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Geodesic and Newtonian Vector Fields and Symmetries of Mechanical Systems

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Abstract: Geodesic vector fields and other distinguished vector fields on a Riemann manifold were used in the study of free motions on such a manifold, and we applied the geometric Hamilton–Jacobi theory for the search of geodesic vector fields from Hamilton–Jacobi vector fields and the same for closed vector fields. These properties were appropriately extended to the framework of Newtonian and generalised Newtonian systems, in particular systems defined by Lagrangians of the mechanical type and velocity-dependent forces. Conserved quantities and a generalised concept of symmetry were developed, particularly for Killing vector fields. Nonholonomic constrained Newtonian systems were also analysed from this perspective, as well as the relation among Newtonian vector fields and Hamilton–Jacobi equations for conformally related metrics.

Keywords: geodesic vector fields; eikonal; geometric Hamilton–Jacobi; external forces in mechanical systems

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1. Introduction

The active interface between mathematics and theoretical physics during the last few years has been very useful for both disciplines. This interaction is very efficient, and in particular, the modern tools of differential geometry have been used to deal with old problems of classical and quantum mechanics, offering a new perspective to analyse conceptual problems, as, for instance, the existence and non-uniqueness of the solution of the inverse problem for a given system of second-order differential equations describing a time evolution. However, many physical theories have also motivated mathematicians, a good example being the theory of group representations, because of its usefulness in the symmetry theory of quantum mechanics.

Riemannian geometry is very relevant in the geometric reformulation of classical systems of a mechanical type, a Riemann metric allowing us the definition, in an intrinsic way, of kinetic energy. Free motions of holonomic constrained systems correspond to geodesic curves of their associated Levi-Civita connection, and the physical systems under the action of external forces can also be dealt with, the curves of motion corresponding to integral curves of an appropriately defined dynamical vector field. In this sense, the Riemannian approach is closer to the elementary and original ideas of Newton on the description of mechanical systems.

The importance of geodesics leads to the study of geodesic vector fields in a Riemann manifold, i.e., vector fields whose integral curves are geodesics. These vector fields are used in the study of Yamabe solitons (see, e.g., [1]) and are also useful in many other fields of physics, where they appear as solutions of the eikonal equation and even in medical

imaging [2]. Unfortunately, these autoparallel vector fields have scarcely been studied in the literature, and we hope that the application of well-known ingredients of classical mechanics, as the Hamilton–Jacobi equation, conveniently reformulated in a geometric way, may shed some light on the characterisation and determination of such an interesting class of vector fields. This is the main aim of the first part of this paper.

Once the study of these notions on the geodesic flow is performed, it is natural to ask about their possible extension to more general second-order vector fields, in particular those coming from mechanical systems subjected to external forces. We developed this extension including the case of nonholonomic mechanical systems with nonholonomic constraints. Then, we studied not only the geodesic case, for which many results are known, but also the unknown case of mechanical and generalized mechanical systems, with external forces possibly depending on the velocity.

The structure of this article is as follows. In Section 2, we quickly review the theory of the Levi-Civita connection on a Riemann manifold, mainly to fix the notation to be used. A quick review of the geometric approach to Lagrangian and Hamiltonian systems is given, and the geodesic flow appears as the flow of a Hamiltonian vector field. In Section 3, we review the properties of some distinguished vector fields in Riemannian manifolds with a special emphasis on geodesic vector fields, which are our main aim. The geometric approach to the Hamilton–Jacobi theory, whose fundamental ideas were developed in a series of articles elaborated by the authors and their coworkers, is reviewed in Section 4, where the concept of the Hamilton–Jacobi vector field is also introduced. The search for Hamilton–Jacobi vector fields for the second-order geodesic vector field and their relations with autoparallel vector fields are studied in Section 5, and the relation between a geodesic vector field, its tangent lift, and the second-order geodesic vector field is studied in Section 6. General Newtonian dynamical systems, the existence of Noether conserved quantities, and Newtonian vector fields and their properties are studied in Section 7, and an application for studying some relations between two conformally related metrics is studied in Section 8. Conclusions and an outlook on future work are given in Section 9. Finally, in order for the paper to be self-contained, we added an Appendix A with some definitions and mathematical results on tensor fields along maps.

2. Notation, Basic Definitions, and Geometric Preliminaries

2.1. Riemannian Manifolds and Levi-Civita Connection

Let M be an n -dimensional differentiable manifold. A Riemannian metric on M is a two-times covariant symmetric tensor field g , which is positive definite, hence non-degenerate. The pair (M, g) is called a Riemann manifold.

Non-degeneracy implies that the map $\widehat{g} : TM \rightarrow T^*M$ from the tangent bundle, $\tau_M : TM \rightarrow M$, to the cotangent bundle, $\pi_M : T^*M \rightarrow M$, defined by $\langle \widehat{g}(v), w \rangle = g(v, w)$, where $v, w \in T_x M$, is a vector bundle isomorphism. As \widehat{g} is a fibred map over the identity on M , it induces a natural map between the $C^\infty(M)$ modules of sections of the tangent and cotangent bundles, denoted by the same letter $\widehat{g} : \mathfrak{X}(M) \rightarrow \Omega^1(M)$: $\langle \widehat{g}(X), Y \rangle = g(X, Y)$, i.e., $\widehat{g}(X) = i(X)g$.

Given a local chart (U, q^1, \dots, q^n) on M , we can consider the coordinate basis of $\mathfrak{X}(U)$, usually denoted $\{\partial/\partial q^j \mid j = 1, \dots, n\}$, and its dual basis for $\Omega^1(U)$, $\{dq^j \mid j = 1, \dots, n\}$. Then, any $v \in T_q M$, with $q \in U$, is $v = \sum_{j=1}^n v^j (\partial/\partial q^j)_q$, and any $\zeta \in T_q^* M$ is $\zeta = \sum_{j=1}^n p_j dq^j|_q$, with $v^j = \langle dq^j|_q, v \rangle$ and $p_j = \langle \zeta, (\partial/\partial q^j)_q \rangle$ being the usual velocities and momenta.

The local coordinate expression for the Riemannian metric g in the open set U is:

$$g = \sum_{i,j=1}^n g_{ij}(q) dq^i \otimes dq^j, \text{ with } g_{ij}(q) = g(q) \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right), \quad i, j = 1, \dots, n = \dim M. \quad (1)$$

The arc-length of a curve $\gamma : \mathbb{R} \rightarrow M, \gamma(t) = (q^1(t), \dots, q^n(t))$, between the points $\gamma(t_1)$ and $\gamma(t_2)$ is given by

$$s(\gamma) = \int_{t_1}^{t_2} \|\dot{\gamma}(t)\| dt = \int_{t_1}^{t_2} \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt = \int_{t_1}^{t_2} \sqrt{\sum_{i,j=1}^n g_{ij}(\gamma(t)) \dot{q}^i(t) \dot{q}^j(t)} dt, \tag{2}$$

i.e., the classical local expression for the arc-length ds is given by $ds^2 = \sum_{i,j=1}^n g_{ij} dq^i dq^j$.

Recall (see, e.g., [3,4]) that a linear connection ∇ on a manifold M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, which is \mathbb{R} -linear on both inputs, $C^\infty(M)$ -linear on the first one, and satisfies the rule $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. It is easy to prove that $\nabla_X Y|_U$ depends only on X and Y on $U \subset M$; hence, in a local coordinate system (U, x^1, \dots, x^n) , the connection symbols Γ_{jk}^i are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) = \sum_{l=1}^n \Gamma_{ji}^l \frac{\partial}{\partial x^l}, \quad i, j = 1, \dots, n, \tag{3}$$

while $\nabla_X Y$ is given by

$$\nabla_X Y = \nabla_{\sum_{i=1}^n X^i \frac{\partial}{\partial x^i}} \left(\sum_{j=1}^n Y^j \frac{\partial}{\partial x^j} \right) = \sum_{i,k=1}^n X^i \left(\frac{\partial Y^k}{\partial x^i} + \sum_{j=1}^n Y^j \Gamma_{ji}^k \right) \frac{\partial}{\partial x^k}. \tag{4}$$

Recall that $\nabla_X Y$ is known as the covariant derivative of the vector field Y with respect to the vector field X . Observe from the above local expression that the value of $\nabla_X Y$ at a point $q \in M$ only depends, with respect to X , on X_q .

Moreover, each linear connection has associated a (1,2) tensor field \mathbb{T} , called the torsion tensor, which is skew-symmetric in the two subscripts, and it is defined as follows:

$$\mathbb{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{5}$$

A connection ∇ on a Riemann manifold (M, g) is metric when, given $X, Y, Z \in \mathfrak{X}(M)$, we have that

$$\mathcal{L}_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{6}$$

where \mathcal{L}_X denotes Lie derivative with respect to the vector field X . This is equivalent to saying that $\nabla_X g = 0$. Recall that the operator ∇_X , for $X \in \mathfrak{X}(M)$, can be extended as a derivation that commutes with the tensor contractions, on all the tensor fields on M . In particular, $\nabla_X g$ is defined by the expression:

$$\nabla_X(g(Y, Z)) = (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z), Y, Z \in \mathfrak{X}(M),$$

because g is non-degenerate. See, e.g., [3,4] for more details on the covariant derivative of tensor fields.

In the particular case of (6) in which the two arguments Y and Z coincide, that is, $Y = Z$, we obtain

$$\mathcal{L}_X(g(Y, Y)) = g(\nabla_X Y, Y) + g(Y, \nabla_X Y) = 2g(\nabla_X Y, Y). \tag{7}$$

It is a fundamental result that every Riemann manifold (M, g) is endowed with a uniquely determined torsionless metric connection ∇ , called the Levi-Civita connection, which is given by the so-called Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= \mathcal{L}_X(g(Y, Z)) + \mathcal{L}_Y(g(Z, X)) - \mathcal{L}_Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned} \tag{8}$$

for every $X, Y, Z \in \mathfrak{X}(M)$. In terms of local coordinates on M , the Koszul formula gives

$$2g\left(\nabla_{\partial/\partial q^i}\left(\frac{\partial}{\partial q^j}\right), \frac{\partial}{\partial q^k}\right) = \frac{\partial g_{jk}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k}, \quad i, j, k = 1, \dots, n,$$

and, by the above definition of connection symbols (usually called second-class Christoffel symbols),

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^n g^{il} \left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right), \quad i, j, k = 1, \dots, n, \tag{9}$$

where $\sum_{j=1}^n g^{ij} g_{jk} = \delta^i_k$. On the other hand, the vanishing of the torsion tensor is equivalent to the symmetry property of the Christoffel symbols: $\Gamma^i_{jk} = \Gamma^i_{kj}$. As indicated above, these symbols Γ^i_{jk} locally determine $\nabla_X Y$ for every pair of vector fields $X, Y \in \mathfrak{X}(M)$ by making use of (4).

2.2. Geodesics: Definition and Second-Order Geodesic Vector Field

Let $\gamma : I \rightarrow M$ be a curve in the manifold M . For each $t \in I$, $\dot{\gamma}(t) \in T_{\gamma(t)}M$ denotes the tangent vector to the curve γ at the point $\gamma(t)$. This shows that we can define a curve $\dot{\gamma}$ on TM , $\dot{\gamma} : I \rightarrow TM$, by $\dot{\gamma}(t) = \frac{d}{ds}\gamma(s)|_{s=t}$. As $\tau_M \circ \dot{\gamma} = \gamma$, the curve $\dot{\gamma}$ defines a vector field along γ , i.e., $\dot{\gamma} \in \mathfrak{X}(\gamma)$. However, we remark that the curve $\dot{\gamma}$ on TM can also be considered as the lift to the tangent bundle TM of the curve γ . To distinguish both roles of $\dot{\gamma}$, when necessary, we will denote such a lifted curve as $\gamma^T(t) = (\gamma(t), \dot{\gamma}(t))$ instead of simply $\dot{\gamma}$. Observe that

$$\gamma^T(t) = T_t\gamma\left(\frac{d}{ds}\right)_t.$$

Let ∇ be a connection on a manifold M . It is a remarkable fact (see, e.g., [3,4] for details, or have a look at the expression (4) above) that the value of $\nabla_X Y$ at a point $x \in M$, $(\nabla_X Y)(x)$, only depends, with respect to X , on the value of X at such a point and, with respect to Y , on the value of Y along any segment of the curve representing the tangent vector $X(x) \in T_xM$.

Then, for each $v \in T_xM$, we can define $\nabla_v Y = (\nabla_X Y)(x)$, where X is any vector field $X \in \mathfrak{X}(M)$ such that $X(x) = v$. Moreover, given a curve γ in M , if $Y \in \mathfrak{X}(\gamma)$, we can define $\nabla_{\dot{\gamma}} Y \in \mathfrak{X}(\gamma)$ by $(\nabla_{\dot{\gamma}} Y)(t) = \nabla_{\dot{\gamma}(t)} Y$, while when $Y \in \mathfrak{X}(M)$, $\nabla_{\dot{\gamma}} Y \in \mathfrak{X}(\gamma)$ is defined as $(\nabla_{\dot{\gamma}} Y)(t) = \nabla_{\dot{\gamma}(t)}(Y \circ \gamma)$. Similarly, if $X, Y \in \mathfrak{X}(\gamma)$, we can define $\nabla_X Y \in \mathfrak{X}(\gamma)$ as the vector field along γ given by $(\nabla_X Y)(t) = \nabla_{X(t)} Y$.

This allows us to introduce the concept of the parallelism of a vector field along a curve as follows: a vector field, $Y \in \mathfrak{X}(\gamma)$, along the curve $\gamma : I \rightarrow M$, is parallel along γ if $\nabla_{\dot{\gamma}(t)} Y = 0$, for all $t \in I$. Observe that this expression is correct by the comments in the previous paragraph.

In the particular case of $Y = \dot{\gamma} \in \mathfrak{X}(\gamma)$, every curve γ such that its velocity vector field is parallel along the curve, that is $\nabla_{\dot{\gamma}(t)} \dot{\gamma} = 0$ for all $t \in I$, is called a geodesic.

Definition 1. We call geodesics of the connection ∇ the curves γ in M whose velocity vector field is parallel along the curve, $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

In the case of a Riemann manifold (M, g) and its Levi-Civita connection ∇ , the geodesic curves are uniquely defined by the Riemannian metric g , and they are local extrema of the distance function as defined in (2).

Recalling the expression (4), given a curve $\gamma : I \rightarrow M$, $\gamma(s) = (q^1(s), \dots, q^n(s))$, which we can assume to be parametrised by its arc-length s , that is $\|\dot{\gamma}(s)\| = 1$ for every s , we have

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \sum_{l=1}^n \left(\sum_{j=1}^n \dot{q}^j \frac{\partial \dot{q}^l}{\partial q^j} + \sum_{j,k=1}^n \Gamma_{jk}^l \dot{q}^j \dot{q}^k \right) \frac{\partial}{\partial q^l} \circ \gamma = \sum_{l=1}^n \left(\ddot{q}^l + \sum_{j,k=1}^n \Gamma_{jk}^l \dot{q}^j \dot{q}^k \right) \frac{\partial}{\partial q^l} \circ \gamma.$$

Then, the local equation of the geodesic lines, $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, is the second-order differential equation:

$$\ddot{q}^i + \sum_{j,k=1}^n \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n. \tag{10}$$

If we consider a reparametrisation of a solution of (10) with a generic parameter τ defined, as a given function of s , by

$$\frac{ds}{d\tau} = \zeta(\tau),$$

we have

$$\frac{d}{ds} = \frac{1}{\zeta} \frac{d}{d\tau} \implies \frac{d^2}{ds^2} = \frac{1}{\zeta} \frac{d}{d\tau} \left(\frac{1}{\zeta} \frac{d}{d\tau} \right) = -\frac{d\zeta/d\tau}{\zeta^3} \frac{d}{d\tau} + \frac{1}{\zeta^2} \frac{d^2}{d\tau^2},$$

and the corresponding differential Equation (10) for geodesics in such a generic parametrisation is

$$\frac{d^2 q^i}{d\tau^2} + \sum_{j,k=1}^n \Gamma_{jk}^i \frac{dq^j}{d\tau} \frac{dq^k}{d\tau} = \lambda(\tau) \frac{dq^i}{d\tau}, \quad i = 1, \dots, n, \tag{11}$$

with

$$\lambda(\tau) = \frac{d^2 s}{d\tau^2} \left(\frac{ds}{d\tau} \right)^{-1}, \quad \text{or equivalently,} \quad \lambda(s) = -\frac{d^2 \tau}{ds^2} \left(\frac{ds}{d\tau} \right)^2. \tag{12}$$

Hence, as $\lambda = 0$ if, and only if, $\tau = as + b$, we can conclude that, as indicated in [5–7], a geodesic curve is parametrised by a parameter that is an affine function of its arc-length, $\tau = as + b$, where $a, b \in \mathbb{R}$, i.e., a geodesic is essentially parametrised by its arc-length s .

2.3. Lagrangian Dynamical Systems

A Lagrangian dynamical system is a pair (M, L) , where M is a differentiable manifold, the configuration manifold, and $L : TM \rightarrow \mathbb{R}$ is the Lagrangian function. The associated dynamical system is defined with the following elements (see [8,9] for details):

1. The Cartan Lagrangian one-form $\theta_L = d^v L = dL \circ S$, with S being the vertical endomorphism on the tangent bundle TM .
2. The Cartan Lagrangian two-form $\omega_L = -d\theta_L$.
3. The energy function $E_L = \Delta L - L$, with Δ the Liouville vector field on TM .
4. Assuming that ω_L is a symplectic form (see below), a unique vector field $\Gamma \in \mathfrak{X}(TM)$ solution to the Hamilton dynamical equation is defined:

$$i(\Gamma)\omega_L = dE_L. \tag{13}$$

The vector field Γ is called the Lagrangian dynamical vector field, and it is the Hamiltonian vector field, defined by the energy function E_L , for the Hamiltonian system (TM, ω_L, E_L) . Furthermore, Γ is an SODE vector field in TM , that is its integral curves $\sigma : \mathbb{R} \rightarrow TM$ are canonical lifts to TM of curves $\gamma : \mathbb{R} \rightarrow M$, i.e., $\sigma = (\gamma, \dot{\gamma}) = \gamma^T$.

In local natural coordinates, (q^i, v^i) , of TM , the expressions of the above elements are:

$$\theta_L = \sum_{i=1}^n \frac{\partial L}{\partial v^i} dq^i, \quad \omega_L = - \sum_{i,j=1}^n \left(\frac{\partial^2 L}{\partial q^j \partial v^i} dq^j \wedge dq^i + \frac{\partial^2 L}{\partial v^j \partial v^i} dv^j \wedge dq^i \right), \quad E_L = \sum_{i=1}^n v^i \frac{\partial L}{\partial v^i} - L,$$

and the dynamical second-order vector field is:

$$\Gamma = \sum_{i=1}^n \left(v^i \frac{\partial}{\partial q^i} + \Gamma^i(q, v) \frac{\partial}{\partial v^i} \right), \text{ with } \sum_{i=1}^n \frac{\partial^2 L}{\partial v^j \partial v^i} \Gamma^i = - \sum_{i=1}^n \frac{\partial^2 L}{\partial v^j \partial q^i} v^i + \frac{\partial L}{\partial q^j}, j = 1, \dots, n.$$

The differential equations for the integral curves, $\sigma = (\alpha, \beta)$, of Γ are

$$\dot{\alpha}^i = \beta^i, \quad \dot{\beta}^i = \Gamma^i,$$

giving $\ddot{q}^i = \Gamma^i(q, v)$, that is a second-order differential equation.

The condition for the Lagrangian two-form ω_L to be symplectic is related to the partial Hessian of L with respect to the velocities: ω_L is symplectic if, and only if, $\omega_L^n \neq 0$, which, in local coordinates, reads

$$\det \left(\frac{\partial^2 L}{\partial v^j \partial v^i} \right) \neq 0.$$

As a consequence, the coefficients Γ^i are univocally determined if ω_L is symplectic.

2.4. The Geodesic Flow as a Lagrangian Dynamical System

If γ is a geodesic curve in the Riemannian manifold (M, g) , its lift to TM , $\gamma^T(t) = (\gamma(t), \dot{\gamma}(t))$, is an integral curve of a vector field $\Gamma \in \mathfrak{X}(TM)$ (see, e.g., [5,10]); this Γ is the Hamiltonian dynamical vector field defined by the Lagrangian $L = T_g$, which gives the Lagrangian system (TM, ω_{T_g}, T_g) , i.e., Γ_g is the unique solution of the Hamiltonian equation $i(\Gamma_g)\omega_{T_g} = dT_g$, where the kinetic energy function, $T_g : TM \rightarrow \mathbb{R}$, is defined by

$$T_g(v) = \frac{1}{2} g(v, v), v \in TM; \quad T_g(q, v) = \frac{1}{2} \sum_{i,j=1}^n g_{ij}(q) v^i v^j, \tag{14}$$

and the Cartan one-form θ_{T_g} is given by

$$\theta_{T_g}(q, v) = g_q(v, \cdot), \quad \theta_{T_g} = \sum_{i=1}^n g_{ij}(q) v^i dq^j, \tag{15}$$

the associated symplectic two-form on TM being $\omega_{T_g} = -d\theta_{T_g}$.

The dynamical vector field corresponding to the Lagrangian system (M, T_g) is the unique solution to the equation:

$$i(\Gamma_g)\omega_{T_g} = dT_g, \tag{16}$$

whose local expression is given by

$$\Gamma_g = \sum_{i=1}^n v^i \frac{\partial}{\partial q^i} - \sum_{i,j,k=1}^n \Gamma_{jk}^i v^j v^k \frac{\partial}{\partial v^i}. \tag{17}$$

The integral curves of Γ_g are liftings of curves in M that are solutions of the system of differential equations for geodesic curves, $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$, i.e., the system (10).

Observe that θ_{T_g} is such that, if $\mathcal{L}_{X^c}\theta_{T_g} = 0$, for a vector field $X \in \mathfrak{X}(M)$ and X^c its complete lift to TM , then from the relation $\mathcal{L}_{X^c}\theta_{T_g} = \theta_{\mathcal{L}_{X^c}T_g} = \theta_{T_{\mathcal{L}_{X^c}g}}$, we obtain that $\theta_{T_{\mathcal{L}_{X^c}g}} = 0$, and this is only possible when X is a Killing vector field, that is if, and only if, $\mathcal{L}_X g = 0$. Moreover, if $\mathcal{L}_{X^c}\omega_{T_g} = 0$, i.e., $\mathcal{L}_{X^c}\theta_{T_g}$ is closed, then $\theta_{T_{\mathcal{L}_{X^c}g}}$ is a closed semibasic one-form, and therefore basic, and this implies that $\theta_{T_{\mathcal{L}_{X^c}g}} = 0$, i.e., $\mathcal{L}_{X^c}\theta_{T_g} = 0$; we have that $\mathcal{L}_{X^c}\omega_{T_g} = 0$ if, and only if, X is a Killing vector field.

2.5. Hamiltonian Dynamical Systems

Going to T^*M , we know that there is a canonical symplectic structure coming from the symplectic form ω_M defined by $\omega_M = -d\theta_M$, where θ_M is the Liouville one-form in T^*M . See [11,12] for details. Their local expressions in natural coordinate systems, (q^i, p_i) , are:

$$\theta_M = \sum_{i=1}^n p_i dq^i, \quad \omega_M = \sum_{i=1}^n dq^i \wedge dp_i.$$

Given a function, the Hamiltonian, $H : T^*M \rightarrow \mathbb{R}$, we can obtain a dynamical vector field $X_H \in \mathfrak{X}(T^*M)$ defined univocally by the equation $i(X_H)\omega_M = dH$. Locally, the expression of the integral curves, $\sigma(t) = (q(t), p(t))$, of this vector field is

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \dots, n,$$

which are known as Hamilton dynamical equations.

2.6. The Geodesic Flow as a Hamiltonian Dynamical System

We study in this paragraph the case of the geodesic flow corresponding to the Lagrangian system (TM, T_g) , where g is a Riemannian metric on M .

Associated with this system in TM , there is a Hamiltonian system in T^*M obtained through the Legendre map; see [11,12] for details.

Given the geodesic Lagrangian system (TM, T_g) on the Riemann manifold (M, g) , the associated Hamiltonian system is (T^*M, H_g) , with Hamiltonian $H_g(q, p) = \sum_{i,j=1}^n \frac{1}{2} g^{ij}(q) p_i p_j$,

where $\sum_{j=1}^n g^{ij} g_{jk} = \delta_k^i$; hence, (g^{ij}) is the corresponding matrix of the metric in the fibres of T^*M .

The Hamiltonian vector field for (T^*M, H_g) , $X_g \in \mathfrak{X}(T^*M)$, is given by

$$X_g = \sum_{i,j=1}^n g^{ij}(q) p_i \frac{\partial}{\partial q^j} - \sum_{i,j,k=1}^n \frac{\partial g^{ij}}{\partial q^k} p_i p_j \frac{\partial}{\partial p_k}.$$

Thus, the Hamilton differential equations of a geodesic $\sigma : \mathbb{R} \rightarrow T^*M$, $\sigma(t) = (q(t), p(t))$, are

$$\dot{q}^k = \sum_{i=1}^n g^{ik}(q) p_i, \quad \dot{p}_k = \sum_{i,j=1}^n \frac{\partial g^{ij}}{\partial q^k} p_i p_j, \quad k = 1, \dots, n.$$

As $p_i = \sum_{j=1}^n g_{ij} \dot{q}^j$, the second set of these equations can be reduced, by making use of the expression of the Christoffel symbols, to Equations (10).

3. Distinguished Vector Fields in a Riemann Manifold

The aim of this section is to review some known properties of Killing vector fields and obtain new ones related to geodesic, and other kinds, of vector fields.

3.1. Definitions and Initial Relations

There are distinguished classes of vector fields on a Riemann manifold (M, g) . If ∇ denotes its associated Levi-Civita connection, then:

Definition 2. (i) We say that a vector field $X \in \mathfrak{X}(M)$ is a conformal Killing vector field if there exists a function $\psi \in C^\infty(M)$ such that $\mathcal{L}_X g = 2\psi g$. A conformal Killing vector field $X \in \mathfrak{X}(M)$ is homothetic when the function ψ is a constant.

- (ii) We say that a vector field $X \in \mathfrak{X}(M)$ is a Killing vector field if X is such that $\mathcal{L}_X g = 0$, i.e., the local flow of the vector field X preserves g .
- (iii) We say that a vector field $X \in \mathfrak{X}(M)$ is an autoparallel vector field if $\nabla_X X = 0$.
- (iv) We say that a vector field $X \in \mathfrak{X}(M)$ is a geodesic vector field if each integral curve of X is a geodesic curve of the associated Levi-Civita connection.

It is well known that the set of conformal Killing vector fields is a Lie algebra called the conformal Lie algebra [13], which contains two Lie subalgebras, the homothetic Lie subalgebra and the Killing Lie algebra, because if $\mathcal{L}_X g = 2\psi_1 g$ and $\mathcal{L}_Y g = 2\psi_2 g$, then

$$\mathcal{L}_{[X,Y]} g = \mathcal{L}_X(\mathcal{L}_Y g) - \mathcal{L}_Y(\mathcal{L}_X g) = 2\mathcal{L}_X(\psi_2 g) - 2\mathcal{L}_Y(\psi_1 g) = 2(\mathcal{L}_X(\psi_2) - \mathcal{L}_Y(\psi_1))g.$$

In particular, when X and Y are homothetic, then $\mathcal{L}_{[X,Y]} g = 0$.

As indicated in [5], one can see that a Killing vector field $X \in \mathfrak{X}(M)$ has constant length if, and only if, every integral curve of X is a geodesic, i.e., X is a geodesic vector field (see, e.g., Lemma 3 in [14]). We can also see that, as will be proven later on, if the vector field $X \in \mathfrak{X}(M)$ is autoparallel, then $g(X, X)$ is constant along the integral curves of X , that is $\mathcal{L}_X(g(X, X)) = 0$.

The condition for $X \in \mathfrak{X}(M)$ to be a Killing vector field, $\mathcal{L}_X g = 0$, can be rewritten, using only norms, as

$$(\mathcal{L}_X g)(Y, Y) = 0, \quad \forall Y \in \mathfrak{X}(M), \tag{18}$$

because

$$2(\mathcal{L}_X g)(Y, Z) = (\mathcal{L}_X g)(Y + Z, Y + Z) - (\mathcal{L}_X g)(Y, Y) - (\mathcal{L}_X g)(Z, Z), \quad \forall Y, Z \in \mathfrak{X}(M).$$

For the local expression of Killing vector fields, let $X \in \mathfrak{X}(M)$ be the vector field with local expression in a local chart:

$$X = \sum_{i=1}^n X^i(q) \frac{\partial}{\partial q^i}, \tag{19}$$

then the Lie derivative with respect to X of the metric tensor field g is

$$\mathcal{L}_X g = \sum_{i,j,k=1}^n X^k \frac{\partial g_{ij}}{\partial q^k} dq^i \otimes dq^j + \sum_{i,j,k=1}^n g_{ij} \left(\frac{\partial X^i}{\partial q^k} dq^k \otimes dq^j + \frac{\partial X^j}{\partial q^k} dq^i \otimes dq^k \right),$$

or using the symmetry property of the metric tensor field,

$$\mathcal{L}_X g = \sum_{i,j=1}^n \left(\sum_{k=1}^n X^k \frac{\partial g_{ij}}{\partial q^k} + \sum_{k=1}^n g_{ik} \frac{\partial X^k}{\partial q^j} + \sum_{k=1}^n g_{jk} \frac{\partial X^k}{\partial q^i} \right) dq^i \otimes dq^j, \tag{20}$$

which shows that the condition for X to be a Killing vector field, i.e., $\mathcal{L}_X g = 0$, is written in local coordinates as

$$\sum_{i,j,k=1}^n \left(X^k \frac{\partial g_{ij}}{\partial q^k} + g_{ik} \frac{\partial X^k}{\partial q^j} + g_{jk} \frac{\partial X^k}{\partial q^i} \right) dq^i \otimes dq^j = 0.$$

Thus, the local conditions for a vector field $X \in \mathfrak{X}(M)$ to be a Killing symmetry are:

$$\sum_{k=1}^n \left(X^k \frac{\partial g_{ij}}{\partial q^k} + g_{ik} \frac{\partial X^k}{\partial q^j} + g_{jk} \frac{\partial X^k}{\partial q^i} \right) = 0, \quad i, j = 1, \dots, n. \tag{21}$$

We can show (see next subsection) that a vector field $X \in \mathfrak{X}(M)$ is a geodesic vector field if, and only if, it is an autoparallel vector field, i.e., if $\nabla_X X = 0$, because the vector

field X is a geodesic vector field if, and only if, its integral curves γ are geodesics, i.e., they satisfy the condition $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ (see, e.g., [1]).

It has also been remarked in [5] that, for each nonvanishing function $f \in C^\infty(M)$, if $X \in \mathfrak{X}(M)$ is an autoparallel vector field, then $Y = fX$ satisfies $\nabla_Y Y = (Xf)Y$. However, each integral curve of the vector field X coincides with one of fX , up to a reparametrisation, and therefore, the preceding property can be used to derive that each integral curve of a vector field X can be reparametrised to be a geodesic curve if, and only if, $\nabla_X X = fX$, where f is a nonvanishing function $f \in C^\infty(M)$. Moreover, it was proven in [5] that all the integral curves of a vector field $X \in \mathfrak{X}(M)$ can be transformed by a Sundman transformation (see [5] and the references therein) into geodesic curves of the metric g if, and only if, X is a *pregeodesic vector field*, that is there exists a function $f \in C^\infty(M)$ such that $\nabla_X X = fX$. This leads to the following definition:

Definition 3. We say that a vector field $X \in \mathfrak{X}(M)$ is a *generalised geodesic vector field*, sometimes also-called *pregeodesic vector field*, if there is a nonvanishing function $f \in C^\infty(M)$ such that $\nabla_X X = fX$.

Observe that an integral curve of such a vector field is such that its acceleration is proportional to its velocity. It was proven in [1] that a nonzero generalised geodesic vector field of constant length is a geodesic vector field.

Using that the metric g provides an identification of $\mathfrak{X}(M)$ and $\Omega^1(M)$, and as closed and exact one-forms are distinguished ones, we can correspondingly introduce the definition of closed and gradient vector fields [15] and some of their properties:

Definition 4. We say that a vector field $X \in \mathfrak{X}(M)$ is a *closed vector field* when its corresponding one-form $i(X)g$ is closed, $d(i(X)g) = 0$. In the special case of exact one-forms, we say that the vector field X is a *gradient vector field*: if $i(X)g = df$, the vector field X is said to be the *gradient of f* .

As an immediate application of (6), we see the above-mentioned property: if a vector field $X \in \mathfrak{X}(M)$ is autoparallel, then $g(X, X)$ is constant along the integral curves of X , that is

$$\mathcal{L}_X(g(X, X)) = 0. \tag{22}$$

Having in mind also that $(\mathcal{L}_X g)(Y, Z) = \mathcal{L}_X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z])$, we obtain from (6) and the torsionless condition for the Levi-Civita connection that

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \tag{23}$$

In particular, for $Z = Y$, we have

$$(\mathcal{L}_X g)(Y, Y) = 2g(\nabla_Y X, Y). \tag{24}$$

Similarly,

$$d(i(X)g)(Y, Z) = \mathcal{L}_Y g(X, Z) - \mathcal{L}_Z g(X, Y) - g(X, [Y, Z]), \tag{25}$$

and then, as the connection is metric,

$$d(i(X)g)(Y, Z) = g(\nabla_Y X, Z) + g(X, \nabla_Y Z) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) - g(X, [Y, Z]),$$

and as the connection is torsionless, such an expression reduces to

$$d(i(X)g)(Y, Z) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y), \tag{26}$$

which can also be rewritten as

$$d(i(X)g)(Y, Z) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y) = 2g(\nabla_Y X, Z) - (\mathcal{L}_X g)(Y, Z); \tag{27}$$

therefore, when X is a closed vector field,

$$g(\nabla_Y X, Z) = \frac{1}{2}(\mathcal{L}_X g)(Y, Z). \tag{28}$$

Some of these expressions will be used in the sequel.

3.2. Some Properties of Geodesic Vector Fields

For the sake of completeness, we give here a brief account of some known results on geodesic vector fields and references relating them to other specific classes of vector fields. The proofs we include are slightly different from the original ones in the given references (see, e.g., [5] and the references therein):

1. Our first remark is that a vector field $X \in \mathfrak{X}(M)$ is an autoparallel vector field if, and only if, its integral curves γ are geodesics, i.e., they satisfy the equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, and then, when lifted to the tangent bundle TM , are integral curves of Γ_g , the second-order geodesic vector field.

In fact, suppose that $\nabla_X X = 0$, and let $\gamma : I \subset \mathbb{R} \rightarrow M$ be an integral curve of X , that is $\dot{\gamma} = X \circ \gamma$. Then:

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = \nabla_{(X \circ \gamma)(t)} (X \circ \gamma)(t) = (\nabla_X X)(\gamma(t)) = 0,$$

and therefore, γ is a geodesic curve in M . If, on the contrary, we suppose that every integral curve of X satisfies the equation $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$ and let $p \in M$ and $\gamma : I \subset \mathbb{R} \rightarrow M$ be the integral curve of X with initial condition p , we have

$$(\nabla_X X)(p) = \nabla_{X(p)} X = \nabla_{\dot{\gamma}(0)} \dot{\gamma}(t) = (\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t))|_{t=0} = 0.$$

However, as this is true for every $p \in M$, we have $\nabla_X X = 0$.

Because of this property, the autoparallel vector fields are also-called *geodesic vector fields* [16]. In the definitions we introduced at the beginning of the section, for a vector field, it is the same to be geodesic as to be autoparallel. We remark that, for each real number a , if X is a geodesic vector field, then aX is geodesic also.

2. There are some relations between autoparallel and Killing vector fields, as the following ones.

It was proven in [14], Lemma 3, and it was used in [7,17] that a Killing vector field $X \in \mathfrak{X}(M)$ has constant length if, and only if, every integral curve of X is a geodesic, that is X is a geodesic vector field. We can give a little more general proof:

Proof. (a) Using that the connection ∇ is metric and torsionless, we have, on the one hand, that, for arbitrary vector fields $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned} \mathcal{L}_X(g(X, Y)) &= (\mathcal{L}_X g)(X, Y) + g(\mathcal{L}_X X, Y) + g(X, \mathcal{L}_X Y) \\ &= (\mathcal{L}_X g)(X, Y) + g(X, \nabla_X Y) - g(X, \nabla_Y X), \end{aligned}$$

because $\mathcal{L}_X X = 0$ and, according to (5), $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$. However, by making use of (7) with an interchange of the roles of X and Y ,

$$\mathcal{L}_X(g(X, Y)) = (\mathcal{L}_X g)(X, Y) + g(X, \nabla_X Y) - \frac{1}{2}\mathcal{L}_Y(g(X, X)), \tag{29}$$

and on the other hand,

$$\mathcal{L}_X(g(X, Y)) = g(\nabla_X X, Y) + g(X, \nabla_X Y). \tag{30}$$

Comparing both expressions, we have that, for arbitrary vector fields $X, Y \in \mathfrak{X}(M)$,

$$g(\nabla_X X, Y) = (\mathcal{L}_X g)(X, Y) - \frac{1}{2} \mathcal{L}_Y(g(X, X)).$$

(b) If $X = Y$, the expression (30) reduces to $\mathcal{L}_X(g(X, X)) = 2g(\nabla_X X, X)$ and this relation shows that if X is a geodesic vector field, $\nabla_X X = 0$, then $\mathcal{L}_X(g(X, X)) = 0$, that is the norm of X is constant along the integral curves of X , in agreement with Property (22).

(c) Moreover, if X is a Killing vector field, we have that $g(\nabla_X X, Y) = -\frac{1}{2} \mathcal{L}_Y(g(X, X))$, and consequently, as Y is an arbitrary vector field, we see that, then, X is a geodesic vector field, i.e., $\nabla_X X = 0$, if, and only if, $g(X, X)$ is a real constant, as we wanted. \square

3. As far as the more general class of vector fields in a Riemannian manifold is concerned, that of the so-called *generalised geodesic vector fields*, usually also-called *pregeodesic vector fields*, which are those such that there exists a function f satisfying $\nabla_X X = fX$, a remarkable property was proven in [1]: a generalised geodesic vector field X of constant length on a Riemannian manifold is a geodesic vector field, because if φ is the function defined by $\varphi = \frac{1}{2}g(X, X)$, as $\nabla_X X = fX$, we have that

$$\mathcal{L}_X(\varphi) = \frac{1}{2}g(\nabla_X X, X) + \frac{1}{2}g(X, \nabla_X X) = 2f\varphi,$$

and equivalently, $2f = \mathcal{L}_X(\log \varphi)$, which shows that as φ is constant, then $f = 0$; therefore, X is a geodesic vector field.

3.3. Another Interesting Property of Killing Vector Fields

First of all, we establish a lemma whose result is to be used later on:

Lemma 1. *Let (M, g) be a Riemannian manifold. The vector field $X \in \mathfrak{X}(M)$ is a Killing vector field if, and only if, we have that $g(\nabla_Y X, Y) = 0$, for every vector field $Y \in \mathfrak{X}(M)$.*

Proof. Consider the following chain of identities for $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} g(\nabla_Y X, Y) &= g(\nabla_X Y + [Y, X], Y) = g(\nabla_X Y, Y) - g([X, Y], Y) \\ &= \frac{1}{2} \mathcal{L}_X(g(Y, Y)) - \left(\frac{1}{2} \mathcal{L}_X(g(Y, Y)) - \frac{1}{2} (\mathcal{L}_X g)(Y, Y) \right) \\ &= \frac{1}{2} (\mathcal{L}_X g)(Y, Y), \end{aligned}$$

obtained by using the properties of the Levi-Civita connection. This shows that the condition $g(\nabla_Y X, Y) = 0$, for every vector field $Y \in \mathfrak{X}(M)$, is equivalent to $(\mathcal{L}_X g)(Y, Y) = 0$, for every vector field $Y \in \mathfrak{X}(M)$, and therefore, Condition (18) for a Killing vector field holds; this proves the result of the lemma. \square

There is a kind of Noether theorem associated with a Killing vector field $X \in \mathfrak{X}(M)$, to which we can associate a one-form $I_X \in \Omega^1(M)$. We recall that, given a one-form $\alpha \in \Omega^1(M)$, we can define an associated, linear in the fibres, function $\widehat{\alpha} : TQ \rightarrow \mathbb{R}$, by $\widehat{\alpha}(q, v) = \langle \alpha_q, v \rangle$. In particular, the one-form $\alpha = \alpha_i(q) dq^i$ has associated the function $\widehat{\alpha}(q, v) = \alpha_i(q) v^i$. Therefore, the one-form I_X has associated the function $\widehat{I}_X \in C^\infty(TM)$, and then, we can prove that if $X \in \mathfrak{X}(M)$ is a Killing vector field, the function \widehat{I}_X is invariant along geodesics. More specifically, we have:

Proposition 1. Let be (M, g) a Riemannian manifold and $X \in \mathfrak{X}(M)$ a Killing vector field. If $\gamma : \mathbb{R} \rightarrow M$ is a geodesic curve and $\mathbf{I}_X = i(X)g \in \Omega^1(M)$, then the function $\widehat{\mathbf{I}}_X$ is constant along the lifted to TM curve γ^T , that is

$$\frac{d}{dt}(\widehat{\mathbf{I}}_X \circ \dot{\gamma}) = 0.$$

Proof. We have:

$$\frac{d}{dt}(\widehat{\mathbf{I}}_X \circ \dot{\gamma}) = \frac{d}{dt}g(X \circ \gamma, \dot{\gamma}) = \nabla_{\dot{\gamma}}(g(X, \dot{\gamma})) = g(\nabla_{\dot{\gamma}}X, \dot{\gamma}) + g(X, \nabla_{\dot{\gamma}}\dot{\gamma}) = 0,$$

and both summands on the right-hand side are null, the first one as a consequence of the preceding lemma and the second one because γ is a geodesic. \square

In classical mechanics, the differential one-form $\mathbf{I}_X = i(X)g \in \Omega^1(M)$ is related to the so-called *momentum form* associated with the vector field X for geodesic motion. For instance, if $M = \mathbb{R}^3$ with the usual metric, the infinitesimal generators of translations and rotations give rise to Killing vector fields whose associated momenta are the linear momentum form and the angular momentum form. More specifically, recall the important property (see, e.g., [18]) that, if X is a vector field on M , $X \in \mathfrak{X}(M)$, and X^c denotes its complete lift, $X^c \in \mathfrak{X}(TM)$, then

$$\mathcal{L}_{X^c}T_g = T_{\mathcal{L}_Xg}, \tag{31}$$

because, by using the definition of the Lie derivative of the function T_g and that if ϕ_t is the flow of X and the flow of X^c is $T\phi_t$, we have that, for all $v \in TM$,

$$\begin{aligned} \mathcal{L}_{X^c}T_g(v) &= \frac{d}{dt}T_g \circ T\phi_t(v)|_{t=0} = \frac{d}{dt} \left(\frac{1}{2}g(T\phi_t(v), T\phi_t(v)) \right) |_{t=0} \\ &= \frac{1}{2} \frac{d}{dt}(\phi_t^*g)(v, v)|_{t=0} = \frac{1}{2}(\mathcal{L}_Xg)(v, v) = T_{\mathcal{L}_Xg}(v). \end{aligned}$$

Consequently, $X \in \mathfrak{X}(M)$ is a Killing vector field for the Riemann structure g if, and only if, $X^c \in \mathfrak{X}(TM)$ is a symmetry for the corresponding kinetic energy function defined by g , and if we consider the symplectic manifold (TM, ω_{T_g}) , when $X \in \mathfrak{X}(M)$ is a Killing vector field, then the vector field $X^c \in \mathfrak{X}(TM)$ is a Hamiltonian vector field, because $\mathcal{L}_{X^c}\theta_{T_g} = \theta_{X^cT_g} = 0$; then, $\mathcal{L}_{X^c}\omega_{T_g} = \omega_{X^cT_g} = 0$. Its Hamilton function is the linear in velocities function corresponding to the one-form $i(X^c)\theta_{T_g}$, because $\mathcal{L}_{X^c}\theta_{T_g} = 0$ implies that $i(X^c)\omega_{T_g} = d(i(X^c)\theta_{T_g})$.

In order to clarify the meaning of the function $\widehat{i(X)g} : TM \rightarrow \mathbb{R}$, we remark that, if $X \in \mathfrak{X}(M)$ is given in a chart by (19), namely

$$X = \sum_{i=1}^n X^i(q) \frac{\partial}{\partial q^i} \in \mathfrak{X}(M),$$

then the coordinate expression of its complete lift $X^c \in \mathfrak{X}(TM)$ is

$$X^c = \sum_{i=1}^n X^i \frac{\partial}{\partial q^i} + \sum_{i,j=1}^n v^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial v^i}, \tag{32}$$

and that of the one-form $i(X)g$ on M is

$$i(X)g = \sum_{i,j=1}^n g_{ij}(q) X^j dq^i, \tag{33}$$

whose associated function $\widehat{i(X)g} \in C^\infty(TM)$ is

$$\widehat{i(X)g}(q, v) = \sum_{i,j=1}^n g_{ij}(q) X^j(q) v^i. \tag{34}$$

We remark also that, as indicated above, the contraction of the complete lift vector field X^c with the semibasic one-form:

$$\theta_{T_g}(q, v) = \sum_{i,j=1}^n g_{ij}(q) v^j dq^i, \tag{35}$$

is given by

$$(i(X^c)\theta_{T_g})(q, v) = \sum_{i,j=1}^n g_{ij}(q) X^i(q) v^j, \tag{36}$$

and therefore, $\widehat{i(X)g} = i(X^c)\theta_{T_g}$.

In this sense, the one-form $\mathbf{I}_X = i(X)g \in \Omega^1(M)$ corresponds to a function on TM that is the Hamiltonian function of the vector field X^c , as shown by a comparison of (33) with (36).

For instance, in the particular case of $M = \mathbb{R}^3$ endowed with the Euclidean metric, if X is the vector field:

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},$$

the infinitesimal generator of rotations along the third axis, such a vector field is a Killing vector field, and we obtain the one-form $\mathbf{I}_X = y dx - x dy$, whose associated function $\widehat{\mathbf{I}_X}(x, y, z, v_x, v_y, v_z) = y v_x - x v_y$ is constant along a geodesic line, i.e., the corresponding constant of the motion is the third component of the angular momentum.

With our understanding of Γ_g as the dynamical Lagrangian vector field of the Lagrangian system (M, T_g) , the function $\widehat{i(X)g}$ is the Noether invariant corresponding to the symmetry X^c of the Lagrangian T_g . This is because $\widehat{i(X)g} = i(X^c)\theta_g$, as we have shown above.

Note also that, as the coordinate expression of the vertical lift $X^v \in \mathfrak{X}(TM)$ of the vector field given by (19) is

$$X^v = \sum_{i=1}^n X^i(q) \frac{\partial}{\partial v^i},$$

we have

$$\mathcal{L}_{X^v}T_g = \sum_{i=1}^n X^i(q) g_{ij}(q) v^j = \widehat{i(X)g}.$$

3.4. Killing Vector Fields and the Geodesic Flow

In Equation (31) and the following paragraphs of Section 3.3, we proved that, if X is a Killing vector field on the Riemannian manifold (M, g) , then its complete lift to TM , X^c , is a symmetry of the kinetic energy function T_g , and consequently we have that $\mathcal{L}_{X^c}\theta_{T_g} = 0$ and $\mathcal{L}_{X^c}\omega_{T_g} = 0$. From these properties, we have the following:

Proposition 2. *Let be $X \in \mathfrak{X}(M)$ a Killing vector field on (M, g) , $X^c \in \mathfrak{X}(TM)$ its complete lift to TM , and ϕ_t^X the local flow of X . We have the following:*

- (i) *If $\Gamma_g \in \mathfrak{X}(TM)$ is the second-order geodesic vector field, then $\mathcal{L}_{X^c}\Gamma_g = 0$.*
- (ii) *If $\gamma : \mathbb{R} \rightarrow M$ is a geodesic curve, then $\phi_t^X \circ \gamma$ is also a geodesic curve.*
- (iii) *If $Y \in \mathfrak{X}(M)$ is a geodesic vector field, i.e., $\nabla_Y Y = 0$, then $(\phi_t^X)_* Y$ is a geodesic vector field, that is*

$$\nabla_{(\phi_t^X)_* Y} (\phi_t^X)_* Y = 0.$$

Proof. (i) From Relation (16), $i(\Gamma_g)\omega_{T_g} = dT_g$, we have

$$\mathcal{L}_{X^c} \left(i(\Gamma_g)\omega_{T_g} \right) = i(\mathcal{L}_{X^c}\Gamma_g)\omega_{T_g} + i(\Gamma_g)\mathcal{L}_{X^c}\omega_{T_g} = i(\mathcal{L}_{X^c}\Gamma_g)\omega_{T_g}.$$

However, $\mathcal{L}_{X^c}dT_g = d(\mathcal{L}_{X^c}T_g) = 0$, by Equation (31). Hence, $i(\mathcal{L}_{X^c}\Gamma_g)\omega_{T_g} = 0$, and ω_{T_g} being a symplectic form, we have that $\mathcal{L}_{X^c}\Gamma_g = 0$.

(ii) If γ is a geodesic curve, then γ^T is an integral curve of Γ_g . From (i), we have that $(T\phi_t^X)_*\Gamma_g = \Gamma_g$; hence, if γ^T is an integral curve of Γ_g , then $(T\phi_t^X) \circ \gamma^T$ is an integral curve of $(T\phi_t^X)_*\Gamma_g = \Gamma_g$.

We have then that $\phi_t^X \circ \gamma$ is a geodesic because $(T\phi_t^X) \circ \gamma^T = (\phi_t^X \circ \gamma)^T$.

(iii) If $Y \in \mathfrak{X}(M)$ is a geodesic vector field, then its integral curves γ are geodesics. However, the integral curves of $(\phi_t^X)_*Y$ are of the form $\phi_t^X \circ \gamma$, which, by Item (ii), are geodesics; hence, $(\phi_t^X)_*Y$ is a geodesic vector field by Item 1 in Section 3.2, because all its integral curves are geodesic curves. \square

4. Hamilton–Jacobi Vector Fields

There are several methods to integrate systems of differential equations, and one of the most significant is to reduce the dimension of the manifold where the vector field corresponding to the system of differential equation is defined. Generically, this method allows us to obtain not all the solutions, but only a subset. As an instance with these ideas, a modern geometric approach to the Hamilton–Jacobi theory was developed in [19]. Even if such a theory is always developed in a Hamiltonian context, we can also consider the framework of the tangent bundle of the given Riemann manifold (M, g) and extend the theory to a Lagrangian setting. In some cases, we can obtain what is called a complete solution to the Hamilton–Jacobi problem and, through it, all the solutions of the initial equation if the full set of reduced equations can be solved.

In the next subsections, we develop a quick survey of the general theory of Hamilton–Jacobi reduction and the particular cases of the Lagrangian and Hamiltonian formalism with their differences and relations.

4.1. Geometric Hamilton–Jacobi Theory

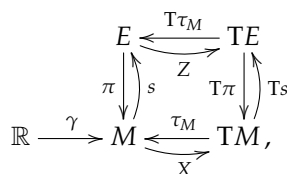
Let $\pi : E \rightarrow M$ be a differentiable vector bundle and $Z \in \mathfrak{X}(E)$. We say that a pair $(X, s) \in \mathfrak{X}(M) \times \Gamma(\pi)$, where $\Gamma(\pi)$ denotes the set of sections of π , i.e., the set of maps $s : M \rightarrow E$ such that $\pi \circ s = \text{id}_M$, is a Hamilton–Jacobi pair for the vector field Z if $X \underset{s}{\approx} Z$, that is $X \in \mathfrak{X}(M)$ is s -related with Z . Equivalently, $Z \circ s = Ts \circ X$.

Suppose that (X, s) is a Hamilton–Jacobi pair for Z and γ is an integral curve of X , $\dot{\gamma} = X \circ \gamma$, then we have that

$$\overline{s \circ \dot{\gamma}} = Ts \circ \dot{\gamma} = Ts \circ X \circ \gamma = Z \circ s \circ \gamma = Z \circ (s \circ \gamma),$$

and then, the curve $s \circ \gamma$ in E is an integral curve of Z . It is easy to see that the converse property is true: if Z and (X, s) are such that every integral curve of X lifted to E by s is an integral curve of Z , then $X \underset{s}{\approx} Z$, which can be seen by using that, for every point $p \in M$, there is an integral curve of X with the initial condition at p .

We can summarize the situation in the following diagram:



where the initial data are the vector field Z , the Hamilton–Jacobi pair is (X, s) , and the diagram is commutative.

4.2. Lagrangian Theory

In the Lagrangian case, the initial bundle E is a tangent bundle $E = TQ$, $\tau_Q : TQ \rightarrow Q$, $Z \in \mathfrak{X}(TQ)$, and $(X, s) \in \mathfrak{X}(Q) \times \mathfrak{X}(Q)$; hence, the section s is another vector field, $s = Y \in \mathfrak{X}(Q)$, and we have that, if (X, Y) is a Hamilton–Jacobi pair, then $X \underset{Y}{\approx} Z$, or equivalently, $Z \circ Y = TY \circ X$. The corresponding commutative diagram is

$$\begin{array}{ccc}
 TQ & \xleftarrow{\tau_{TQ}} & T(TQ) \\
 \tau_Q \downarrow & \begin{array}{c} Y \\ \uparrow \\ Z \end{array} & \begin{array}{c} T\tau_Q \\ \uparrow \\ TY \end{array} \\
 \mathbb{R} \xrightarrow{\gamma} & Q & \xleftarrow{\tau_Q} TQ \\
 & & \begin{array}{c} X \\ \uparrow \end{array}
 \end{array}$$

and therefore, if γ is an integral curve of X , then $Y \circ \gamma$ is an integral curve of Z .

A particular case, very relevant in the Lagrangian framework, is when the vector field $Z \in \mathfrak{X}(TQ)$ is an SODE, a second-order differential equation, vector field. Then, as indicated above, (X, Y) with $X \in \mathfrak{X}(TQ)$ and $Y \in \mathfrak{X}(Q)$ is a Hamilton–Jacobi pair if $Z \circ Y = TY \circ X$. However, if γ is an integral curve of X , i.e., $\dot{\gamma} = X \circ \gamma$, the relatedness condition shows that the curve $Y \circ \gamma$ in TQ is an integral curve of the Z , but as Z is assumed to be a SODE vector field, its integral curves are canonical lifts of curves in Q ; hence, we have that, necessarily, $Y \circ \gamma = \dot{\gamma}$, because the curve $Y \circ \gamma$ in TQ is an integral curve of Z , and this implies that $Y = X$, because γ is an arbitrary integral curve of X . Thus, in this case, given Z , we look for a vector field X such that $Z \circ X = TX \circ X$, or equivalently, $X \underset{X}{\approx} Z$, and the relevant commutative diagram is

$$\begin{array}{ccc}
 TQ & \xleftarrow{\tau_{TQ}} & T(TQ) \\
 \tau_Q \downarrow & \begin{array}{c} X \\ \uparrow \\ Z \end{array} & \begin{array}{c} T\tau_Q \\ \uparrow \\ TX \end{array} \\
 \mathbb{R} \xrightarrow{\gamma} & Q & \xleftarrow{\tau_Q} TQ \\
 & & \begin{array}{c} X \\ \uparrow \end{array}
 \end{array}$$

This is a specific characteristic of the Hamilton–Jacobi problem in the case of second-order differential equations: we have only one vector field, which plays a double role, first, as a vector field on Q and, second, as a section for τ_Q , as solution of the Hamilton–Jacobi problem associated with a vector field Z .

4.3. Hamiltonian Theory

In this case, the initial bundle is the cotangent bundle $\pi_Q : T^*Q \rightarrow Q$, $Z \in \mathfrak{X}(T^*Q)$, and $(X, s) \in \mathfrak{X}(Q) \times \Omega^1(Q)$, i.e., the section s is a one-form on Q , $s = \alpha \in \Omega^1(Q)$; we have that (X, α) is a Hamilton–Jacobi pair for Z if, and only if, $X \underset{\alpha}{\approx} Z$, or equivalently, $Z \circ \alpha = T\alpha \circ X$. The corresponding commutative diagram is

$$\begin{array}{ccc}
 T^*Q & \xleftarrow{\tau_{T^*Q}} & T(T^*Q) \\
 \pi_Q \downarrow & \begin{array}{c} \alpha \\ \uparrow \\ Z \end{array} & \begin{array}{c} T\pi_Q \\ \uparrow \\ T\alpha \end{array} \\
 \mathbb{R} \xrightarrow{\gamma} & Q & \xleftarrow{\tau_Q} TQ \\
 & & \begin{array}{c} X \\ \uparrow \end{array}
 \end{array}$$

We can ask ourselves if there is any relation between the vector field X and the one-form α . Actually, from the relation $Z \circ \alpha = T\alpha \circ X$ and as α is a section of π_Q , we have that

$$T\pi_Q \circ Z \circ \alpha = T\pi_Q \circ T\alpha \circ X = T(\pi_Q \circ \alpha) \circ X = X.$$

Thus, we have that

$$X = T\pi_Q \circ Z \circ \alpha. \tag{37}$$

In this case, we have two relevant specific situations. Recall first that a cotangent bundle $\pi_Q : T^*Q \rightarrow Q$, with $\dim Q = n$, is endowed with a canonical one-form θ , called the Liouville one-form, such that $\alpha^*\theta = \alpha$, for each $\alpha \in \Omega^1(Q)$. Moreover, the two-form $\omega = -d\theta$ is a symplectic form. Then, the first relevant situation is when the one-form α of the Hamilton–Jacobi pair (X, α) is a closed (and therefore, locally exact) one-form, $\alpha = dU$, where $U : Q \rightarrow \mathbb{R}$ is a differentiable function. This is equivalent to saying that the image of α is a Lagrangian submanifold of the natural symplectic manifold (T^*Q, ω) , because it is an n -dimensional submanifold such that, since $\alpha^*\omega = -\alpha^*d\theta = -d(\alpha^*\theta) = -d\alpha$, by the definition of the canonical one-form θ of T^*Q , we see that, as α is a closed one-form, we have $\alpha^*\omega = 0$, i.e., the image of α is a Lagrangian submanifold of T^*Q .

The second relevant situation is when the considered vector field $Z \in \mathfrak{X}(T^*Q)$ is a Hamiltonian vector field, namely $i(Z)\omega = dH$, with $H : T^*Q \rightarrow \mathbb{R}$ being a differentiable function, the Hamiltonian function associated with Z . In this case, we have the following result:

Proposition 3. *Suppose that $Z \in \mathfrak{X}(T^*Q)$ is a Hamiltonian vector field, with Hamiltonian function H , i.e., $i(Z)\omega = dH$, and that (X, α) is a Hamilton–Jacobi pair for Z with α a closed one-form, $d\alpha = 0$. Then, $\alpha^*dH = d(H \circ \alpha) = 0$.*

Proof. From $i(Z)\omega = dH$, we have that $\alpha^*(i(Z)\omega) = \alpha^*dH$. However, using that $Z \circ \alpha = T\alpha \circ X$ because (X, α) a Hamilton–Jacobi pair for Z , the left-hand side of this relation is

$$\alpha^*(i(Z)\omega) = i(X)\alpha^*\omega, \tag{38}$$

because, for any vector field $Y \in \mathfrak{X}(Q)$,

$$\alpha^*(i(Z)\omega)(Y) = \omega(Z \circ \alpha, T\alpha \circ Y) = \omega(T\alpha \circ X, T\alpha \circ Y) = \alpha^*\omega(X, Y) = (i(X)\alpha^*\omega)(Y).$$

We proved before that the closedness of the one-form α implies that $\alpha^*\omega = 0$, and from the relation $\alpha^*(i(Z)\omega) = \alpha^*(dH)$, we obtain that $d(H \circ \alpha) = \alpha^*dH = 0$. \square

Consequently, when Q is connected, locally, $H \circ \alpha = \text{constant}$, and if, also locally, $\alpha = dU$ by the Poincaré lemma, then $H \circ dU = \text{constant}$, that is

$$H\left(q, \frac{\partial U}{\partial q^i}\right) = \text{constant},$$

which is the classical Hamilton–Jacobi equation. Recall that, as indicated in (37), the vector field X can be recovered from the vector field Z and the one-form α as $X = T\pi_Q \circ Z \circ \alpha$.

5. Hamilton–Jacobi Vector Fields for the Second-Order Geodesic Vector Field

In this section, we fix our attention on the particular case of Hamilton–Jacobi vector fields for a second-order geodesic vector field according to the theory developed in [19], but not only in a Hamiltonian context, but also in the Lagrangian setting for systems of a mechanical type developed in the framework of the tangent bundle of the given Riemann manifold (M, g) (see, e.g., [5]).

Recall that, as indicated in the preceding section, a Hamilton–Jacobi pair for a vector field in the tangent bundle TQ reduces to only a vector field in Q playing a double role. More explicitly, we establish the following definition:

Definition 5. *Given an SODE vector field $Z \in \mathfrak{X}(TQ)$, a vector field $X \in \mathfrak{X}(Q)$ is said to be a Hamilton–Jacobi vector field for Z if, for every integral curve $\gamma : \mathbb{R} \rightarrow Q$ of X , the curve $X \circ \gamma : \mathbb{R} \rightarrow TQ$ is an integral curve of Z .*

In [19–22], some useful properties of these Hamilton–Jacobi vector fields were studied. Recall that, alternatively, a vector field $X \in \mathfrak{X}(M)$ is a Hamilton–Jacobi vector field for an

SODE vector field Z if, and only if, $TX \circ X = Z \circ X$, i.e., X and Z are X -related vector fields, $X \approx Z$, or equivalently, if the following diagram is commutative:

$$\begin{array}{ccccc}
 & & TQ & \xleftarrow{T\tau_Q} & T(TQ) \\
 & & \uparrow \tau_Q & & \uparrow T\tau_Q \\
 & & X & \xrightarrow{TX} & Z \\
 \mathbb{R} & \xrightarrow{\gamma} & Q & \xleftarrow{\tau_Q} & TQ \\
 & & \downarrow X & & \downarrow X
 \end{array}$$

Let now (M, g) be an n -dimensional Riemann manifold and ∇ denote its associated Levi-Civita connection. The relation among autoparallel vector fields and Hamilton–Jacobi vector fields for the second-order geodesic vector fields Γ is given in the following statement:

Proposition 4. *A vector field $X \in \mathfrak{X}(M)$ is an autoparallel vector field on a Riemann manifold (M, g) , that is $\nabla_X X = 0$, if, and only if, it is a Hamilton–Jacobi vector field for the second-order geodesic vector field Γ of g , i.e., $\Gamma \circ X = TX \circ X$.*

Proof. Suppose that the vector field $X \in \mathfrak{X}(M)$ is autoparallel. If $p \in M$ and γ is the integral curve of the vector field X , $\dot{\gamma} = X \circ \gamma$, with initial condition $\gamma(0) = p$, then we know that the tangent lift γ^T of γ from M to TM is an integral curve of Γ , and hence, as $\dot{\gamma} = X \circ \gamma$, $X \circ \gamma$ is an integral curve of Γ ; therefore, Γ and X are X -related, i.e., X is a Hamilton–Jacobi vector field for Γ .

Conversely, suppose that γ is the integral curve with initial condition $p \in M$ of a vector field $X \in \mathfrak{X}(M)$, i.e., $\dot{\gamma} = X \circ \gamma$. If X is a Hamilton–Jacobi vector field for Γ , then γ^T is an integral curve of Γ . Hence, γ is a geodesic curve, and then, $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$, which implies that $\nabla_{X_p} X = 0$. However, as $p \in M$ is arbitrary, we have $\nabla_X X = 0$, i.e., and X is autoparallel. \square

Observe that, in that case, the curve $(X \circ \gamma)^T$ is given in local coordinates as

$$\left(\gamma^i, \sum_{j=1}^n \dot{\gamma}^j \left(\frac{\partial X^i}{\partial q^j} \right) \circ \gamma \right). \tag{39}$$

Hence, the local condition for X to be geodesic is

$$\sum_{j=1}^n \dot{\gamma}^j \left(\frac{\partial X^i}{\partial q^j} \right) \circ \gamma = \left(- \sum_{j,k=1}^n \Gamma_{jk}^i X^j X^k \right) \circ \gamma, \quad i = 1, \dots, n,$$

for every integral curve γ of X . Equivalently:

$$\left(\sum_{j=1}^n X^j \frac{\partial X^i}{\partial q^j} \right) \circ \gamma = \left(- \sum_{j,k=1}^n \Gamma_{jk}^i X^j X^k \right) \circ \gamma, \quad i = 1, \dots, n. \tag{40}$$

As there are integral curves of X passing through every point in M , we have

$$\sum_{j=1}^n X^j \frac{\partial X^i}{\partial q^j} = - \sum_{j,k=1}^n \Gamma_{jk}^i X^j X^k, \quad i = 1, \dots, n, \tag{41}$$

as the set of local conditions for a vector field X in M to be a geodesic vector field.

Now, we go to the problem of how to find geodesic vector fields because Equation (41) is not easy to solve, nor is it even easy to know whether it has solutions. As far as closed Hamilton–Jacobi fields are concerned, we have the following:

Proposition 5. (Hamilton–Jacobi equation)

Let $X \in \mathfrak{X}(M)$ be a closed vector field on a Riemann manifold (M, g) , i.e., satisfying the condition $d(i(X)g) = 0$. Then, X is an autoparallel vector field, $\nabla_X X = 0$, i.e., X is a geodesic vector field, if, and only if, the function $T_g \circ X$ on M satisfies $d(T_g \circ X) = 0$, where $T_g : TM \rightarrow \mathbb{R}$ is the kinetic energy function defined by (14).

Proof. For each vector field $Y \in \mathfrak{X}(M)$, as $T_g \circ X = \frac{1}{2}g(X, X)$, we have:

$$d(T_g \circ X)(Y) = \mathcal{L}_Y(T_g \circ X) = \frac{1}{2}\mathcal{L}_Y(g(X, X)) = g(\nabla_Y X, X), \tag{42}$$

where use has been made of (7).

Now, taking into account (26), i.e., that

$$d(i(X)g)(Y, Z) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y),$$

when the vector field X is closed and $Z = X$, we have

$$g(\nabla_Y X, X) = g(\nabla_X X, Y), \tag{43}$$

and then, (42) shows that $d(T_g \circ X)(Y) = g(\nabla_X X, Y)$. Therefore, as $Y \in \mathfrak{X}(M)$ is an arbitrary vector field, we proved that, for closed vector fields X , i.e., such that $d(i(X)g) = 0$,

$$d(T_g \circ X) = 0 \iff \nabla_X X = 0.$$

□

As indicated above, the closedness condition $d(i(X)g) = 0$ tells us that the image of the differential one-form $i(X)g$ on T^*M is a Lagrangian submanifold of T^*M endowed with its canonical symplectic structure. Of course, as each exact form is closed, the closedness condition $d(i(X)g) = 0$ is satisfied for the particular case of gradient vector fields.

According to the Poincaré lemma, condition $d(T_g \circ X) = 0$ is equivalent to saying that the function $T_g \circ X$ is locally constant, and this is the standard Hamilton–Jacobi equation. In fact, in local coordinates, it reads as

$$\sum_{i,j=1}^n g_{ij}X^i X^j = \text{constant}.$$

When X is a gradient vector field, $X = \text{grad } U$, then $X = \sum_{i,j=1}^n g^{ij} \frac{\partial U}{\partial x^i} \frac{\partial}{\partial x^j}$, and as

$$T_g \circ X = \frac{1}{2} \sum_{i,j=1}^n g_{ij}X^i X^j = \frac{1}{2} \sum_{i,k=1}^n g^{ik} \frac{\partial U}{\partial x^i} \frac{\partial U}{\partial x^k},$$

the Hamilton–Jacobi equation is the standard one:

$$\frac{1}{2} \sum_{i,k=1}^n g^{ik} \frac{\partial U}{\partial x^i} \frac{\partial U}{\partial x^k} = \text{constant}.$$

This last equation provides a method for finding Hamilton–Jacobi gradient vector fields for the second-order geodesic dynamical vector field Γ_g of the metric g , by solving the preceding partial differential equation for the function U . This is, once again, another particular expression of the Hamilton–Jacobi equation.

From a complete solution of this PDE, we can obtain all the geodesics of the metric by integrating vector fields on M , that is systems of first-order ordinary differential equations.

Observe that, if X is a gradient vector field, then X is a geodesic vector field, $\nabla_X X = 0$, if, and only if, $g(X, X) = \text{constant}$, because if X is a gradient, then it satisfies $d(i(X)g) = 0$.

Therefore, by taking into account the three conditions for vector fields in a Riemann manifold:

$$(a) \nabla_X X = 0, \quad (b) g(X, X) = \text{constant}, \quad (c) d(i(X)g) = 0,$$

we proved the following logical relations for vector fields:

$$b \wedge c \Rightarrow a, \quad b \wedge a \Rightarrow c, \quad c \Rightarrow (a \Leftrightarrow b), \quad b \Rightarrow (a \Leftrightarrow c).$$

In the case where X is a gradient, it satisfies c) and the corresponding logical relations.

Finally, we remark that, if $\nabla_X X = 0$, the length of X , the function $g(X, X)$, is constant along the integral curves of X , but this does not imply that $g(X, X)$ must be a real constant. For example, in the ordinary Euclidean plane (x, y) , the vector field $X = y \partial/\partial x$ is a geodesic vector field, but $g(X, X)$ is not a constant.

6. Hamilton–Jacobi Vector Fields and Complete Lift of Vector Fields

In Section 4.2, we saw that, when looking for Hamilton–Jacobi vector fields for an SODE $Z \in \mathfrak{X}(TM)$, we must take as the section $s : M \rightarrow TM$ appearing in a Hamilton–Jacobi pair (X, s) , the same Hamilton–Jacobi vector field $X \in \mathfrak{X}(M)$. Hence, the Hamilton–Jacobi pair is (X, X) . We analyse a similar situation where this also happens.

Let $X \in \mathfrak{X}(M)$ be a vector field and $X^c \in \mathfrak{X}(TM)$ denote its complete lift from M to TM . We can ask whether (X, X) is a Hamilton–Jacobi pair for X^c , i.e., whether $X^c \circ X = TX \circ X$.

Proposition 6. *If $X \in \mathfrak{X}(M)$ is a vector field on a Riemann manifold (M, g) , then $X^c \circ X = TX \circ X$, and therefore, X is a Hamilton–Jacobi vector field for X^c .*

Proof. In local coordinates (q^i) for M and the induced ones (q^i, v^i) for TM , respectively, the local expressions of X and its complete lift are

$$X(q) = \sum_{j=1}^n X^j(q) \frac{\partial}{\partial q^j}, \quad X^c(q, v) = \sum_{j=1}^n X^j(q) \frac{\partial}{\partial q^j} + \sum_{j,k=1}^n v^j \frac{\partial X^k}{\partial q^j} \frac{\partial}{\partial v^k}.$$

Hence,

$$X^c \circ X = \left(\sum_{j=1}^n X^j \frac{\partial}{\partial q^j} + \sum_{j,k=1}^n X^j \frac{\partial X^k}{\partial q^j} \frac{\partial}{\partial v^k} \right),$$

which is equal to $TX \circ X$, as it can be easily computed. Consequently, $X \approx_X X^c$, i.e., (X, X) is a Hamilton–Jacobi pair. \square

For a geodesic vector field X on a Riemann manifold (M, g) , for which $\nabla_X X = 0$, we have that X is a Hamilton–Jacobi vector field for Γ_g and for X^c . The results of Proposition 4 and Proposition 6 are summarised as follows:

Proposition 7. *Let $X \in \mathfrak{X}(M)$ be a geodesic vector field on a Riemann manifold (M, g) and Γ_g the second-order geodesic vector field on TM . Then, $\Gamma_g \circ X = X^c \circ X$, and X is a Hamilton–Jacobi vector field for X^c and also for Γ_g .*

7. General Newtonian Dynamical Systems and Newtonian Vector Fields

Up to now, we have studied geodesic motions on Riemann manifolds; from the point of view of dynamical systems, they are called free systems. In the coming sections, we extend our study to motions under the action of external forces acting on the system, i.e., forced geodesic motions. We will define Newtonian systems and give the dynamical equations,

the notion of symmetry, and a kind of Noether theorem. We will finish the section by introducing the idea of Newtonian vector fields and the Hamilton–Jacobi equation. We will follow the notation used in [5].

7.1. Newtonian Systems and Their Dynamical Equations

Let (M, g) be an n -dimensional Riemann manifold, and consider a semibasic one-form F called the *force form* or *work form*. The dynamics is given by the SODE vector field Γ solution of the dynamical equation:

$$i(\Gamma)\omega_{T_g} - dT_g = F, \tag{44}$$

instead of the corresponding one for the free case, $i(\Gamma_g)\omega_{T_g} = dT_g$. When the one-form F is basic, $F = \sum_{i=1}^n F_i(q) dq^i$, the system is said to be Newtonian. In this last case, given the force form F , then $\Gamma = \Gamma_g + X_F$, where Γ_g is the second-order geodesic vector field defined by (16) and $X_F = -\sum_{i,j=1}^n g^{ij} F_i \frac{\partial}{\partial v^j}$ is the local expression of the *vector field of force*, which is the vertical lift to TM of the vector field Z_F on M such that $-i(Z_F)g = F$, i.e., $X_F = Z_F^v$. The local expression of Z_F is $Z_F = \sum_{i,j=1}^n Z^i \partial/\partial q^j$, with $F = -\sum_{j=1}^n F_j dq^j = -\sum_{i,j=1}^n g_{ij} Z^i dq^j$.

Moreover, when F is exact, $F = -\tau_M^*(dV)$ with $V : M \rightarrow \mathbb{R}$, we say that F is a force of potential type, and then, if we introduce a Lagrange function on TM by $L = T_g - V$, and therefore $E_L = T_g + V$, the dynamical vector field Γ is the solution of

$$i(\Gamma)\omega_L = dE_L. \tag{45}$$

When the one-form F can be written as a sum $F = -\tau_M^*(dV) + \bar{F}$, if we define the Lagrangian function L as $L = T_g - V$, the dynamical Equation (44) becomes

$$i(\Gamma)\omega_L - dE_L = \bar{F}. \tag{46}$$

See [23] for a more detailed study on Lagrangian systems with external forces.

Going back to the above general case, as indicated in [5], the projections on M of the integral curves of the SODE vector field Γ corresponding to Equation (44) satisfy the second-order differential equation:

$$\ddot{q}^i = -\sum_{k,l=1}^n \Gamma_{kl}^i \dot{q}^k \dot{q}^l + Z^i, \quad i = 1, \dots, n, \tag{47}$$

and a curve $\gamma : I \subset \mathbb{R} \rightarrow M$ is a solution of this system if, and only if, it satisfies the following condition:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \gamma, \tag{48}$$

as we can prove by expressing both members of the last equation in local coordinates. The trajectories $\gamma : I \rightarrow M$ of the mechanical system defined on (M, g) we are studying are therefore solutions of the dynamical Equation (48), which only uses tensorial objects on the base manifold M

We denote this system by (M, g, F) and collect this notion in the following:

Definition 6. A Newtonian dynamical system is given by (M, g, F) , where (M, g) is a Riemann manifold and F is a basic one-form on TM , and its dynamical equation is

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \gamma, \tag{49}$$

where $Z_F \in \mathfrak{X}(M)$ is the vector field such that $i(Z_F)g = -F$. The solution curves to this equation are called trajectories of the system.

Observe that, when $F = 0$, then we obtain the equation of geodesic curves, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Recall that we say that (M, g, F) is a simple mechanical-type system if F is an exact one-form, $F = -\tau_M^*(dV)$.

We are also interested in the relevant case in which the external force F is a semibasic one-form, which can also be seen as a one-form along the map $\tau_M : TM \rightarrow M$, i.e., $F : TM \rightarrow T^*M$ is such that $\pi_M \circ F = \tau_M$. In this more general case of generalised Newtonian systems for which F may be strictly semibasic, we must use the formalism of tensor fields along maps, and a short summary of this theory is summarised in the Appendix A. Now, F is not a basic one-form on TM , but a semibasic one. The vector field Γ solution of the dynamical Equation (44) when F is semibasic is said to be the *second-order dynamical vector field* associated with the given system (M, g, F) , and it is given by

$$\Gamma = \Gamma_g + X_F,$$

where $X_F \in \mathfrak{X}(TM)$ is the vertical vector field of force whose local coordinate expression is

$$X_F(q, v) = - \sum_{i,j=1}^n g^{ij}(q) F_j(q, v) \frac{\partial}{\partial v^i}. \tag{50}$$

In order to recover in an intrinsic way such a vector field, observe first that there is one vector field along τ_M , $Z_F \in \mathfrak{X}(\tau_M)$, such that $i_{Z_F}g = -F$, where $i_{Z_F}g$ is the one-form on TM defined for $Z \in \mathfrak{X}(\tau_M)$ by [24,25]

$$(i_{Z_F}g)(Y)(q, v) = g_q(Z(q, v), \tau_{M*}(q,v)(Y(q, v))), \quad \forall Y \in \mathfrak{X}(TM). \tag{51}$$

Remark that $i_{Z_F}g$ is semibasic because when Y is a τ_M vertical vector field, then, according to the definition (51), $(i_{Z_F}g)(Y) = 0$.

Then, the vector field X_F is defined as the vertical lift to TM of the mentioned $Z_F \in \mathfrak{X}(\tau_M)$, a vector field along τ_M , that is:

$$X_F(q, v) = (\lambda \circ Z_F)(q, v) = \lambda_q^{(q,v)}(Z_F(q, v)),$$

where $\lambda_q^{(q,v)} : T_qM \rightarrow T_{(q,v)}TM$ is the usual vertical lifting.

The local coordinate expressions of Z_F and F are, respectively,

$$Z_F = \sum_{i,j=1}^n Z^i \left(\frac{\partial}{\partial q^i} \circ \tau_M \right), \quad F = - \sum_{j=1}^n g_{ij} Z^i dq^j,$$

where, now, Z^i are functions on TM . The vector field of force X_F is as defined above, and its coordinate expression is

$$X_F = \sum_{i,j=1}^n Z^i(q, v) \frac{\partial}{\partial v^i}, \quad Z^i(q, v) = - \sum_{j=1}^n g^{jj}(q) F_j(q, v).$$

The trajectories of (M, g, F) are the projections on M of the integral curves of Γ and satisfy the system of second-order differential equations

$$\ddot{q}^i = - \sum_{k,l=1}^n \Gamma_{kl}^i(q) \dot{q}^k \dot{q}^l + Z^i(q, v), \quad i = 1, \dots, n. \tag{52}$$

This system corresponds to the SODE vector field:

$$\Gamma(q, v) = \sum_{i=1}^n v^i \frac{\partial}{\partial q^i} + \sum_{i,j,k=1}^n \left(-\Gamma_{jk}^i(q) v^j v^k + Z^i(q, v) \right) \frac{\partial}{\partial v^i}. \tag{53}$$

A curve $\gamma : I \subset \mathbb{R} \rightarrow M$ is a solution of the system (52) if, and only if, it satisfies the following condition:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \dot{\gamma}, \tag{54}$$

as we can prove by expressing both members of the equation in local coordinates.

We remark first that, if $Z \in \mathfrak{X}(TM)$, i.e., $Z : TM \rightarrow TM$ is such that $\tau_M \circ Z = \tau_M$, then $Z \circ \dot{\gamma} : I \rightarrow TM$ is such that $\tau_M \circ (Z \circ \dot{\gamma}) = (\tau_M \circ Z) \circ \dot{\gamma} = \tau_M \circ \dot{\gamma} = \dot{\gamma}$, and therefore, $Z \circ \dot{\gamma}$ is a vector field along γ , that is $Z \circ \dot{\gamma} \in \mathfrak{X}(\gamma)$. Consequently, both sides of the equality (54) are vector fields along the curve γ .

The trajectories $\gamma : I \rightarrow M$ of the mechanical system defined on (M, g, F) we are studying are therefore solutions of the dynamical Equation (54).

As basic one-forms are also semibasic ones, the apparent difference between the dynamical Equations (48) and (54) may seem a surprise. Note, however, that, given a vector field $Z_F \in \mathfrak{X}(M)$ as in (48), it has associated a $\tilde{Z}_F \in \mathfrak{X}(TM)$ by means of $\tilde{Z}_F = Z_F \circ \tau_M$, and using such \tilde{Z}_F in (54), we see that $\tilde{Z}_F \circ \dot{\gamma} = Z_F \circ \tau_M \circ \dot{\gamma} = Z_F \circ \dot{\gamma}$, which shows the identity of the dynamical Equations (48) and (54) for basic force one-forms. More explicitly:

Definition 7. A generalised Newtonian dynamical system is given by (M, g, F) , where (M, g) is a Riemann manifold and F is a semibasic one-form on TM , and its dynamical equation is

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \dot{\gamma}, \tag{55}$$

where Z_F is the vector field along τ_M defined by $i(Z_F)g = -F$.

As we have shown in the previous comments, this definition includes as a particular case the case of Newtonian dynamical systems. From now on, we will study the generalised systems and only comment on the particular case when necessary. This situation corresponds to the case where the forces on a mechanical system depend not only on the positions, but also on the velocities.

7.2. Conserved Quantities and Dynamical Symmetries

The reduction processes of the integrability problem of a system make mainly use of infinitesimal symmetries and of constants of motion, and sometimes, there exist relations among these ingredients when a compatible geometric structure is known, using results similar to those of the well-known Noether’s theorem.

The concept of (infinitesimal) symmetry plays a relevant role. We have seen that, for free geodesic motions, the symmetries are given by the Killing vector fields, because if $X \in \mathfrak{X}(M)$ is a Killing vector field, then its complete lift $X^c \in \mathfrak{X}(TM)$ preserves both the symplectic form ω_{T_g} and the free Lagrangian T_g . Moreover, $\mathcal{L}_{X^c} \theta_{T_g} = 0$ shows that the vector field X^c is Hamiltonian with corresponding Hamiltonian function $i(X^c) \theta_{T_g}$, which is, moreover, a conserved quantity for the second-order geodesic vector field Γ_g , as we showed in Proposition 1 and the associated comments.

We can ask for the vector fields playing in the case of forced geodesic motions the role of Killing vector fields in the case of free geodesic flow, that is we look for a generalisation of Killing vector fields for this kind of Newtonian systems.

We devote the first part of this section to a kind of Noether theorem for these Newtonian, and even for generalised Newtonian, dynamical systems. According to Definition 7 and the previous comments, the Newtonian systems can be considered as a particular case of generalised Newtonian dynamical systems; hence, we develop our results for these vector fields and only comment on the particular case when there are some specific differences.

Given a vector field $X \in \mathfrak{X}(M)$, consider the one-form $i(X)g$ on M and its associated function $\widehat{\mathbf{I}}_X = \widehat{i(X)g}$ defined on TM as $\widehat{i(X)g}(q, v) = g(X(q), v)$. We study under what conditions the function $\widehat{\mathbf{I}}_X$ is constant along the lifts γ^T to TM of the trajectories γ of a generalised Newtonian system (M, g, F) , that is $\gamma^T(t) = (\gamma(t), \dot{\gamma}(t))$. Then, if γ is a trajectory of the dynamical system on M defined by (M, g, F) , i.e., it is the projection onto M of an integral curve of the associated dynamical SODE vector field Γ , and thus, γ satisfies (55), we have that

$$\begin{aligned} \frac{d}{dt}\widehat{\mathbf{I}}_X(\gamma^T(t)) &= \frac{d}{dt}g(X(\gamma(t)), \dot{\gamma}(t)) = \nabla_{\dot{\gamma}(t)}(g(\dot{\gamma}(t), X(\gamma(t)))) \\ &= g(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), X(\gamma(t))) + g(\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}X(\gamma(t))). \end{aligned}$$

Hence, if F is a semibasic one-form on M , and $Z_F \in \mathfrak{X}(\tau_M)$ is the vector field along τ_M such that $i_{Z_F}g = -F$, then

$$\frac{d}{dt}\widehat{\mathbf{I}}_X(\gamma^T(t)) = g(Z_F, X)(\gamma^T(t)) + g(\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}X(\gamma(t))). \tag{56}$$

We can change the expression of the second element of the right-hand side to a more suitable one with the following lemma:

Lemma 2. *Let be (M, g) a Riemannian manifold and $T_g : TM \rightarrow \mathbb{R}$ the associated kinetic energy. For any $X, Y \in \mathfrak{X}(M)$, we have that*

$$(\mathcal{L}_{X^c}T_g)(Y) = g(\nabla_Y X, Y).$$

Proof. Using that $\mathcal{L}_{X^c}T_g = T_{\mathcal{L}_X g}$ and the torsionless property of the Levi-Civita connection, we have that

$$\begin{aligned} (\mathcal{L}_{X^c}T_g)(Y) &= T_{\mathcal{L}_X g}(Y) = \frac{1}{2}(\mathcal{L}_X g)(Y, Y) = \frac{1}{2}\mathcal{L}_X(g(Y, Y)) - g(\mathcal{L}_X Y, Y) \\ &= \frac{1}{2}\nabla_X(g(Y, Y)) - g(\mathcal{L}_X Y, Y) = g(\nabla_X Y, Y) - g(\mathcal{L}_X Y, Y) \\ &= g(\nabla_Y X, Y) + g(\mathcal{L}_X Y, Y) - g(\mathcal{L}_X Y, Y) \\ &= g(\nabla_Y X, Y), \end{aligned}$$

from which the result of the lemma follows. \square

As a consequence, we have the following result:

Proposition 8. *Let be (M, g, F) a generalised Newtonian dynamical system and $X \in \mathfrak{X}(M)$ and $X^c \in \mathfrak{X}(TM)$ its complete lift. Then, the function $\widehat{i(X)g} : TM \rightarrow \mathbb{R}$ is constant along the lifts γ^T to TM of the trajectories γ of the given system on M if, and only if,*

$$\mathcal{L}_{X^c}T_g - i(X^c)F = 0. \tag{57}$$

Proof. Considering the two sides of the Equation (56), we remark that if $Z_F \in \mathfrak{X}(\tau_M)$ is such that $i_{Z_F}g = -F$,

$$g(Z_F(\gamma^T(t)), X(\gamma(t))) = ((i(Z_F)g)(X))(\gamma^T(t)) = -(i(X^c)F)(\gamma^T(t)),$$

while, as a result of the preceding lemma,

$$g(\nabla_{\dot{\gamma}(t)}X(\gamma(t)), \dot{\gamma}(t)) = (\mathcal{L}_{X^c}T_g)(\dot{\gamma}(t)),$$

and therefore, by (56), we see that the function $\widehat{i(X)g}$ is constant along the lifts γ^T to TM of the trajectories γ if, and only if,

$$\mathcal{L}_{X^c}T_g - i(X^c)F = 0,$$

because the trajectory γ is arbitrary. \square

Particularly interesting cases are those corresponding to Killing vector fields $X \in \mathfrak{X}(M)$, for which $\mathcal{L}_{X^c}T_g = T_{\mathcal{L}_X g} = 0$, and then, Condition (57) reduces to $i(X^c)F = 0$, or what is equivalent $g(Z_F, X) = 0$, that is, at every vector $v_x \in TM$, the vector $Z_F(v_x)$ is orthogonal to $X(x)$. One says that the Killing vector field $X \in \mathfrak{X}(M)$ satisfying the condition $i(X^c)F = 0$ is a *Killing symmetry* of the generalised Newtonian system (M, g, F) and the function $\widehat{i(X)g}$ is the *associated conserved quantity*. Summing up, we obtained the following proposition:

Proposition 9. *Let be (M, g, F) a generalised Newtonian dynamical system, $Z_F \in \mathfrak{X}(\tau_M)$ such that $i_{Z_F}g = -F$, $X \in \mathfrak{X}(M)$ a Killing vector field, and $X^c \in \mathfrak{X}(TM)$ its complete lift. Then, the function $\widehat{i(X)g} : TM \rightarrow \mathbb{R}$ is constant along the lifts γ^T to TM of the trajectories γ of the given system if, and only if, the semibasic one-form F vanishes on X^c , i.e., $i(X^c)F = 0$.*

Comments:

1. Observe that $i(X^c)F = i(X)F = -g(Z_F, X)$; hence, the condition in the above proposition is that Z_F and X are orthogonal.
2. If F is a basic one-form, that is (M, g, F) is a Newtonian dynamical system, the result is the same.
3. In the case of the geodesic systems, $F = 0$, the condition $g(X, Z_F) = 0$ is trivially satisfied, and we obtain the result of Proposition 1 as a particular case.
4. If the system is a conservative simple mechanical one and its potential function is $V : M \rightarrow \mathbb{R}$, then $Z_F = \text{grad } V$ and $F = -dV$. In this situation, the condition $i(X^c)F = 0$ is $\mathcal{L}_X V = 0$, that is the potential function V is invariant by the vector field X .

Now, we go to the dynamical symmetries of a Newtonian system.

Recall that a vector field $Y \in \mathfrak{X}(TM)$ is a dynamical symmetry for the dynamics $\Gamma \in \mathfrak{X}(TM)$ if $[Y, \Gamma] = 0$, i.e., the flow of the vector field Y preserves Γ .

Definition 8. *A vector field $X \in \mathfrak{X}(M)$ is called a Lie symmetry of the dynamics defined by the generalised Newtonian system (M, g, F) , when X^c is a dynamical symmetry of the associated dynamical vector field Γ , namely $[X^c, \Gamma] = 0$.*

To characterise the dynamical symmetries of a generalised Newtonian system (M, g, F) , we need to have an expression of the dynamical vector field Γ . We know that the curves γ in M solutions to the dynamical Equation (55), when lifted to TM , are the integral curves of the SODE vector field Γ , the uniquely defined solution to the equation:

$$i(\Gamma)\omega_{T_g} - dT_g = F, \tag{58}$$

then applying the exterior differential to both sides, we obtain that the SODE vector field Γ satisfies

$$\mathcal{L}_\Gamma \omega_{T_g} = dF, \tag{59}$$

and therefore, Γ is not a locally Hamiltonian vector field when F is not closed, but only when F is closed.

With this interpretation of the dynamical vector field, we can obtain Proposition 8 and Proposition 9 in this different approach for the generalised Newtonian system (M, g, F) :

Proposition 10. (i) Given a vector field $X \in \mathfrak{X}(M)$, the function $\widehat{i(X)g}$ is a conserved quantity for the generalised Newtonian system (M, g, F) if, and only if, $\mathcal{L}_{X^c}(T_g) = i(X^c)F$.

(ii) If $X \in \mathfrak{X}(M)$ is a Killing vector field, then the function $\widehat{i(X)g}$ is a conserved quantity for the generalised Newtonian system (M, g, F) if, and only if, $i(X^c)F = 0$.

Proof. (i) The contraction with the complete lift X^c of the vector field $X \in \mathfrak{X}(M)$ of both sides of Relation (58) leads to

$$i(X^c)i(\Gamma)\omega_{T_g} = i(X^c)dT_g + i(X^c)F. \tag{60}$$

The left-hand side can be rewritten as

$$\omega_{T_g}(\Gamma, X^c) = -d\theta_{T_g}(\Gamma, X^c) = -\mathcal{L}_\Gamma(\theta_{T_g}(X^c)) + \mathcal{L}_{X^c}(\theta_{T_g}(\Gamma)) + \theta_{T_g}([\Gamma, X^c]),$$

and having in mind that

$$\theta_{T_g}(\Gamma) = \sum_{i,j=1}^n g_{ij}(q)v^i v^j = 2T_g, \quad \theta_{T_g}(X^c) = \sum_{i,j=1}^n g_{ij}(q)v^j X^i = \widehat{i(X^c)g},$$

and that $[\Gamma, X^c]$ is a vertical vector field on which θ_{T_g} vanishes, the relation (60) becomes

$$-\mathcal{L}_\Gamma(\widehat{i(X^c)g}) + \mathcal{L}_{X^c}(T_g) - i(X^c)F = 0,$$

and therefore, $\widehat{i(X^c)g}$ is a conserved quantity if, and only if, $\mathcal{L}_{X^c}(T_g) = i(X^c)F$.

(ii) In the particular case of a Killing vector field, as $\mathcal{L}_{X^c}(T_g) = 0$, we have that $\widehat{i(X^c)g}$ is a conserved quantity if, and only if, $i(X^c)F = 0$. \square

As far as the dynamical symmetries are concerned, we have the following:

Proposition 11. Given a generalised Newtonian system (M, g, F) , a Killing vector field $X \in \mathfrak{X}(M)$ is a Lie symmetry of the dynamical vector field $\Gamma \in \mathfrak{X}(TM)$ solution of (58) if, and only if, $\mathcal{L}_{X^c}F = 0$.

Proof. Using the relation:

$$i([X^c, \Gamma])\omega_{T_g} = (\mathcal{L}_{X^c} \circ i(\Gamma) - i(\Gamma) \circ \mathcal{L}_{X^c})\omega_{T_g},$$

we obtain

$$i([X^c, \Gamma])\omega_{T_g} = \mathcal{L}_{X^c}(dT_g + F) - i(\Gamma)\omega_{T_{\mathcal{L}_{X^c}g}} = \mathcal{L}_{X^c}F,$$

because, as X is a Killing vector, $\mathcal{L}_{X^c}T_g = 0$ and $\mathcal{L}_{X^c}\omega_{T_g} = 0$. Since ω_{T_g} is nondegenerate, $i([X^c, \Gamma])\omega_{T_g}$ vanishes if, and only if, $[X^c, \Gamma]$ does, and then, the result of the proposition follows. \square

Another approach to a similar result, but related to the case of conserved quantities, is the following property:

Proposition 12. For a generalised Newtonian system (M, g, F) , if a Killing vector field $X \in \mathfrak{X}(M)$ is such that $i(X^c)F = 0$, then X is a Lie symmetry of Γ if, and only if, $i(X^c)dF = 0$.

Proof. We first remark about an important relation that we can derive by making use of the relation $\mathcal{L}_{X^c}\theta_{T_g} = \theta_{\mathcal{L}_{X^c}T_g} = \theta_{T_{\mathcal{L}_{X^c}g}}$, from which we see that

$$i(X^c)\omega_{T_g} = -i(X^c)d\theta_{T_g} = -\mathcal{L}_{X^c}\theta_{T_g} + d(i(X^c)\theta_{T_g}) = -\theta_{T_{\mathcal{L}_{X^c}g}} + d(i(X^c)\theta_{T_g}). \tag{61}$$

As the dynamical vector field Γ satisfies (59), we can write

$$i(X^c)dF = i(X^c)\mathcal{L}_\Gamma\omega_{T_g} = \mathcal{L}_\Gamma(i(X^c)\omega_{T_g}) + i([X^c, \Gamma])\omega_{T_g},$$

and from the expression (61) for $i(X^c)\omega_{T_g}$, the preceding equation becomes

$$i(X^c)dF = -\mathcal{L}_\Gamma(\theta_{T_{\mathcal{L}_X g}}) + \mathcal{L}_\Gamma(d(i(X^c)\theta_{T_g})) + i([X^c, \Gamma])\omega_{T_g};$$

if $X \in \mathfrak{X}(M)$ is a Killing vector field, $\mathcal{L}_X g = 0$ reduces the preceding relation to

$$i(X^c)dF = \mathcal{L}_\Gamma(\widehat{di(X)g}) + i([X^c, \Gamma])\omega_{T_g} = \mathcal{L}_\Gamma(d(i(X^c)\theta_{T_g})) + i([X^c, \Gamma])\omega_{T_g}.$$

This shows that, if $i(X^c)F = 0$ and, therefore, $\mathcal{L}_\Gamma(\widehat{di(X)g}) = 0$, we have that $[X^c, \Gamma] = 0$ if, and only if, $i(X^c)dF = 0$. Recall that $i(X^c)F = 0$ and $i(X^c)dF = 0$ imply that $\mathcal{L}_{X^c}F = 0$. \square

Comments:

1. We remark about the difference between Proposition 9 and Proposition 12 for the characterisation of the vector field X as a generator of a conserved quantity or as a dynamical symmetry.
2. In the case of a conservative simple mechanical system, that is $F = -dV$, if X is a Killing vector field, we have that:
 - (a) If $\mathcal{L}_X V = 0$, then the quantity $\widehat{i(X)g}$ is conserved by the dynamics;
 - (b) If $\mathcal{L}_X dV = 0$, then the vector field X^c is a dynamical symmetry of Γ . This condition on V and X is equivalent to $d\mathcal{L}_X V = 0$, which is a condition weaker than Condition (a). In fact, we have that Condition (a) implies Condition (b), then we can have a dynamical symmetry X such that $\widehat{i(X)g}$ is not a conserved quantity. In these circumstances, the vector fields X satisfying Condition (a) are usually called Noether symmetries. Obviously, a Noether symmetry is a dynamical symmetry, but not the contrary.

As we said above, see [23] for a comparison of the obtained results and used methods.

7.3. Newtonian Vector Fields

Similar to the definition of geodesic vector fields, we introduce the following definitions:

Definition 9. (i) A Newtonian vector field for the Newtonian system (M, g, F) is a vector field $X \in \mathfrak{X}(M)$ satisfying the equation:

$$\nabla_X X = Z_F, \tag{62}$$

where the vector field $Z_F \in \mathfrak{X}(M)$ is such that $i(Z_F)g = -F$.

(ii) A generalised Newtonian vector field for the generalised Newtonian system (M, g, F) is a vector field $X \in \mathfrak{X}(M)$ such that

$$\nabla_X X = Z_F \circ X, \tag{63}$$

where $Z_F \in \mathfrak{X}(\tau_M)$ is such that $i_{Z_F}g = -F$.

We remark that, when $Z_F \in \mathfrak{X}(M)$, we can replace Z_F on the right-hand side of (62) by $(Z_F \circ \tau_M) \circ X$, and this fact identifies (62) with (63), because the remark preceding Definition 7 leads to considering as equivalent both definitions (62) and (63), and hereafter, we only consider the generalised case and Condition (63).

These concepts play an interesting role when studying relations among geodesics for conformal metrics in the next section.

The first interesting property of this kind of vector fields is similar to the characterisation of the geodesic vector fields for the second-order geodesic vector field.

Proposition 13. A vector field $X \in \mathfrak{X}(M)$ is a generalised Newtonian vector field for the generalised Newtonian system (M, g, F) if, and only if, its integral curves satisfy the dynamical equation $\nabla_{\dot{\gamma}}\dot{\gamma} = Z_F \circ \dot{\gamma}$.

Proof. Let $X \in \mathfrak{X}(M)$, and suppose that its integral curves $\gamma : I \rightarrow M$ satisfy the dynamical Equation (54). If $p \in M$ and $\gamma : I \rightarrow M$ is the integral curve of X with the initial condition $\gamma(0) = p$, we have that $\dot{\gamma} = X \circ \gamma$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = Z_F \circ \dot{\gamma}$. Then,

$$(\nabla_X X)(p) = \nabla_{X_p} X = \nabla_{\dot{\gamma}(0)} \dot{\gamma}(t) = Z_F(\dot{\gamma}(0)) = (Z_F \circ X)(\gamma(0)) = (Z_F \circ X)(p).$$

However, the point $p \in M$ is arbitrary; hence, $\nabla_X X = Z_F \circ X$.

Conversely, suppose that $X \in \mathfrak{X}(M)$ satisfies the equation $\nabla_X X = Z_F \circ X$, and let $\gamma : I \rightarrow M$ be an integral curve of X , that is $\dot{\gamma} = X \circ \gamma$. Then,

$$(\nabla_X X)(\gamma(t)) = (Z_F \circ X)(\gamma(t));$$

hence, as γ is arbitrary, $\nabla_{\dot{\gamma}}\dot{\gamma} = Z_F \circ \dot{\gamma}$. \square

We can extend some results on geodesic vector fields contained in Proposition 2 to the case of generalised Newtonian vector fields. In fact, recall first that, according to Proposition 12, if $X \in \mathfrak{X}(M)$ is a Killing vector field of the Riemann manifold (M, g) and $X^c \in \mathfrak{X}(TM)$ denotes its complete lift, then the vector field X is a Lie symmetry of Γ if, and only if, $i(X^c)dF = 0$. Two important properties are then the following:

Proposition 14. (i) Let be $X \in \mathfrak{X}(M)$ a Lie symmetry of the generalised Newtonian system (M, g, F) and ϕ_t^X its local flow. If $\gamma : \mathbb{R} \rightarrow M$ is a trajectory of the generalised Newtonian system (M, g, F) , then $\phi_t^X \circ \gamma$ is also a trajectory.

(ii) If $Y \in \mathfrak{X}(M)$ is a generalised Newtonian vector field for (M, g, F) , i.e., $\nabla_Y Y = Z_F \circ Y$, then $(\phi_t^X)_* Y$ is also a generalised Newtonian vector field for the same system, that is

$$\nabla_{(\phi_t^X)_* Y} (\phi_t^X)_* Y = Z_F \circ ((\phi_t^X)_* Y).$$

Proof. (i) Let γ be a trajectory of the system (M, g, F) , then γ^T is an integral curve of Γ . As $[X^c, \Gamma] = 0$, we have that $(T\phi_t^X)_* \Gamma = \Gamma$. Hence being γ^T an integral curve of Γ , we have that $(T\phi_t^X) \circ \gamma^T$ is an integral curve of $(T\phi_t^X)_* \Gamma = \Gamma$. Thus, $\phi_t^X \circ \gamma$ is a trajectory of the system because $(T\phi_t^X) \circ \dot{\gamma} = (\phi_t^X \circ \dot{\gamma})$.

(ii) Let $Y \in \mathfrak{X}(M)$ be a generalised Newtonian vector field, then its integral curves γ are trajectories of the system. However, the integral curves of $(\phi_t^X)_* Y$ are of the form $\phi_t^X \circ \gamma$, which, by Item (i), are also trajectories; hence, $(\phi_t^X)_* Y$ is a Newtonian vector field by Proposition 13. \square

The geodesic systems are characterized by $F = 0$, and then, all the results on geodesic vector fields appear as particular cases of Newtonian systems. Moreover, in these geodesic cases, the condition $\mathcal{L}_{X^c} T_g = 0$ is equivalent to $\mathcal{L}_X g = 0$, that is X is a Killing vector field, and the above results correspond to those in Proposition 2.

In [23], several different situations with symmetries of a Lagrangian system with external forces were studied. We only work here with geodesic systems whose Lagrangian is T_g and forced geodesic systems, where external forces depending on the velocities may appear, and we study them as generalised Newtonian systems. In the coming sections, we will try to generalise the ideas of geodesic vector fields and apply them to conformal metrics.

7.4. Hamilton–Jacobi Vector Fields for Generalised Newtonian Systems

This section is devoted to generalising to the framework of generalised Newtonian systems the results obtained in Section 5.

Consider a generalised Newtonian system (M, g, F) as above, and let $\Gamma = \Gamma_g + X_F$, with $X_F = Z_F^v$, the corresponding dynamical SODE vector field. We say that a vector field

$X \in \mathfrak{X}(M)$ is a *Hamilton–Jacobi vector field* for the system if, for every integral curve γ of X , we have that $X \circ \gamma$ is an integral curve of Γ . As in the geodesic case, we have the following equivalence:

Proposition 15. *A vector field $X \in \mathfrak{X}(M)$ is a generalised Newtonian vector field for the generalised Newtonian system (M, g, F) if, and only if, it is a Hamilton–Jacobi vector field for the same system.*

Proof. Suppose that $X \in \mathfrak{X}(M)$ is a Hamilton–Jacobi vector field for the generalised Newtonian system (M, g, F) , and let γ be an integral curve of X . Then, $\dot{\gamma} = X \circ \gamma$ is an integral curve of Γ . Hence, γ satisfies the equation $\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \dot{\gamma}$. By Proposition 13, we have that $\nabla_X X = Z_F \circ X$, because γ is arbitrary, and consequently, X is a generalised Newtonian vector field.

Conversely, if $X \in \mathfrak{X}(M)$ is a generalised Newtonian vector field, then, for any integral curve γ , we have that $\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \dot{\gamma}$, and therefore, $\dot{\gamma}$ is an integral curve of Γ . \square

Apart from the definition, we have no explicit characterisation of the generalised Newtonian vector fields, but, using the preceding proposition, we can try to obtain a local equation for them.

Suppose that $X \in \mathfrak{X}(M)$ is a Hamilton–Jacobi vector field for the generalised Newtonian system (M, g, F) , and let γ be an integral curve of X . Then, $(X \circ \gamma)' = \Gamma \circ X \circ \gamma$, that is $TX \circ \dot{\gamma} = \Gamma \circ X \circ \gamma$. In local natural coordinates, we have:

$$\sum_{j=1}^n \left(\frac{\partial X^i}{\partial q^j} \right)_{\gamma} \dot{\gamma}^j = \left(- \sum_{j,k=1}^n \Gamma_{jk}^i X^j X^k \right)_{\gamma} + Z^i \circ \dot{\gamma}, \quad i = 1, \dots, n,$$

for every integral curve γ of X . Equivalently:

$$\left(\sum_{j=1}^n X^j \frac{\partial X^i}{\partial q^j} \right) \circ \gamma = \left(- \sum_{j,k=1}^n \Gamma_{jk}^i X^j X^k \right) \circ \gamma + Z^i \circ \dot{\gamma}, \quad i = 1, \dots, n.$$

However, as $\dot{\gamma} = X \circ \gamma$ and there are integral curves through every point in M , we have

$$\sum_{j=1}^n X^j \frac{\partial X^i}{\partial q^j} = - \sum_{j,k=1}^n \Gamma_{jk}^i X^j X^k + Z^i \circ X, \quad i = 1, \dots, n, \tag{64}$$

as the local condition for a vector field X in M to be a generalised Newtonian vector field. Observe that, if the external force vanishes, $F = 0$, hence $Z^i = 0$, then we obtain the corresponding expression (41) for geodesic vector fields.

In parallel with Proposition 5, we have the following result for a generalised Newtonian system (M, g, F) :

Proposition 16. *(Hamilton–Jacobi equation)*

Let be $X \in \mathfrak{X}(M)$ a closed vector field, i.e., satisfying the condition $d(i(X)g) = 0$, and (M, g, F) a general Newtonian system. The following conditions are equivalent:

- (i) X is a generalised Newtonian vector field, $\nabla_X X = Z_F \circ X$.
- (ii) The function $T_g \circ X$ on M satisfies the condition $d(T_g \circ X) = -F \circ X$, where $T_g : TM \rightarrow \mathbb{R}$ is the kinetic energy function defined by (14).

Proof. We know (see (42)) that, for each vector field $Y \in \mathfrak{X}(M)$, we have:

$$d(T_g \circ X)(Y) = g(\nabla_Y X, X).$$

Now, as indicated in Section 5, taking into account that X is closed and using (26), we have that $g(\nabla_Y X, X) = g(\nabla_X X, Y)$. Consequently, if X is a generalised Newtonian vector

field, using (63), $d(T_g \circ X)(Y) = g(\nabla_X X, Y) = g(Z_F \circ X, Y) = -(F \circ X)(Y)$. However, $Y \in \mathfrak{X}(M)$ is an arbitrary vector field, then for closed vector fields, i.e., such that $d(i(X)g) = 0$, we have that

$$d(T_g \circ X) = -F \circ X. \tag{65}$$

If, conversely, the vector field $X \in \mathfrak{X}(M)$ satisfies (65), then, from Equation (42), we obtain that $g(\nabla_Y X, X) = -(F \circ X)(Y)$, and using (43), because X is a closed vector field, this can be rewritten as $g(\nabla_X X, Y) = -(F \circ X)(Y)$. However, Z_F satisfies $i_{Z_F}g = -F$, then we obtain $g(\nabla_X X, Y) = g(Z_F \circ X, Y)$; and as the vector field Y is arbitrary, this implies that $\nabla_X X = Z_F \circ X$, i.e., the vector field X is a generalised Newtonian vector field. \square

The expression (65) is the Hamilton–Jacobi equation for a generalised Newtonian system (M, g, F) for an associated closed generalised Newtonian vector field X , and as

$$d(T_g \circ X) = d(X^*T_g) = X^*dT_g = -F \circ X,$$

we can rewrite such an equation as

$$d(X^*T_g) + F \circ X = 0, \tag{66}$$

which, in the case of F being an exact one-form, gives the usual expression of the Hamilton–Jacobi equation for simple mechanical systems.

In local coordinates Equation (66) gives

$$\frac{1}{2}d\left(\sum_{i,j=1}^n g_{ij}(q)X^i(q)X^j(q)\right) + \sum_{j=1}^n F_j(q, X(q))dq^j = 0.$$

7.5. The Case of Nonholonomic Generalised Newtonian Systems

Consider now a generalised Newtonian system (M, g, F) , with $F \in \Omega^1(\tau_M)$, and suppose the system is constrained to move on a submanifold $C \subset TM$, with $\tau_M(C) = M$. Then, we have a nonholonomic system, and the dynamical equation is obtained supposing there exists a constraint force $R \in \mathfrak{X}(\tau_M)$, which forces the system to move on the submanifold C . Hence, the dynamical equation is

$$\nabla_{\dot{\gamma}}\dot{\gamma} = Z_F \circ \dot{\gamma} + R \circ \dot{\gamma} \tag{67}$$

and by the application of the d’Alembert nonholonomic principle (see [10] for more details), we impose that the constraint force $R \in \mathfrak{X}(\tau_M)$ is such that its vertical lift $R^v \in \mathfrak{X}(TM)$ is orthogonal to C : for every $(q, u) \in TM$, we suppose that

$$R(q, u) \in (V_{(q,u)}TM \cap T_{(q,u)}C)^\perp.$$

Recall that $R(q, u) \in T_qM$.

Comment: For every $(q, u) \in C$, we used the notation $(V_{(q,u)}TM \cap T_{(q,u)}C)_q$ to denote the subspace of T_qM such that its vertical lifting gives $(V_{(q,u)}TM \cap T_{(q,u)}C)$. As $(V_{(q,u)}TM \cap T_{(q,u)}C)_q$ is a linear subspace of T_qM , the meaning of its orthogonal complement with respect to the metric g is clear.

One of the most-interesting cases is when the constraint submanifold C is defined by the annihilation of a family of functions $\phi^\alpha : TM \rightarrow \mathbb{R}$, $\alpha = 1, \dots, h$, independent with respect to the velocities, that is such that $d^v\phi^1, \dots, d^v\phi^h$ are linearly independent at every point of C (recall that $d^v\phi = d\phi \circ S$, where S is the vertical endomorphism in TM). Locally, this condition is equivalent to saying that, for a natural coordinate system on TM , (q^i, v^i) , they satisfy that

$$\text{rank}\left(\frac{\partial(\phi^1, \dots, \phi^h)}{\partial(v^1, \dots, v^n)}\right) = h,$$

at every point of C .

In this situation, in order to satisfy the d’Alembert nonholonomic principle, see [10], we have that there exist functions $\lambda_\alpha : TM \rightarrow \mathbb{R}$, called Lagrange multipliers, such that

$$R = \sum_{\alpha=1}^h \lambda_\alpha Z_\alpha = \lambda_1 Z_1 + \dots + \lambda_h Z_h,$$

where the vector fields along τ_M , $Z_\alpha \in \mathfrak{X}(\tau_M)$, are defined by $i_{Z_\alpha} g = -d^v \phi^\alpha$, and the dynamical equation is:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Z_F \circ \dot{\gamma} + \sum_{\alpha=1}^h \lambda_\alpha Z_\alpha.$$

To solve this equation for the trajectory γ and the Lagrange multipliers λ_α , we need to add the equations $\phi^1 = 0, \dots, \phi^h = 0$, which are the constraint functions defining C . Observe that we also have the constraint form $F_R \in \Omega^1(\tau_M)$, defined by $i_R g = -F_R$, whose expression is given by:

$$F_R = - \sum_{\alpha=1}^h \lambda_\alpha d^v \phi^\alpha.$$

Going to conserved quantities, we can apply Propositions 8 and 9 to the dynamical Equation (67) and obtain, by direct application, the following results for a nonholonomic system defined by (M, g, F) and the constraint submanifold C , with the above notations:

Proposition 17. *Let be $X \in \mathfrak{X}(M)$ and $X^c \in \mathfrak{X}(TM)$ its complete lift. Then, the function $\widehat{i(X)g} : TM \rightarrow \mathbb{R}$ is constant along the lifts γ^T to TM of the trajectories γ of the given system on M if, and only if,*

$$\mathcal{L}_{X^c} Tg - i(X^c)F - i(X^c)F_R = 0. \tag{68}$$

Proposition 18. *Let be $X \in \mathfrak{X}(M)$ a Killing vector field and $X^c \in \mathfrak{X}(TM)$ its complete lift. Then, the function $\widehat{i(X)g} : TM \rightarrow \mathbb{R}$ is constant along the lifts γ^T to TM of the trajectories γ of the given system if, and only if, the semibasic one-form $F + F_R$ vanishes on X^c , i.e., $i(X^c)(F + F_R) = 0$.*

Proposition 19. *Let $X \in \mathfrak{X}(M)$ be a Killing vector field and $X^c \in \mathfrak{X}(TM)$ its complete lift. Suppose that $i(X^c)F_R = 0$. Then, the function $\widehat{i(X)g} : TM \rightarrow \mathbb{R}$ is constant along the lifts γ^T to TM of the trajectories γ of the given system if, and only if, the semibasic one-form F vanishes on X^c , i.e., $i(X^c)F = 0$.*

If the constraint submanifold C is $C = \{(q, u) \in TM \mid \phi^1 = 0, \dots, \phi^h = 0\}$, the condition $i(X^c)F_R = 0$ can be reduced to

$$0 = i(X^c)F_R = \sum_{\alpha=1}^h \lambda_\alpha i(X^c)d^v \phi^\alpha = \sum_{\alpha=1}^h \lambda_\alpha i(X)d^v \phi^\alpha,$$

because $d^v \phi^\alpha$ are semibasic forms. Hence, in order to satisfy the condition $i(X^c)F_R = 0$, it is enough to satisfy $i(X)d^v \phi^\alpha = 0, \alpha = 1, \dots, h$. This is equivalent to saying that $g(X, R) = 0$ at every point of C .

With respect to the Newtonian and Hamilton–Jacobi vector fields, we can extend to this case Propositions 13, 14, and 16 and Hamilton–Jacobi Equation (66) without change, but including the constraint force.

8. Application to Conformal Metrics

8.1. Conformal Metrics: Definitions and Geodesics

Definition 10. *Two metrics g and \bar{g} on the manifold M are conformally related if there exists a function $\varphi : M \rightarrow \mathbb{R}$ such that $\bar{g} = \exp(2\varphi) g$.*

This definition allows us to establish an equivalence relation in the set of metrics. A conformal structure is an equivalence class of metrics. The covariant derivatives with respect to both metrics g and \bar{g} are related by (see [5] for details):

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + (\mathcal{L}_X \varphi)Y + (\mathcal{L}_Y \varphi)X - g(X, Y) \operatorname{grad}_g \varphi \\ &= \nabla_X Y + (\mathcal{L}_X \varphi)Y + (\mathcal{L}_Y \varphi)X - \bar{g}(X, Y) \operatorname{grad}_{\bar{g}} \varphi. \end{aligned} \tag{69}$$

In the case of $X = Y$, the above expression reduces to

$$\bar{\nabla}_X X = \nabla_X X + 2(\mathcal{L}_X \varphi)X - g(X, X) \operatorname{grad}_g \varphi. \tag{70}$$

The new Christoffel symbols of the second kind of the Levi-Civita connection are related to the previous ones by (see [5]):

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_j^i \frac{\partial \varphi}{\partial q^k} + \delta_k^i \frac{\partial \varphi}{\partial q^j} - \sum_{l=1}^n g_{jk} g^{il} \frac{\partial \varphi}{\partial q^l}, \quad i, j, k = 1, \dots, \dim M, \tag{71}$$

where δ_k^i denotes the Kronecker delta symbol.

How related are the geodesic curves of conformally related Riemannian metrics? This is the question we will study now. Recall that, as we stated in Section 2, given a Riemann manifold (M, g) , the geodesic curves are the curves γ in M whose velocity vector field $\dot{\gamma}$ is parallel along the curve, $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$. They satisfy that $g(\dot{\gamma}(t), \dot{\gamma}(t))$ is a constant function, and we can parametrise the curves with the arc parameter, then $g(\dot{\gamma}(t), \dot{\gamma}(t)) = 1$. Locally, the geodesics are the solutions to the system of second-order differential Equations (10).

$$\ddot{q}^i + \sum_{j,k=1}^n \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \dots, n, \tag{72}$$

where the functions Γ_{jk}^i are the Christoffel symbols corresponding to the Levi-Civita connection of the Riemannian metric g . This equation corresponds to an SODE vector field on the tangent bundle TM , the so-called second-order geodesic vector field, denoted by Γ_g , whose local expression is given by

$$\Gamma_g = \sum_{i=1}^n v^i \frac{\partial}{\partial q^i} - \sum_{i,j,k=1}^n \Gamma_{jk}^i v^j v^k \frac{\partial}{\partial v^i}.$$

As indicated above Γ_g is the only vector field on TM satisfying (16):

$$i(\Gamma_g)\omega_g = dT_g,$$

that is Γ is the Hamiltonian dynamical vector field defined by the Lagrangian dynamical system, (TM, ω_g, T_g) , where the kinetic energy function T_g is defined in (14) and ω_g is the associated Cartan two-form.

Now, we go to the expression of the geodesic curves of the conformal metric $\bar{g} = \exp(2\varphi) g$.

As the Christoffel symbols corresponding to $\bar{g} = \exp(2\varphi) g$ are given by (71), we have that

$$\sum_{i=1}^n \bar{\Gamma}_{jk}^i(q) \frac{\partial}{\partial v^i} = \sum_{i=1}^n \Gamma_{jk}^i(q) \frac{\partial}{\partial v^i} + 2 \sum_{i,k=1}^n \frac{\partial \varphi}{\partial q^k} v^k v^i \frac{\partial}{\partial v^i} - \sum_{i,j,k,l=1}^n g_{jk} v^j v^k g^{il} \frac{\partial \varphi}{\partial q^l} \frac{\partial}{\partial v^i},$$

which gives

$$\Gamma_{\bar{g}} = \Gamma_g + Z^v,$$

where

$$Z^v = 2 \sum_{i,k=1}^n \frac{\partial \varphi}{\partial q^k} v^k v^i \frac{\partial}{\partial v^i} - \sum_{i,j,k,l=1}^n g_{jk} v^j v^k g^{il} \frac{\partial \varphi}{\partial q^l} \frac{\partial}{\partial v^i},$$

namely

$$Z^v = \left(2 \sum_{i,k=1}^n \frac{\partial \varphi}{\partial q^k} v^k v^i \frac{\partial}{\partial q^i} \right)^v - \left(\sum_{i,j,k,l=1}^n g_{jk} v^j v^k g^{il} \frac{\partial \varphi}{\partial q^l} \frac{\partial}{\partial q^i} \right)^v.$$

We can rewrite it as $Z = Z_1 + Z_2$, with

$$Z_1 = 2 \sum_{i,k=1}^n \frac{\partial \varphi}{\partial q^k} v^k v^i \frac{\partial}{\partial q^i} = 2 \widehat{d\varphi} \mathbb{I}_{TM} \in \mathfrak{X}(\tau_M),$$

where $\mathbb{I}_{TM} : TM \rightarrow TM$ is the identity map, seeing it as a vector field along τ_M , and

$$Z_2 = - \sum_{i,j,k,l=1}^n g_{jk} v^j v^k g^{il} \frac{\partial \varphi}{\partial q^l} \frac{\partial}{\partial q^i} = - \exp(-2\varphi) \text{grad } \varphi = \frac{1}{2} \text{grad}(\exp(-2\varphi)),$$

because $\exp(2\varphi) g_{jk} v^j v^k = 1$, as we consider the geodesic curves parametrised by the parameter arc defined by \bar{g} .

We obtained in this way that the geodesic curves of the metric $\bar{g} = \exp(2\varphi) g$ are the trajectories of a generalised Newtonian system (M, g, F) , where $F = F_1 + F_2$ with

$$F_1 = -2 \widehat{d\varphi} i_{\mathbb{I}_{TM}} g, \quad F_2 = -\frac{1}{2} d(\exp(-2\varphi)).$$

As the one-form F_2 is exact, following the terminology of [23] and Expression (46), we can consider the system as the Lagrangian system on M with Lagrangian function $L = T_{\bar{g}} + \frac{1}{2}(\exp(-2\varphi))$ and external force F_1 .

We remark that, in the expression of F_1 , $\mathbb{I}_{TM} : TM \rightarrow TM$ appears as a vector field along τ_M . This \mathbb{I}_{TM} corresponds to the classical evolution operator for functions defined on the configuration space M , that is, for $f : M \rightarrow \mathbb{R}$,

$$\frac{d}{dt} f = \mathbb{I}_{TM}(f) = \sum_{i=1}^n \frac{\partial f}{\partial q^i} v^i = \widehat{df}.$$

If γ is a trajectory of the system, that is $\gamma^T = (\gamma, \dot{\gamma})$ is an integral curve of the SODE dynamical vector field Γ_L , then

$$\frac{d}{dt} \Big|_{\gamma(t)} f = \mathbb{I}_{TM}(\gamma(t), \dot{\gamma}(t)) f = \sum_{i=1}^n \frac{\partial f}{\partial q^i} \Big|_{\gamma(t)} \dot{\gamma}^i(t).$$

See [26] and the references therein for a detailed exposition on the evolution operator and its extension to functions defined on the phase space.

8.2. Newtonian Vector Fields and Hamilton–Jacobi Equation for Geodesics of the Metric $\bar{g} = \exp(2\varphi) g$

We know that the geodesic vector fields $X \in \mathfrak{X}(M)$ of the metric \bar{g} satisfy the equation

$$\bar{\nabla}_X X = 0, \tag{73}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection for the Riemann structure \bar{g} .

However, we proved that the geodesic curves of the metric $\bar{g} = \exp(2\varphi) g$ are the trajectories of a generalised Newtonian system (M, g, F) with

$$F = F_1 + F_2 = -2 \widehat{d\varphi} i_{\mathbb{I}_{TM}} g - \frac{1}{2} d(\exp(-2\varphi)),$$

and then, according to (63), the associated Newtonian vector fields $X \in \mathfrak{X}(M)$ satisfy the equation

$$\nabla_X X = Z \circ X = Z_1 \circ X + Z_2 \circ X = 2(\widehat{d\varphi} \mathbb{I}_{TM}) \circ X + \frac{1}{2} \text{grad}(\exp(-2\varphi)) \circ X.$$

If we look for the corresponding Hamilton–Jacobi equation, corresponding to (65), we have

$$dT_g \circ X = -F \circ X = 2\widehat{d\varphi} i(X)g + \frac{1}{2}d(\exp(-2\varphi)),$$

or equivalently,

$$d\left(T_g + \frac{1}{2}(\exp(-2\varphi))\right) \circ X = 2\widehat{d\varphi} \circ X i(X)g, \tag{74}$$

which is the classical Hamilton–Jacobi equation for a simple mechanical system with external force given by Z_1 . Compare with similar expressions in [27].

If we consider our system as the Lagrangian system defined on the Riemann manifold (M, g) by the Lagrangian function $L = T_g + \frac{1}{2}(\exp(-2\varphi))$ and with an external force one-form $F_1 = -2\widehat{d\varphi} i_{TM}g$, then Equation (74) can be written as

$$dE_L \circ X = 2\widehat{d\varphi} \circ X i(X)g,$$

where E_L is the Lagrangian energy corresponding to L . This last expression is the classical Hamilton–Jacobi equation in Lagrangian form for our Lagrangian system. See [19,27] for a comparison.

9. Conclusions and Future Work

We developed a detailed study from the perspective of the geometric mechanics of the properties of geodesic vector fields, i.e., autoparallel vector fields, on a Riemann manifold. Geodesic motion then corresponds to free motions of mechanical systems on a Riemann manifold, and therefore, the relevant tools in classical mechanics, as Hamilton–Jacobi theory, must shed light on the properties on geodesic motions and related vector fields. As the autoparallel vector fields are very helpful in such a study, one can try to introduce an analogous concept for classical systems of the mechanical type, also-called natural Lagrangian systems, or even for systems under the action of additional external forces, even in the case of velocity-dependent forces. We identified autoparallel vector fields on a Riemann manifold (M, g) as Hamilton–Jacobi vector fields for the second-order geodesic vector field of the given metric g , and furthermore, we showed that closed vector fields are autoparallel vector fields and satisfy $d(T_g \circ X) = 0$, the classical Hamilton–Jacobi equation. We also remarked that each vector field $X \in \mathfrak{X}(M)$ is a Hamilton–Jacobi vector field for its complete lift X^c and that each geodesic vector field $X \in \mathfrak{X}(M)$ is a Hamilton–Jacobi vector field for both its complete lift X^c and the second-order geodesic vector field of the given metric g .

These properties were appropriately extended to the framework of Newtonian and generalised Newtonian systems, in particular systems defined by Lagrangians of the mechanical type. Conserved quantities and a generalised concept of symmetry were developed, particularly for the case of Killing vector fields. The role of geodesic vector fields is now played by the so-called Newtonian vector fields, and their properties were studied, in particular the relation with Hamilton–Jacobi theory. The properties of generalised Newtonian systems subjected to nonholonomic constraints were also analysed from this perspective, and the relation among Newtonian vector fields and Hamilton–Jacobi equations for conformally related metrics was displayed. This last property is a consequence of the understanding of the geodesic flow of a conformal metric as a generalised mechanical system on the Riemann manifold of the initial metric.

There are other interesting systems to continue studying with the same methods. First is the particular case where the velocity-dependent forces come from a Rayleigh

function and the study of the associated dissipated quantities. See [23] for another approach and [28,29] for several generalizations and nice examples.

Finally, there is another geometric way of dealing with dissipative systems, which comes from contact mechanics. We hope that the application of similar techniques to those of our approach may be interesting. Therefore, we will study one of the simplest mechanical contact systems, where the Lagrangian has, in addition to the kinetic energy coming from the metric, a velocity-dependent potential with an extra variable, usually called s , to describe the dissipation, that is $L = T_g - V(q, v, s)$. The variational principle is due to Herglotz /9see [30] and the references therein), and the dynamical equations contain another term coming from the potential, the gradient as in the usual situation, and a dissipation term. Furthermore, another dynamical equation for the dissipation variable is needed, $\dot{s} = L(q, v, s)$. See [31,32] for other approaches. We also aim to use in the near future all the geometric machinery here developed to deal with these contact dissipative systems and complete the application to the study of conformal metrics, started in Section 8, together with the utilisation of the Sundman transformation; see [5,33].

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Appendix A. Sections along Maps and External Forces

Recall the concept of *section along a map*, [24,25,34–36]. Let $\pi : E \rightarrow M$ be a fibre bundle and $\phi : N \rightarrow M$ a differentiable map. A section along ϕ is a map $\sigma : N \rightarrow E$ such that $\pi \circ \sigma = \phi$.

They are in a 1-1 correspondence with sections of the induced bundle ϕ^*E

$$\begin{array}{ccc}
 \phi^*E & \xrightarrow{\phi[\pi]} & E \\
 \phi^*\pi \downarrow & \nearrow \sigma & \downarrow \pi \\
 N & \xrightarrow{\phi} & M
 \end{array}$$

where

$$\phi^*E = \{(n, e) \in N \times E \mid \phi(n) = \pi(e)\} \subset N \times E.$$

The set of sections along ϕ will be noted $\Sigma_\phi(\pi)$. When E is a vector bundle the set $\Sigma_\phi(\pi)$ is endowed with a $C^\infty(N)$ -module structure.

In particular we will be interested in the vector bundles $\tau_M : TM \rightarrow M$, $\pi_M^p : (T^*M)^{\wedge p} \rightarrow M$, $\rho_M^p = (T^*M)^{\wedge p} \otimes TM \rightarrow M$ for $p \in \mathbf{N}$, and in these cases we will denote, respectively,

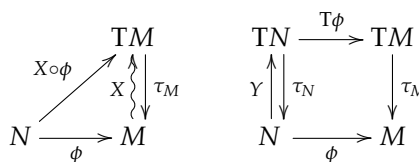
$$\mathfrak{X}(\phi) = \Sigma_\phi(\tau_M), \quad \Omega^p(\phi) = \Sigma_\phi(\pi_M^p), \quad V^p(\phi) = \Sigma_\phi(\rho_M^p).$$

When $N = M$ and $\phi = \text{id}$ the set $\mathfrak{X}(\text{id})$ coincides with $\mathfrak{X}(M)$ and the set $\Omega^p(\text{id})$ reduces to $\Omega^p(M)$.

As an example, if $\gamma : \mathbb{R} \rightarrow M$ is a curve on M , the tangent vectors $\dot{\gamma}$ define a section $\dot{\gamma} : \mathbb{R} \rightarrow TM$ of τ_M along γ . The restriction of $X \in \mathfrak{X}(M)$ on the curve γ , that is the composition $X \circ \gamma$, defines also a vector field along γ .

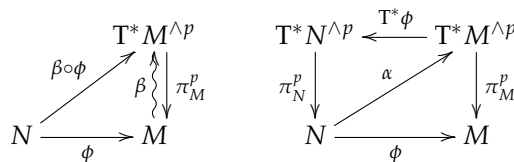
The generalisation of these examples is:

Let ϕ be a map from N to M . A vector field $Y \in \mathfrak{X}(N)$ defines a vector field along ϕ by $T\phi \circ Y \in \mathfrak{X}(\phi)$. Similarly, when $X \in \mathfrak{X}(M)$ the restriction $X \circ \phi$ of X on the image by ϕ is a vector field along ϕ . The above vector fields X and Y are said to be ϕ -related when $X \circ \phi$ and $T\phi \circ Y$ coincide along ϕ .



If β is a p -form in M , the restriction $\beta \circ \phi$ of β on the image $\phi(N)$ of ϕ is a p -form along ϕ .

Given $\alpha \in \Omega^p(\phi)$, $T^*\phi \circ \alpha$ is a p -form in N . The pull-back by ϕ of $\beta \in \Omega^p(M)$ is obtained by iteration of both processes $\phi^*(\beta) = T^*\phi \circ \beta \circ \phi$.



Local expressions:

When E is a vector bundle and $\{\sigma_\alpha\}$ is a local basis of $\Sigma(\pi)$, then $\{\sigma_\alpha \circ \phi\}$ is a local basis of $\Sigma_\phi(\pi)$, and $\sigma \in \Sigma_\phi(\pi)$ can be written as

$$\sigma = \zeta^\alpha(\sigma_\alpha \circ \phi) \quad \zeta^\alpha \in C^\infty(N).$$

In the above case, taking local coordinates (z^A) in N and (x^i) in M we have

$$\begin{aligned} X \in \mathfrak{X}(\phi) & \quad X = X^i(z^A) \left(\frac{\partial}{\partial x^i} \circ \phi \right) \\ \alpha \in \Omega^p(\phi) & \quad \alpha = \alpha_{i_1 \dots i_p}(z^A) (dx^{i_1} \circ \phi) \wedge \dots \wedge (dx^{i_p} \circ \phi). \end{aligned}$$

Observe that X^i , $\alpha_{i_1 \dots i_p}$ and $L_{i_1 \dots i_p}^i$ are functions in N .

When $N = TM$ and ϕ is the projection τ_M the vector fields and forms along τ_M are written as

$$\begin{aligned} X &= X^i(x, v) \left(\frac{\partial}{\partial x^i} \circ \tau_M \right) \\ \alpha &= \alpha_{i_1 \dots i_p}(x, v) (dx^{i_1} \circ \tau_M) \wedge \dots \wedge (dx^{i_p} \circ \tau_M) \end{aligned}$$

Vector fields along ϕ act on functions on M giving rise to functions on N . If $X \in \mathfrak{X}(\phi)$ and $n \in N$ then $X(n)$ is a tangent vector to M at the point $\phi(n)$ which acts on a function $h \in C^\infty(M)$ by $(Xh)(n) = X(n)h$. The Leibniz rule for tangent vectors implies that

$$X(h_1 h_2) = \phi^* h_1 Xh_2 + \phi^* h_2 Xh_1.$$

A map satisfying this property is called a ϕ^* -derivation (of degree 0).

Pidello and Tulczyjev generalised in [37] the theory of Frölicher and Nijenhuis for these new derivations:

Definition A1. Given a differentiable map $\phi : N \rightarrow M$, a ϕ^* -derivation of degree r of scalar on M is a \mathbb{R} -linear map $D : \Omega(M) \rightarrow \Omega(N)$ satisfying

$$\begin{aligned} D(\Omega^p(M)) &\subset \Omega^{p+r}(N) \\ D(\alpha \wedge \beta) &= D\alpha \wedge \phi^* \beta + (-1)^{pr} \phi^* \alpha \wedge D\beta \end{aligned}$$

for $\beta \in \Omega^q(M)$ and $\alpha \in \Omega^p(M)$. It is said to be of type i_* when $Dg = 0 \forall g \in C^\infty(M)$.

For instance, given a vector field along $\phi : N \rightarrow M$, $X \in \mathfrak{X}(\phi)$, a type i_* ϕ -derivation $i_X : \Omega^p(M) \rightarrow \Omega^{p-1}(N)$ of degree -1 is defined by $i_X g = 0, \forall g \in C^\infty(M)$ and if $\omega \in \Omega^p(M)$,

$$(i_X \omega)_z(v_1, \dots, v_{p-1}) = \omega_{\phi(z)}(X_z, \phi_{*z} v_1, \dots, \phi_{*z} v_{p-1})$$

where $v_1, \dots, v_{p-1} \in T_z(N)$.

By a type d_* ϕ -derivation of degree r we mean a ϕ -derivation such that $D \circ d_{(M)} = (-1)^r d_{(N)} \circ D$. An example of such a type ϕ -derivation, d_X , is defined by

$$d_X = i_X \circ d_{(M)} + d_{(N)} \circ i_X,$$

where $d_{(M)}$ stands for the operator of exterior differentiation in M . This is of type d_* , i.e.,

$$d_X \circ d_{(M)} = d_{(N)} \circ d_X.$$

Note that when $X \in \mathfrak{X}(\text{id}_M) \equiv \mathfrak{X}(M)$ the id_M -derivations i_X and d_X are but the contraction or inner product $i(X)$ (or i_X) and the Lie derivative \mathcal{L}_X , respectively. For this reason, i_X and d_X will be called *contraction* and *Lie derivative*, respectively.

Appendix A.1. The Case of a Riemannian Manifold

We are especially interested in the case of vector fields and one-forms along the projection $\tau_M : TM \rightarrow M$ on a Riemann manifold (M, g) . The metric induces natural isomorphisms between the C^∞ -modules of vector fields $\mathfrak{X}(M)$ and one-forms $\Omega^1(M)$. They are given by the musical isomorphisms between the tangent and cotangent bundles

$$\flat : TM \rightarrow T^*M, \quad \sharp : T^*M \rightarrow TM,$$

one inverse of the other, defined by

$$\flat(u_p) = i(u_p)g, \quad \sharp(\alpha_p) = i(\alpha_p)g^{-1},$$

and extended by C^∞ -linearity to the sections $\mathfrak{X}(M)$ and $\Omega^1(M)$. We use the same notation, \flat and \sharp , for the maps on the bundles and the sections, as usual.

These isomorphisms can be extended in the same way to the spaces of vector fields and one-forms along τ_M . In fact, given $\tau_M : TM \rightarrow M$, $X \in \mathfrak{X}(\tau_M), \alpha \in \Omega^1(\tau_M)$, we have

$$\begin{aligned} (\flat(X)(p, u))(p, v) &= (i_{X(p, u)}g)(v) = g_p(X(p, u), v) \\ (\sharp(\alpha)(p, u))(p, \beta) &= (i_{\alpha(p, u)}g_p^{-1})(\beta) = g_p^{-1}(\alpha(p, u), \beta) \end{aligned}$$

for $(p, v) \in T_pM$ and $(p, \beta) \in T_p^*M$.

For sections along τ_M in general, $X \in \mathfrak{X}(\tau_M)$, $\alpha \in \Omega^1(\tau_M)$, and $(p, u) \in T_pM$, we have:

$$\flat(X)(p, u) = i_{X(p,u)}g_p \in T_p^*M, \quad \sharp(\alpha)(p, u) = i_{\alpha(p,u)}g^{-1} \in T_pM,$$

and there is a duality relation between $\mathfrak{X}(\tau_M)$ and $\Omega^1(\tau_M)$ given by

$$\langle \alpha, X \rangle : TM \rightarrow \mathbb{R}, \quad \langle \alpha, X \rangle(p, u) = \langle \alpha(p, u), X(p, u) \rangle.$$

Appendix A.2. External Forces Depending on Velocities

The use we make of the above notions is the following: the elements of $\Omega^1(\tau_M)$ are the forms of force, or external forces, acting on a system in the case they depend not only on the positions but on the velocities also. Then the elements of $\mathfrak{X}(\tau_M)$ are the vector forces depending on the positions and velocities. The musical isomorphisms give the way to pass from one to the other depending on the form of the equations we use.

For the Lagrangian formulation, in the phase space TM , we use expressions with $\Omega^1(\tau_M)$, but for Riemannian expressions, in the configuration manifold M , is more suitable to use $\mathfrak{X}(\tau_M)$. Musical isomorphisms and the duality relation helps us to pass from one formulation to the other.

Observe that if $\gamma : \mathbb{R} \rightarrow M$ is a curve, $\dot{\gamma} : \mathbb{R} \rightarrow TM$ its tangent prolongation and $X \in \mathfrak{X}(\tau_M)$, then $X \circ \dot{\gamma} : \mathbb{R} \rightarrow TM$ is a vector field along γ , that is $X \circ \dot{\gamma} \in \mathfrak{X}(\gamma)$, because $\tau_M \circ \dot{\gamma} = \gamma$.

On the same way we have that if $\alpha \in \Omega^1(\tau_M)$, then $\alpha \circ \dot{\gamma} \in \Omega^1(\gamma)$.

Appendix A.3. External Forces as Semibasic Forms on TM

Usually in the study of Lagrangian systems we say that an external force is a semibasic one-form on TM , that is $\alpha \in \Omega^1(TM)$ such that $i_X\alpha = 0$ for any vertical vector field on TM . The local expression of such kind of forces is

$$\alpha = \alpha_j(x, v)dx^j,$$

in a natural coordinate system (x, v) of TM . In the case that the coordinates $\alpha_j(x, v)$ of α do not depend on the velocities, $\alpha_j(x)$, then the form α is basic and there exists a one-form $\hat{\alpha} \in \Omega^1(M)$ such that $\alpha = \tau_M^*\hat{\alpha}$. The set of semibasic one-forms on TM is denoted by $\Omega_1^1(TM)$ with the subindex indicating that the contraction with a vertical vector field is null.

Is there any relation between the sets $\Omega_1^1(TM)$, of semibasic one-forms and $\Omega^1(\tau_M)$, the set of one-forms along the map $\tau_M : TM \rightarrow M$? Let see how it goes:

Let $\alpha \in \Omega_1^1(TM)$ and $(p, u) \in TM$. Then $\alpha(p, u) \in T_{(p,u)}^*(TM)$. We can consider that $\alpha(p, u)$ acts on T_pM as

$$\bar{\alpha}(p, u) : (p, v) \rightarrow \alpha(p, u)((p, u), W), \quad W \in T_{(p,u)}(TM), \quad (\tau_M)_*(W) = v.$$

Observe that this is well defined because α is semibasic, hence if $(\tau_M)_*(W) = (\tau_M)_*(\bar{W}) = v$, then $W - \bar{W}$ is vertical hence $\alpha(p, u)((p, u), W) = \alpha(p, u)((p, u), \bar{W})$.

On the other side $\tau_M \circ \bar{\alpha}(p, u) = \tau_M(p, u)$, hence we have defined a map $\bar{\alpha} : TM \rightarrow T^*M$ which is an element of $\Omega^1(\tau_M)$, that is, we have a map:

$$\Omega_1^1(TM) \rightarrow \Omega^1(\tau_M),$$

which is $C^\infty(TM)$ -linear. This map is an isomorphism with inverse given by the following map: if $\beta \in \Omega^1(\tau_M)$, and $((p, u), W) \in T_{(p,u)}(TM)$, then take

$$\hat{\beta}((p, u), W) = \beta(p, u)((\tau_M)_*(W)),$$

then, clearly, $\hat{\beta} \in \Omega_1^1(TM)$. That this last one is the inverse of the other is easy to prove. Hence we have proven that both $C^\infty(TM)$ -modules are canonically isomorphic.

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