On \(q\)-Hermite–Hadamard Inequalities via \(q – h\)-Integrals

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Abstract: This paper aims to find Hermite–Hadamard-type inequalities for a generalized notion of integrals called \(q – h\)-integrals. Inequalities for \(q\)-integrals can be deduced by taking \(h = 0\) and are connected with several known results of \(q\)-Hermite–Hadamard inequalities. In addition, we analyzed \(q – h\)-integrals, \(q\)-integrals, and the corresponding inequalities for symmetric functions.

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1. Introduction

In mathematical analysis, integrals and derivatives play an important role. Their application to almost all fields of science and engineering is impressive. There are many new types of derivatives/integrals that have contributed to the literature. For example, fractional order derivatives/ integrals, contour integrals, \(q\)-derivatives/integrals, \(h\)-derivatives/integrals, etc. These new generalized definitions of derivatives are frequently used to enhance the classical results. For example, fractional derivatives/integrals and \(q\)-integrals have been applied to extend and generalize the classical theory of integral and differential equations. For a detailed study on \(q\)-calculus, we refer readers to the works in [1,2].

Inequalities composed of ordinary derivatives and integrals are of great importance for researchers. There are many well-known integral inequalities that have been extensively studied since their appearance. For example, the Ostrowski inequality, Gruss inequality, Opial inequality, and Hermite–Hadamard inequality are commonly studied. Recently, in [3–7], the authors studied Hermite–Hadamard inequalities for \(q\)-integrals.

Our goal in this paper is to derive Hermite–Hadamard-type inequalities for \(q – h\)-integrals. We follow the same pattern of proof as that adopted in [3]. The results of our paper not only generalize the results of [3] but also produce inequalities for \(q\)-integrals and \(h\)-integrals in an implicit form. Some inequalities are given for symmetric convex functions explicitly.

Next, we provide the definition of the convex function and state the Hermite–Hadamard inequality for convex functions.

Definition 1. Let a real valued function \(f\) be defined on a finite interval \(I\) of a real line \(\mathbb{R}\). The function \(f\) is said to be convex on \(I\) if the following inequality holds

\[f(ta + (1 – t)b) \leq tf(a) + (1 – t)f(b)\]

for \(t \in [0, 1]\), \(a, b \in I\).
Convex functions are equivalently studied using the Hermite–Hadamard inequality and are stated as follows:

**Theorem 1 ([8]).** Let \( f : I \rightarrow \mathbb{R} \) be a convex function defined on an interval \( I \subseteq \mathbb{R} \) and \( a, b \in I \), where \( a < b \). Then, the following inequality holds:

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]  

A convex function defined on the interval \([a, b]\) along with a function \( g \), which is non-negative, integrable, and symmetric about \( \frac{a + b}{2} \), gives a generalization of the Hermite–Hadamard inequality. It is stated in the following theorem:

**Theorem 2 ([9]).** Let \( f : I \rightarrow \mathbb{R} \) be a convex function defined on an interval \( I \subseteq \mathbb{R} \) and \( a, b \in I \), where \( a < b \). If \( g \) is non-negative, integrable, and symmetric about \( \frac{a + b}{2} \), then the following inequality holds:

\[
f \left( \frac{a + b}{2} \right) \int_{a}^{b} g(x) \, dx \leq \int_{a}^{b} f(x) g(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_{a}^{b} g(x) \, dx.
\]  

Some recently published results related to the inequalities stated in the above two theorems for \( q \)-integrals can be found in [10–12]. The rest of the paper is organized as follows. In the next section, we provide the definitions of the \( q \)-derivative, \( h \)-derivative, \( q - h \)-derivative, \( q \)-derivative on an interval in the form of \( q_a \)- and \( q_b \)-derivatives, as well as the corresponding definite integrals. We also define analogously \( q_a - h \) and \( q_b - h \)-derivatives and definite integrals. We observe that for symmetric functions, \( q_a \)- and \( q_b \)-integrals become identical. In Section 3, we provide Hermite–Hadamard-type inequalities using \( q_a - h \) and \( q_b - h \)-integrals. We also present some special cases of these inequalities for symmetric functions, as well as connections with the results independently proved in [3,13].

2. Preliminaries and Definitions

Below, we define the notions of the \( q \)-derivative, \( q_a \)-derivative, \( q_b \)-derivative, \( q - h \)-derivative, \( q \)-integral, \( q_a \)-integral, \( q_b \)-integral, and \( q - h \)-integral.

**Definition 2 ([14]).** Let \( 0 < q < 1 \) and \( f : I \rightarrow \mathbb{R} \) be a continuous function. Then, the expression

\[ D_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x} \]  

is the well-known \( q \)-derivative of the function \( f(x) \) and the expression

\[ D_q f(x) = \frac{f(x + h) - f(x)}{h} \]  

is the well-known \( h \)-derivative of the function \( f(x) \).

The following definition of the \( q - h \)-derivative unifies the definitions of the \( q \)-derivative and \( h \)-derivative:
Definition 3 ([15]). Let $0 \leq q \leq 1$, $h \in \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be a continuous function. Then, the $q - h$ derivative of $f$ is defined by

\[
C_h D_q f(x) = \frac{q^h f(x) - f(qx + (1-q)h)}{h}, \quad x \neq q \frac{qh}{1-q} = w,
\]

and

\[
C_h D_q f(w) = \lim_{x \to w} C_h D_q f(x).
\]

For $h = 0$ in (5), the $q - h$-derivative reduces to the $q$-derivative, as defined in (3). For $q = 1$ in (5), the $q - h$-derivative reduces to the $h$-derivative, as defined in (4). Next, we give the definition of $q$-derivatives on an interval $[a, b]$, which are called the $q_a$-derivative and $q_b$-derivative defined on the interval $[a, b]$.

Definition 4 ([13,16]). Let $f : I \subset [a, b] \rightarrow \mathbb{R}$ be a continuous function and let $x \in [a, b]$. Then, the expressions

\[
a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a
\]

and

\[
b D_q f(x) = \frac{f(x) - f(qx + (1-q)b)}{(1-q)(x-b)}, \quad x \neq b,
\]

are called the $q_a$-derivative and $q_b$-derivative, respectively, on $[a, b]$ of function $f$ at $x$. In addition, at $x = a$ we have $a D_q f(a) = \lim_{x \to a} a D_q f(x)$ and at $x = b$ we have $b D_q f(b) = \lim_{x \to b} b D_q f(x)$.

The $q$-definite integrals on the interval $[a, b]$ are called the $q_a$-definite integral and $q_b$-definite integral and are defined as follows:

Definition 5 ([13,16]). Let $f : I \subset [a, b] \rightarrow \mathbb{R}$ be a continuous function and let $x \in [a, b]$. Then, the $q_a$-definite integral and $q_b$-definite integral on $[a, b]$ are given in the following expressions:

\[
\int_a^x f(t)_{a} d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a),
\]

(8)

\[
\int_x^b f(t)_{b} d_q t = (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).
\]

(9)

For $a = 0$, one can obtain the Jackson $q$-integral formula from (8). If $f$ is symmetric about $\frac{a+q}{2}$, from (8), one can see that

\[
\int_a^x f(t)_{a} d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)x).
\]

(10)

From (9) and (10), it can be noted that $\int_a^b f(t)_{a} d_q t = \int_a^b f(t)_{b} d_q t$. Next, we give the definition of the $q - h$-derivative on the interval $[a, b]$ in the form of the $q_a - h$-derivative and $q_b - h$-derivative. In addition, we define the $q - h$-integral on the interval $[a, b]$ in the form of the $q_a - h$-integral and $q_b - h$-integral.

Definition 6. For a continuous function $f : I \rightarrow \mathbb{R}$ and $q \in (0,1)$, the $q_{a} - h$-derivative of $f$ at $x \in [a, b]$ is defined by the expression

\[
C_{h} D_{q}^{a} f(x) = \frac{f(x) - f(qx + (1-q)a + qh)}{(1-q)(x-a) - qh}, \quad x \neq a \frac{(1-q) + qh}{1-q} := x_{o}.
\]

(11)
Analogously, let the \( q_{b-h} \)-derivative of \( f \) at \( x \in [a, b] \) be

\[
C_h D_q^{b} f(x) = \frac{f(x) - f(qx + (1 - q)b + qh)}{(1 - q)(x - b) - qh}, \quad x \neq \frac{b(1 - q) +qh}{1 - q} := y_0. \tag{12}
\]

In addition, \( C_h D_q^{b} f(x_0) = \lim_{x \to x_0} C_h D_q^{b} f(x) \) and \( C_h D_q^{b} f(y_0) = \lim_{x \to y_0} C_h D_q^{b} f(x) \).

For \( h = 0 \), Definition 6 reduces to Definition 4.

**Definition 7.** Let \( 0 < q < 1 \) and \( f : I = [a, b] \to \mathbb{R} \) be a continuous function. Then, the \( q_{a-h} \)-integral and \( q_{b-h} \)-integral on \( I \) denoted by \( I_{q-a}^{b} f \) and \( I_{q-b}^{a} f \) are defined as follows:

\[
I_{q-a}^{b} f(x) := \int_{a}^{x} f(t) d_q^{b} t = ((1 - q)(x - a) + qh) \sum_{n=0}^{\infty} q^n f(q^n x + (1 - q^n)a + nq^n h), \quad x > a,
\]

\[
I_{q-b}^{a} f(x) := \int_{b}^{x} f(t) d_q^{b} t = ((1 - q)(b - x) + qh) \sum_{n=0}^{\infty} q^n f(q^n x + (1 - q^n)b + nq^n h), \quad x < b.
\]

For \( h = 0 \), Definition 7 reduces to Definition 5.

**Example 1.** Let \( g(x) = x \) and \( 0 < q < 1 \). Then, we have

\[
\int_{a}^{b} g(x) dq^{b} x = ((1 - q)(b - a) + qh) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n)a + nq^n h) \tag{13}
\]

\[
= ((1 - q)(b - a) + qh) \sum_{n=0}^{\infty} q^n ((1 - q^n)a + nq^n h)
\]

\[
= ((1 - q)(b - a) + qh) \left( \frac{b + aq}{1 - q} + h \sum_{n=0}^{\infty} nq^n \right)
\]

and

\[
\int_{a}^{b} g(x) dq^{b} x = ((1 - q)(b - a) + qh) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b + nq^n h) \tag{14}
\]

\[
= ((1 - q)(b - a) + qh) \sum_{n=0}^{\infty} q^n ((1 - q^n)b + nq^n h)
\]

\[
= ((1 - q)(b - a) + qh) \left( \frac{a + bq}{1 - q} + h \sum_{n=0}^{\infty} nq^n \right).
\]

3. \( q - h \)-Hermite–Hadamard Inequalities

In this section, we prove the \( q - h \)-Hermite–Hadamard inequality and the varieties of the \( q - h \)-Hermite–Hadamard-type inequalities. We find some \( q \)-Hermite–Hadamard inequalities in special cases. Throughout this paper, we consider the sum of the series \( \sum_{n=0}^{\infty} nq^{2n} \) equal to \( S \).
Theorem 3. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex and differentiable function on \((a, b)\) and \(0 \leq a < b\). Then, we have the following inequality for the \(q_{a-h}\)-integrals:
\[
    f \left( \frac{qa + b}{1 + q} \right) \left( 1 - q \right) (b - a) + \frac{1}{1 - q} \left( (1 - q) (b - a) + qh \right) hS \leq \int_a^b f(x) \, dq_x^a x \leq \left( (1 - q) (b - a) + qh \right) \left( \frac{qf(a) + f(b)}{1 - q^2} + \frac{f(b) - f(a)}{b - a} \right) hS.
\]

Proof. Since \( f \) is a differentiable function on \((a, b)\), let us denote by \( T \), the function describing the tangent line to \( f \) at point \( \frac{qa + b}{1 + q} \in (a, b) \). Then,
\[
    T(x) = f \left( \frac{qa + b}{1 + q} \right) + f' \left( \frac{qa + b}{1 + q} \right) \left( x - \frac{qa + b}{1 + q} \right).
\]

Since \( f \) is a convex function, the following inequality must hold
\[
    f \left( \frac{qa + b}{1 + q} \right) + f' \left( \frac{qa + b}{1 + q} \right) \left( x - \frac{qa + b}{1 + q} \right) \leq f(x),
\]
for all \( x \in (a, b) \). Taking the \(q_{a-h}\)-integral on both sides, we obtain the following inequality:
\[
    \int_a^b \left( f \left( \frac{qa + b}{1 + q} \right) + f' \left( \frac{qa + b}{1 + q} \right) \left( x - \frac{qa + b}{1 + q} \right) \right) hS dx \leq \int_a^b f(x) hS dx.
\]

The left-hand side of the above inequality is calculated as follows: From Example 1, we have
\[
    \int_a^b x \, dq_x^a x = \left( (1 - q) (b - a) + qh \right) \left( \frac{b + a q}{1 - q} + hS \right),
\]
and
\[
    \int_a^b dq_x^a x = \frac{(1 - q) (b - a) + qh}{1 - q}.
\]

By using (18) and (19) in (17), we obtain the first inequality of (15). Let us denote by \( K \) the function representing the line connecting the points \((a, f(a))\) and \((b, f(b))\) as follows:
\[
    K(x) = f(a) + \frac{f(b) - f(a)}{b - a} (x - a).
\]

Since \( f \) is a convex function on \([a, b]\), we have that \( f(x) \leq K(x), x \in [a, b] \). Hence the following inequality holds
\[
    f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a} (x - a),
\]
for all \( x \in [a, b] \). Taking the \(q_{a-h}\)-integrals on both sides, we obtain the following inequality:
\[
    \int_a^b f(x) \, dq_x^a x \leq \int_a^b \left( f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right) hS dx.
\]

By using Example 1 in (21), the second inequality in (15) is obtained. The proof is completed. \( \Box \)

Remark 1. Under the assumptions of the aforementioned theorem, one can obtain the following results:
(i) If \( h = 0 \) in Theorem 3, we obtain the following inequality, which is independently proved in Theorem 6 [3]:

\[
f \left( \frac{qa + b}{1 + q} \right) \leq \frac{1}{b - a} \int_a^b f(x)dh^q_x x \leq \frac{qf(a) + f(b)}{1 + q}. \tag{22}
\]

Moreover, for \( f \) to be symmetric about \( \frac{a + b}{2} \), we have that the \( q \)-\( h \)-integral also satisfies the inequality in (22).

(ii) If \( h = 0 \) and \( q \to 1 \) in Theorem 3, we obtain the Hermite–Hadamard inequality:

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{23}
\]

Some more versions of the \( q - h \)-Hermite–Hadamard inequality are stated and proved in the following results.

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be a convex and differentiable function on \((a, b)\) and \( 0 \leq a < b \). Then, we have the following inequality for the \( q_{b-h} \)-integrals:

\[
f \left( \frac{a + qb}{1 + q} \right) \left( 1 - q \right)(b - a) + qh + f' \left( \frac{a + qb}{1 + q} \right) \left( 1 - q \right)(b - a) + qh \right) hS \tag{24}
\]

\[
\leq \int_a^b f(x)h^q_x x \leq ((1 - q)(b - a) + qh) \left( \frac{f(a) + qf(b)}{1 - q^2} + \frac{f(b) - f(a)}{b - a} hS \right).
\]

**Proof.** Since \( f \) is a differentiable function on \((a, b)\), let us denote by \( T_1 \) the function describing the tangent line to \( f \) at point \( \frac{a + qb}{1 + q} \in (a, b) \). We have that \( T_1(x) \leq f(x), x \in [a, b] \), i.e., the following inequality holds

\[
f \left( \frac{a + qb}{1 + q} \right) + f' \left( \frac{a + qb}{1 + q} \right) \left( x - \frac{a + qb}{1 + q} \right) \leq f(x), \tag{25}
\]

for all \( x \in [a, b] \). Taking the \( q_{b-h} \)-integral on both sides, we obtain the following inequality:

\[
\int_a^b \left( f \left( \frac{a + qb}{1 + q} \right) + f' \left( \frac{a + qb}{1 + q} \right) \left( x - \frac{a + qb}{1 + q} \right) \right) \mu^q_{h} dx \leq \int_a^b f(x) \mu^q_{h} dx,
\]

which takes the following form:

\[
f \left( \frac{a + qb}{1 + q} \right) \int_a^b \mu^q_{h} dx + f' \left( \frac{a + qb}{1 + q} \right) \int_a^b \left( x - \frac{a + qb}{1 + q} \right) \mu^q_{h} dx \leq \int_a^b f(x) \mu^q_{h} dx. \tag{26}
\]

From Example 1, we have

\[
\int_a^b x \mu^q_{h} dx = ((1 - q)(b - a) + qh) \left( \frac{a + bq}{1 - q^2} + hS \right), \tag{27}
\]

and

\[
\int_a^b \mu^q_{h} dx = \frac{(1 - q)(b - a) + qh}{1 - q}. \tag{28}
\]
By using (27) and (28) in the above inequality in (26), one can obtain the first inequality of (24). On the other hand, by taking the $q_{a-h}$-integrals on both sides of the inequality in (20), we have

\[
\int_a^b f(x)_{h}d^q_{a}x \leq \int_a^b \left( f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)_{h}d^q_{a}x. \tag{29}
\]

By using (18) and (19) in the above inequality in (29), one can obtain the second inequality of (24). \qed

**Remark 2.** Under the assumptions of Theorem 3, one can obtain the following results:

(i) If $h = 0$ in Theorem 4, we obtain the following inequality, which is independently proved in Theorem 12 [13]:

\[
f\left( \frac{a + qb}{1 + q} \right) \leq \frac{1}{b - a} \int_a^b f(x)_{0}d^q_{a}x \leq \frac{f(a) + qf(b)}{1 + q}. \tag{30}
\]

Moreover, for $f$ to be symmetric about $\frac{a + b}{2}$, we have that the $q_{b}$-integral also satisfies the inequality in (30).

(ii) If $h = 0$ and $q \to 1$ in Theorem 4, we obtain the Hermite–Hadamard inequality (23).

(iii) By adding the inequalities in (22) and (30), one can obtain Corollary 14 [13].

**Theorem 5.** Under the assumptions of Theorem 3, the following inequality also holds:

\[
f\left( \frac{a + b}{2} \right) \frac{(1 - q)(b - a) + qh}{1 - q} + f'\left( \frac{a + b}{2} \right)\left( (1 - q)(b - a) + qh \right) \left( hS + \frac{b - a}{2(1 + q)} \right) \tag{31}
\]

\[
\leq \int_a^b f(x)_{h}d^q_{a}x \leq ((1 - q)(b - a) + qh) \left( \frac{qf(a) + f(b)}{1 - q^2} + \frac{f(b) - f(a)}{b - a}hS \right).
\]

**Proof.** Since $f$ is a differentiable function on $(a, b)$, let us denote by $T_2$ the function describing the tangent line to $f$ at point $\frac{a + b}{2} \in (a, b)$. We have that $T_2(x) \leq f(x), x \in [a, b]$, i.e., the following inequality holds

\[
f\left( \frac{a + b}{2} \right) + f'\left( \frac{a + b}{2} \right)\left( x - \frac{a + b}{2} \right) \leq f(x), \tag{32}
\]

for all $x \in [a, b]$. Taking the $q_{a-h}$-integral on both sides, we obtain the following inequality:

\[
f\left( \frac{a + b}{2} \right) \int_a^b h_{a}d^q_{a}x + f'\left( \frac{a + b}{2} \right) \int_a^b \left( x - \frac{a + b}{2} \right) h_{a}d^q_{a}x \leq \int_a^b f(x)_{a}d^q_{a}x. \tag{33}
\]

By using (18) and (19) in the above inequality, we obtain the first inequality of (31). The second inequality of (31) has already been proved in Theorem 3. \qed

**Remark 3.** Under the assumptions of Theorem 5, one can obtain the following results:

(i) If $h = 0$ in Theorem 5, we obtain the following inequality, which is independently proved in Theorem 9 [3]:

\[
f\left( \frac{a + b}{2} \right) + f'\left( \frac{a + b}{2} \right) \frac{(b - a)(1 - q)}{2(1 + q)} \leq \frac{1}{b - a} \int_a^b f(x)_{0}d^q_{a}x \leq \frac{qf(a) + f(b)}{1 + q}. \tag{34}
\]

Moreover, for $f$ to be symmetric about $\frac{a + b}{2}$, we have that the $q_{b}$-integral also satisfies the inequality in (34).
(ii) If \( h = 0 \) and \( q \to 1 \) in Theorem 5, we obtain the Hermite–Hadamard inequality.

The \( q_b-h \)-integral version of the above results is given in the following theorem.

**Theorem 6.** Under the assumptions of Theorem 4, the following inequality also holds:

\[
\begin{align*}
&f \left( \frac{a + b}{2} \right) \left( 1 - q \right) (b - a) + \frac{q h}{1 - q} \left( (1 - q) (b - a) + qh \left( hS - \frac{b - a}{2(1 + q)} \right) \right) \leq \int_a^b f(x) q_b^h dx \leq \left( (1 - q) (b - a) + qh \left( \frac{f(a) + qf(b)}{1 - q^2} + \frac{f(b) - f(a)}{b - a} hS \right) \right) \int_a^b f(x) q_b^h dx. \tag{35}
\end{align*}
\]

**Proof.** Taking the \( q_b-h \)-integral on both sides of the inequality in (32), we have

\[
\begin{align*}
&f \left( \frac{a + b}{2} \right) \int_a^b q_b^h x + f' \left( \frac{a + b}{2} \right) \int_a^b \left( x - \frac{a + b}{2} \right) q_b^h x \leq \int_a^b f(x) q_b^h x. \tag{36}
\end{align*}
\]

By using (27) and (28) in the above inequality, we obtain the first inequality of (35). The second inequality of (35) has already been proved in Theorem 4. \( \square \)

**Remark 4.** Under the assumptions of Theorem 5, one can obtain the following results:

i If \( h = 0 \) in Theorem 6, we obtain the following inequality:

\[
\begin{align*}
f \left( \frac{a + b}{2} \right) - f' \left( \frac{a + b}{2} \right) \left( b - a \right) \frac{1 - q}{2(1 + q)} \leq \frac{1}{b - a} \int_a^b f(x) q_b^h x \leq \frac{f(a) + qf(b)}{1 + q} \tag{37}
\end{align*}
\]

Moreover, for \( f \) to be symmetric for \( \frac{a + b}{2} \), we have that the \( q_b \)-integral also satisfies the inequality in (30).

ii If \( h = 0 \) and \( q \to 1 \) in Theorem 6, we obtain the Hermite–Hadamard inequality.

iii By adding the inequalities in (34) and (37), one can obtain Corollary 15 [13].

4. Conclusions

In this paper, we have studied Hermite–Hadamard inequalities for a composite definition of \( q \)- and \( h \)-integrals, the so-called \( q - h \)-integrals. These inequalities are called \( q - h \)-Hermite–Hadamard inequalities. From the established inequalities, one can obtain the main results in [3,13]. Inequalities for \( q \)-integrals are further studied for symmetric functions. The definition of the \( q - h \)-integrals can be used to generalize classical and new inequalities of Riemann integrals.

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