Article

On the Separation Principle in Dynamic Output Controller Design for Uncertain Linear Systems

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Abstract: In this paper, the problem of designing the improper dynamic output controllers for uncertain linear multi-input/multi-output continuous-time systems is considered. In the principal part, the system state coordinate transform is outlined and matrix variable structures are found, defining the solution based on the structured linear matrix inequalities and linear matrix equality. Of principal novelty is the framework related to the system uncertainties with newly introduced inequality structures and their partial factorization. The main results are shown in detail by the example to characterize potential application of the method for quadratic stabilization of uncertain linear systems by the improper dynamic output controllers.

Keywords: dynamic output controllers; performances of separation principle; uncertain systems; matching conditions; linear matrix inequalities; linear matrix equality

1. Introduction

The output dynamic controllers (ODC) are preferred to deal with the unavailability of measurement of the system state variables, but the design of improper ODC means a bilinear matrix inequality (BMI) problem, which is generally non-convex [1,2]. To obtain conditions based on linear matrix inequalities (LMI), some constraints on the design are defined [3,4], or proper ODCs are considered [5]. The accompanying formulations of the asymptotic system tracking using ODCs [6], distributed $H_2$ control by dynamic output feedback [7], and control for multi-agent systems by ODCs [8] represent the range of related applications. It is natural to use ODCs in the parallel distributed form for systems with Takagi–Sugeno fuzzy models [9–11]. A brief survey of design, potentially using a non-fragile ODCs, is given in [12], ODC forms in network control structures are focused in [13], some interesting ODC applications are presented in [14,15].

In this article, new design conditions for improper ODCs for uncertain linear multi-input and multi-output (MIMO) systems are derived. To construct two separable LMIs conditioned by one linear matrix equality (LME), the principle of the system state coordinate transform in design of improper ODCs is made transparent, to build the theoretical robustness base adaptable to uncertain MIMO systems. By applying the quadratic Lyapunov function approach and the matching conditions of the system uncertainties, a quasi-separation structure of stabilization conditions is proposed in terms of a set of LMIs, together with a conditional LME, which gives a less conservative result in ODC design.

The remaining part of the article is organized as follows: Section 2 deals with ODC structures and the representative design task conditions, when applying the separation principle in the design task. Section 3 focuses on the closed-loop control problems for uncertain linear MIMO systems with relevant ODCs synthesis and Section 4 presents shortly the task adaptation to proper ODCs design. The performance of the proposed technique and the system parameter roles are illustrated by carrying out a detailed numerical example in Section 5, supporting an outline of points of view and conclusions in Section 6.
The used notations reflect usual conventionality so \( x^T \) and \( X^T \) denote the transpose of the vector \( x \) and matrix \( X \), \( \rho(X) \) signifies the eigenvalue spectrum of a square matrix \( X \); for a symmetric square matrix, \( X < 0 \) means that \( X \) is negative definite, \( * \) represents a block inferred by the matrix symmetry, the symbol \( I_n \) indicates the \( n \)-th order unit matrix, \( \mathbb{R} \) qualifies the set of real numbers, and \( \mathbb{R}^{n \times r} \) refers to the set of \( n \times r \) real matrices.

2. Separation Principle in Design of Improper ODCs

In general, a linear continuous-time MIMO system is given by the state-space equations

\[
\begin{align*}
\dot{q}(t) &= Aq(t) + Bu(t), \\
y(t) &= Cq(t),
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}, C \in \mathbb{R}^{m \times n} \) and \( q(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^r, y(t) \in \mathbb{R}^m \) are the state vector, the input vector, and the output vector, respectively.

Considering the square system (1) and (2), the related improper linear continuous-time MIMO ODC can be defined as

\[
\begin{align*}
\dot{p}(t) &= Jp(t) + Ky(t), \\
u(t) &= Lp(t) + Hy(t),
\end{align*}
\]

where the matrix parameters are \( J \in \mathbb{R}^{n \times n} \) (the controller system matrix), \( K \in \mathbb{R}^{n \times m} \) (the controller input matrix), \( L \in \mathbb{R}^{r \times n} \) (the controller output matrix), \( H \in \mathbb{R}^{r \times n} \) (the controller direct link matrix), while \( p(t) \in \mathbb{R}^n \) is the controller internal state and \( r = m \). Note, if \( H = 0 \), then the ODC is proper.

Considering the equilibrium control, then (1) to (4) imply a description of the form

\[
\begin{bmatrix}
\dot{q}(t) \\
\dot{p}(t)
\end{bmatrix} =
\begin{bmatrix}
A + BHC & BL \\
KC & J
\end{bmatrix}
\begin{bmatrix}
q(t) \\
p(t)
\end{bmatrix} =
\begin{bmatrix}
Mq(t) \\
Mp(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
y(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
C & 0 \\
HC & L
\end{bmatrix}
\begin{bmatrix}
q(t) \\
p(t)
\end{bmatrix} =
\begin{bmatrix}
Nq(t) \\
Np(t)
\end{bmatrix},
\]

where

\[
M =
\begin{bmatrix}
A + BHC & BL \\
KC & J
\end{bmatrix},
N =
\begin{bmatrix}
C & 0 \\
HC & L
\end{bmatrix}.
\]

With the goal to obtain a separable structure, the following theorem is proposed.

**Theorem 1.** The improper ODCs (3) and (4) and the square system (1) and (2) exist if there exist symmetric positive-definite matrices \( P_1, Y_1 \in \mathbb{R}^{n \times n} \) and matrices \( K_1 \in \mathbb{R}^{n \times m}, Z_1 \in \mathbb{R}^{m \times n}, X \in \mathbb{R}^{r \times m}, X_1 \in \mathbb{R}^{r \times m}, H \in \mathbb{R}^{m \times m} \) such that

\[
P_1 = P_1^T > 0, \quad Y_1 = Y_1^T > 0,
\]

\[
- P_1 \begin{bmatrix}
I_n \\
- Y_1
\end{bmatrix} < 0,
\]

\[
P_1 A + A^T P_1 + K_1 C + C^T K_1^T < 0,
\]

\[
A Y_1 + Y_1 A^T + B Z_1 + Z_1^T B^T + B X_1 C + C^T X_1^T B^T < 0,
\]

\[
C Y_1 = X C.
\]

If the above set of conditions is feasible, then it terminates the solution

\[
K = -(P_1 - Y_1^{-1})^{-1} K_1, \quad L = Z_1 Y_1^{-1}, \quad H = X_1 X^{-1},
\]

\[
J = (P_1 - Y_1^{-1})^{-1} (A^T Y_1^{-1} + P_1 A + P_1 B H C Y_1^{-1} + P_1 B L) - K C.
\]
Proof. In this case, the Lyapunov function can be set as
\[ v(q(t), p(t)) = [q^T(t) \ p^T(t)] P [q(t) \ p(t)] > 0 \] (15)
and must be applied to become the time derivative
\[ \dot{v}(q(t), p(t)) = [\dot{q}^T(t) \ \dot{p}^T(t)] P [q(t) \ p(t)] + [q^T(t) \ p^T(t)] P [\dot{q}(t) \ \dot{p}(t)] < 0, \] (16)
\[ [q^T(t) \ p^T(t)] (M^T P + PM) [q(t) \ p(t)] < 0, \] (17)
respectively, which implies
\[ M^T P + PM < 0, \quad P = P^T = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} > 0. \] (18)
Thus, using the Schur complement with respect to (18), it yields
\[ \begin{align*}
Y_1^{-1} &= P_1 - P_2 P_3^{-1} P_2^T > 0, \\
P_1 &= P_1^T > 0, \\
P_3 &= P_3^T > 0, \\
Y_1 &= Y_1^T > 0,
\end{align*} \] (19)
\[ P_1 - Y_1^{-1} = P_2 P_3^{-1} P_2^T > 0, \] (20)
respectively, where \( P_1, P_2, P_3, Y_1 \in \mathbb{R}^{n \times n}. \)
Using the matrices \( P_1 \) and \( Y_1^{-1} \), the transformation of the state variables can be defined
\[ \begin{bmatrix} q(t) \\
p(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -P_3^{-1} P_2 \end{bmatrix} \begin{bmatrix} q_1(t) \\
p_1(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -P_3^{-1} P_2 \end{bmatrix} \begin{bmatrix} Y_1 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_2(t) \\
p_2(t) \end{bmatrix}, \] (21)
\[ \begin{bmatrix} q(t) \\
p(t) \end{bmatrix} = T_1 T_2 \begin{bmatrix} q_2(t) \\
p_2(t) \end{bmatrix}, \] (22)
respectively, where
\[ T_1 = \begin{bmatrix} I & 0 \\ 0 & -P_3^{-1} P_2 \end{bmatrix}, \quad T_2 = \begin{bmatrix} Y_1 & I \\ 0 & 0 \end{bmatrix} \] (23)
and substituting (22) into (17), yields
\[ \begin{bmatrix} q_2^T(t) \\
p_2^T(t) \end{bmatrix} T_2^T T_1^T (M^T P + PM) T_1 T_2 \begin{bmatrix} q_2(t) \\
p_2(t) \end{bmatrix} < 0, \] (24)
\[ T_2^T T_1^T (M^T P + PM) T_1 T_2 < 0, \] (25)
respectively.
Thus, the following can be obtained
\[ T_2^T T_1^T M^T P T_1 T_2 \]
\[ = \begin{bmatrix} Y_1 & Y_1 \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -P_3^{-1} P_2 \end{bmatrix} M^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -P_3^{-1} P_2 \end{bmatrix} T_2 \]
\[ = \begin{bmatrix} Y_1 & Y_1 \\ I & 0 \end{bmatrix} \begin{bmatrix} A^T + C^T H^T B^T & C^T K^T P_1 \\ 0 & L^T B^T \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} Y_1 & I \\ 0 & 0 \end{bmatrix} \] (26)
\[ = \begin{bmatrix} Y_1 A^T + Y_1 C^T H^T B^T & Y_1 C^T K^T - Y_1 P_2 P_3^{-1} f^T \\ A^T & C^T K^T \end{bmatrix} \begin{bmatrix} I & P_1 \\ 0 & P_2 \end{bmatrix} \]
and denoting
\[ f_1^T = P_2 P_3^{-1} f^T, \quad K_1 = K^T P_2^T, \quad L_1 = -P_2 P_3^{-1} L^T, \] (27)
then
\[ T_2^T P_1^T M^T P T_1 T_2 = \begin{bmatrix} Y_1 A^T + Y_1 C^T H^T B^T + Y_1 L_1^T B^T & \Omega_{12} \\ A^T P_1 + C^T K_1^T & \end{bmatrix}, \] (28)
where
\[ \Omega_{12} = Y_1 A^T P_1 + Y_1 C^T H^T B^T P_1 + Y_1 L_1^T B^T P_1 + Y_1 C^T K_1^T - Y_1 J_1^T P_2^T. \] (29)

Setting
\[ \Omega_{12} + A = 0, \] (30)
then (25) implies
\[ T_2^T T_1^T (M^T P + P M) T_1 T_2 = \begin{bmatrix} A Y_1 + Y_1 A^T + Y_1 L_1^T B^T + B L_1 Y_1 + \Pi_{11} & 0 \\ 0 & A^T P_1 + C^T K_1^T + P A + K_1 C \end{bmatrix} \] \[ \prec 0, \] (31)
where
\[ \Pi_{11} = BH CY_1 + Y_1 C^T H^T B^T, \] (32)
which can be expressed by two matrix inequalities
\[ A Y_1 + Y_1 A^T + B L_1 Y_1 + Y_1 L_1^T B^T + B H C Y_1 + Y_1 C^T H^T B^T \prec 0, \] (33)
\[ P_1 A + A^T P_1 + K_1 C + C^T K_1^T \prec 0, \] (34)
where (34) defines (10).

It is evident that (33) is a bilinear matrix inequality with respect to matrix variables \( Y_1 \) and \( H \), (34) is a linear matrix inequality with respect to \( P_1 \) and \( K_1 \), and (33) needs to be linearized.

Writing
\[ B H C Y_1 = B H X X^{-1} C Y_1 = B X_1 C, \] (35)
when defining that
\[ X_1 = H X, \quad X^{-1} C = C Y_1^{-1}, \quad Z_1 = L_1 Y_1, \] (36)
the bilinear matrix inequality (33) is transformed to the linear matrix inequality
\[ A Y_1 + Y_1 A^T + B Z_1 + Z_1^T B^T + B X_1 C + C^T X_1^T B^T \prec 0, \] (37)
conditioned by the matrix equality
\[ C Y_1 = X C. \] (38)

It is possible to verify that the Schur complement form (19) implies also one from the symmetric matrix
\[ \begin{bmatrix} P_1 & -(P_1 - Y_1^{-1}) \\ -(P_1 - Y_1^{-1}) & P_1 - Y_1^{-1} \end{bmatrix} \prec 0, \] (39)
because it yields
\[ P_1 - (P_1 - Y_1^{-1})(P_1 - Y_1^{-1})^{-1}(P_1 - Y_1^{-1}) = P_1 - P_1 + Y_1^{-1} = Y_1^{-1} \succ 0 \] (40)
and the following forms of \( P_2 \) and \( P_3 \) result in
\[ -P_2 = P_3 = P_1 - Y_1^{-1} \succ 0. \] (41)
These conditions eliminate the existing degree of freedom and support the solutions implied by (27)

\[ J^T_1 = -J^T, \quad K^T_1 = -K^T(P_1 - Y_1^{-1}), \quad L^T_1 = L^T, \]

whilst (30) gives

\[ J_1 = -(P_1 - Y_1^{-1})^{-1}(A^T + P_1 A Y_1 + P_1 B H C + P_1 B L_1 Y_1 + K_1 C Y_1) Y_1^{-1} \]

\[ = -(P_1 - Y_1^{-1})^{-1}(A^T Y_1^{-1} + P_1 B H C Y_1^{-1} + P_1 A + P_1 B L + K_1 C). \]

which implies (14).

As a consequence, the inequality (41) has to be satisfied, which introduces an additive matrix inequality structure

\[
\begin{bmatrix}
-P_1 & I_n \\
I_n & -Y_1
\end{bmatrix}
\preceq 0,
\]

forming (9), where \( P_1 \) and \( Y_1 \) are LMI variables. This concludes the proof. \( \square \)

3. Synthesis of ODCs for Uncertain Systems

The class of uncertain continuous-time linear MIMO systems is given as

\[
\dot{q}(t) = (A + \Delta A(t))q(t) + (B + \Delta B(t))u(t),
\]

\[
\begin{bmatrix}
\Delta A(t) \\ \Delta B(t)
\end{bmatrix}
= V \Xi(t) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},
\]

\[
\Xi^T(t) \Xi(t) \preceq I_m,
\]

\[
y(t) = Cq(t),
\]

where the input \( u(t) \in \mathbb{R}^r \), the output \( y(t) \in \mathbb{R}^m \), and the system state \( q(t) \in \mathbb{R}^n \). The matrix parameters are of the following relations \( C \in \mathbb{R}^{m \times n} \), \( B, \Delta B(t) \in \mathbb{R}^{n \times r} \), \( A, \Delta A(t) \in \mathbb{R}^{n \times n} \), \( V \in \mathbb{R}^{n \times p} \), \( W_1 \in \mathbb{R}^{p \times n} \), \( W_2 \in \mathbb{R}^{p \times m} \), and \( r = m = p \).

**Definition 1** ([16]). *If the uncertainties of the system (45) satisfy for all \( t \) the condition (46), where \( V \in \mathbb{R}^{n \times p} \), \( W_1 \in \mathbb{R}^{p \times n} \), \( W_2 \in \mathbb{R}^{p \times m} \) are known real matrices characterizing the uncertainty structure, \( \Xi(t) \in \mathbb{R}^{n \times p} \) is a time varying matrix satisfying the bound (47), and the elements of \( \Xi(t) \) are Lebesgue measurable, then the uncertainties \( \Delta A(t) \) and \( \Delta B(t) \) are matching uncertainties.*

Considering the equilibrium control, then (45) to (49) and (3) and (4) imply the description

\[
\begin{bmatrix}
\dot{q}(t) \\
\dot{p}(t)
\end{bmatrix} = \begin{bmatrix} A + \Delta A(t) + (B + \Delta B(t))HC & (B + \Delta B(t))L \\ KC & J \end{bmatrix} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix},
\]

\[
(M + \Delta M(t)) \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ HC & L \end{bmatrix} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = N \begin{bmatrix} q(t) \\ p(t) \end{bmatrix},
\]

where \( M \) and \( N \) are given as in (7) and

\[
\Delta M(t) = \begin{bmatrix} \Delta A(t) + \Delta B(t)HC & \Delta B(t)L \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} V \Xi(t)W_{1c} & V \Xi(t)W_{2c} \\ 0 & 0 \end{bmatrix},
\]

whilst

\[
W_{1c} = W_1 + W_2 HC, \quad W_{2c} = W_2 L.
\]

To work under matching uncertainties, the following lemmas are applicable.
Theorem 2. The improper ODC (3) and (4) for the uncertain square system (45)–(49) exist if for a

Lemma 2

where $Q$, $R$, $S$ are of compatible dimensions and symmetries, then

(a) if $Q^{-1}$ exists, a Schur complement of $Q$ in $\Omega$ is $S_Q = R - T^T Q^{-1} T$,

(b) if $R^{-1}$ exists, a Schur complement of $R$ in $\Omega$ is $S_R = Q - T R^{-1} T^T$.

Given (49)–(52), the quadratic stability of the closed-loop system is proven.

If this set of LMIs conditioned by the equality (59) is feasible, then it terminates the solution

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Using the quadratic Lyapunov function (15), then (17) and (18) imply

\[
\begin{bmatrix} q^T(t) & p^T(t) \end{bmatrix} \begin{bmatrix} M^T P + P M + \Delta M^T(t) P + P \Delta M(t) \end{bmatrix} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} < 0,
\]

respectively, where a positive definite symmetric matrix $P$ reflects (18).
According to the condition (54), it can be obtained

\[
P \Delta M(t) + \Delta M(t) P
= P \left[ V \Xi(t) W_{1c} \ V \Xi(t) W_{2c} \right] + \left[ V \Xi(t) W_{1c} \ V \Xi(t) W_{2c} \right]^T P
= P \left[ \begin{bmatrix} V \big[ \Xi(t) \big] \ W_{1c} \ W_{2c} \big[ \Xi(t) \big] \end{bmatrix} \right] + \left[ \begin{bmatrix} V \big[ \Xi(t) \big] \ W_{1c} \ W_{2c} \big[ \Xi(t) \big] \end{bmatrix} \right]^T P
\]

(66)

\[
\leq \gamma P \left[ \begin{bmatrix} V \big[ \Xi(t) \big] \ W_{1c} \ W_{2c} \big[ \Xi(t) \big] \end{bmatrix} \right] + \gamma^{-1} \begin{bmatrix} \gamma^{-1} \begin{bmatrix} W_{1c} \ W_{2c} \end{bmatrix} \end{bmatrix} \right] P
\]

where disparities follow from the inequalities (47) and (54).

With the transform matrix \(T_1\) (Equation (23)), it yields (more details can be seen when comparing with (26))

\[
T_1^T P \begin{bmatrix} V V^T \\ 0 \end{bmatrix} P T_1 = \begin{bmatrix} P_1 & P_2 \\ -P_2 P_3^{-1} P_2^T & -P_2 \end{bmatrix} \begin{bmatrix} V V^T \\ 0 \end{bmatrix} P_1 - P_2 P_3^{-1} P_2^T
\]

(67)

\[
= \begin{bmatrix} P_1 V V^T P_1 & -P_1 V V^T P_2 P_3^{-1} P_2^T \\ -P_2 P_3^{-1} P_2^T V V^T P_1 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} P_1 V V^T P_1 & -P_1 V V^T (P_1 - Y_1^{-1}) \\ -(P_1 - Y_1^{-1}) V V^T P_1 & 0 \end{bmatrix}
\]

then

\[
T_2^T T_1^T P \begin{bmatrix} V V^T \\ 0 \end{bmatrix} P T_1 T_2
= Y_1 Y_1 \begin{bmatrix} P_1 V V^T P_1 & -P_1 V V^T (P_1 - Y_1^{-1}) \\ I & 0 \end{bmatrix} Y_1 I
\]

(68)

\[
= Y_1 Y_1 \begin{bmatrix} P_1 V V^T P_1 & P_1 V V^T \\ I & 0 \end{bmatrix} Y_1 I
\]

\[
= Y_1 P_1 V V^T + V V^T P_1 Y_1 - Y_1 P_1 V V^T P_1 Y_1 V V^T P_1
\]

\[
P_1 V V^T
\]

and

\[
T_2^T T_1^T \begin{bmatrix} W_{1c}^T \\ W_{2c}^T \end{bmatrix} \begin{bmatrix} 0 \\ W_{1c} \\ W_{2c} \end{bmatrix} = T_1 T_2
\]

(69)

\[
= Y_1 Y_1 \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} W_{1c}^T \\ W_{2c}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -P_2 P_3^{-1} \end{bmatrix} \begin{bmatrix} Y_1 I \\ Y_1 I \end{bmatrix}
\]

\[
= Y_1 Y_1 \begin{bmatrix} W_{1c}^T + P_1 C^T H^T W_{2c} - Y_1 P_2 P_3^{-1} L^T W_{2c}^T \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
\]

Thus, using the Schur complement property and combining (31) and (66) together with (68) and (69), then (65) can be rewritten as

\[
\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 \\ * & \Lambda_{22} & \Lambda_{23} & 0 \\ * & * & -\gamma I_m & 0 \\ 0 & 0 & 0 & -\gamma I_{n-m} \end{bmatrix} \preceq 0, \quad \Rightarrow \quad \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ * & \Lambda_{22} & \Lambda_{23} \\ * & * & -\gamma I_m \end{bmatrix} \preceq 0
\]
where

\[ \Lambda_{11}^c = \Lambda_{11} - \gamma Y_1 P_1 V V^T P_1 Y_1, \]
\[ \Lambda_{11} = A Y_1 + Y_1 A^T + B L Y_1 + Y_1 L_1^T B^T + B H C Y_1 + Y_1 C^T H^T B^T + \gamma V V^T P_1 Y_1, \]
\[ \Lambda_{12} = Y_1 A^T P_1 + Y_1 C^T H^T B^T P_1 + Y_1 L_1^T B^T P_1 + Y_1 C^T K_1^T - Y_1 f_1^T P_1 + A + \gamma P_1 V V^T, \]
\[ \Lambda_{13} = Y_1 W_1^T + Y_1 C^T H^T W_2 - Y_1 P_2 P_3^{-1} L^T W_2^T, \]
\[ \Lambda_{22} = \Lambda_{22} + \gamma P_1 V V^T P_1, \]
\[ \Lambda_{13} = Y_1 W_1^T + C^T H^T W_2. \]

Since \( \gamma Y_1 P_1 V V^T P_1 Y_1 \) in (71) is symmetric and positive semi-definite, (70) and (74) imply

\[
\begin{bmatrix}
\Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 \\
* & \Lambda_{22} & \Lambda_{23} & \gamma P_1 V \\
* & * & -\gamma I_m & 0 \\
* & * & * & -\gamma I_m \\
\end{bmatrix} < 0 \tag{77}
\]

and setting \( \Lambda_{12}^T = 0 \), then (77) can be reduced to the form

\[
\begin{bmatrix}
\Lambda_{11} & Y_1 W_1^T + Y_1 C^T H^T W_2 - Y_1 P_2 P_3^{-1} L^T W_2^T & 0 \\
* & W_1^T + C^T H^T W_2^T & \gamma P_1 V \\
* & * & -\gamma I_m \\
* & * & * & -\gamma I_m \\
\end{bmatrix} < 0. \tag{78}
\]

Thus, using (27) and (36) adapted by (35), then (58) is defined by using the inequality

\[
\begin{bmatrix}
\Lambda_{11} & Y_1 W_1^T + C^T X_1 W_2^T + Z_1^T W_2^T & 0 \\
* & W_1^T + C^T H^T W_2^T & \gamma P_1 V \\
* & * & -\gamma I_m \\
* & * & * & -\gamma I_m \\
\end{bmatrix} < 0. \tag{79}
\]

Exploiting the setting \( \Lambda_{12}^T = 0 \) and the notation (41) and (42), it implies from (73) that

\[ J = (P_1 - Y_1^{-1})^{-1}(A^T + P_1 A Y_1 + P_1 B H C + P_1 B L Y_1 + K_1 C Y_1 + \gamma V V^T P_1) Y_1^{-1}, \tag{80} \]

which determines (60). This concludes the proof. \( \square \)

4. Synthesis of Proper ODCs

Based on the above relationships, the following Corollaries are immediate.

**Corollary 1.** Since \( \Theta_{12i} = 0 \), setting \( H = 0 \), the design of proper ODCs for uncertainty-free systems (1) and (2) is related to the simplified matrix inequalities implied by (8)–(12)

\[ P_1 = P_1^T > 0, \quad Y_1 = Y_1^T > 0, \quad \begin{bmatrix} -P_1 & I_n \\ I_n & -Y_1 \end{bmatrix} < 0, \tag{81} \]
\[ P_1 A + A^T P_1 + K_1 C + C^T K_1^T < 0, \quad A Y_1 + Y_1 A^T + B Z_1 + Z_1^T B^T < 0, \tag{82} \]

where \( P_1, Y_1 \in \mathbb{R}^{n \times n}, K_1 \in \mathbb{R}^{n \times m}, Z_1 \in \mathbb{R}^{r \times n} \), whilst the controller parameters are given with relation to (61) and (62) as

\[ K = -(P_1 - Y_1^{-1})^{-1} K_1, \quad L = Z_1 Y_1^{-1}, \tag{83} \]
\[ J = (P_1 - Y_1^{-1})^{-1}(A^T Y_1^{-1} + P_1 A + P_1 B L) - K C. \tag{84} \]
Corollary 2. Analogously, considering \( H = 0 \), the design of proper ODCs for uncertain systems (45)–(49) is related to the simplified matrix inequalities, implying from (56)–(61)

\[
P_1 = P_1^T > 0, \quad Y_1 = Y_1^T > 0, \quad \begin{bmatrix} -P_1 & I_n \\ I_n & -Y_1 \end{bmatrix} < 0, \tag{85}
\]

\[
\begin{bmatrix}
\Lambda_{11} & Y_1 W_1^T + C^T X_1^T W_2^T + Z_1^T W_2^T \\
* & -\gamma I_m
\end{bmatrix} < 0 \tag{86}
\]

\[
\Lambda_{11} = A Y_1 + Y_1 A^T + B Z_1 + Z_1^T B^T + \gamma U_1 V V^T + \gamma V V^T U_1, \tag{87}
\]

\[
\Lambda_{22} = P_1 A + A^T P_1 + K_1 C + C^T K_1^T, \tag{88}
\]

and \( P_1, Y_1, U_1 \in \mathbb{R}^{n \times n}, K_1 \in \mathbb{R}^{n \times m}, Z_1 \in \mathbb{R}^{r \times n} \), whilst the controller parameters are given with relation to (61) and (62) as

\[
K = -(P_1 - Y_1^{-1})^{-1} K_1, \quad L = Z_1 Y_1^{-1}, \tag{89}
\]

\[
J = (P_1 - Y_1^{-1})^{-1} ((A^T + \gamma V V^T P_1) Y_1^{-1} + P_1 A + P_1 B L) - KC. \tag{90}
\]

Note that these expressions are defined all in terms of \( P_1 \) and \( Y_1 \).

5. Illustrative Example

To illustrate the effectiveness of the proposed method, the design principle is applied to the system model (45)–(49) defined by the chemical reactor nominal parameters [19]

\[
A = 
\begin{bmatrix}
1.380 & -0.208 & 6.715 & -5.676 \\
-0.581 & 4.290 & 0 & 6.75 \\
1.067 & 4.273 & -6.654 & 5.893 \\
0.048 & 4.273 & 1.343 & -2.104
\end{bmatrix}, \quad B = 
\begin{bmatrix}
0 & 0 \\
5.679 & 0 \\
1.136 & -3.146 \\
1.136 & 0
\end{bmatrix}, \quad C^T = 
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix},
\]

\[
V^T = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad W_1 = 
\begin{bmatrix}
0.02 & 0 & 0 & 0 \\
0 & 0.01 & 0 & 0
\end{bmatrix}, \quad W_2 = 
\begin{bmatrix}
0.2 & 0 \\
0 & 0.2
\end{bmatrix},
\]

where \( A \) is not Hurwitz. The matrix parameters of matching uncertainties define that four additive parametric changes act on the system, namely \( \Delta a_{21}(t) \in (-0.01, +0.01), \Delta a_{32}(t) \in (-0.1, +0.1), \Delta b_{12}(t) \in (-0.2, +0.2), \Delta b_{23}(t) \in (-0.2, +0.2) \), which can occur in arbitrary combinations of the allowable amplitudes and elements. Mainly the changes in these positions affect the robustness of the poles of the closed loop.

Solving (56)–(61) by applying the SeDuMi package [20] in the MATLAB environment, the feasible task with the tuning parameter \( \delta = 15 \) admits a solution with the following variables

\[
P_1 = 
\begin{bmatrix}
2.7693 & 0.0290 & 0.0802 & -0.0830 \\
0.0290 & 0.6064 & 0.0961 & -0.3954 \\
0.0802 & -0.0961 & 0.7537 & -0.0615 \\
-0.0830 & 0.3954 & 0.0615 & 3.9132
\end{bmatrix} > 0, \quad K_1 = 
\begin{bmatrix}
-9.8725 & 10.5900 \\
-2.0899 & 14.8928 \\
-6.4392 & -8.6633 \\
1.9267 & 1.4324
\end{bmatrix},
\]

\[
Y_1 = 
\begin{bmatrix}
6.5341 & 1.4077 & -1.2840 & 0.8904 \\
1.4077 & 7.0436 & -1.4077 & 0.0000 \\
-1.2840 & -1.4077 & 6.5341 & 0.8904 \\
0.8904 & 0.0000 & 0.8904 & 3.5131
\end{bmatrix} > 0, \quad X = 
\begin{bmatrix}
5.2501 & 1.7808 \\
0.8904 & 3.5131
\end{bmatrix},
\]

\[
Z_1 = 
\begin{bmatrix}
5.5822 & -10.0013 & -4.8276 & -0.0179 \\
-4.5994 & 1.3116 & 3.7012 & 1.2332
\end{bmatrix}, \quad X_1 = 
\begin{bmatrix}
0.7546 & -0.0179 \\
8.3005 & 1.2332
\end{bmatrix},
\]
Using matrix relations (61) and (62), the rest of the ODC gains are computed as follows:

\[ L = \begin{bmatrix} 1.0725 & -1.8168 & -0.9133 & -0.0454 \\ 0.8255 & 0.1759 & 0.7740 & -0.0544 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1582 & -0.0853 \\ 5.5539 & 18.2406 & -9.7850 & -44.8285 \end{bmatrix}, \]

\[ K = \begin{bmatrix} 3.5316 & -5.4200 \\ 3.2276 & 44.1842 \\ 11.5027 & 24.8169 \\ -0.1234 & 4.4668 \end{bmatrix}, \quad J = \begin{bmatrix} -2.1728 & -0.0189 & 3.6749 & -0.5433 \\ 5.5539 & 18.2406 & -9.7850 & -44.8285 \\ -9.8079 & -0.0821 & -24.5868 & 16.9435 \\ 1.7001 & 2.0946 & 0.6577 & -7.1565 \end{bmatrix}, \]

where the eigenvalue spectrum of \( J \) is

\[ \rho(J) = \{-5.4922, -18.0733, -14.2955 \pm 5.2437i\}. \]

It is a direct way to construct \( N \) and Hurwitz \( M \) as follows:

\[ N = \begin{bmatrix} 1.0000 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0.1582 & 0 & 0.1582 & -0.0853 & 1.0725 & -1.8168 & -0.9133 & -0.0454 \\ 1.6646 & 0 & 1.6646 & -0.4927 & 0.8255 & 0.1759 & 0.7740 & -0.0544 \end{bmatrix}, \]


where

\[ \rho(M) = \{-4.9274, -18.9560, -1.6232 \pm 2.7060i, -7.4579 \pm 13.0688i, -13.4664 \pm 2.4952i\}. \]

It can seen that controller dynamics as well as closed-loop system dynamics are quadratically stable. It is possible to verify that the closed-loop dynamics is stable also for the working interval, defined by the parametric uncertainties. From the set of eigenvalues, it can be found that the proposed method has acceptable system responses, which correspond to the limits implied by the system’s quadratic stability.

For the sake of clarity, only the stability of the solution is illustrated in the example. Interested readers have the opportunity to compare the results obtained using other methodologies [6,19,21] and find the forced mode structure [22], to see the design problem in the whole complexity.

6. Concluding Remarks

In the paper, a novel method is proposed for design of an improper ODC for the linear uncertain MIMO continuous-time systems. The stability conditions of the dynamic output feedback control are formulated using Lyapunov theory and LMI-based construction combined with an additional matrix equality, conditioning the LMI-based synthesis of ODCs for improper structures. What is specific is the tuning positive scalar, enabling searching for less conservative results. Corollaries outline potential modifications to solve this problem by defining the proper ODC structures. Of principal novelty are the LMIs, related to the system uncertainties with newly formulated structures and partial factorization.

In addition, within the presented design approach, the results provide any additive information on robustness and transfer function matrix norms, which can support the usefulness of the results and their interpretation in the fault tolerant control. Nevertheless, the standard algorithm would be useful to support tasks addressing multi-agent system
control by CDSs and the network controlling levels. The obtained results also show that the proposed approach is able to generate competitive solutions for this class of systems and controllers, the illustrative example confines to the theoretical design points. To the best of authors’ knowledge, no comparable theoretical results are available.

Regarding future works, the ideas presented can be potentially extended to a class of fractional-order systems, switching systems, and interconnected network systems. Choosing the principle of $D$-stability region in the design task is also an interesting problem for improving the proposed method for an improper ODC synthesis, to satisfy additional $H_{\infty}$ robustness constraints [23].

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**Abbreviations**
The following abbreviations are used in this manuscript:

- LMI: Linear Matrix Inequality
- LME: Linear Matrix Equality
- MIMO: Multiple-Input Multiple-Output
- ODC: Output Dynamic Controller
- SeDuMi: Self Dual Minimization

**Notations**
The following basic notations are used in this manuscript:

$q(i), u(i), y(i), p(i)$ state, input and output vectors, controller state vector

$A, B, C$ nominal system matrix parameters

$\Delta A(t), \Delta B(t)$ system uncertainty matrix parameters

$M, N$ closed-loop system parameters

$J, K, L, H$ nominal controller matrix parameters

$H, K_1, P_1, X, X_1, Y_1, Z_1$ matrix variables of LMIs

$I_n, \gamma$ identity matrix, scalar tuning parameter

All other notations are defined in the given context fluently.

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