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On the Study of Rainbow Antimagic Connection Number of Comb Product of Friendship Graph and Tree

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Abstract: Given a graph G with vertex set $V(G)$ and edge set $E(G)$, for the bijective function $f(V(G)) \rightarrow \{1, 2, \ldots, |V(G)|\}$, the associated weight of an edge $xy \in E(G)$ under f is $w(xy) = f(x) + f(y)$. If all edges have pairwise distinct weights, the function f is called an edge-antimagic vertex labeling. A path P in the vertex-labeled graph G is said to be a rainbow $x - y$ path if for every two edges $xy, x'y' \in E(P)$ it satisfies $w(xy) \neq w(x'y')$. The function f is called a rainbow antimagic labeling of G if there exists a rainbow $x - y$ path for every two vertices $x, y \in V(G)$. We say that graph G admits a rainbow antimagic coloring when we assign each edge xy with the color of the edge weight $w(xy)$. The smallest number of colors induced from all edge weights of antimagic labeling is the rainbow antimagic connection number of G, denoted by $rac(G)$. This paper is intended to investigate non-symmetrical phenomena in the comb product of graphs by considering antimagic labeling and optimizing rainbow connection, called rainbow antimagic coloring. In this paper, we show the exact value of the rainbow antimagic connection number of the comb product of graph $\mathcal{F}_n \rhd T_m$, where $\mathcal{F}_n$ is a friendship graph with order $2n + 1$ and $T_m \in \{P_m, S_m, Br_{m,p}, S_{m,m}\}$, where P_m is the path graph of order m, S_m is the star graph of order $m + 1$, $Br_{m,p}$ is the broom graph of order $m + p$ and $S_{m,m}$ is the double star graph of order $2m + 2$.

Keywords: rainbow antimagic connection number; antimagic labeling; rainbow coloring; comb product of graphs; graph theory; discrete mathematics

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1. Introduction

Let G and H be two connected graphs. Let v be a vertex of graph H. The comb product between graphs G and H, denoted by $G \rhd H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i-th copy of H at the vertex v to the i-th vertex of G. In this study, the graph definition used is based on Chartrand et al. [1].

The study of the graph in this research is focused on rainbow antimagic coloring, which is a combination of the concepts of antimagic labeling and rainbow coloring. Rainbow connection definitions can be found in [2,3]. Let G be a connected graph, the edge coloring of G with the function $f(E(G)) \rightarrow \{1, 2, \ldots, k\}, k \in E(G)$ is k-coloring of graph G, where adjacent edges can be colored with the same color. Rainbow $u - v$ path is the path in G if no two edges are the same color. The graph G is a rainbow connection if every $u, v \in V(G)$ has a rainbow path. The edge coloring on G has a rainbow connection called rainbow coloring. The minimum colors to make G rainbow-connected is called the rainbow connection number of G and is denoted by $rc(G)$. Rainbow coloring is an interesting study and has found many results, including [4,5].

Rainbow vertex coloring and rainbow total coloring are other variants of rainbow coloring. Rainbow vertex coloring was introduced in [6] and rainbow vertex coloring results can be found in [7,8]. Total rainbow coloring results can be seen in [9,10]. Wallis et al. [11]

introduced graph labeling. Hartsfield and Ringel [12] introduced antimagic labeling. The antimagic labeling of graph G is a bijective function of the edge set $E(G)$ to $\{1, 2, ..., |E(G)|\}$ and $w(v) = \Sigma_{e \in E(v)} f(e)$, and $E(v)$ is the set of edges incident with the vertex v for vertices $u, v \in V(G)$, $w(u) \neq w(v)$.

Antimagic labeling has had several results, including those of Baca et al. in [13–16]. Dafik et al. contributed to the antimagic labeling in [17]. In addition, antimagic labeling results can also be found in [18,19]. The concept of combining graph labeling and graph coloring was initiated by Arumugam et al. [20]. The bijective function from edge set $E(G)$ to $\{1, 2, ...|E(G)|\}$ is $w(v) = \Sigma_{e \in E(v)} f(e)$, and $E(v)$ is the set of edges that are incident to vertices v, for every $v \in V(G)$ The bijective function f for two adjacent vertices $u, v \in V(G), w(u) \neq w(v)$ is called antimagic labeling. The coloring of the vertices on G with the vertices of v colored with $w(v)$ is the local antimagic label. If we consider the local antimagic labeling chromatic number, then it is called local antimagic coloring.

Motivated by the combination performed by Arumugam, in [21], Dafik et al. took the initiative to combine the concepts of antimagic labeling and rainbow coloring on graphs into a new concept, namely, rainbow antimagic coloring. Septory et al. determined the lower bound of rainbow antimagic connection number for any connected graph.

Lemma 1 ([22])**.** *Let G be any connected graph. Let $rc(G)$ and $\Delta(G)$ be the rainbow connection number of G and the maximum degree of G, respectively. $rac(G) \geq \max\{rc(G), \Delta(G)\}$.*

While Dafik et al., has two theorems about characterizing the existence of rainbow $u - v$ path of any graph of $diam(G) \leq 2$, the result of rainbow antimagic connection number of friendship graph and any tree is shown in the following Theorem.

Theorem 1 ([21])**.** *Let G be a connected graph of diameter $diam(G) \leq 2$. Let f be any bijective function from $V(G)$ to the set $\{1, 2, \ldots, |V(G)|\}$; there exists a rainbow $u - v$ path.*

Theorem 2 ([21])**.** *For $n \geq 2$, $rac(\mathcal{F}_n) = 2n$.*

Theorem 3 ([21])**.** *For T_m, being any tree of order $m \geq 3$, $rac(T_m) = m - 1$.*

Some other results about rainbow antimagic connection number can be found in [21–24]. In this paper, we will study the rainbow antimagic connection number of graph $\mathcal{F}_n \rhd T_m$ where $\mathcal{F}_n$ is a friendship graph with order $2n + 1$ and $T_m \in \{P_m, S_m, Br_{m,p}, S_{m,m}\}$, where P_m is the path graph of order m, S_m is the star graph of order $m + 1$, $Br_{m,p}$ is the broom graph of order $m + p$ and $S_{m,m}$ is the double star graph of order $2m + 2$.

2. Results

In this section, we will show our new results about rainbow antimagic connection number on those above graphs stated in a lemma and theorem. Our strategy is firstly determined with the lower bound rainbow antimagic connection number of $rac(\mathcal{F}_n \rhd T_m)$. Finally, we show the exact values of $rac(\mathcal{F}_n \rhd P_m)$, $rac(\mathcal{F}_n \rhd S_m)$, $rac(\mathcal{F}_n \rhd Br_{m,p})$ and $rac(\mathcal{F}_n \rhd S_{m,m})$.

Lemma 2. *Let $\mathcal{F}_n \rhd T_m$ be a comb product of friendship graph $\mathcal{F}_n$ and T_m be any tree of order $m \geq 2$. The lower bound of $rac(\mathcal{F}_n \rhd T_m) \geq rac(T_m)(|V(\mathcal{F}_n)|)$.*

Proof of Lemma 2. Graph $\mathcal{F}_n \rhd T_m$ is the comb product of two graphs $\mathcal{F}_n$ and T_m, and o is the vertex of T_m. Graph $\mathcal{F}_n \rhd T_m$ is obtained by taking one copy of $\mathcal{F}_n$ and $|V(\mathcal{F}_n)|$ copies of T_m and grafting o from the i-th copy of T_m at the i-th vertex of $\mathcal{F}_n$. By this definition, it implies that graph $\mathcal{F}_n \rhd T_m$ contains graph $\mathcal{F}_n$ and $|V(\mathcal{F}_n)|$ copies of graph T_m. We can determine $rac(\mathcal{F}_n \rhd T_m)$ by finding $rac(\mathcal{F}_n)$ and $rac(T_m)$. Based on Theorem 2, we have $rac(\mathcal{F}_n) = 2n$. Based on Theorem 3, we have $rac(T_m) = E(T_m) = m - 1$; so, every edge of all i-th copies of T_m has a different color. Thus, $rac(\mathcal{F}_n \rhd T_m) \geq (rac(T_m)(|V(\mathcal{F}_n)|)$. □

Theorem 4. *For $n, m \geq 3$, $rac(\mathcal{F}_n \triangleright P_m) = 2nm + m - 2n - 1$.*

Proof of Theorem 4. Graph $\mathcal{F}_n \triangleright P_m$ is a connected graph with vertex set $V(\mathcal{F}_n \triangleright P_m) = \{a\} \cup \{x_i, y_i, 1 \leq i \leq n\} \cup \{x_{ij}, y_{ij}, 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{z_j, 1 \leq j \leq m-1\}$ and edge set $E(\mathcal{F}_n \triangleright P_m) = \{ax_i, ay_i, x_iy_i, x_ix_1, y_iy_1, 1 \leq i \leq n\} \cup \{az_1\} \cup \{z_jz_{j+1}, 1 \leq j \leq m-2\} \cup \{x_{ij}x_{ij+1}, y_{ij}y_{ij+1}, 1 \leq i \leq n, 1 \leq j \leq m-2\}$. The cardinality of $|V(\mathcal{F}_n \triangleright P_m)| = 2nm + m - 1$ and the cardinality of $|E(\mathcal{F}_n \triangleright P_m)| = n + m + 2nm$.

To prove the rainbow antimagic connection number of $rac(\mathcal{F}_n \triangleright P_m)$, first, we have to show the lower bound of $rac(\mathcal{F}_n \triangleright P_m)$. Based on Lemma 2, we have rac $(\mathcal{F}_n \triangleright P_m) \geq rac(P_m)(|V(\mathcal{F}_n)|)$. Since $rac(P_m) = m-1$, $rac(\mathcal{F}_n \triangleright P_m) \geq 2nm + m - 2n - 1$.

Secondly, we have to show the upper bound of $rac(\mathcal{F}_n \triangleright P_m)$. Define the vertex labeling $f(V(\mathcal{F}_n \triangleright P_m)) \to \{1, 2, ..., 2nm + m - 1\}$ as follows:

$$
\begin{aligned}
f(a) &= 2n+1 \\
f(x_i) &= 2n+1-i, \text{ for } 1 \leq i \leq n \\
f(y_i) &= 4n+2-i, \text{ for } 1 \leq i \leq n \\
f(z_j) &= 2nj+2n+j+1, \text{ for } 1 \leq j \leq m-1 \\
f(x_{ij}) &= \begin{cases} 2n-2i+1, & \text{for } 1 \leq i \leq n, j=1 \\ 2nj+j+i, & \text{for } 1 \leq i \leq n, 2 \leq j \leq m-1 \end{cases} \\
f(y_{ij}) &= \begin{cases} 2i, & \text{for } 1 \leq i \leq n, j=1 \\ 2nj+2n+j-i+1, & \text{for } 1 \leq i \leq n, 2 \leq j \leq m-1 \end{cases}
\end{aligned}
$$

The edge weights of the above vertex labeling f can be presented as

$$
\begin{aligned}
w(ax_i) &= 4n+2+i, \text{ for } 1 \leq i \leq n \\
w(ay_i) &= 6n+3-i, \text{ for } 1 \leq i \leq n \\
w(az_1) &= 6n+3 \\
w(z_jz_{j+1}) &= 6n+4nj+2j+3, \text{ for } 1 \leq j \leq m-2 \\
w(x_ix_{i1}) &= 4n+2-i, \text{ for } 1 \leq i \leq n \\
w(y_iy_{i1}) &= 4n+4-i, \text{ for } 1 \leq i \leq n \\
w(x_{ij}x_{ij+1}) &= \begin{cases} 6n+3-i & \text{for } 1 \leq i \leq n, j=1 \\ 2n+4nj+2j+2i+1 & \text{for } 1 \leq i \leq n,\ 2 \leq j \leq m-2 \end{cases} \\
w(y_{ij}y_{ij+1}) &= \begin{cases} 6n+i+3 & \text{for } 1 \leq i \leq n, j=1 \\ 6n+4nj+2j-2i+3 & \text{for } 1 \leq i \leq n,\ 2 \leq j \leq m-2 \end{cases}
\end{aligned}
$$

It is easy to see that the above edge weight will induce a rainbow antimagic coloring of graph $\mathcal{F}_n \triangleright P_m$. Based on Theorem 3, $rac(P_m) = m-1$; since $E(P_m) = m-1$, the weight of each edge in graph P_m is different. Therefore, the sum of the weights on $|V(\mathcal{F}_n)|$ copies of graph P_m is $(|V(\mathcal{F}_n)|)(|E(P_m)|) = 2nm + m - 2n - 1$. Based on the description above, we have that the distinct weight of graph $(\mathcal{F}_n \triangleright P_m)$ is $2nm + m - 2n - 1$. It implies that the edge weights of $f(V(\mathcal{F}_n \triangleright P_m)) \to \{1, 2, ..., 2nm + m - 1\}$ induce a rainbow antimagic coloring of $2nm + m - 1$ colors. Thus, $rac(\mathcal{F}_n \triangleright P_m) \leq 2nm + m - 2n - 1$. Comparing the two bounds, we have the exact value of $rac(\mathcal{F}_n \triangleright P_m) = 2nm + m - 2n - 1$.

The next step is to evaluate to prove the existence of a rainbow $u-v$ path $\mathcal{F}_n \triangleright P_m$. Based on the definition of graph $\mathcal{F}_n \triangleright P_m$, graph $\mathcal{F}_n \triangleright P_m$ contains one graph $\mathcal{F}_n$ and $|V(\mathcal{F}_n)|$ copies of P_m; so, we can evaluate the rainbow $u-v$ path of graph $\mathcal{F}_n \triangleright P_m$ by evaluating the rainbow $u-v$ path on graph $\mathcal{F}_n$ and graph P_m. Since $diam(\mathcal{F}_n) = 2$, based on Theorem 1, there is a rainbow $u-v$ path for every $u, v \in V(\mathcal{F}_n)$. Based on Theorem 3, $rac(P_m) = m-1$; since P_m has $m-1$ edges, there is a rainbow $u-v$ path for every $u, v \in V(P_m)$. Therefore, according to the explanation, it can be seen that there is a rainbow $u-v$ path for every $u, v \in V(\mathcal{F}_n \triangleright P_m)$. □

The illustration of a rainbow antimagic coloring of graph $\mathcal{F}_n \triangleright P_m$ can be seen in Figure 1.

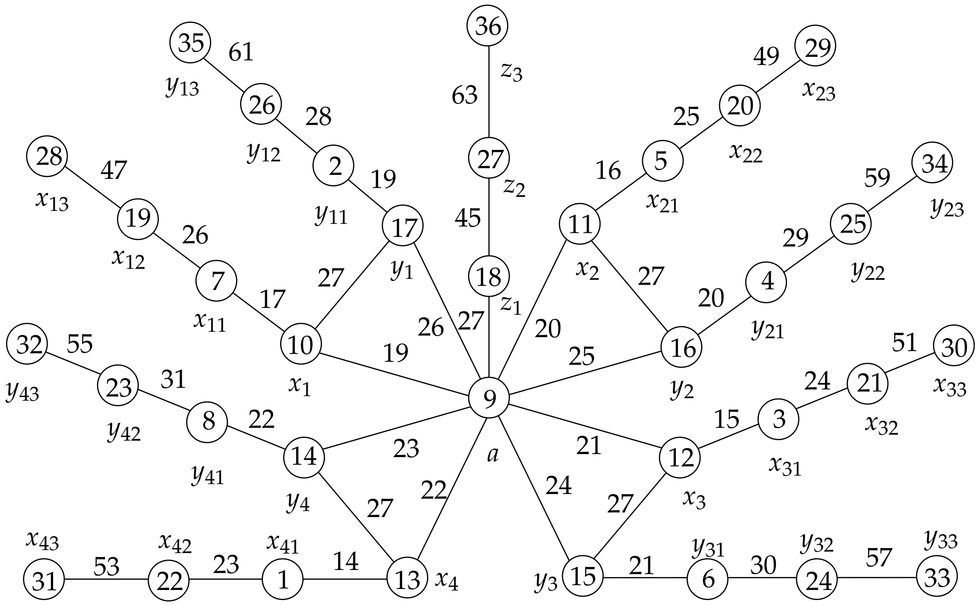

Figure 1. The illustration of rainbow antimagic coloring of graph $\mathcal{F}_4 \triangleright P_4$.

Theorem 5. *For* $n \geq 3, m = 2n-1$, $rac(\mathcal{F}_n \triangleright S_m) = 2nm + m$.

Proof of Theorem 5. Graph $\mathcal{F}_n \triangleright S_m$ is a connected graph with vertex set $V(\mathcal{F}_n \triangleright S_m) = \{a\} \cup \{x_i, y_i, 1 \leq i \leq n\} \cup \{z_j, 1 \leq j \leq m\} \cup \{x_{ij}, y_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(\mathcal{F}_n \triangleright S_m) = \{ax_i, ay_i, x_iy_i, 1 \leq i \leq n\} \cup \{az_j, 1 \leq j \leq m\} \cup \{x_ix_{ij}.y_iy_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$. The cardinality of $|V(\mathcal{F}_n \triangleright S_m)| = 2n + m + 2nm + 1$ and the cardinality of $|E(\mathcal{F}_n \triangleright S_m)| = 3n + m + 2nm$.

To prove the rainbow antimagic connection number of $rac(\mathcal{F}_n \triangleright S_m)$, first, we have to show the lower bound of $rac(\mathcal{F}_n \triangleright S_m)$. Based on Lemma 2, we have rac $(\mathcal{F}_n \triangleright S_m) \geq rac(S_m)(|V(\mathcal{F}_n)|)$. Since $rac(S_m) = m$, $rac(\mathcal{F}_n \triangleright S_m) \geq 2nm + m$.

Secondly, we have to show the upper bound of $rac(\mathcal{F}_n \triangleright S_m)$. Define the vertex labeling $f(V(\mathcal{F}_n \triangleright S_m)) \to \{1, 2, ..., 2n + m + 2nm + 1\}$ as follows:

$$f(a) = 2n + 1$$
$$f(x_i) = 4n + i, \text{ for } 1 \leq i \leq n$$
$$f(y_i) = 6n + 1 - i, \text{ for } 1 \leq i \leq n$$
$$f(z_j) = \begin{cases} 8n & \text{for } j = 1 \\ 2j - 1 & \text{for } 2 \leq j \leq \lceil \frac{m}{2} \rceil \\ 2j + 1 & \text{for } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m \end{cases}$$
$$f(x_{ij}) = \begin{cases} 4n + 2 - 2i & \text{for } 1 \leq i \leq n, j = 1 \\ 1 & \text{for } i = 1, j = 2 \\ 6n + j - 2 & \text{for } i = 1, 3 \leq j \leq m \\ 6n + m + j - 3 & \text{for } i = 2, 2 \leq j \leq 3 \\ 6n + m + j - 2 & \text{for } i = 2, 4 \leq j \leq m \\ 6n + im + j - m - i & \text{for } 3 \leq i \leq n, 2 \leq j \leq m \end{cases}$$
$$f(y_{ij}) = \begin{cases} 2i & \text{for } 1 \leq i \leq n, j = 1 \\ 2nm + 2n + m + i + j - im & \text{for } 1 \leq i \leq n, 2 \leq j \leq m \end{cases}$$

The edge weights of the above vertex labeling f can be presented as

$$w(ax_i) = 6n + 1 + i, \text{ for } 1 \leq i \leq n$$

$$w(ay_i) = 8n + 2 - i, \text{ for } 1 \le i \le n$$

$$w(az_j) = \begin{cases} 10n+1 & \text{for } j = 1 \\ 2n+2j & \text{for } 2 \le j \le \lceil \frac{m}{2} \rceil \\ 2n+2j+2 & \text{for } \lceil \frac{m}{2} \rceil + 1 \le j \le m \end{cases}$$

$$w(x_i x_{ij}) = \begin{cases} 8n+2-i & \text{for } 1 \le i \le n, j = 1 \\ 4n+2 & \text{for } i = 1, j = 2 \\ 10n+j+i-2 & \text{for } i = 1, 3 \le j \le m \\ 10n+m+j+i-3 & \text{for } i = 2, 2 \le j \le 3 \\ 10n+m+j+i-2 & \text{for } i = 2, 4 \le j \le m \\ 10n+im+j-m & \text{for } 3 \le i \le n, 2 \le j \le m \end{cases}$$

$$w(y_i y_{ij}) = \begin{cases} 6n+1+i & \text{for } 1 \le i \le n, j = 1 \\ 2nm+8n+m+j-im+1 & \text{for } 1 \le i \le n, 2 \le j \le m \end{cases}$$

It is easy to see that the above edge weight will induce a rainbow antimagic coloring of graph $\mathcal{F}_n \triangleright S_m$. Based on Theorem 3, $rac(S_m) = m$; since $E(S_m) = m$, the weight of each edge in graph S_m is different. Therefore, the sum of the weights on $|V(\mathcal{F}_n)|$ copies of graph S_m is $(|V(\mathcal{F}_n)|)(|E(S_m)|) = 2nm + m$. Based on the description above, we have that the distinct weight of graph $(\mathcal{F}_n \triangleright S_m)$ is $2nm + m$. It implies that the edge weights of $f(V(\mathcal{F}_n \triangleright S_m)) \rightarrow \{1, 2, ..., 2n + m + 2nm + 1\}$ induce a rainbow antimagic coloring of $2nm + m$ colors. Thus, $rac(\mathcal{F}_n \triangleright S_m) \le 2nm + m$. Comparing the two bounds, we have the exact value of $rac(\mathcal{F}_n \triangleright S_m) = 2nm + m$.

The next step is to evaluate to prove the existence of a rainbow $u - v$ path $\mathcal{F}_n \triangleright S_m$. Based on the definition of graph $\mathcal{F}_n \triangleright S_m$, graph $\mathcal{F}_n \triangleright S_m$ contains one graph $\mathcal{F}_n$ and $|V(\mathcal{F}_n)|$ copies of S_m; so, we can evaluate the rainbow $u - v$ path of graph $\mathcal{F}_n \triangleright S_m$ by evaluating the rainbow $u - v$ path on graph $\mathcal{F}_n$ and graph S_m. Since $diam(\mathcal{F}_n) = 2$, based on Theorem 1, there is a rainbow $u - v$ path for every $u, v \in V(\mathcal{F}_n)$. Based on Theorem 3, $rac(S_m) = m$; since S_m has m edges, there is a rainbow $u - v$ path for every $u, v \in V(S_m)$. Therefore, according to the explanation, it can be seen that there is a rainbow $u - v$ path for every $u, v \in V(\mathcal{F}_n \triangleright S_m)$. □

The illustration of a rainbow antimagic coloring of graph $\mathcal{F}_3 \triangleright S_5$ can be seen in Figure 2.

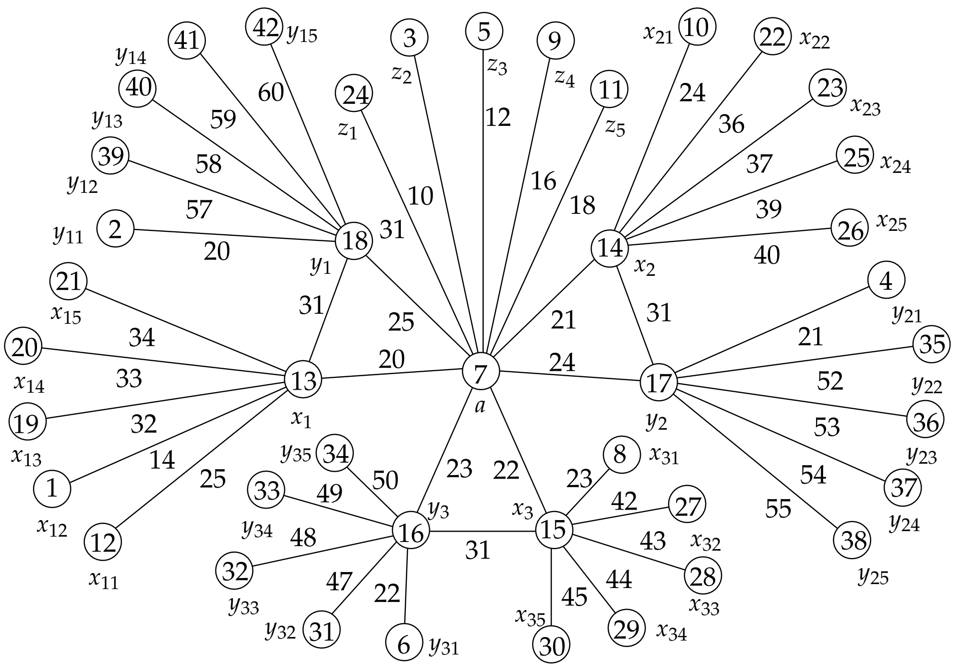

Figure 2. The illustration of rainbow antimagic coloring of graph $\mathcal{F}_3 \triangleright S_5$.

Theorem 6. *For $n, m, p \ge 3$, $rac(\mathcal{F}_n \triangleright Br_{m,p}) = 2nm + 2np + m + p - 2n - 1$.*

Proof of Theorem 6. Graph $\mathcal{F}_n \rhd Br_{m,p}$ is a connected graph with vertex set $V(\mathcal{F}_n \rhd Br_{m,p}) = \{a\} \cup \{x_i, y_i, 1 \leq i \leq n\} \cup \{x_{ij}, y_{ij}, 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{z_j, 1 \leq j \leq m-1\} \cup \{z_{mk}, b_{jk}, c_{jk}, 1 \leq j \leq m-1, 1 \leq k \leq p\}$ and edge set $E(\mathcal{F}_n \rhd Br_{m,p}) = \{ax_i, ay_i, x_ix_{i1}, y_iy_{i1}, x_iy_i, 1 \leq i \leq n\} \cup \{az_1\} \cup \{z_jz_{j+1}, 1 \leq j \leq m-2\} \cup \{x_{ij}x_{ij+1}, y_{ij}y_{ij+1}, 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \{z_mz_{mk}, 1 \leq k \leq p\} \cup \{x_{im}b_{ik}, y_{im}c_{ik}, 1 \leq i \leq n, 1 \leq k \leq p\}$. The cardinality of $|V(\mathcal{F}_n \rhd Br_{m,p})| = 2nm + m + 3mp - 3p$ and the cardinality of $|E(\mathcal{F}_n \rhd Br_{m,p})| = n + 2nm + m - 1 + p + 2np$.

To prove the rainbow antimagic connection number of $rac(\mathcal{F}_n \rhd Br_{m,p})$, first, we have to show the lower bound of $rac(\mathcal{F}_n \rhd Br_{m,p})$. Based on Lemma 2, we have $rac(\mathcal{F}_n \rhd Br_{m,p}) \geq rac(Br_{m,p})(|V(\mathcal{F}_n)|)$. Since $rac(Br_{m,p}) = m + p - 1$, $rac(\mathcal{F}_n \rhd Br_{m,p}) \geq 2nm + 2np + m + p - 2n - 1$.

Secondly, we have to show the upper bound of $rac(\mathcal{F}_n \rhd Br_{m,p})$. Define the vertex labeling $f(V(\mathcal{F}_n \rhd Br_{m,p})) \to \{1, 2, ..., 2nm + m + 3mp - 3p\}$ as follows:

$$
\begin{aligned}
f(a) &= 2n + 1 \\
f(x_i) &= 2n + 1 - i, \text{ for } 1 \leq i \leq n \\
f(y_i) &= 4n + 2 - i, \text{ for } 1 \leq i \leq n \\
f(z_j) &= 2nj + 2n + j + 1, \text{ for } 1 \leq j \leq m-1 \\
f(x_{ij}) &= \begin{cases} 2n - 2i + 1, & \text{for } 1 \leq i \leq n, j = 1 \\ 2nj + j + i, & \text{for } 1 \leq i \leq n, 2 \leq j \leq m-1 \end{cases} \\
f(y_{ij}) &= \begin{cases} 2i, & \text{for } 1 \leq i \leq n, j = 1 \\ 2nj + 2n + j - i + 1, & \text{for } 1 \leq i \leq n, 2 \leq j \leq m-1 \end{cases} \\
f(z_{m-1k}) &= 2nm + 2np + m + k, \text{ for } 1 \leq k \leq p \\
f(b_{ik}) &= 2nm + m + ip + k - p, \text{ for } 1 \leq i \leq n, 1 \leq k \leq p \\
f(c_{ik}) &= 2nm + 2np + m + k - ip, \text{ for } 1 \leq i \leq n, 1 \leq k \leq p
\end{aligned}
$$

The edge weights of the above vertex labeling f can be presented as

$$
\begin{aligned}
w(ax_i) &= 4n + 2 + i, \text{ for } 1 \leq i \leq n \\
w(ay_i) &= 6n + 3 - i, \text{ for } 1 \leq i \leq n \\
w(az_1) &= 6n + 3 \\
w(z_jz_{j+1}) &= 6n + 4nj + 2j + 3, \text{ for } 1 \leq j \leq m-2 \\
w(x_ix_{i1}) &= 4n + 2 - i, \text{ for } 1 \leq i \leq n \\
w(y_iy_{i1}) &= 4n + 4 - i, \text{ for } 1 \leq i \leq n \\
w(x_{ij}x_{ij+1}) &= \begin{cases} 6n + 3 - i & \text{for } 1 \leq i \leq n, j = 1 \\ 2n + 4nj + 2j + 2i + 1 & \text{for } 1 \leq i \leq n,\ 2 \leq j \leq m-3 \end{cases} \\
w(y_{ij}y_{ij+1}) &= \begin{cases} 6n + i + 3 & \text{for } 1 \leq i \leq n, j = 1 \\ 6n + 4nj + 2j - 2i + 3 & \text{for } 1 \leq i \leq n,\ 2 \leq j \leq m-3 \end{cases} \\
w(z_{m-1}z_{m-1k}) &= 4nm + 2np + 2m + k, \text{ for } 1 \leq k \leq p \\
w(x_{im-1}b_{ik}) &= 4nm + 2m + i + ip + k - 2n - p - 1, \text{ for } 1 \leq i \leq n, 1 \leq k \leq p \\
w(y_{im-1}c_{ik}) &= 4nm + 2np + 2m + k - i - ip, \text{ for } 1 \leq i \leq n, 1 \leq k \leq p
\end{aligned}
$$

It is easy to see that the above edge weight will induce a rainbow antimagic coloring of graph $\mathcal{F}_n \rhd Br_{m,p}$. Based on Theorem 3, $rac(Br_{m,p}) = m + p - 1$; since $E(Br_{m,p}) = m + p - 1$, the weight of each edge in graph $Br_{m,p}$ is different. Therefore, the sum of the weights on $|V(\mathcal{F}_n)|$ copies of graph $Br_{m,p}$ is $(|V(\mathcal{F}_n)|)(|E(Br_{m,p})|) = 2nm + 2np + m + p - 2n - 1$. Based on the description above, we have that the distinct weight of graph $(\mathcal{F}_n \rhd S_m)$ is $2nm + 2np + m + p - 2n - 1$. It implies that the edge weights of $f(V(\mathcal{F}_n \rhd Br_{m,p})) \to \{1, 2, ..., 2nm + 3mp + m - 3p\}$ induce a rainbow antimagic coloring of $2nm + 2np + m +$

$p - 2n - 1$ colors. Thus, $rac(\mathcal{F}_n \rhd Br_{m,p}) \leq 2nm + 2np + m + p - 2n - 1$. Comparing the two bounds, we have the exact value of $rac(\mathcal{F}_n \rhd Br_{m,p}) = 2nm + 2np + m + p - 2n - 1$.

The next step is to evaluate to prove the existence of a rainbow $u - v$ path $\mathcal{F}_n \rhd Br_{m,p}$. Based on the definition of graph $\mathcal{F}_n \rhd Br_{m,p}$, graph $\mathcal{F}_n \rhd Br_{m,p}$ contains one graph $\mathcal{F}_n$ and $|V(\mathcal{F}_n)|$ copies of $Br_{m,p}$; so, we can evaluate the rainbow $u - v$ path of graph $\mathcal{F}_n \rhd Br_{m,p}$ by evaluating the rainbow $u - v$ path on graph $\mathcal{F}_n$ and graph $Br_{m,p}$. Since $diam(\mathcal{F}_n) = 2$, based on Theorem 1, there is a rainbow $u - v$ path for every $u, v \in V(\mathcal{F}_n)$. Based on Theorem 3 $rac(Br_{m,p}) = m + p - 1$, since $Br_{m,p}$ has $m + p - 1$ edges, there is a rainbow $u - v$ path for every $u, v \in V(Br_{m,p})$. Therefore, according to the explanation, it can be seen that there is a rainbow $u - v$ path for every $u, v \in V(\mathcal{F}_n \rhd Br_{m,p})$. □

The illustration of a rainbow antimagic coloring of graph $\mathcal{F}_n \rhd Br_{m,p}$ can be seen in Figure 3.

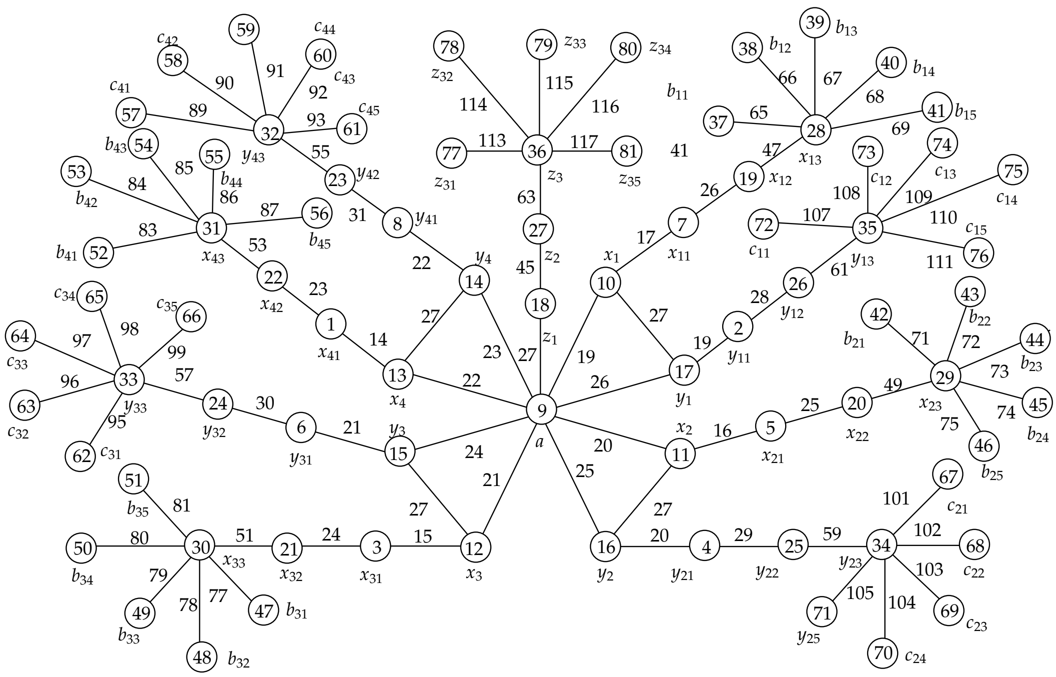

Figure 3. The illustration of rainbow antimagic coloring of graph $\mathcal{F}_4 \rhd Br_{4,5}$.

Theorem 7. *For* $n \geq 3, m = 2n - 2, rac(\mathcal{F}_n \rhd S_{m,m}) = 4nm + 2n + 2m + 1$.

Proof of Theorem 7. Graph $\mathcal{F}_n \rhd S_{m,m}$ is a connected graph with vertex set $V(\mathcal{F}_n \rhd S_{m,m}) = \{a\} \cup \{x_i, y_i, 1 \leq i \leq n\} \cup \{z_j, 1 \leq j \leq m\} \cup \{x_{ij}, y_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{a_0\} \cup \{a_k, 1 \leq k \leq m\} \cup \{b_i, c_i, 1 \leq j \leq n\} \cup \{b_{ik}, c_{ik}, 1 \leq i \leq n, 1 \leq k \leq m\}$ and edge set $E(\mathcal{F}_n \rhd S_{m,m}) = \{ax_i, ay_i, x_iy_i, 1 \leq i \leq n\} \cup \{az_j, 1 \leq j \leq m\} \cup \{aa_0\} \cup \{a_0a_k, 1 \leq k \leq m\} \cup \{x_ix_{ij}, y_iy_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{x_ib_i, y_ic_i, 1 \leq i \leq n\} \cup \{b_ib_ik, c_ic_{ik}, 1 \leq i \leq n, 1 \leq k \leq k\}$. The cardinality of $|V(\mathcal{F}_n \rhd S_{m,m})| = 4n + 2m + 4nm + 2$ and the cardinality of $|E(\mathcal{F}_n \rhd S_{m,m})| = 5n + 2m + 4nm + 1$. To prove the rainbow antimagic connection number of $rac(\mathcal{F}_n \rhd S_{m,m})$, first, we have to show the lower bound of $rac(\mathcal{F}_n \rhd S_{m,m})$. Based on Lemma 2, we have $rac(\mathcal{F}_n \rhd S_{m,m}) \geq rac(S_{m,m})(|V(\mathcal{F}_n)|)$. Since $rac(S_{m,m}) = 2m + 1$, $rac(\mathcal{F}_n \rhd S_{m,m}) \geq 4nm + 2n + 2m + 1$.

Secondly, we have to show the upper bound of $rac(\mathcal{F}_n \rhd S_{m,m})$. Define the vertex labeling $f(V(\mathcal{F}_n \rhd S_{m,m})) \rightarrow \{1, 2, ..., 4n + 2m + 4nm + 2\}$ as follows:

$$f(a) = 2n + 1$$
$$f(x_i) = 4n + i, \text{ for } 1 \leq i \leq n$$

$$f(y_i) = 6n + 1 - i, \text{ for } 1 \leq i \leq n$$

$$f(z_j) = \begin{cases} 2j+1 & \text{for } 1 \leq j \leq \lfloor \frac{m}{2} \rfloor \\ 2j+3 & \text{for } \lceil \frac{m}{2} \rceil \leq j \leq m \end{cases}$$

$$f(x_{ij}) = \begin{cases} 4n+2-2i & \text{for } 1 \leq i \leq n, j = 1 \\ 1 & \text{for } i = 1, j = 2 \\ 6n+j-2 & \text{for } i = 1, 3 \leq j \leq m \\ 6n+m+j-2 & \text{for } i = 2, 2 \leq j \leq 3 \\ 6n+m+j-1 & \text{for } i = 2, 4 \leq j \leq m \\ 6n+im+j-m-i & \text{for } 3 \leq i \leq n, 2 \leq j \leq m \end{cases}$$

$$f(y_{ij}) = \begin{cases} 2i & \text{for } 1 \leq i \leq n, j = 1 \\ 2nm+4n+m+j-im+1 & \text{for } 1 \leq i \leq n, 2 \leq j \leq m \end{cases}$$

$$f(a_0) = 8n$$

$$f(b_i) = \begin{cases} 6n+m-1 & \text{for } i = 1 \\ 6n+im & \text{for } 2 \leq i \leq n \end{cases}$$

$$f(c_i) = 2nm + 4n + 2m + 2 - im, \text{ for } 1 \leq i \leq n$$

$$f(a_k) = 2nm + 4n + 2m + 2 + k, \text{ for } 1 \leq k \leq m$$

$$f(b_{ik}) = \begin{cases} 2nm+4n+m+2+k & \text{for } i = 1, 1 \leq k \leq m, \\ 2nm+4n+m+im+2+k & \text{for } 2 \leq i \leq n, 1 \leq k \leq m \end{cases}$$

$$f(c_{ik}) = 4nm + 4n + 2m + k + 2 - im, \text{ for } 1 \leq i \leq n, 1 \leq k \leq m$$

The edge weights of the above vertex labeling f can be presented as

$$w(ax_i) = 6n + i + 1, \text{ for } 1 \leq i \leq n$$

$$w(ay_i) = 8n + 1 - i, \text{ for } 1 \leq i \leq n$$

$$w(x_iy_i) = 10n + 1, \text{ for } 1 \leq i \leq n$$

$$w(az_j) = \begin{cases} 2n+2j+2 & \text{for } 1 \leq j \leq \lfloor \frac{m}{2} \rfloor \\ 2n+2j+4 & \text{for } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m \end{cases}$$

$$w(x_ix_{ij}) = \begin{cases} 8n+2-i & \text{for } 1 \leq i \leq n, j = 1 \\ 4n+2 & \text{for } i = 1, j = 2 \\ 10n+j+i-2 & \text{for } i = 1, 3 \leq j \leq m \\ 10n+m+j+i-2 & \text{for } i = 2, 2 \leq j \leq 3 \\ 10n+m+j+i-1 & \text{for } i = 2, 4 \leq j \leq m \\ 10n+j+im+i-m-1 & \text{for } 3 \leq i \leq n, 2 \leq j \leq m \end{cases}$$

$$w(y_iy_{ij}) = \begin{cases} 6n+i+1 & \text{for } 1 \leq i \leq n, j = 1 \\ 2nm+10n+m+j+2-i-im & \text{for } 1 \leq i \leq n, 2 \leq j \leq m \end{cases}$$

$$w(aa_0) = 10n + 1$$

$$w(x_ib_i) = \begin{cases} 10n+m+i-1 & \text{for } i = 1 \\ 10n+im+i & \text{for } 2 \leq i \leq n \end{cases}$$

$$w(y_ic_i) = 2nm + 10n + 2m + 3 - i - im, \text{ for } 1 \leq i \leq n$$

$$w(a_0a_k) = 2nm + 12n + 2m + 2 + k, \text{ for } 1 \leq k \leq m$$

$$w(b_ib_{ik}) = \begin{cases} 2nm+10n+2m+1+k & \text{for } i = 1, 1 \leq k \leq m, \\ 2nm+10n+m+2im+2+k & \text{for } 2 \leq i \leq n, 1 \leq k \leq m \end{cases}$$

$$w(c_ic_{ik}) = 6nm + 8n + 4m + k + 4 - 2im, \text{ for } 1 \leq i \leq n, 1 \leq k \leq m$$

It is easy to see that the above edge weight will induce a rainbow antimagic coloring of graph $\mathcal{F}_n \triangleright S_{m,m}$. Based on Theorem 3, $rac(S_{m,m}) = 2m + 1$; since $E(S_{m,m}) = 2m + 1$, the weight of each edge in graph $S_{m,m}$ is different. Therefore, the sum of the weights on $|V(\mathcal{F}_n)|$ copies of graph $S_{m,m}$ is $(|V(\mathcal{F}_n)|)(|E(S_{m,m})|) = 4nm + 2n + 2m + 1$. Based on the description above, we have that the distinct weight of graph $(\mathcal{F}_n \triangleright S_{m,m})$ is $4nm + 2n +$

$2m+1$. It implies that the edge weights of $f(V(\mathcal{F}_n \rhd S_{m,m})) \to \{1, 2, ..., 4n+2m+4nm+2\}$ induce a rainbow antimagic coloring of $4nm+2n+2m+1$ colors. Thus, $rac(\mathcal{F}_n \rhd S_{m,m}) \leq 4nm+2n+2m+1$. Comparing the two bounds, we have the exact value of $rac(\mathcal{F}_n \rhd S_{m,m}) = 4nm+2n+2m+1$.

The next step is to evaluate to prove the existence of a rainbow $u-v$ path $\mathcal{F}_n \rhd S_{m,m}$. Based on the definition of graph $\mathcal{F}_n \rhd S_{m,m}$, graph $\mathcal{F}_n \rhd S_{m,m}$ contains one graph $\mathcal{F}_n$ and $|V(\mathcal{F}_n)|$ copies of $S_{m,m}$; so, we can evaluate the rainbow $u-v$ path of graph $\mathcal{F}_n \rhd S_{m,m}$ by evaluating the rainbow $u-v$ path on graph $\mathcal{F}_n$ and graph $S_{m,m}$. Since $diam(\mathcal{F}_n) = 2$, based on Theorem 1, there is a rainbow $u-v$ path for every $u, v \in V(\mathcal{F}_n)$. Based on Theorem 3, $rac(S_{m,m}) = 2m+1$; since $S_{m,m}$ has $2m+1$ edges, there is a rainbow $u-v$ path for every $u, v \in V(S_{m,m})$. Therefore, according to the explanation, it can be seen that there is a rainbow $u-v$ path for every $u, v \in V(\mathcal{F}_n \rhd S_{m,m})$. $\square$

3. Conclusions

We have studied the rainbow antimagic coloring of the comb product of a friendship graph with any tree graph. Based on the result, we have a new lower bound of rainbow antimagic connection number for the comb product of a friendship graph with any tree $\mathcal{F}_n \rhd T_m$ and the exact value of the rainbow antimagic connection number of graph $\mathcal{F}_n \rhd T_m$, where T_m is path P_m, star S_m, broom $Br_{m,p}$ and double star $S_{m,m}$. However, if it is not a tree, it is still difficult to determine the exact value of the rainbow antimagic connection number. Therefore, this study raises an open problem:

Determine the exact value of the rainbow antimagic connection number of graph $G \rhd H$ where H is not a tree.

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