Article

Noncommutative Bispectral Algebras and Their Presentations

Brian D. Vasquez Campos 1,2 and Jorge P. Zubelli 1,2,*

1 IMPA, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro 22460-320, RJ, Brazil
2 Mathematics Department, Khalifa University, Abu Dhabi 127788, United Arab Emirates
* Correspondence: zubelli@gmail.com

Abstract: We prove a general result on presentations of finitely generated algebras and apply it to obtain nice presentations for some noncommutative algebras arising in the matrix bispectral problem. By “nice presentation”, we mean a presentation that has as few as possible defining relations. This, in turn, has potential applications in computer algebra implementations and examples. Our results can be divided into three parts. In the first two, we consider bispectral algebras with the eigenvalue in the physical equation to be scalar-valued for 2×2 and 3×3 matrix-valued eigenfunctions. In the third part, we assume the eigenvalue in the physical equation to be matrix-valued and draw an important connection with Spin Calogero–Moser systems. In all cases, we show that these algebras are finitely presented. As a byproduct, we answer positively a conjecture of F. A. Grünbaum about these algebras.

Keywords: bispectral problem; Calogero–Moser systems; presentations of finitely generated algebras; completely integrable systems

1. Introduction

In this work, we characterize the symmetry structure of a noncommutative version of the bispectral problem [1]. The latter refers to families of eigenfunctions ψ(x, z) of an operator \( L = L(x, \partial_x) \), with \( z \)-dependent eigenvalue parameter, that are also eigenfunctions for some nontrivial operator \( B = B(z, \partial_z) \) with an \( x \)-dependent eigenvalue. We shall refer to the differential equations involving the operator \( L = L(x, \partial_x) \) as the physical equations.

In the commutative (or scalar) case, the bispectral problem already displays unexpected connections to different areas [1–3]. One of the most important connections is that a remarkable set of bispectral Schrödinger operators \( L = -\partial^2_x + U(x) \) are obtained when \( U(x) \) is a rational solution of the KdV equation [4]. The abundance of connections is even more pronounced in the matrix case. See [2,5–11] and references therein. In the theory of infinite dimensional systems and solitons the study of the symmetries led to a deeper understanding of the structure of these equations. See for example [12,13].

Characterizing the algebraic structure of the solutions to a problem through presentations is a major task in many areas. In our context, this consists in looking for a set of generators in such a way that the relations among them are as simple as possible [14–16].

We address this problem for some algebras associated to the noncommutative bispectral problem, which in turn is connected to the Spin Calogero–Moser system [17,18].

Let \( K \) be a field. A presentation of a \( K \)-algebra \( A \) comprises a set \( S \) of generators so that every element of the algebra can be written as a polynomial in these generators and a set \( I \) of relations among those generators. We then say \( A \) has a presentation \( K \cdot \langle S \mid I \rangle \). Interesting conjectures concerning presentations of some noncommutative algebras were proposed in connection with the interplay of matrix-valued orthogonal polynomials [18,19] and the bispectral problem [20]. Only one of the conjectures proposed in [19] was solved in [21]. In [18] the algebras involved are bispectral algebras while in [19] the algebras involved are algebras of differential operators associated to matrix-valued orthogonal polynomials. This
article solves the conjectures concerning noncommutative bispectral algebras presented in [18].

In the present incarnation of the bispectral problem, we consider the triples \((L, \psi, B)\) satisfying systems of equations

\[
\begin{align*}
L\psi(x, z) &= \psi(x, z)F(z) \\
(\psi B)(x, z) &= \theta(x)\psi(x, z)
\end{align*}
\]

(1)

with \(L = L(x, \partial_x), B = B(z, \partial_z)\) linear matrix differential operators, i.e., \(L\psi = \sum_{l=0}^{L} a_l(x) \cdot \partial_x^l \psi, \psi B = \sum_{l=0}^{m} \partial_z^l \psi \cdot b_l(z)\). The functions \(a_l, b_l, F, \theta\) and the nontrivial common eigenfunction \(\psi\) are in principle compatible sized matrix-valued functions. A triple \((L, \psi, B)\) satisfying (1) is called a bispectral triple.

The main goal of this article is to give a presentation of each (bispectral) algebra using its generators and some relations among them. Thus, describing the ideal of relations, we give three examples of bispectral algebras to illustrate a general theorem of presentations of finitely generated algebras. For a given eigenvalue function the corresponding algebra of matrix eigenvalues is characterized. In the former two cases, the eigenvalue \(F(z)\) is scalar valued and in the last case the eigenvalue \(\theta(x)\) is matrix valued. These results give positive answers to the three conjectures in [18]. We use the software Singular and Maxima to obtain a set of generators and nice relations among them and after that, we prove that in fact, this set of nice relations are enough to give presentations for these algebras.

To obtain the algebras involved in the mentioned conjectures arising from the bispectral context, we consider a normalized operator \(L = L(x, \partial_x), L = \sum_{l=0}^{L} a_l(x)\partial_x^l\) with \(a_1\) constant and scalar, \(a_{l-1} = 0\). We are interested in the bispectral pairs associated to \(L = L(x, \partial_x), i.e.,\), the algebra

\[
\mathfrak{A}(\psi) = \{\theta \in M_N(\mathbb{C})[x] \mid \exists B = B(z, \partial_z), (\psi B)(x, z) = \theta(x)\psi(x, z)\}.
\]

(2)

Notice that we fixed \(L\) normalized and consider the algebra of bispectral pairs to \(L\). However, this algebra depends on \(L\).

Since the operators \(L\) and \(B\) are acting on opposite directions, we have a generalized version of the ad-conditions of Duistermaat and Grunbaum [1]. See [22] for the proof in this context. As a consequence, we can consider the algebra \(\mathfrak{A}(\psi)\) as a subalgebra of the matrix polynomial algebra \(M_N(\mathbb{C})[x]\).

We shall now make precise the three conjectures from [18]:

1. Consider the matrix-valued function

\[
\psi_1(x, z) = e^{xz} \begin{pmatrix} z - x^{-1} & x^{-2} \\ 0 & z - x^{-1} \end{pmatrix}
\]

and observe that \(L_1 \psi_1 = -z^2 \psi_1\) for the operator

\[
L_1 = -\partial_x^2 + 2 \begin{pmatrix} x^{-2} & -2x^{-3} \\ 0 & x^{-2} \end{pmatrix}.
\]

Conjecture 1. The algebra of all matrix-valued polynomials \(\theta(x)\) for which there exists some operator \(B\) such that

\[(\psi_1 B)(x, z) = \theta(x)\psi_1(x, z)\]

is the algebra of all polynomials of the form

\[
\begin{pmatrix} r_{11}^{11} & r_{12}^{11} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} r_{11}^{11} & r_{12}^{11} \\ 0 & r_{11}^{11} \end{pmatrix} x + \begin{pmatrix} r_{11}^{11} & r_{12}^{11} & r_{12}^{12} \\ r_{11}^{11} & r_{12}^{11} & r_{12}^{12} \end{pmatrix} x^2 + \begin{pmatrix} r_{22}^{11} & r_{11}^{11} & r_{12}^{12} \\ r_{22}^{11} & r_{11}^{11} & r_{12}^{12} \end{pmatrix} x^3 + x^4 p(x),
\]
where \( p \in M_2(\mathbb{C})[x] \) and all the variables \( r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{12}, r_3^{11}, r_3^{12}, r_3^{22} \in \mathbb{C} \) are arbitrary. Furthermore, look for a nice presentation in terms of generators and relations.

2. Consider the matrix-valued function

\[
\psi_2(x,z) = e^{xz} \begin{pmatrix} z - x^{-1} & x^{-2} & -x^{-3} \\ 0 & z - x^{-1} & x^{-2} \\ 0 & 0 & z - x^{-1} \end{pmatrix}
\]

and observe that \( L_2 \psi_2 = -z^2 \psi_2 \) for the operator

\[
L_2 = -\partial_x^2 + 2 \begin{pmatrix} x^{-2} & -2x^{-3} & 3x^{-4} \\ 0 & x^{-2} & -2x^{-3} \\ 0 & 0 & x^{-2} \end{pmatrix}.
\]

**Conjecture 2.** The algebra of all matrix-valued polynomials \( \theta(x) \) for which there exists some operator \( B \) such that

\[
(\psi_2 B)(x,z) = \theta(x) \psi_2(x,z)
\]

is the algebra of all polynomials of the form

\[
\begin{aligned}
&\begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ 0 & r_0^{11} & r_0^{12} \\ 0 & 0 & r_0^{11} \end{pmatrix} x^0 + \begin{pmatrix} r_0^{11} & r_1^{12} & r_1^{13} \\ r_0^{11} & r_0^{12} & r_1^{13} \\ 0 & r_1^{11} & r_1^{13} \end{pmatrix} x^1 \\
&\quad+ \begin{pmatrix} r_0^{11} & r_0^{12} & r_2^{12} \\ r_0^{11} & r_0^{12} & r_0^{13} \\ r_2^{11} & r_1^{12} & r_1^{13} \end{pmatrix} x^2 + \begin{pmatrix} r_0^{11} & r_3^{11} & r_2^{12} \\ r_0^{11} & r_0^{12} & r_2^{13} \\ r_3^{11} & r_1^{12} & r_1^{13} \end{pmatrix} x^3 \\
&\quad+ \begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ r_0^{11} & r_0^{12} & r_3^{13} \\ r_0^{11} & r_0^{12} & r_0^{13} \end{pmatrix} x^4 + \begin{pmatrix} r_0^{11} & r_0^{12} & r_0^{13} \\ r_0^{11} & r_0^{12} & r_3^{12} \\ r_3^{11} & r_0^{12} & r_0^{13} \end{pmatrix} x^5 + x^6 p(x),
\end{aligned}
\]

where \( p \in M_3(\mathbb{C})[x] \) and all the variables \( r_0^{11}, r_0^{12}, \ldots, r_3^{33} \in \mathbb{C} \) are arbitrary. Furthermore, look for a nice presentation in terms of generators and relations.

3. Consider the matrix-valued function

\[
\psi_3(x,z) = \frac{e^{xz}}{(x-2)x^z} \begin{pmatrix} x^2z^2 - 2x^2z^2 - 2x^2z^2 + 3xz^2 + 2x^2 - 2 \\ x^2z^2 - 2x^2z^2 - x^2z^2 + 2x^2 - 2 \\ x^2z^2 - 2x^2z^2 + 2x^2 - 2 \end{pmatrix},
\]

it is easy to check that \( \psi_3 B_3 = \theta \psi_3 \) for

\[
B_3 = \partial_z^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \partial_z \begin{pmatrix} 0 & 0 & 0 \\ -2z + 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \partial_z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6z - 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}
\]

and

\[
\theta(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.
\]

**Conjecture 3.** The algebra of all matrix-valued polynomials \( F(z) \) for which there exists some operator \( L \) such that

\[
(L \psi_3)(x,z) = \psi_3(x,z) F(z)
\]
Finally, \( \alpha \)

**Definition 3.** Let \( C \) be a noncommutative ring and \( A \) a subring of \( C \). We say that an element \( \alpha \) is integral over \( A \) if there exists a presentation with \( \Lambda \) finite and finitely presented such that \( f(\alpha) = 0 \). Furthermore, we say that \( \beta \in C \) is integral over \( \alpha \) if \( \beta \) is integral over \( A \cdot \langle \alpha \rangle \).

**Remark 1.** Note that Definition 3 is consistent with Definition 1 since in this case \( A = \mathbb{K}, \) \( S = \{x_\lambda | \lambda \in \Lambda \} \) and \( \mathbb{K} \) is a field. In particular, the elements of \( A \) commute with the elements of \( S \) and we have Equation (3).

**Definition 4.** Let \( A \) be a \( \mathbb{K} \)-algebra. A presentation for an algebra \( A \) is a triple \( (\mathbb{K} \langle \langle x_\lambda \rangle_{\lambda \in \Lambda} \rangle, f, I) \) such that \( I \subset A \) is an ideal and \( f: \mathbb{K} \langle \langle x_\lambda \rangle_{\lambda \in \Lambda} \rangle / I \to A \) is an isomorphism. Furthermore, we say that \( A \) is finitely generated if there exists a presentation with \( \Lambda \) finite and finitely presented if there exists a presentation with \( \Lambda \) finite and the ideal \( I \) is generated by finitely many elements.

**Remark 2.** If an algebra \( A \) has presentation \( (\mathbb{K} \langle \langle x_\lambda \rangle_{\lambda \in \Lambda} \rangle, f, I) \) we can identify \( A \) with the algebra generated by the variables \( x_\lambda, \lambda \in \Lambda \) satisfying the relations given in the ideal \( I \). We will denote this algebra by

\[
\mathbb{K} \langle \langle x_\lambda \rangle_{\lambda \in \Lambda} \mid I = 0 \rangle := \mathbb{K} \langle \langle x_\lambda \rangle_{\lambda \in \Lambda} \mid P(\langle x_\lambda \rangle_{\lambda \in \Lambda}) = 0, \forall P \in I \rangle.
\]

In [22] we studied the bispectral algebra \( \mathbb{K} \langle \psi_k \rangle \) associated the operator \( L_k \) and eigenfunction \( \psi_k \), for every \( k \in \{1,2,3\} \). We gave an explicit expression for the operator \( B = B(z, \partial_z) \) associated to the matrix eigenvalue \( \theta \). However, we did not give a characterization in terms of generators and relations for \( \mathbb{K} \langle \psi_k \rangle \), which is the last part of the Conjectures 1–3. The main goal of the present work is to prove characterization in terms of generators and relations for \( \mathbb{K} \langle \psi_k \rangle \), for every \( k \in \{1,2,3\} \) and conclude positive answers to the Conjectures 1–3.
Theorems 2–4 complete the positive answers to the Conjectures 1–3 of [18] about three bispectral full rank 1 algebras. Moreover, these algebras are Noetherian and finitely generated because they are contained in the $N \times N$ matrix polynomial ring $M_N(K[x])$. The characterization in terms of generators and relations is an important tool to understand the algebraic structure of the bispectral pairs of a given normalized operator. If the eigenfunction satisfies the condition $\psi B = 0$ implies $B = 0$ for every linear differential operator $B = B(z, \partial_z)$ and $\theta \psi = 0$ implies $\theta = 0$ for every matrix-valued polynomial then the algebra of bispectral pairs of the operator $L$ and $\mathfrak{a}(\psi)$ are isomorphic. For more details see [22].

The plan of this article is as follows: In Section 2, we consider noncommutative finitely generated algebras which are countably generated as the left module over a subalgebra and prove Theorem 1 about their presentations. In Section 3, we give a positive answer to the first conjecture in [18] by applying Theorem 1 to obtain the presentation for an algebra with an integral element over a nilpotent one. In Section 4, we give positive answer to the second conjecture in [18], by applying Theorem 1 to obtain the presentation for an algebra with nilpotent and idempotent associated elements. Finally, in Section 5, we give positive answer to the third conjecture in [18] by applying Theorem 1 to obtain the presentation for an algebra with two integral elements over one nilpotent and one idempotent.

2. Presentations for Finitely Generated Algebras

In this section, we face the presentation problem and obtain a method to tackle it. This method was motivated by a result used in the work presented in [23].

**Theorem 1** (Presentation of finitely generated algebras). Let $A$ be a finitely generated $\mathbb{K}$-algebra by $\beta_1, \beta_2, \ldots, \beta_n$ such that:

- There exist an ideal $I$ of $\mathbb{K} \cdot \langle a_1, a_2, \ldots, a_n \rangle$ and an epimorphism of algebras
  $$f : \mathbb{K} \cdot \langle a_1, a_2, \ldots, a_n \rangle / I \to A,$$
  $$f(x) = \beta_i,$$

- There exists a subalgebra $\mathbb{K} \subset R \subset \mathbb{K} \cdot \langle a_1, a_2, \ldots, a_n \rangle / I$ such that $\mathbb{K} \cdot \langle a_1, a_2, \ldots, a_n \rangle / I$
  is a free left $R$-module generated by $\{x_i\}_{j=0}^\infty$, i.e.,
  $$\mathbb{K} \cdot \langle a_1, a_2, \ldots, a_n \rangle / I = \bigoplus_{j=0}^\infty Rx_j.$$

- $f |_R : R \to A$ is a monomorphism.
- The set $\{f(x_j)\}_{j=0}^\infty$ is a basis for $A$ as a left $f(R)$-module.

Then, $f$ is an isomorphism.

**Proof.** It is enough to prove that $f$ is injective. Pick $x \in \ker(f)$ and write $x = \sum_{j=0}^m r_j x_j$, then $0 = f(x) = \sum_{j=0}^m f(r_j)f(x_j)$. However, since $\{f(x_j)\}_{j=0}^\infty$ is a basis for $A$ as a left $f(R)$-module we have $f(r_j) = 0$, for $0 \leq j \leq m$. Here we use that $f |_R : R \to A$ is a monomorphism to conclude $r_j = 0$, for $0 \leq j \leq m$ and $x = 0$. \(\square\)

**Remark 3.** The theorem guarantees a presentation of $A$ in terms of generators and relations through the isomorphism $f$, i.e.,
\[A = \mathbb{K} \cdot \langle \beta_1, \beta_2, \ldots, \beta_n | P(\beta_1, \beta_2, \ldots, \beta_n) = 0, \forall P \in I \rangle.\]

This theorem is a method to find out presentations for finitely generated algebras. Nevertheless, we need to choose generators for the algebra and look for relations among them. Furthermore, we must seek for an intermediate $\mathbb{K}$-algebra $\mathbb{K} \subset S \subset A$ and a linearly independent set $\{y_j\}_{j=0}^\infty$ of $A$ such that $A = \bigoplus_{j=0}^\infty Sy_j$. 


In the following sections, we shall apply this method to obtain presentations for some noncommutative bispectral algebras.

3. An Algebra with an Integral Element over a Nilpotent One

In this section, we consider an algebra generated by two elements, one of them nilpotent. The statement of the theorem is as follows.

**Theorem 2.** Let \( A(\psi) \) be the sub-algebra of \( M_2(\mathbb{C})[x] \) of the form

\[
\left(\begin{array}{cc}
r_{11}^2 & r_{12}^2 \\
r_0^1 & r_1^1
\end{array}\right) + \left(\begin{array}{cc}
r_{11}^2 & r_{12}^2 \\
r_0^1 & r_1^1
\end{array}\right)x + \left(\begin{array}{cc}
r_{11}^2 & r_{12}^2 \\
r_0^1 & r_1^1
\end{array}\right)x^2 + \left(\begin{array}{cc}
r_{11}^2 & r_{12}^2 \\
r_0^1 & r_1^1
\end{array}\right)x^3 + x^4p(x),
\]

where \( p \in M_2(\mathbb{C})[x] \) and all the variables \( r_{11}^1, r_{12}^2, r_1^1, r_{12}^1, r_1^2, r_2^1, r_3^1, r_2^2, r_3^2, r_3^3 \in \mathbb{C} \). Then, we have the presentation \( A(\psi) = \mathbb{C} \cdot \langle a_0, a_1 \mid I(\psi_1) = 0 \rangle \) with the ideal \( I \) given by

\[
I(\psi_1) := \langle a_0^2 + a_0a_1a_0 - 3a_1a_0a_1 + a_0a_1^2 + a_1^2a_0 \rangle.
\]

**Proof.** The idea of the proof is to consider a basis for the vector space \( A(\psi) \cap \oplus_{j=0}^{j=3} M_2(\mathbb{C}[x]) \) of polynomials in \( A(\psi_1) \) of degree less or equal to 3 and observe that this basis generates the algebra \( A(\psi_1) \). After that, we look for remarkable elements on the basis that generate the others and obtain some set of relations. Finally, we verify the hypothesis of Theorem 1 to obtain the proof of the assertion.

Note that \( A(\psi) \) is generated by \( \beta_0 = e_{12}, \beta_1 = 1x + e_{21}x^2, \beta_2 = e_{12}x + e_{11}x^2, \beta_3 = e_{12}x + e_{22}x^2, \beta_4 = e_{12}x^2, \beta_5 = e_{12}x - e_{21}x^3, \beta_6 = e_{11}x^3, \beta_7 = e_{12}x^3, \beta_8 = e_{22}x^3 \).

Moreover, we can eliminate the variables \( \beta_i \) for \( 2 \leq j \leq 8 \). In fact, \( \beta_2 = \beta_0\beta_3, \beta_3 = \beta_1\beta_0, \beta_4 = \beta_0\beta_1\beta_0, \beta_5 = \frac{\beta_0\beta_1\beta_0 - \beta_2^2}{\beta_1}, \beta_6 = \frac{\beta_0\beta_1\beta_0 - \beta_0\beta_2^2}{\beta_1}, \beta_7 = \frac{\beta_0\beta_1\beta_0 - \beta_0\beta_2^2}{\beta_1}, \beta_8 = \frac{\beta_0\beta_1\beta_0 - \beta_0\beta_2^2}{\beta_1} \).

Furthermore, we are going to check the presentation using Theorem 1. We begin with some general results before we conclude the proof:

**Proposition 1.** Let \( A \) be a \( \mathbb{K} \)-algebra. Suppose that \( \beta_0 \in A \) is a nilpotent element of degree 2, then

\[
\left\{ \beta_i^j \mid j \geq 0 \right\} \cup \left\{ \beta_i^j\beta_0 \mid j \geq 0 \right\} \cup \left\{ \beta_i^j\beta_0\beta_1 \mid j \geq 0 \right\} \cup \left\{ \beta_i^j\beta_0\beta_1\beta_0 \mid j \geq 0 \right\}
\]

is a linearly independent set over \( \mathbb{K} \) if and only if

\[
\left\{ \beta_i^j\beta_0 \mid j \geq 0 \right\} \cup \left\{ \beta_i^j\beta_0\beta_1 \beta_0 \mid j \geq 0 \right\}
\]

is a linearly independent set over \( \mathbb{K} \).

**Proof.** Clearly the condition is sufficient. We consider the expression:

\[
\sum_{j=0}^{n} a_j\beta_i^j + \sum_{j=0}^{n} b_j\beta_i^j\beta_0 + \sum_{j=0}^{n} c_j\beta_i^j\beta_0\beta_1 + \sum_{j=0}^{n} d_j\beta_i^j\beta_0\beta_1\beta_0 = 0
\]

for \( a_j, b_j, c_j, d_j \in \mathbb{K}, n \in \mathbb{N} \).

Multiply by \( \beta_0 \) on the right and using that \( \beta_0^2 = 0 \), we obtain:

\[
\sum_{j=0}^{n} a_j\beta_i^j\beta_0 + \sum_{j=0}^{n} c_j\beta_i^j\beta_0\beta_1\beta_0 = 0.
\]
If we assume that \( \left\{ \beta_1^j \beta_0 \mid j \geq 0 \right\} \cup \left\{ \beta_1^j \beta_0 \beta_1 \beta_0 \mid j \geq 0 \right\} \) is linearly independent we have \( a_j = c_j = 0 \) and (4) reduces to:

\[
\sum_{j=0}^{n} b_j \beta_1^j \beta_0 + \sum_{j=0}^{n} d_j \beta_1^j \beta_0 \beta_1 \beta_0 = 0.
\]

Again, using this assumption we have \( b_j = d_j = 0 \). With this fact, we obtain the necessity. □

**Proposition 2.** Taking the elements \( \beta_0 \) and \( \beta_1 \) in \( \mathbb{A}(\phi_1) \) we obtain that

\[
\left\{ \beta_1^j \beta_0 \mid j \geq 0 \right\} \cup \left\{ \beta_1^j \beta_0 \beta_1 \beta_0 \mid j \geq 0 \right\}
\]

is a linearly independent set.

**Proof.** Note that \( \beta_1^j \beta_0 = e_{12} x^j + j e_{22} x^{j+1} \) and \( \beta_1^j \beta_0 \beta_1 \beta_0 = e_{12} x^{j+2} + j e_{22} x^{j+3} \). Consider the expression:

\[
\sum_{j=0}^{n} a_j \beta_1^j \beta_0 + \sum_{j=0}^{n} b_j \beta_1^j \beta_0 \beta_1 \beta_0 = 0.
\]

Replacing the previous relations, we obtain:

\[
\sum_{j=0}^{n} a_j (e_{12} x^j + j e_{22} x^{j+1}) + \sum_{j=0}^{n} b_j \beta_1^j (e_{12} x^{j+2} + j e_{22} x^{j+3}) = 0.
\]

Using the entries of the matrix, we obtain:

\[
\sum_{j=0}^{n} a_j x^j + \sum_{j=0}^{n} b_j x^{j+2} = 0 \quad \text{and} \quad \sum_{j=0}^{n} ja_j x^j + \sum_{j=0}^{n} j b_j x^{j+3} = 0.
\]

Equivalently,

\[
\sum_{j=0}^{n} a_j x^j + \sum_{j=2}^{n+2} b_{j-2} x^j = 0 \quad \text{and} \quad \sum_{j=0}^{n} ja_j x^j + \sum_{j=2}^{n+2} (j-2) b_{j-2} x^j = 0.
\]

Hence,

\[
a_0 + a_1 x + \sum_{j=2}^{n} (a_j + b_{j-2}) x^j + b_{n-1} x^{n+1} + b_n x^{n+2} = 0 \quad \text{and}
\]

\[
a_1 x + \sum_{j=2}^{n} (ja_j + (j-2) b_{j-2}) x^j + (n-1) b_{n-1} x^{n+1} + nb_n x^{n+2} = 0.
\]

Therefore,

\[
a_0 = a_1 = b_{n-1} = b_n = 0, \quad \left( \begin{array}{cc} 1 & 1 \\ j & j-2 \end{array} \right) \left( \begin{array}{c} a_j \\ b_{j-2} \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), 2 \leq j \leq n.
\]

Since \( \det \left( \begin{array}{cc} 1 & 1 \\ j & j-2 \end{array} \right) = -2 \neq 0 \) we have \( a_j = b_{j-2} = 0, \ 2 \leq j \leq n \) and

\[
\left\{ \beta_1^j \beta_0 \mid j \geq 0 \right\} \cup \left\{ \beta_1^j \beta_0 \beta_1 \beta_0 \mid j \geq 0 \right\}
\]

is linearly independent. □
Lemma 1. Consider the algebra $\mathbb{K} \cdot \langle a_0, a_1 \rangle / I(\psi_1)$ with

$$I(\psi_1) = \langle a_0^2 a_1^3 + a_0 a_1 a_0 - 3a_1 a_0 a_1 + a_0 a_1^2 + a_1^2 a_0 \rangle$$

then $\{1, a_0, a_0 a_1, a_0 a_1 a_0 \}$ is a system of generators for $\mathbb{K} \cdot \langle a_0, a_1 \rangle / I(\psi_1)$ as a free left $R$-module, with $R = \mathbb{K} \cdot \langle a_1 \rangle / I(\psi_1)$.

Proof. Define $M = R \oplus R \cdot a_0 \oplus R \cdot a_0 a_1 \oplus R \cdot a_0 a_1 a_0$. We have to see that $\mathbb{K} \cdot \langle a_0, a_1 \rangle / I = M$. It is enough to show that $M$ is invariant under left and right multiplication by $a_0$ and $a_1$.

- $a_1 M \subset M$. It is clear, since $a_1 \in R$.
- $M a_0 \subset M$. In fact, $M a_0 \subset R \cdot a_0 \oplus R \cdot a_0 a_1 a_0 \subset M$.
- $M a_1 \subset M$.

Since $a_0 a_1^2 = -a_1^3 - a_1^2 a_0 + 3a_1 a_0 a_1 - a_0 a_1 a_0$, we have

$$a_0 a_1^2 a_0 = -a_1^3 a_0 + 3(a_1 a_0)^2$$

and

$$0 = -a_0 a_1^2 a_0 - a_0 a_1^2 a_0 + 3(a_0 a_1)^2.$$  

Furthermore,

$$a_0 a_1^3 = -a_1^4 - a_1^2 (a_0 a_1) + 3a_1 (a_0 a_1^2) - (a_0 a_1)^2.$$  

Hence,

$$3(a_0 a_1)^2 = a_0 a_1^3 + a_0 a_1^2 a_0 = a_0 a_1^3 - a_1^2 a_0 + 3(a_1 a_0)^2$$

$$= -a_1^4 - a_1^2 (a_0 a_1) + 3a_1 (a_0 a_1^2) - (a_0 a_1)^2 - a_1^2 a_0 + 3(a_1 a_0)^2.$$  

Equivalently,

$$4(a_0 a_1)^2 = -a_1^4 - a_1^2 a_0 - a_1^2 (a_0 a_1) + 3a_1 (a_0 a_1^2) + 3(a_1 a_0)^2.$$  

However,

$$a_1 a_0 a_1^2 = -a_1^4 - a_1^2 a_0 - 3a_1^2 (a_0 a_1) - (a_1 a_0)^2.$$  

Thus,

$$4(a_0 a_1)^2 = -a_1^4 - a_1^2 a_0 - a_1^2 (a_0 a_1) + 3a_1^4 - 3a_1^2 a_0 + 9a_1^2 (a_0 a_1) - 3(a_1 a_0)^2 + 3(a_1 a_0)^2$$

$$= -4a_1^4 - 4a_1^3 a_0 + 8a_1^2 (a_0 a_1).$$  

Therefore,

$$(a_0 a_1)^2 = -a_1^4 - a_1^2 a_0 + 2a_1^2 (a_0 a_1).$$  

This implies that $(a_0 a_1)^2 \subset M$, $a_0 a_1^2 \subset M$. Since $M$ is a left $R$-module, we have $M a_1 \subset R a_1 \oplus R a_0 a_1 \oplus R a_0 a_1 a_0 \oplus R(a_0 a_1)^2 \subset M$.

- $a_0 M \subset M$.

We claim that $a_0 a_n^2 \subset M$ for every $n \in \mathbb{N}$. For $n = 0$ is clear. Assume this for some $n \in \mathbb{N}$ and note that $a_0 a_n^{n+1} = (a_0 a_n^2) a_1 \in M a_1 \subset M$. The claim follows by induction. In particular, $a_0 R \subset M$. Thus, $a_0 M \subset a_0 R \oplus a_0 R a_0 \oplus a_0 R a_0 a_1 \oplus a_0 R a_0 a_1 a_0 \subset R \oplus R \cdot a_0 \oplus R \cdot a_0 a_1 \oplus R \cdot a_0 a_1 a_0 \subset M$.

Finally, we conclude with the proof of the nice presentation. Define

$$f : \mathbb{C} \cdot \langle a_0, a_1 \rangle / I \rightarrow A(\psi_1),$$

$$f(\alpha) = \beta_1,$$

the previous lemma guarantees the existence of a subalgebra $R = \mathbb{C} \cdot \langle a_1 \rangle / I(\psi_1)$ and a system of generators $\{1, a_0, a_0 a_1, a_0 a_1 a_0\}$ for $\mathbb{C} \cdot \langle a_0, a_1 \rangle / I(\psi_1)$ as a free left $R$-module. Furthermore, $f |_R : R \rightarrow A(\psi_1)$ is a monomorphism.
Proposition 2 implies that \( \{1, \beta_0, \beta_0\beta_1, \beta_0\beta_1\beta_0\} \) is a linearly independent set over \( \mathbb{C} \). Consequently, we are under the hypothesis of Theorem 1 and \( f \) is an isomorphism.

Putting together Lemma 1, Propositions 1 and 2, we conclude the proof of Theorem 2.

4. An Algebra with Nilpotent and Idempotent Associated Elements

In this section, we consider an algebra generated by two associated elements, one of them nilpotent and the other idempotent. The statement of the theorem is as follows.

**Theorem 3.** Let \( \mathbb{A}(\psi_2) \) the sub-algebra of \( M_3(\mathbb{C})[x] \) of the form

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix} + x
\begin{pmatrix}
\beta_{11} & \beta_{12} & \beta_{13} \\
\beta_{21} & \beta_{22} & \beta_{23} \\
\beta_{31} & \beta_{32} & \beta_{33}
\end{pmatrix} + x^2
\begin{pmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33}
\end{pmatrix} + x^3
\begin{pmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{pmatrix} + x^4
\begin{pmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33}
\end{pmatrix} + x^5
\begin{pmatrix}
\zeta_{11} & \zeta_{12} & \zeta_{13} \\
\zeta_{21} & \zeta_{22} & \zeta_{23} \\
\zeta_{31} & \zeta_{32} & \zeta_{33}
\end{pmatrix},
\]

where \( p \in M_3(\mathbb{C})[x] \) and all the variables \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \epsilon_{ij}, \zeta_{ij} \in \mathbb{C} \) are arbitrary.

Then, we have the presentation \( \mathbb{A}(\psi_2) = \mathbb{C} \cdot \langle \alpha_{23}, \alpha_{32} \mid I(\psi_2) = 0 \rangle \) with

\[
I(\psi_2) = \langle \alpha_{23}^2, \alpha_{32}^2 - \alpha_{32}, (\alpha_{32}3)^2 \alpha_{23} - 4\alpha_{32}a_3^2 \rangle.
\]

**Proof.** The idea of the proof is to consider a basis for the vector space \( \mathbb{A}(\psi_2) \cap \mathbb{D}_0 = M_3(\mathbb{C}[x]) \) of polynomials in \( \mathbb{A}(\psi_2) \) of degree less or equal to 5 and observe that this basis generates the algebra \( \mathbb{A}(\psi_2) \). After that, we look for remarkable elements on the basis that generate the others and obtain some set of relations. Finally, we verify the hypothesis of Theorem 1 to obtain proof of the assertion.

Note that \( \mathbb{A}(\psi_2) \) is generated by \( \beta_0 = 1, \beta_1 = \alpha_{23} - \alpha_{32}x + \alpha_{32}x^2 + \alpha_{31}x^3, \beta_2 = \alpha_{23} - \alpha_{32}x + \alpha_{32}x^2 + \alpha_{31}x^3, \beta_3 = \alpha_{23} - \alpha_{32}x + \alpha_{32}x^2 + \alpha_{31}x^3, \beta_4 = \alpha_{23} - \alpha_{32}x + \alpha_{32}x^2 + \alpha_{31}x^3, \beta_5 = \alpha_{23} - \alpha_{32}x + \alpha_{32}x^2 + \alpha_{31}x^3 \), and \( \beta_6 = \alpha_{23} - \alpha_{32}x + \alpha_{32}x^2 + \alpha_{31}x^3 \).

In fact, \( \beta_0 = \beta_2, \beta_1 = 1/2[\beta_3\beta_2\beta_3 - \beta_3\beta_2\beta_3], \beta_2 = 1/2[\beta_3\beta_2\beta_3 + \beta_3\beta_2\beta_3], \beta_3 = 1/2[\beta_3\beta_2\beta_3 + \beta_3\beta_2\beta_3], \beta_4 = 1/2[\beta_3\beta_2\beta_3 + \beta_3\beta_2\beta_3], \beta_5 = 1/2[\beta_3\beta_2\beta_3 + \beta_3\beta_2\beta_3], \beta_6 = 1/2[\beta_3\beta_2\beta_3 + \beta_3\beta_2\beta_3]. \)

The hypothesis of Theorem 1 is obtained by verifying that \( \mathbb{A}(\psi_2) \) is a free \( \mathbb{C} \)-module.
\[ \beta_{23} = \beta_2^2 \beta_3 \beta_2^2 \beta_3 - 1/2 \beta_2^2 \beta_3 \beta_2 \beta_3 \beta_2^2, \beta_{24} = -\beta_2^2 \beta_3 \beta_3 \beta_2^2, \beta_{25} = 1/2 \beta_2 \beta_2^2 \beta_3 \beta_2^2 \beta_3 - \beta_2^2 \beta_3 \beta_3 \beta_2 + 1 \beta_2 \beta_2^2 \beta_3 \beta_2^2 \beta_3, \beta_{26} = -1/2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3 + \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3 - 1/2 \beta_2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 + \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3, \beta_{27} = -1/2 \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 + \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 + \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2, \beta_{28} = 1/2 \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3 - \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3, \beta_{30} = 1/2 \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 - \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2, \beta_{36} = 1/2 \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 + 1/2 \beta_2^2 (\beta_3 \beta_2^2)^2 - \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 + 1/2 (\beta_2 \beta_3^2)^2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 - \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2, \beta_{35} = 1/2 \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2 - \beta_2 \beta_2^2 \beta_3 \beta_3 \beta_2^2 \beta_3^2. \]

Furthermore, we are going to check the presentation using Theorem 1. We begin with some general results:

**Lemma 2.** Let \( A \) be a \( \mathbb{K} \)-algebra. Suppose that \( \beta_2 \in A \) is a nilpotent element of degree \( D \geq 3 \). Suppose that

\[
\{ \beta_2^{D-1}(\beta_3 \beta_2)^j | j \geq 0 \}
\]

is a linearly independent set over \( \mathbb{K} \). Then, \( \{ \beta_2^{D-1}(\beta_3 \beta_2)^j | j \geq 0, 1 \leq k \leq D - 2 \} \) is linearly independent over \( \mathbb{K} \).

**Proof.** Consider the expression

\[
\sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{jk} \beta_2^{D-2}(\beta_3 \beta_2)^j \beta_2^k = 0. \tag{5}
\]

Multiplying by \( \beta_2^{D-3} \) on the right:

\[
\sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{jk} \beta_2^{D-2}(\beta_3 \beta_2)^j \beta_2^{k-2} = 0. \tag{6}
\]

However, \( \{ \beta_2^{D-1}(\beta_3 \beta_2)^j | j \geq 0 \} \) is linearly independent over \( \mathbb{K} \). Consequently, \( c_{j1} = 0 \) for \( 0 \leq j \leq n \).

Thus, (5) reduces to

\[
\sum_{j=1}^{n} \sum_{k=2}^{D-2} c_{jk} \beta_2^{D-2}(\beta_3 \beta_2)^j \beta_2^{k-2} = 0. \tag{7}
\]

Assume that,

\[
\sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{jk} \beta_2^{D-2}(\beta_3 \beta_2)^j \beta_2^{k} = 0. \tag{8}
\]

Multiplying by \( \beta_2^{D-2-k_0} \) on the right:

\[
\sum_{j=1}^{n} c_{jk_0} \beta_2^{D-2}(\beta_3 \beta_2)^j \beta_2^{D-2} = 0. \tag{9}
\]

However, \( \{ \beta_2^{D-1}(\beta_3 \beta_2)^j | j \geq 0 \} \) is linearly independent over \( k \). Consequently, \( c_{jk_0} = 0 \) for \( 1 \leq j \leq n \). In consequence

\[
\sum_{j=1}^{n} \sum_{k=k_0+1}^{D-2} c_{jk} \beta_2^{D-2}(\beta_3 \beta_2)^j \beta_2^k = 0. \tag{10}
\]

Since the case \( k_0 = 1 \Rightarrow k_0 = 2 \), was seen we have that \( c_{jk} = 0 \) for \( 1 \leq j \leq n, 1 \leq k \leq D - 2 \). \( \square \)
\textbf{Proposition 3.} Let $A$ be a $K$-algebra. Suppose that $\beta_2 \in A$ is a nilpotent element of degree $D \geq 3$, then

\[
\left\{ \beta_2^i(\beta_3^j) \beta_3^k \mid 0 \leq i \leq D-1, j \geq 0 \right\} \cup \left\{ \beta_2^i(\beta_3^j) \beta_2^k \mid 0 \leq i \leq D-1, j \geq 1, 1 \leq k \leq D-2 \right\}
\]

\[
\cup \left\{ \beta_2^i \beta_3^j \mid 0 \leq i \leq D-1, j \geq 0 \right\},
\]

is a linearly independent set over $K$ if and only if

\[
\left\{ \beta_2^{D-1}(\beta_3^j) \beta_2^{D-2} \mid j \geq 0 \right\}
\]

is a linearly independent set over $K$.

\textbf{Proof.} The sufficiency of the statement is clear. To show the necessity, we consider the expression

\[
\sum_{i=0}^{n} \sum_{j=0}^{D-1} a_{ij} \beta_2^i(\beta_3^j) \beta_3^k + \sum_{i=0}^{n} \sum_{j=1}^{D-1} b_{ij} \beta_2^i(\beta_3^j) \beta_3^k + \sum_{i=0}^{n} \sum_{j=0}^{D-1} \sum_{k=1}^{D-2} c_{ijk} \beta_2^i(\beta_3^j) \beta_2^k = 0, \tag{11}
\]

where $a_{ij}, b_{ij}, c_{ijk} \in K$, $n \geq 0$.

We have to see that $a_{ij} = b_{ij} = c_{ijk} = 0$.

We are going to see that

\[
\sum_{i=0}^{n} \sum_{j=1}^{D-1} b_{ij} \beta_2^i(\beta_3^j) \beta_3^k + \sum_{i=0}^{n} \sum_{j=1}^{D-1} b_{ij} \beta_2^i(\beta_3^j) \beta_3^k + \sum_{i=0}^{n} \sum_{j=1}^{D-1} \sum_{k=1}^{D-2} c_{ijk} \beta_2^i(\beta_3^j) \beta_2^k = 0 \tag{12}
\]

for some $0 \leq l \leq D - 1$ implies that $a_{ij} = b_{ij} = c_{ijk} = 0$.

For $l = 0$, we have Equation (11). Multiplying by $\beta_2^{D-1}$ on the left and on the right:

\[
\sum_{i=0}^{n} \sum_{j=0}^{D-1} b_{ij} \beta_2^{D-1}(\beta_3^j) \beta_2^{D-1} = \sum_{i=0}^{n} \sum_{j=0}^{D-1} b_{ij} \beta_2^{D-1}(\beta_3^j) \beta_2^{D-2} = 0. \tag{13}
\]

Nevertheless, $\left\{ \beta_2^{D-1}(\beta_3^j) \beta_2^{D-2} \mid j \geq 0 \right\}$ is linearly independent over $K$. Thus, $b_{0j} = 0$ for $0 \leq j \leq n$.

This reduces (11) to

\[
\sum_{i=0}^{n} \sum_{j=1}^{D-1} b_{ij} \beta_2^i(\beta_3^j) \beta_3^k + \sum_{i=0}^{n} \sum_{j=1}^{D-1} \sum_{k=1}^{D-2} c_{ijk} \beta_2^i(\beta_3^j) \beta_2^k = 0. \tag{14}
\]

Multiplying by $\beta_2^{D-1}$ on the left:

\[
\sum_{j=0}^{n} a_{0j} \beta_2^{D-1}(\beta_3^j) \beta_3^k + \sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{0jk} \beta_2^{D-2}(\beta_3^j) \beta_2^k = 0. \tag{15}
\]

Multiplying by $\beta_2^{D-2}$ on the right:

\[
\sum_{j=0}^{n} a_{0j} \beta_2^{D-1}(\beta_3^j) \beta_2^{D-2} = 0. \tag{16}
\]

Consequently, $a_{0j} = 0$ for $0 \leq j \leq n$. Since $\left\{ \beta_2^{D-1}(\beta_3^j) \beta_2^{D-2} \mid j \geq 0 \right\}$ is linearly independent over $K$. 

This reduces (15) to
\[
\sum_{j=0}^{n} \sum_{k=1}^{D-2} c_{0jk} \beta_{2}^{D-2}(\beta_{3}\beta_{2})^j / \beta_{2}^k = 0. 
\tag{17}
\]

However, by Lemma 2, \( \{ \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k \mid j \geq 0, 1 \leq k \leq D - 2 \} \) is linearly independent over \( \mathbb{K} \). Thus, \( c_{0jk} = 0 \) for \( 1 \leq j \leq n, 1 \leq k \leq D - 2 \).

Assume (12) for \( l \) and multiply this by \( \beta_{2}^{D-l-1} \) on the left:
\[
\sum_{j=0}^{n} a_{ij} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j + \sum_{j=0}^{n} b_{ij} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j + \sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{jk} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k = 0. 
\tag{18}
\]

Multiplying by \( \beta_{2}^{D-1} \) on the right:
\[
\sum_{j=1}^{n} b_{ij} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k = 0. 
\tag{19}
\]

Nevertheless, \( \{ \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k \mid j \geq 0 \} \) is linearly independent over \( \mathbb{K} \). In consequence, \( b_{ij} = 0 \) for \( 0 \leq j \leq n \).

Therefore, (18) reduces to:
\[
\sum_{j=0}^{n} a_{ij} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j + \sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{jk} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k = 0. 
\tag{20}
\]

Multiplying by \( \beta_{2}^{D-2} \) on the right:
\[
\sum_{j=0}^{n} a_{ij} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^{D-2} = 0. 
\tag{21}
\]

However, \( \{ \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^{D-2} \mid j \geq 0 \} \) is linearly independent over \( \mathbb{K} \). Consequently, \( a_{ij} = 0 \) for \( 0 \leq j \leq n \).

Therefore,
\[
\sum_{j=1}^{n} \sum_{k=1}^{D-2} c_{jk} \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k = 0. 
\tag{22}
\]

Nevertheless, by Lemma 2, \( \{ \beta_{2}^{D-1}(\beta_{3}\beta_{2})^j / \beta_{2}^k \mid j \geq 0, 1 \leq k \leq D - 2 \} \) is linearly independent over \( \mathbb{K} \). Thus, \( c_{ik} = 0 \) for \( 1 \leq j \leq n, 1 \leq k \leq D - 2 \).

Hence, we obtain (12) for \( l + 1 \). Then (12) is valid for \( 0 \leq l \leq D - 1 \), i.e., \( a_{ij} = b_{ij} = c_{ijk} = 0. \)

\[ \square \]

**Lemma 3.** Consider the algebra \( \mathbb{K} \cdot \langle a_2, a_3 \rangle / I(\psi_2) \) with
\[
I(\psi_2) = \langle a_2^3, a_2^3 - a_3, (a_3a_2)^2 a_3 - 4a_3a_2^2 a_3 \rangle
\]
then \( \{ (a_3a_2)^n \mid n \geq 0 \} \cup \{ (a_3a_2)^n a_3 \mid n \geq 0 \} \cup \{ (a_3a_2)^n a_2 \mid n \geq 0 \} \) is a system of generators for \( \mathbb{K} \cdot \langle a_2, a_3 \rangle / I(\psi_2) \) as a free left \( R \)-module, with \( R = \mathbb{K} \cdot \langle a_2 \rangle / I(\psi_2) \).

**Proof.** Define \( M = \bigoplus_{n=0}^{\infty} R \cdot (a_3a_2)^n \oplus \bigoplus_{n=0}^{\infty} R \cdot (a_3a_2)^n a_3 \oplus \bigoplus_{n=1}^{\infty} R \cdot (a_3a_2)^n a_2 \). We have to see that \( \mathbb{K} \cdot \langle a_2, a_3 \rangle / I(\psi_2) = M \). It is enough to show that \( M \) is invariant under left and right multiplication by \( a_2 \) and \( a_3 \).

- \( a_2 M \subset M \).
  Since \( a_2 \in R \).
- \( M a_2 \subset M \).
Since \( R(a_3a_2)^n a_2 \subset M, (a_3a_2)^n a_3 a_2 = (a_3a_2)^{n+1} \in M, \) for \( n \geq 0 \) and \((a_3a_2)^n a_2 \) \( a_2 = 0 \in M \) for \( n \geq 1 \). Then \( M_{a_2} \subset \bigoplus_{n=0}^{\infty} R \cdot (a_3a_2)^{n+1} \oplus \bigoplus_{n=0}^{\infty} R \cdot (a_3a_2)^{n} a_2 \subset M.

- \( M_{a_3} \subset M. \)

Note that \((a_3a_2)^n a_3 = (a_3a_2)^{n-1} a_3 + (a_3a_2)^n a_3 = \frac{1}{2} (a_3a_2)^{n+1} a_3 \) for every \( n \geq 1 \), then \( M_{a_3} \subset \bigoplus_{n=0}^{\infty} R \cdot (a_3a_2)^{n} a_3 + \bigoplus_{n=1}^{\infty} R \cdot (a_3a_2)^{n} a_2 \subset M. \)

- \( M_{a_3} \subset M. \)

Finally, we conclude with the proof of the nice presentation. Define

\[
\begin{align*}
\phi : C \cdot \langle a_2, a_3 \rangle / I(\psi_2) & \to \Lambda(\psi_2), \\
\phi(\overline{x}j) & = \beta_{ij},
\end{align*}
\]

the previous lemma guarantees the existence of a subalgebra \( R = C \cdot \langle a_2 \rangle / I(\psi_2) \) and a system of generators \( \{ (a_3a_2)^n \mid n \geq 0 \} \cup \{ (a_3a_2)^n a_3 \mid n \geq 0 \} \cup \{ (a_3a_2)^n a_2 \mid n \geq 0 \} \) for \( C \cdot \langle a_2, a_3 \rangle / I(\psi_2) \) as a free left \( R \)-module. Furthermore \( \phi \mid R : R \to \Lambda(\psi_2) \) is a monomorphism.

Since \( \beta_2^2 = 2 \beta_3^2 \beta_2 \beta_2 = 2^{n-1} e_{13} x^{n+1} \) for \( n \geq 1 \) applying Proposition 3 with \( D = 3 \), we obtain

\[
\begin{align*}
\{ \beta_2^2 (\beta_3 \beta_2) / \beta_3 \mid 0 \leq i \leq D-1, j \geq 0 \} & \cup \{ \beta_2^2 (\beta_3 \beta_2) / \beta_2 \mid 0 \leq i \leq D-1, j \geq 1 \} \\
& \cup \{ \beta_2^2 (\beta_3 \beta_2) / \beta_2 \mid 0 \leq i \leq D-1, j \geq 0 \},
\end{align*}
\]

is a linearly independent set over \( C. \)

Putting together Proposition 3, Lemmas 2 and 3 we conclude the proof of Theorem 3.

5. An Algebra with Two Integral Elements over One Nilpotent and One Idempotent

In this section, we consider an algebra generated by four elements. This algebra is linked to the Spin Calogero–Moser systems whose relation with bispectrality can be found in [17]; see also [24].

We now consider the case when both "eigenvalues" \( F \) and \( \theta \) are matrix valued. Let

\[
\phi_3(x, z) = \frac{e^{\frac{1}{2}z}}{(x - 2) x^2} \left( x^2 z^2 - 2 x^2 - x^2 z + 3 x z + 2 x - 2 \right)^{\frac{1}{2}}
\]

\[
\frac{1}{x^2} - 2 x z - x + 1
\]
and 
\[ L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_z^2 + \begin{pmatrix} 0 & -\frac{1}{x-2}x^2 \\ -\frac{1}{x-2}x^2 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} -\frac{1}{x^2(x-2)^2} & \frac{x-1}{x(x-2)^2} \\ \frac{x-1}{x^2(x-2)^2} & -\frac{2x}{x^2(x-2)^2} \end{pmatrix}, \]
then \( L\psi_3 = \psi_3F \) with 
\[ F(z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} z^2. \]
On the other hand, it is easy to check that \( \psi_3B = \theta \psi_3 \) for
\[ B = \partial_\nu^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} -\frac{1}{x^2} & \frac{1}{2(x-1)^2} \\ \frac{1}{2(x-1)^2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{z-1}{6z-3} \\ \frac{1}{z-1} \end{pmatrix} \]
and
\[ \theta(x) = \begin{pmatrix} x \\ x(x-2) \end{pmatrix}. \]
In this opportunity, we characterize the algebra \( \mathcal{A}(\psi_3) \) of all polynomial \( F \) such that there exist \( L = L(x, \partial_x) \) with \( L\psi_3 = \psi_3F \).

**Theorem 4.** Let \( \mathcal{A}(\psi_3) \) be the sub-algebra of \( M_2(\mathbb{C})[z] \) of the form
\[ \begin{pmatrix} a & c \\ b-a & e \end{pmatrix} \begin{pmatrix} \frac{z}{2} \\ z \end{pmatrix} + \begin{pmatrix} a-b & c+a-b \\ d & e \end{pmatrix} z^2 + z^3 p(z), \]
where \( p \in M_2(\mathbb{C})[z] \) and all the variables \( a, b, c, d, e \) are arbitrary.
Then, we have the presentation \( \mathcal{A}(\psi_3) = \mathbb{C} \cdot (\theta_1, \theta_3, \theta_4, \theta_5 \mid I = 0) \) with
\[ I(\psi_3) = (\theta_1^2 - \theta_1, \theta_3^2 - \theta_3, \theta_4^2 - \theta_4, \theta_5^2 - \theta_5, \theta_1 \theta_5 \theta_3 - \frac{1}{2} \theta_4 \theta_3 + \frac{1}{2} \theta_3 \theta_4 + \frac{1}{2} \theta_2^2 + \theta_3 \theta_4 - \theta_1 \theta_5 - \theta_3 \theta_5), \]
\[ \theta_1 \theta_3 - \theta_3 + \theta_4 - \theta_5 - 3 \theta_3 \theta_4 + \frac{3}{2} \theta_2 \theta_3 + 3 \theta_2 \theta_4 + 3 \theta_3 \theta_4 + \theta_3 \theta_5, \]
\[ \theta_5 \theta_3 - \theta_4 \theta_1 + \theta_4 \theta_3 - \theta_3 \theta_5 - \theta_5 \theta_4 + 3 \theta_2 \theta_3 \theta_5 - \theta_3 \theta_4 - \theta_5 \theta_4 \theta_1 - \theta_5 \theta_4 \theta_2 + \theta_3 \theta_4 \theta_4 + 3 \theta_3 \theta_4 + \theta_3 \theta_5 \theta_1, \]
\[ \theta_4 \theta_3 \theta_5 + \theta_4 \theta_3 \theta_4 - \theta_3 \theta_5 \theta_4 + \theta_3 \theta_4 \theta_1). \]

**Proof.** The idea of the proof is to consider a basis for the vector space \( \mathcal{A}(\psi_3) \cap \oplus_{j=0}^2 M_2(\mathbb{C}[x]) \), of polynomials in \( \mathcal{A}(\psi_3) \) of degree less or equal to 2 and observe that this basis generates the algebra \( \mathcal{A}(\psi_3) \). After that, we look for remarkable elements in the basis that generate the others and obtain some set of relations. Finally, we verify the hypothesis of Theorem 1 to obtain proof of the assertion.

In order to check the presentation using Theorem 1, we start with a result about the generators of a free \( \mathbb{K} \)-vector space.

**Lemma 4.** Consider the \( \mathbb{K} \)-algebra \( \mathbb{K} \cdot (\theta_1, \theta_3, \theta_4, \theta_5) / I(\psi_3) \) with \( \mathbb{K} \) a central field of characteristic 0 and
\[ I(\psi_3) = (\theta_1^2 - \theta_1, \theta_3^2 - \theta_3, \theta_4^2 - \theta_4, \theta_5^2 - \theta_5, \theta_1 \theta_5 \theta_3 - \frac{1}{2} \theta_4 \theta_3 + \frac{1}{2} \theta_3 \theta_4 + \frac{1}{2} \theta_2^2 + \theta_3 \theta_4 - \theta_1 \theta_5 - \theta_3 \theta_5), \]
\[ \theta_1 \theta_3 - \theta_3 + \theta_4 - \theta_5 - 3 \theta_3 \theta_4 + \frac{3}{2} \theta_2 \theta_3 + 3 \theta_2 \theta_4 + 3 \theta_3 \theta_4 + \theta_3 \theta_5, \]
\[ \theta_5 \theta_3 - \theta_4 \theta_1 + \theta_4 \theta_3 - \theta_3 \theta_5 - \theta_5 \theta_4 - 3 \theta_2 \theta_3 \theta_5 + \theta_3 \theta_4 - \theta_5 \theta_4 \theta_1 - \theta_5 \theta_4 \theta_2 + \theta_3 \theta_4 \theta_4 + 3 \theta_3 \theta_4 + \theta_3 \theta_5 \theta_1, \]
\[ \theta_4 \theta_3 \theta_5 + \theta_4 \theta_3 \theta_4 - \theta_3 \theta_5 \theta_4 + \theta_3 \theta_4 \theta_1). \]
Then, \( \{ \theta_4, \theta_3, \theta_1 \} \cup \{ \theta_5^{n} | n \geq 0 \} \cup \{ \theta_6^{n} | n \geq 0 \} \cup \{ \theta_{5}^{n} \theta_1 | n \geq 1 \} \cup \{ \theta_{6}^{n} \theta_1 | n \geq 1 \} \cup \{ \theta_{5}^{n} \theta_4 | n \geq 1 \} \cup \{ \theta_{6}^{n} \theta_4 | n \geq 1 \} \) is a system of generators for \( K \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \rangle / I(\varphi_3) \) as a free \( K \)-vector space.

**Proof.** Define \( M = K \cdot \theta_1 \oplus K \cdot \theta_3 \oplus K \cdot \theta_4 \oplus \bigoplus_{n=0}^{\infty} K \cdot \theta_5^n \oplus \bigoplus_{n=1}^{\infty} K \cdot \theta_6^n \oplus \bigoplus_{n=1}^{\infty} K \cdot \theta_5^n \theta_3 \oplus \bigoplus_{n=1}^{\infty} K \cdot \theta_6^n \theta_3 \oplus \bigoplus_{n=1}^{\infty} K \cdot \theta_5^n \theta_4 \oplus \bigoplus_{n=1}^{\infty} K \cdot \theta_6^n \theta_4 \). We have to see that \( K \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \rangle / I = M \). It is enough to show that \( M \) is invariant under left and right multiplication by \( \theta_1, \theta_3, \theta_4 \) and \( \theta_5 \).

- \( M \theta_5 \subset M \).
  Note that \( \theta_3 \theta_4 \theta_5 = 0 \in M, \theta_4 \theta_1 \theta_5 = -\theta_5^2 \theta_1 - \theta_3 \theta_4 + \theta_3^2 \in M \). On the other hand \( \theta_1 \theta_4 \theta_5 = 0 \in M, \theta_3 \theta_5 \theta_4 \in M, \theta_6^n M \in M \) for every \( n \geq 1 \), \( \theta_6^n \theta_3 \theta_5 = 0 \in M \) for every \( n \geq 0 \), \( \theta_6^n \theta_4 \theta_5 = 0 \in M \) for every \( n \geq 1 \).

- \( M \theta_4 \subset M \).
  Note that \( \theta_4 \in M, (\theta_3 \theta_4) \theta_4 = 0 \in M, \theta_4 \theta_1 = \theta_3 \theta_4 + \theta_3 \theta_5 - \theta_3 \theta_4 + \theta_3^2 \) and \( \theta_4 \theta_3 = 2 \theta_4 + \theta_3 \theta_4 + \theta_3^2 - \theta_4 \theta_1 \) imply \( \theta_4 \theta_1 = \frac{1}{2} \theta_3 \theta_3 + \theta_4 + \theta_4^2 - \frac{1}{2} \theta_3 \theta_4 \), hence \( \theta_4 \theta_1 \theta_4 = \theta_3^2 \theta_4 - \theta_3 \theta_4 \theta_4 \in M \).

On the other hand, \( (\theta_1 \theta_4) \theta_4 = \theta_4^2 = 0 \in M, \theta_3 \theta_4 \in M, \theta_4 \theta_4 \in M \) for every \( n \geq 0 \), \( \theta_4 \theta_4 \theta_4 = 0 \in M \) for every \( n \geq 0 \), \( (\theta_3 \theta_4 \theta_4) \theta_4 = 0 \in M \) for every \( n \geq 1 \), \( (\theta_1 \theta_4 \theta_4) \theta_4 = \theta_3 \theta_4 \theta_4 \theta_4 = 0 \in M \) for every \( n \geq 1 \), \( (\theta_1 \theta_4 \theta_4 \theta_4) \theta_4 = 0 \in M \) for every \( n \geq 1 \).

- \( \theta_1 M \subset M \).
  Note that \( \theta_1 \in M \). Since \( \theta_1 \theta_5 = \theta_3 - \theta_5 - \theta_3 + \frac{1}{2} \theta_3 \theta_1 - \frac{1}{2} \theta_3 \theta_3 + 2 \theta_3 \theta_1 + \frac{1}{2} \theta_3 \theta_3 - \frac{1}{2} \theta_3^2 - 3 \theta_3 \theta_1 - \theta_3 \theta_1 \theta_3 \theta_1 \) multiply the \( \theta_4 \theta_3 \) on the right we obtain \( \theta_1 \theta_3 \theta_4 = \theta_3 \theta_3 \theta_3 - \theta_3 \theta_3 \theta_3 - \theta_3 \theta_3 \theta_3 \theta_3 \theta_4 \theta_4 \in M \).

On the other hand, the equation \( \theta_4 \theta_1 = \frac{1}{2} \theta_3 \theta_3 + \theta_4 + \theta_4^2 - \frac{1}{2} \theta_3 \theta_4 \) implies \( \theta_4 \theta_4 \theta_1 = \frac{1}{2} \theta_3 \theta_3 + \theta_4 + \theta_4^2 - \frac{1}{2} \theta_3 \theta_4 \), putting this equation together with the equations

\[
\begin{align*}
\theta_1 \theta_3 \theta_2 \theta_1 + \theta_1 \theta_5 + \theta_3 \theta_5 - \theta_5^2 + \theta_1 \theta_4 - \theta_5^2 \theta_1 - \theta_3 \theta_4 - \theta_3 \theta_5^2 &= 0, \\
\theta_1 \theta_3 \theta_3 - 2 \theta_3 \theta_4 - \theta_1 \theta_5 - \theta_3 \theta_5 + \theta_5^2 + 3 \theta_3 \theta_4 - \theta_1 \theta_4 + \theta_5^2 \theta_1 + \theta_3 \theta_5^2 + 2 \theta_3 \theta_3 \theta_4 + 2 \theta_3 \theta_3 \theta_4 &= 0,
\end{align*}
\]

we obtain:

\[
\theta_1 \theta_4 \theta_1 - \theta_3 \theta_4 - 2 \theta_1 \theta_4 - \theta_1 \theta_5 - \theta_3 \theta_5 + 2 \theta_3 \theta_4 + \theta_3 \theta_5 \theta_4 + \theta_3 \theta_5 \theta_4 + \theta_3 \theta_5^2 + \theta_3 \theta_5^2 - \theta_1 \theta_5^2 + \theta_5^2 = 0.
\]

In particular, \( \theta_1 \theta_4 \theta_1 \in M \).

Moreover, \( \theta_1 \theta_3 \theta_1 + \theta_1 \theta_4 + \theta_1 \theta_5 + \theta_3 \theta_5 - \theta_5^2 \theta_1 - \theta_3 \theta_4 - \theta_3 \theta_5^2 - \theta_5^2 = 0 \) implies \( \theta_1 \theta_5 \theta_1 \in M \).

However, multiplying \( \theta_4 \theta_1 + \theta_4 \theta_3 - 2 \theta_4 - \theta_3 \theta_4 - \theta_5^2 = 0 \) by \( \theta_1 \) on the left we have \( \theta_1 \theta_4 \theta_1 + \theta_1 \theta_4 \theta_3 - 2 \theta_4 \theta_1 - \theta_3 \theta_4 \theta_1 - \theta_5^2 \theta_1 = 0 \). Hence, \( \theta_1 \theta_4 \theta_1 = \theta_5^2 - \theta_3 \theta_4 - \theta_3 \theta_5 - 2 \theta_4 \theta_1 + \theta_3 \theta_4 \theta_1 + \theta_3 \theta_5 \theta_4 + \theta_5^2 \theta_1 + \theta_3 \theta_5^2 + \theta_1 \theta_5 \theta_4 \in M \).

Moreover, \( \theta_1 \theta_3 \theta_4 = \theta_5 \theta_4 \in M \) and \( \theta_1 \theta_5 = \theta_3 - \theta_4 - \theta_5 + \frac{1}{2} \theta_4 \theta_1 - \frac{1}{2} \theta_3 \theta_3 + 2 \theta_3 \theta_1 + \frac{1}{2} \theta_3 \theta_3 - \frac{1}{2} \theta_3^2 - 3 \theta_3 \theta_1 - \theta_3 \theta_1 \theta_3 \theta_1 \). On the other hand, \( \theta_1 \theta_5 = \theta_5 \theta_4 \in M \) for every \( n \geq 1 \), \( \theta_1 \theta_3 \theta_4 \theta_1 \in M \) for every \( n \geq 0 \). Note that \( \theta_1 \theta_5 \theta_1 = -\theta_1 \theta_5^2 - \theta_3 \theta_5 - \theta_3 \theta_5^{-1} \theta_1 + \theta_3 \theta_5^2 \theta_1 + \theta_3 \theta_5^2 + \theta_3 \theta_5^2 + \theta_3 \theta_5^2 + \theta_5^2 = 1 \) in every \( n \geq 2 \) and \( \theta_1 \theta_5 \theta_1 \in M \) imply \( \theta_1 \theta_5 \theta_1 \in M \) for every \( n \geq 1 \).
Since \( \theta_1 \theta_3 \in M \), we have \( \theta_1 \theta_3 \theta_\theta^\theta \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Furthermore, \( \theta_1 (\theta_3 \theta_\theta^\theta) = \theta_1 \theta_\theta^\theta \in M \), for every \( n \geq 1 \) and \( \theta_1 (\theta_3 \theta_\theta^\theta) \theta_4 = (\theta_1 \theta_\theta^\theta) \theta_4 \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Nonetheless, \( \theta_1 \theta_3 \theta_\theta \in M \) then \( \theta_1 (\theta_3 \theta_\theta^\theta) M \), for every \( n \geq 0 \). Note that \( \theta_1 (\theta_3 \theta_\theta^\theta) \theta_4 = \theta_1 \theta_\theta^\theta \theta_4 \subset M \), for every \( n \geq 1 \). Thus \( \theta_1 M \subset M \).

- \( \theta_1 M \subset M \).

Note that \( \theta_1 (\theta_3 \theta_\theta^\theta) = \theta_1 \theta_\theta^\theta \in M \), for every \( n \geq 1 \) and \( \theta_1 (\theta_3 \theta_\theta^\theta) \theta_4 = (\theta_1 \theta_\theta^\theta) \theta_4 \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Nonetheless, \( \theta_1 \theta_3 \theta_\theta \in M \) then \( \theta_1 (\theta_3 \theta_\theta^\theta) \theta_4 \subset M \), for every \( n \geq 0 \). Note that \( \theta_1 (\theta_3 \theta_\theta^\theta) \theta_4 = \theta_1 \theta_\theta^\theta \theta_4 \subset M \), for every \( n \geq 1 \). Thus \( \theta_1 M \subset M \).

- \( \theta_3 \theta_\theta^\theta \in M \).

Note that \( \theta_3 \theta_3 \theta_\theta (\theta_3 \theta_\theta^\theta) = \theta_3 \theta_\theta^\theta \in M \), for every \( n \geq 1 \). Furthermore, \( \theta_3 \theta_\theta^\theta \theta_4 = \theta_3 \theta_\theta^\theta \theta_4 \in M \), for every \( n \geq 1 \) and \( \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = (\theta_3 \theta_\theta^\theta) \theta_4 \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Nonetheless, \( \theta_3 \theta_3 \theta_\theta \in M \) then \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \subset M \), for every \( n \geq 0 \). Note that \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = \theta_3 \theta_\theta^\theta \theta_4 \theta_5 \subset M \), for every \( n \geq 1 \). Thus \( \theta_3 M \subset M \).

- \( \theta_3 \theta_3 \theta_\theta \in M \).

Note that \( \theta_3 \theta_3 \theta_\theta (\theta_3 \theta_\theta^\theta) = \theta_3 \theta_\theta^\theta \in M \), for every \( n \geq 1 \). Furthermore, \( \theta_3 \theta_\theta^\theta \theta_4 = \theta_3 \theta_\theta^\theta \theta_4 \in M \), for every \( n \geq 1 \) and \( \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = (\theta_3 \theta_\theta^\theta) \theta_4 \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Nonetheless, \( \theta_3 \theta_3 \theta_\theta \in M \) then \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \subset M \), for every \( n \geq 0 \). Note that \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = \theta_3 \theta_\theta^\theta \theta_4 \theta_5 \subset M \), for every \( n \geq 1 \). Thus \( \theta_3 M \subset M \).

- \( \theta_3 \theta_3 \theta_\theta \in M \).

Note that \( \theta_3 \theta_3 \theta_\theta (\theta_3 \theta_\theta^\theta) = \theta_3 \theta_\theta^\theta \in M \), for every \( n \geq 1 \). Furthermore, \( \theta_3 \theta_\theta^\theta \theta_4 = \theta_3 \theta_\theta^\theta \theta_4 \in M \), for every \( n \geq 1 \) and \( \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = (\theta_3 \theta_\theta^\theta) \theta_4 \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Nonetheless, \( \theta_3 \theta_3 \theta_\theta \in M \) then \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \subset M \), for every \( n \geq 0 \). Note that \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = \theta_3 \theta_\theta^\theta \theta_4 \theta_5 \subset M \), for every \( n \geq 1 \). Thus \( \theta_3 M \subset M \).

- \( \theta_3 \theta_3 \theta_\theta \in M \).

Note that \( \theta_3 \theta_3 \theta_\theta (\theta_3 \theta_\theta^\theta) = \theta_3 \theta_\theta^\theta \in M \), for every \( n \geq 1 \). Furthermore, \( \theta_3 \theta_\theta^\theta \theta_4 = \theta_3 \theta_\theta^\theta \theta_4 \in M \), for every \( n \geq 1 \) and \( \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = (\theta_3 \theta_\theta^\theta) \theta_4 \in M \theta_\theta^\theta \subset M \), for every \( n \geq 1 \). Nonetheless, \( \theta_3 \theta_3 \theta_\theta \in M \) then \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \subset M \), for every \( n \geq 0 \). Note that \( \theta_3 \theta_3 \theta_\theta^\theta \theta_4 \theta_5 = \theta_3 \theta_\theta^\theta \theta_4 \theta_5 \subset M \), for every \( n \geq 1 \). Thus \( \theta_3 M \subset M \).
On the other hand, \( \theta_3\theta_2\theta_4\theta_3 = 2\theta_3\theta_2^2\theta_4 + \theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_4 + \theta_3\theta_2^2\theta_4 - \theta_2\theta_4 - \theta_2\theta_4 \in M \) for every \( n \geq 0 \) and \( \theta_1\theta_2\theta_4 = -\theta_1\theta_2\theta_4 + 2\theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_4 - \theta_2\theta_4 - \theta_2\theta_4 \in M \) for every \( n \geq 0 \). Hence, \( M\theta_3 \subset M \).

In [22], it was proved that the algebra \( \Gamma \) is generated by the elements

\[
\alpha_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} z^2 / 2,
\]
\[
\alpha_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} z + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} z^2 / 2,
\]
\[
\alpha_3 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} z + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} z^2 / 2, \quad \alpha_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z^2 / 2, \quad \text{and} \quad \alpha_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^2 / 2.
\]

In the following proposition, we look for the generators which give us the presentation.

**Proposition 4.** Define \( \beta_1 = \alpha_1 + \alpha_3, \beta_3 = \alpha_1 - \alpha_3, \beta_4 = 2\alpha_4, \beta_5 = 2\alpha_5 \) then \( \{\beta_4\beta_1, \beta_3, \beta_1\} \cup \{\beta_5^2 \mid n \geq 0\} \cup \{\beta_3^2 \beta_4 \mid n \geq 0\} \cup \{\beta_3 \beta_1^2 \mid n \geq 0\} \cup \{\beta_1^2 \beta_5 \mid n \geq 1\} \cup \{\beta_3 \beta_1^2 \beta_4 \mid n \geq 1\} \cup \{\beta_1 \beta_5^2 \beta_4 \mid n \geq 1\} \) is a linearly independent set over \( K \).

**Proof.** Note that

\[
A = K \cdot \beta_1 + K \cdot \beta_3 + K \cdot \beta_4 + K \cdot \beta_1^2 \beta_4 + \bigoplus_{n=0}^{\infty} K \cdot \beta_3 \beta_1 \beta_4 + \bigoplus_{n=0}^{\infty} K \cdot \beta_5^2 \beta_4 + \bigoplus_{n=1}^{\infty} K \cdot \beta_5^2 \beta_1 \beta_4 + \bigoplus_{n=1}^{\infty} K \cdot \beta_1 \beta_5^2 \beta_4
\]

\[
+ K \cdot \left( \frac{1}{2} \beta_4 \beta_1 - \frac{1}{2} \beta_4 \beta_3 + \beta_4 + \beta_5 \beta_1 + \frac{1}{2} \beta_5 \beta_4 - \frac{1}{2} \beta_5^2 - \beta_3 \beta_4 \right)
\]

\[
+ K \cdot \left( \frac{1}{2} \beta_4 \beta_1 - \frac{1}{2} \beta_4 \beta_3 + \beta_4 + \beta_5 \beta_1 + \frac{1}{2} \beta_5 \beta_4 - \frac{1}{2} \beta_5^2 - \beta_3 \beta_4 \right) \bigoplus_{n=0}^{\infty} K \cdot \left( \beta_5^2 - \beta_5 \beta_4 - \beta_3 \beta_5^2 \right)
\]

\[
+ \bigoplus_{n=0}^{\infty} K \cdot \left( \beta_5 \beta_4 + \beta_5 \beta_4 \right) \bigoplus_{n=1}^{\infty} K \cdot \beta_3 \beta_1 + \beta_5 \beta_4 \bigoplus_{n=1}^{\infty} K \cdot \beta_5 \beta_1 \beta_4
\]

The second equality is given by an isomorphism of \( K \) vector spaces sending \( \{\beta_4\beta_1, \beta_3, \beta_1\} \cup \{\beta_5^2 \mid n \geq 0\} \) \( \{\beta_5 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_1 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_1 \mid n \geq 1\} \cup \{\beta_3 \beta_1^2 \mid n \geq 1\} \cup \{\beta_3 \beta_5 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_1 \mid n \geq 1\} \cup \{\beta_5 \beta_5 \beta_4 \mid n \geq 0\} \) to the set \( \{1, \beta_4, \beta_3, \beta_4 \beta_1 - \frac{1}{2} \beta_4 \beta_3 + \beta_4 + \beta_5 \beta_1 + \frac{1}{2} \beta_5 \beta_4 - \beta_3 \beta_4, \beta_5 \beta_4 \beta_1 - \frac{1}{2} \beta_5 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_1 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_1 \beta_4 + \beta_5 \beta_1 \beta_4 \mid n \geq 0\} \cup \{\beta_5 \beta_1 + \beta_5 \beta_1 \beta_4 \mid n \geq 1\} \cup \{\beta_5 \beta_1 + \beta_5 \beta_1 \beta_4 \mid n \geq 1\} \) which is linearly independent because exactly \( \{1, \beta_4, \beta_3, \beta_1\} \cup \{\beta_3 \beta_4 + \beta_3 \beta_4 \mid n \geq 1\} \) is linearly independent.

Finally, we conclude with the proof of the presentation. Define

\[
f : C \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \rangle / I \rightarrow A(\psi_3),
\]
\[
f(\beta_i) = \beta_i.
\]
Lemma 4 guarantees the existence of the system of generators \( \{ \theta_3 \theta_1, \theta_3, \theta_1 \} \cup \{ \theta_2^n | n \geq 0 \} \cup \{ \theta_2^n \theta_4 | n \geq 0 \} \cup \{ \theta_4^n \theta_1 | n \geq 1 \} \cup \{ \theta_4^n \theta_2 | n \geq 1 \} \cup \{ \theta_4^n \theta_3 | n \geq 1 \} \cup \{ \theta_3 \theta_4^n \theta_1 | n \geq 0 \} \cup \{ \theta_3 \theta_4^n \theta_2 | n \geq 0 \} \cup \{ \theta_3 \theta_4^n \theta_3 | n \geq 1 \} \) for \( C \cdot (\theta_1, \theta_3, \theta_4, \theta_3) / 1(\psi_3) \) as a free \( C \)-vector space. Furthermore, \( f : C \rightarrow A(\psi_3) \) is a monomorphism.

Proposition 4 implies that \( \{ \beta_4 \beta_3, \beta_3, \beta_1 \} \cup \{ \beta_4^n | n \geq 0 \} \cup \{ \beta_4^n \beta_2 | n \geq 0 \} \cup \{ \beta_4^n \beta_3 | n \geq 1 \} \cup \{ \beta_3 \beta_4^n | n \geq 0 \} \cup \{ \beta_3 \beta_4^n \beta_1 | n \geq 1 \} \cup \{ \beta_3 \beta_4^n \beta_2 | n \geq 1 \} \cup \{ \beta_3 \beta_4^n \beta_3 | n \geq 1 \} \) is a linearly independent set over \( C \).

Putting together Lemma 4 and Propositions 4, we conclude the proof of Theorem 4. \( \square \)

6. Conclusions and Final Comments

In this article, we obtained in Theorem 1 a general result for presentations of finitely generated algebras. The theorem can be used to obtain a complete description in terms of generators and relations since it says when a set of relations is enough to characterize a given finitely generated algebra. As an application, we find nice presentations for matrix bispectral algebras and give positive answers for the conjectures presented in [18].

An important role was played by the Ad-condition due to the fact that the matrix-valued operators were acting from opposite directions, since we can consider these algebras as matrix polynomial.

Another research direction would be to investigate the presentations of the full rank 1 algebras, which by Theorem 1 in [22], are finitely generated. As we saw, the examples given in [18] and worked out here, are finitely presented; however, this is not necessarily true for general non-commutative rings.

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