| A\Big((\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_1{}^{-1})^{-1}, (\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_2{}^{-1})^{-1}\Big) - \\
& L^{-1}\Big((\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_1{}^{-1}), (\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_2{}^{-1})\Big)\Big| \\
& \leq \frac{(\mathfrak{y}_2 - \mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2}[2{}_1T - ({}_2T +{}_3 T)],
\end{aligned}
$$

$$
\begin{aligned}
(II)\quad & \Big| A\Big((\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_1{}^{-1})^{-p-2}, (\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_2{}^{-1})^{-p-2}\Big) - \\
& \frac{1}{B_1B_2} L_r^r\Big((\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_1{}^{-1})^{-1}, (\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_2{}^{-1})^{-1}\Big)\Big| \\
& \leq \frac{(\mathfrak{y}_2 - \mathfrak{y}_1)(r+2)}{\mathfrak{y}_1\mathfrak{y}_2}\Big[{}_1T(\bar{\ltimes}_s^{r+1} + \bar{\ltimes}_e^{r+1}) - ({}_2T\mathfrak{y}_1{}^{r+1} +{}_3 T\mathfrak{y}_2{}^{r+1})\Big],
\end{aligned}
$$

$$
\begin{aligned}
(III)\quad & \Big| \frac{u+2s}{u} A\Big((\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_1{}^{-1})^{-\frac{s}{u}-2}, (\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_2{}^{-1})^{-\frac{s}{u}-2}\Big) - \\
& \frac{s+u}{uB_1B_2} L_{\frac{s}{u}}^{\frac{s}{u}}\Big((\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_1{}^{-1})^{-1}, (\bar{\ltimes}_s^{-1} + \bar{\ltimes}_e^{-1} - \mathfrak{y}_2{}^{-1})^{-1}\Big)\Big| \\
& \leq \frac{(\mathfrak{y}_2 - \mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2}\Big[{}_1T(\bar{\ltimes}_s^{\frac{s}{u}+1} + \bar{\ltimes}_e^{\frac{s}{u}+1}) - ({}_2T\mathfrak{y}_1{}^{\frac{s}{u}+1} +{}_3 T\mathfrak{y}_2{}^{\frac{s}{u}+1})\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
{}_1T = {} & \frac{B_2^{-2}}{2}{}_2F_1\left(2,2,3,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{1}{\mathfrak{y}_2}\Big)}\right) + \frac{B_2^{-2}}{2}{}_2F_1\left(2,1,3,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{1}{\mathfrak{y}_2}\Big)}\right) \\
& + \frac{M^{-2}}{2}{}_2F_1\left(2,2,3,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2}\Big)}\right). \\
{}_2T = {} & \frac{B_2^{-2}}{3}{}_2F_1\left(2,2,4,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{1}{\mathfrak{y}_2}\Big)}\right) + \frac{B_2^{-2}}{6}{}_2F_1\left(2,2,4,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{1}{\mathfrak{y}_2}\Big)}\right) \\
& + \frac{M^{-2}}{6}{}_2F_1\left(2,2,4,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2M}\Big)}\right). \\
{}_3T = {} & \frac{B_2^{-2}}{6}{}_2F_1\left(2,2,4,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{1}{\mathfrak{y}_2}\Big)}\right) + \frac{B_2^{-2}}{3}{}_2F_1\left(2,1,4,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{1}{\mathfrak{y}_2}\Big)}\right) \\
& + \frac{M^{-2}}{6}{}_2F_1\left(2,2,4,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2M}\Big)}\right) + \frac{M^{-2}}{2}{}_2F_1\left(2,2,3,\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2\Big(\frac{1}{\ltimes_s}+\frac{1}{\ltimes_e}-\frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2M}\Big)}\right),
\end{aligned}
$$

where B_1, B_2 *are defined earlier and* $M = \frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{(\mathfrak{y}_1+\mathfrak{y}_2)}{2\mathfrak{y}_1\mathfrak{y}_2}$.

Proof. By choosing $\tilde{\mathfrak{F}}(z) = z, \tilde{\mathfrak{F}}(z) = z^{r+2}$ and $\tilde{\mathfrak{F}}(z) = \frac{u}{s+2u}z^{\frac{s}{u}+1}$ in Theorem 8 with $v = 1$ and $\mu = 0$, respectively, we obtain the required result. □

Proposition 2. *Under the assumptions of Theorem 9, we have*

$$(I)\quad \Big|A\Big((\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_1{}^{-1})^{-1},(\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_2{}^{-1})^{-1}\Big)- L^{-1}\Big((\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_1{}^{-1}),(\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_2{}^{-1})\Big)\Big| \leq \frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{\mathfrak{y}_1\mathfrak{y}_2}T_4^{\frac{1}{p}}\left[\left(\frac{2}{q+1}-\frac{1}{(q+2)(q+1)}-\frac{1}{q+2}\right)^{\frac{1}{q}}\right].$$

$$(II)\quad \Big|A\Big((\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_1{}^{-1})^{-p-2},(\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_2{}^{-1})^{-p-2}\Big)- \frac{1}{B_1B_2}L_r^r\Big((\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_1{}^{-1})^{-1},(\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_2{}^{-1})^{-1}\Big)\Big| \leq \frac{(\mathfrak{y}_2-\mathfrak{y}_1)(r+2)}{2\mathfrak{y}_1\mathfrak{y}_2}T_4^{\frac{1}{p}}\Bigg[\bigg(\frac{1}{q+1}(\bar{\ltimes}_s^{rq+q}+\bar{\ltimes}_e^{rq+q})-\frac{1}{(q+2)(q+1)}\mathfrak{y}_1{}^{rq+q} -\frac{1}{q+2}\mathfrak{y}_2{}^{rq+q}\bigg)^{\frac{1}{q}}+\bigg(\frac{1}{q+1}(\bar{\ltimes}_s^{rq+q}+\bar{\ltimes}_e^{rq+q})-\frac{1}{(q+2)(q+1)}\mathfrak{y}_2{}^{rq+q}-\frac{1}{q+2}\mathfrak{y}_1{}^{qr+q}\bigg)^{\frac{1}{q}}\Bigg].$$

$$(III)\quad \Big|\frac{u+2s}{u}A\Big((\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_1{}^{-1})^{-\frac{s}{u}-2},(\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_2{}^{-1})^{-\frac{s}{u}-2}\Big)- \frac{s+u}{uB_1B_2}L_{\frac{s}{u}}^{\frac{s}{u}}\Big((\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}-\mathfrak{y}_1{}^{-1})^{-1},(\bar{\ltimes}_s^{-1}+\bar{\ltimes}_e^{-1}+1\mathfrak{y}_2{}^{-1})^{-1}\Big)\Big| \leq \frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{2\mathfrak{y}_1\mathfrak{y}_2}T_4^{\frac{1}{p}}\Bigg[\bigg(\frac{1}{q+1}(\bar{\ltimes}_s^{\frac{qs}{u}+q}+\bar{\ltimes}_e^{\frac{qs}{u}+q})+\frac{1}{(q+2)(q+1)}\mathfrak{y}_1{}^{\frac{qs}{u}+q}+\frac{1}{q+2}\mathfrak{y}_2{}^{\frac{qs}{u}+q}\bigg)^{\frac{1}{q}} +\bigg(\frac{1}{q+1}(\bar{\ltimes}_s^{\frac{qs}{u}+q}+\bar{\ltimes}_e^{\frac{qs}{u}+q})+\frac{1}{(q+2)(q+1)}\mathfrak{y}_2{}^{\frac{qs}{u}+q}+\frac{1}{vq+2}\mathfrak{y}_1{}^{\frac{qs}{u}+q}\bigg)^{\frac{1}{q}}\Bigg],$$

where T_4 is already defined.

Proof. By choosing $\tilde{\mathfrak{F}}(z)=z,\tilde{\mathfrak{F}}(z)=z^{r+2}$ and $\tilde{\mathfrak{F}}(z)=\frac{u}{s+2u}z^{\frac{s}{u}+1}$ in Theorem 9 with $v=1$ and $\mu=0$, respectively, we obtain our required result. □

Remark 4. *Note that similar interesting inequalities can be established by other Theorems; we omit the proof of the remaining result.*

Error Bounds

At the end, we present some bounds for quadrature schemes of the above integral inequalities. Let $\Delta=\{\bar{\ltimes}_s=u_0<u_1<u_2<\cdots<u_{-1}<u=\bar{\ltimes}_e\}$ be the partition of $[\bar{\ltimes}_s,\bar{\ltimes}_e]\subset(0,\infty)$. We denote

$$\chi_1=\sum_{i=0}\frac{\left(\tilde{\mathfrak{F}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{u_i}\right)^{-1}+\tilde{\mathfrak{F}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{1}{u_{i+1}}\right)^{-1}\right)h_i}{2u_iu_{i+1}}$$

$$\chi_2=\sum_{i=0}\frac{\left(\tilde{\mathfrak{F}}\left(\frac{1}{\bar{\ltimes}_s}+\frac{1}{\bar{\ltimes}_e}-\frac{(u_{i+1}-u_i)}{2u_iu_{i+1}}\right)^{-1}\right)h_i}{u_iu_{i+1}}.$$

$$\int_{B_2}^{B_1}\tilde{\mathfrak{F}}(\frac{1}{u})\mathrm{d}u=\chi_1(\Delta,\tilde{\mathfrak{F}})+R_1(\Delta,\tilde{\mathfrak{F}}),\quad \int_{A_2}^{A_1}\tilde{\mathfrak{F}}(\frac{1}{u})\mathrm{d}u=\chi_2(\Delta,\tilde{\mathfrak{F}})+R_2(\Delta,\tilde{\mathfrak{F}})$$

where $R_1(\Delta,\tilde{\mathfrak{F}})$ and $R_2(\Delta,\tilde{\mathfrak{F}})$ are the remainder terms, and $h_i=u_{i+1}-u_i$, $i=0,1,2,\ldots-1$ and B_1 and B_2 are already defined in previous results.

Proposition 3. *Under the assumptions of Theorem 9, we have*

$$
\begin{aligned}
&|R_1(\Delta, \tilde{\mathfrak{F}})| \\
&\leq \sum_{i=0}^{-1} \frac{(u_{i+1}-u_i)^2}{2(u_{i+1}u_i)^2} T_{4,*}^{\frac{1}{p}} \left[\left(\frac{1}{q+1}(|\tilde{\mathfrak{F}}'(u_0)|^q + |\tilde{\mathfrak{F}}'(u)|^q) - \frac{1}{(q+2)(q+1)}|\tilde{\mathfrak{F}}'(u_i)|^q \right.\right. \\
&\left. - \frac{1}{q+2}|\tilde{\mathfrak{F}}'(u_{i+1})|^q \right)^{\frac{1}{q}} + \left(\frac{1}{q+1}(|\tilde{\mathfrak{F}}'(u_0)|^q + |\tilde{\mathfrak{F}}'(u)|^q) - \frac{1}{(q+2)(q+1)}|\tilde{\mathfrak{F}}'(u_{i+1})|^q \right. \\
&\left.\left. - \frac{1}{q+2}|\tilde{\mathfrak{F}}'(u_i)|^q \right)^{\frac{1}{q}} \right],
\end{aligned}
$$

where

$$
T_{4,*} = \frac{\mathfrak{y}_1\mathfrak{y}_2\left[\left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{u_i} \right)^{1-2p} - \left(\frac{1}{\ltimes_s} + \frac{1}{\ltimes_e} - \frac{1}{u_{i+1}} \right)^{1-2p} \right]}{(u_{i+1}-u_i)(1-2p)}.
$$

Proof. By applying Theorem 9 with $v = 1$ on subinterval $[u_i, u_{i+1}]$ $(i = 0, 1, 2, 3, \ldots, -1)$ of the partition Δ and summing over i from 0 to -1, we obtain our required result. □

Proposition 4. *All the assumptions of Theorem 12 are satisfied, then we have*

$$
\begin{aligned}
&|R_2(\Delta, \tilde{\mathfrak{F}})| \\
&\leq \sum_{i=0}^{-1} \frac{1(u_{i+1}-u_i)^2}{2(pu_{i+1}u_i)^2} \left[C_{6,*}^{\frac{1}{p}} \left(\frac{1}{q+1}(|\tilde{\mathfrak{F}}'(u_0)|^q + |\tilde{\mathfrak{F}}'(u)|^q) - \frac{q+3}{2(q+1)(q+2)}|\tilde{\mathfrak{F}}(u_i)|^q \right.\right. \\
&\left. - \frac{1}{2(q+2)}|\tilde{\mathfrak{F}}(u_{i+1})|^q \right)^{\frac{1}{q}} + C_{7,*}^{\frac{1}{p}} \left(\frac{1}{q+1}(|\tilde{\mathfrak{F}}'(u_0)|^q + |\tilde{\mathfrak{F}}'(u)|^q) \right. \\
&\left.\left. - \frac{q+3}{2(q+1)(q+2)}|\tilde{\mathfrak{F}}(u_{i+1})|^q - \frac{1}{2(q+2)}|\tilde{\mathfrak{F}}(u_i)|^q \right)^{\frac{1}{q}} \right],
\end{aligned}
$$

where

$$
C_{6,*} = \frac{2u_{i+1}u_i}{(u_{i+1}-u_i)(1-2p)} \left[\left(\frac{1}{u_0} + \frac{1}{u} - \frac{1}{u_i} \right)^{1-2p} - \left(\frac{1}{u_0} + \frac{1}{u} - \frac{(u_i+u_{i+1})}{2u_{i+1}u_i} \right)^{1-2p} \right].
$$

$$
C_{7,*} = \frac{2u_iu_{i+1}}{(u_{i+1}-u_i)(1-2p)} \left[\left(\frac{1}{u_0} + \frac{1}{u} - \frac{(u_i+u_{i+1})}{2u_iu_{i+1}} \right)^{1-2p} - \left(\frac{1}{u_0} + \frac{1}{u} - \frac{1}{u_{i+1}} \right)^{1-2p} \right].
$$

Proof. By applying Theorem 12 with $v = 1 =$ on subinterval $[u_i, u_{i+1}]$ $(i = 0, 1, 2, 3, \ldots, -1)$ of the partition Δ and summing over i from 0 to -1, we obtain our required result. □

6. Conclusions

In this study, we have obtained some new variants of Hermite–Hadamard-type inequalities through the Jensen–Mercer inequality for strongly harmonic convex functions in the fractional domain. We have used the Atanagna and Baleanu fractional integral in the Riemann sense, which is developed to overcome the deficiency of Caputo–Fabrizio fractional integrals and derivatives; we have developed the H.H.J.M-type inequalities for strongly convex functions by utilizing the fractional identities. Later on, we discussed some applications as well. Hopefully, the technique of the paper and the idea used in the article will inspire interested readers and opens= a new avenue for further research in the field. In the future, we will consider some quantum analogs of the H. H. J. M-like inequalities involving strong convexity and further extend the work for other classes of convexity.

Author Contributions: Conceptualization, M.Z.J., M.U.A. and M.A.N.; methodology, M.U.A.; software, B.B-.M. and M.Z.J.; validation, M.U.A., M.V.M. and H.B.; formal analysis, M.Z.J., M.U.A., M.V.M., H.B. and A.G.K.; investigation, M.Z.J., M.U.A., M.V.M. and H.B.; writing—original draft preparation, B.B.-M., M.Z.J., M.U.A., M.V.M., H.B., A.G.K. and M.A.N.; writing—review and editing, M.Z.J. and M.U.A. supervision, M.U.A. All authors have read and agreed to the published version of the manuscript.

Funding: Bandar Bin-Mohsin is supported by Researchers Supporting Project number (RSP-2021/158), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Acknowledgments: The authors are thankful to the editor and the anonymous reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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