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Mild Solutions for the Time-Fractional Navier–Stokes Equations with MHD Effects

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Abstract: Recently, various techniques and methods have been employed by mathematicians to solve specific types of fractional differential equations (FDEs) with symmetric properties. The study focuses on Navier-Stokes equations (NSEs) that involve MHD effects with time-fractional derivatives (FDs). The (NSEs) with time-FDs of order $\beta \in (0,1)$ are investigated. To facilitate anomalous diffusion in fractal media, mild solutions and Mittag-Leffler functions are used. In $H^{\delta,r}$, the existence, and uniqueness of local and global mild solutions are proved, as well as the symmetric structure created. Moderate local solutions are provided in J_r. Moreover, the regularity and existence of classical solutions to the equations in J_r. are established and presented.

Keywords: Navier–Stokes equations; Caputo fractional derivative; Mittag-Leffler functions; mild solutions; regularity

MSC: 26A33; 34K37

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1. Introduction

The fractional calculus branch of mathematical analysis, cited in references [1,2], is concerned with the analysis of a large number of interpretations of real or complex number powers as defined by the differentiation operator D and the integration operator J, as well as the development of a calculus for these operators. In fluid mechanics, the Navier-Stokes equation is a partial differential equation that is used to simulate the flow of incompressible fluids [3]. The model is an extended version of the one developed by Swiss mathematician Leonhard Euler in the 18th century to describe the motion of incompressible fluids, as shown in [4]. For incompressible fluid, the Navier-Stokes equation (NSE) is

$$\frac{\partial u}{\partial \varsigma} + (u.\nabla)u - \mu\nabla^2 u = -\frac{1}{\rho}\nabla P + f.$$

The Navier–Stokes equations (NSE) are a family of equations that fundamentally represent how a fluid flows through its environment, since (NSE) explains the movement of an incompressible fluid. For Newtonian fluid flows, which reflect the conservation of momentum and mass [5,6], covering both the lubrication and greasing of ball bearings and large-scale atmospheric movements, the advantage of using (NSE) is that the unsteady (NSE) are directly solved by them, and they are also able to resolve the smallest eddies and temporal scales of turbulence in the flow. Furthermore, they can offer all the data for each instantaneous flow in the flow field. We see that this system has so many occurrences that the existence, regularity and boundary conditions must be explained using the full power of mathematical theories [7,8]. It is interesting to note that Leray conducted an early work that revealed that a boundary value problem for time-dependent (NSE) has an interesting, excellent solution on particular time intervals if the data are properly smooth [9]. Liquid metals and space plasmas are only two examples of the numerous physical substances that fall under the category of Magneto-hydrodynamics (MHD). MHD, also referred to as magneto-fluid dynamics or hydromagnetics, is the examination of the magnetic properties and "behaviour" of electrically conducting fluids. These "magneto-fluids" include, to name a few, plasmas, liquid metals, salt water, and electrolytes, according to Reference [10]. The words "magneto-hydrodynamics" are derived from the words magneto, which means magnetic field, hydro, which means water, and dynamics, which means motion.Hannes Alfven founded the domain of MHD, for in which he was awarded the 1970 Nobel Prize in Physics. MHD equations are a combination of NSE, which describe the motion of fluids and Maxwell's equations for electromagnetics (electric field and magnetic field); these system of equations are coupled together to form the hydro-magnetic or magneto-hydrodynamic system [11]. In this research, applications of MHD in the medical sciences are categorized into four groups. These areas include MHD applications in simple flow, peristaltic flow, pulsatile flow and drag delivery. These groups' respective numerical studies are examined and reported separately. It is really worth stating the significance of Leray's pioneering work in establishing whether the solution of (NS) decays to zero in L^2 as time reaches infinity. His work inspired numerous scholars to examine this topic, and there is now a large and strong body of literature on the subject; we may list a few examples. As a result, we may argue that the decay aspects of this issue are well recognized. As a result, it has piqued the curiosity of researchers during the last few years. The NSE are regarded as crucial mathematical tools for better understanding a variety of real-world problems in disciplines such as thermo-hydraulics, aeronautical sciences, meteorology, the petroleum industry, plasma physics and others. These equations provide a natural characterization of the interaction of a viscous liquid with rigid bodies [12]. Qayyum et al. [13], Rehman et al. [14] and Saeed et al. [15] worked on MHD. Niazi et al. [16], Shafqat et al. [17], Alnahdi [18], Khan [19] and Abuasbeh et al. [20] investigated the existence and uniqueness of the fractional evolution equations. Symmetry analysis is a powerful tool for understanding partial differential equations, especially when working with equations derived from accounting-related mathematical ideas. The secret of nature is symmetry, despite the fact that it is lacking from the majority of natural observations. It is preferable to hide symmetry when unanticipated symmetry-breaking occurs. Finite and infinitesimal symmetry can be divided into two distinct categories, Finite and infinitesimal. Finite symmetry can either be discrete or continuous, while infinitesimal symmetry is always continuous. Discrete finite symmetry refers to symmetry that exists in a finite number of elements, while continuous finite symmetry is a symmetry that is present across all elements in a given system. In the past few decades, fractional calculus has become increasingly important in mathematics. While space is a continuous transformation, natural symmetries such as parity and temporal inversion are discrete. These two types of symmetry provide different perspectives on the same phenomenon and can be used to gain insight into a variety of mathematical problems. Differential equations of fractional order are more appropriate for a variety of physical problems than equations of integer order. FDEs are extensively used in many fields of science, as electrical circuits, viscoelasticity materials, neural networks, engineering, chemistry, control theory, biology, mechanics and physics [21–23]. We draw attention to the fact that over the last three years, FDEs have significantly evolved and are a useful tool for describing certain materials and processes [24,25]. In this paper, we used the previously mentioned explanation. By considering the smooth boundary $\Omega = \mathcal{K}$ in $\mathbb{R}^n$ for $(n \geq 3)$, we investigate the time-fractional NSE:

$$\begin{cases} \partial_\varsigma^\beta u - \nu\Delta u + (u.\nabla)u = -\nabla p + (\dfrac{-\sigma B_0^2 u}{\rho}),\ \varsigma > 0, \\ \nabla.u = 0, \\ u|_{\partial\mathcal{K}} = 0, \\ u(x,\varsigma) = axcos\beta + b\varsigma sin\beta, \end{cases}$$

where the CFD of order $\beta \in (0,1)$ is denoted by $\partial_{\varsigma}^{\beta}$. Motion is magneto-hydrodynamics (MHD), σ is electrical conductivity, B_0 is the magnetic field and ρ is constant due to incompressible fluid. Here, the velocity field at a point $x \in \mathcal{K}$ and time $\varsigma > 0$ is denoted by u =$(u_1(\varsigma,x), u_2(\varsigma,x), u_3(\varsigma,x), ..., u_n(\varsigma,x))$, p = $p(\varsigma,x)$ denotes the pressure term, the kinematics viscosity shown by the symbol ν, ς represents the time, and a = a(x) defines the initial velocity. We set the smooth boundary to be $\mathcal{K}$. This model was modified by replacing the first time derivative with a fractional derivative of order β, where $0 < \beta \leq 1$. There was the first time a fractional derivative of this order was in the model. The flow of fluid is detected at an angle of 45°, then we have

$$u(0,x) = axcos45^{\circ} = \frac{ax}{\sqrt{2}},$$

$$\begin{cases} \partial_{\varsigma}^{\beta} u - \nu \Delta u + (u.\nabla)u = -\nabla P + (\dfrac{-\sigma B_0^2 u}{\rho}), & \varsigma > 0, \\ \nabla.u = 0, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0,x) = \dfrac{ax}{\sqrt{2}}. \end{cases} \tag{1}$$

Hereby, we apply the Helmholts leray projector P_L on Equation (1) to convert the NSE into a time- fractional model.

The operator $-\nu P \Delta$ with Dirichlet boundary conditions is simply the same operator A, just like in the divergence-free function space under consideration. Next, we write (1) in its abstract sense, which is

$$\begin{cases} {}^{C}D_{\varsigma}^{\beta} u(\varsigma) = -Au + F(u,u) + P_L(\dfrac{-\sigma B_0^2 u}{\rho}), & \varsigma > 0, \\ u(0) = \dfrac{ax}{\sqrt{2}}, \end{cases} \tag{2}$$

where $F(u,v) = -P_L(u.\nabla)v$. If the Stokes function A and the Helmholtz–Leray projection P_L seem alike, then the Equation (2) has a similar solution as that of Equation (1). For convenience, we simply write P instead of P_L. The goal of this paper is to demonstrate the presence and distinction of moderate global and local problem solutions of (2) in $H^{\delta,r}$. Additionally, we demonstrate the regularity findings, which indicate that there is just one classical solution if $P(\dfrac{-\sigma B0^2 u}{\rho})$ is Hölder-continuous. In order for Au and ${}^{C}D\varsigma^{\beta}u(\varsigma)$ to be Hölder-continuous in J_r, $u(\varsigma)$ has to be such solution.

The essential idea behind MHD is that magnetic fields have the ability to induce currents in conductive fluids that are in motion, which in turn produces forces on the fluid and modifies the magnetic field. The Navier–Stokes equations for fluid dynamics and Maxwell's equations for electromagnetism combine to provide the set of equations that describe MHD. It is necessary to simultaneously solve these differential equations, either analytically or numerically. The flow of conductive fluid is influenced by a magnetic field. A current that travels through the magnet at a 45° angle as it goes down the screw and enters the magnetic field.

2. Preliminaries

In this section, we set the representations, definitions and introductory information that are used throughout the research [26]. Consider $\Omega = \mathcal{K} = \{(x_1, x_2, ...x_n) : x_n > 0\}$ to be an open subset of $\mathbb{R}^n$, where $n \geq 3$. Let $1 < r < \infty$, then the bounded Hodge projection to $(L^r(\mathcal{K}))^n$ on projection P, whose range is the closure of

$$C_{\sigma}^{\infty}(\mathcal{K}) := \{u \in (C^{\infty}(\mathcal{K}))^n : \nabla.u = 0, \; u \; has \; compact \; subspace \; in \; \mathcal{K}\},$$

and its null space is given by the closure of

$$\{u \in (C^{\infty}(\mathcal{K}))^n : u = \nabla \eta, \quad \eta \in C^{\infty}(\mathcal{K})\}.$$

For clear and simple notations, let $J_r := \overline{C_{\sigma}^{\infty}(\mathcal{K})}^{|.|_r}$, which is a closed subspace of $(L^r(\mathcal{K}))^n$. Additionally, $(W^{m,r}(\mathcal{K}))^n$ is a Sobolev space with the norm $|.|_{m,r}$. $A = -\nu P\Delta$ stands for the Stokes operator in J_r with the domain is $D_r(A) = D_r(\Delta) \cap J_r$; we have

$$D_r(\Delta) = \{u \in (Z^{2,r}(\mathcal{K}))^n : u|_{\partial\mathcal{K}} = 0\}.$$

The closed linear operator -A generates the bounded analytic semigroup $\{e^{-\zeta A}\}$ on J_r. To present our results, we present the traditional power spaces which is associated with -A. For $\delta > 0$ and $u \in J_r$, describe

$$A^{-\delta}u = \frac{1}{\Gamma(\delta)} \int_0^{\infty} \varsigma^{\delta-1} e^{-\varsigma A} u d\varsigma.$$

Then, $A^{-\delta}$ is a one-to-one operator on J_r. A^{δ} is the inverse of $A^{-\delta}$, and set $H^{\delta,r}$ for the range of $A^{-\delta}$ supplemented by norm for $\delta > 0$,

$$|u|_{H^{\delta,r}} = |A^{\delta}u|_r.$$

$e^{-\varsigma A}$ is extended (or restricted) to a bound analytic semigroup on $H^{\delta,r}$ is simple. Let X be a Banach space and J be an $\mathbb{R}$ interval. The family of continuous X-valued functions is given by $C(J, X)$. $C^{\vartheta}(J, X)$ is the family of all Hölder-continuous functions considering exponent ϑ for $0 < \vartheta < 1$. Let $\beta \in (0, 1]$ and $\nu : [0, \infty) \to X$. The fractional integral of order β for a function ν with a lower limit of zero is defined as

$$I_{\varsigma}^{\beta}\nu(\varsigma) = \int_0^{\varsigma} g_{\beta}(\varsigma - s)\nu(s)ds, \ \varsigma > 0.$$

Let the right-hand side be point-wise defined the interval $[0, \infty)$, where g_a tends for Riemann Liouville kernel,

$$g_{\beta}(\varsigma) = \frac{\varsigma^{\beta-1}}{\Gamma(\beta)}, \ \varsigma > 0.$$

Furthermore, the CFD operator of order β is denoted by ${}^{C}D_{\varsigma}^{\beta}$; and defined as

$${}^{C}D_{\varsigma}^{\beta}\nu(\varsigma) = \frac{d}{d\varsigma}[I_{\varsigma}^{1-\beta}(\nu(\varsigma) - \nu(0))] = \frac{d}{d\varsigma}\left(\int_0^{\varsigma} g_{1-\beta}(\varsigma - s)(\nu(\varsigma) - \nu(0))ds\right), \ \varsigma > 0.$$

Generally, for $u : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}^n$. The expression for the CFD of the function u with respect to time is

$$\partial_{\varsigma}^{\beta}u(\varsigma, x) = \partial_{\varsigma}\left(\int_0^{\varsigma} g_{1-\beta}(\varsigma - s)(u(\varsigma, x) - u(0, x))ds\right), \ \varsigma > 0.$$

Let us look have s look at Mittag-Leffler functions

$$\begin{aligned} E_{\beta}(-\varsigma^{\beta}A) &= \int_0^{\infty} \bar{M}_{\beta}(s)e^{-s\varsigma^{\beta}A}ds, \\ E_{\beta,\beta}(-\varsigma^{\beta}A) &= \int_0^{\infty} \beta s\bar{M}_{\beta}(s)e^{-s\varsigma^{\beta}A}ds, \end{aligned}$$

here $\bar{M}_{\beta}(\lambda)$ stands for Mainardi Wright function, which is defined by

$$\bar{M}_{\beta}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^m}{m!\Gamma(1 - \beta(1 + m))}.$$

Proposition 1. *Here are some properties of the Mittag-Leffler special functions*

(i) $E_{\beta,\beta}(-\varsigma^\beta A) = \dfrac{1}{2\pi i}\int_{\Gamma\theta} E_{\beta,\beta}(-\varsigma^\beta \mu)(\mu I + A)^{-1} d\mu;$

(ii) $A^\alpha E_{\beta,\beta}(-\varsigma^\beta A) = \dfrac{1}{2\pi i}\int_{\Gamma\theta} \mu^\alpha E_{\beta,\beta}(-\varsigma^\beta \mu)(\mu I + A)^{-1} d\mu.$

Proof. (i) By having $\int_0^\infty \beta s \bar{M}_\beta(s) e^{-s\varsigma} ds = E_{\beta,\beta}(-\varsigma)$ with Fubini theorem, we introduce

$$\begin{aligned} E_{\beta,\beta}(-\varsigma^\beta A) &= \int_0^\infty \beta s \bar{M}_\beta(s) e^{-s\varsigma^\beta A} ds \\ &= \frac{1}{2\pi i}\int_0^\infty \beta s \bar{M}_\beta(s) \int_{\Gamma\theta} e^{-\mu s \varsigma^\beta}(\mu I + A)^{-1} d\mu\, ds \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\beta,\beta}(-\varsigma^\beta \mu)(\mu I + A)^{-1} d\mu \end{aligned}$$

with the appropriate Γ_θ integral route.

(ii) In the same manner,

$$\begin{aligned} A^\alpha E_{\beta,\beta}(-\varsigma^\beta A) &= \int_0^\infty \beta s \bar{M}_\beta(s) A^\alpha e^{-s\varsigma^\beta A} ds \\ &= \frac{1}{2\pi i}\int_0^\infty \beta s \bar{M}_\beta(s) \int_{\Gamma\theta} \mu^\alpha e^{-\mu s \varsigma^\beta}(\mu I + A)^{-1} d\mu\, ds \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^\alpha E_{\beta,\beta}(-\mu \varsigma^\beta)(\mu I + A)^{-1} d\mu. \end{aligned}$$

The results are gotten are like. □

Lemma 1. *([27]). The operators $E_\beta(-\varsigma^\beta A)$ and $E_{\beta,\beta}(-\varsigma^\beta A)$ are continuous for $\varsigma > 0$, in uniform operator topology. Moreover, they are uniformly continuous on $[r, \infty)$ for $r > 0$.*

Lemma 2. *([28]).*

Take $0 < \beta < 1$, which implies

(i) $\forall\, u \in X$, $\lim_{\varsigma \to 0^+} E_\beta(-\varsigma^\beta A)u = u;$

(ii) $\forall\, u \in D(A)$ *and* $\varsigma > 0$, ${}^C D_\varsigma^\beta E_\beta(-\varsigma^\beta A)u = -A E_\beta(-\varsigma^\beta A)u;$

(iii) $\forall\, u \in X$, $E'_\beta(-\varsigma^\beta A)u = -\varsigma^{\beta-1} A E_{\beta,\beta}(-\varsigma^\beta A)u;$

(iv) $\forall\, \varsigma > 0$, $E_\beta(-\varsigma^\beta A)u = I_\varsigma^{1-\beta}\left(\varsigma^{\beta-1} E_{\beta,\beta}(-\varsigma^\beta A)u\right).$

We introduce the next lemma for the function $h : [0, \infty) \to X$, before presenting the notion of a mild solution of (2). For this, see [6].

Lemma 3.

$$\begin{cases} {}^C D_\varsigma^\beta u(\varsigma) = -Au + F(u,u) + P\left(\dfrac{-\sigma B_0^2 u}{\rho}\right), \ \varsigma > 0, \\ u(0) = \dfrac{ax}{\sqrt{2}}, \end{cases}$$

satisfying solution is

$$u(\varsigma) = \frac{a\varsigma}{\sqrt{2}} + \frac{1}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma - s)^{\beta-1}\Big(-Au(s) + F(u(s), u(s)) + P(\frac{-\sigma B_0^2 u}{\rho})\Big)ds, \ \ for \ \varsigma \geq 0 \tag{3}$$

$$\begin{aligned} u(\varsigma) &= \frac{a\varsigma}{\sqrt{2}} + \frac{1}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma - s)^{\beta-1}(-Au(s))ds \\ &+\frac{1}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma - s)^{\beta-1}F(u(s), u(s))ds + \frac{1}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma - s)^{\beta-1}P(\frac{-\sigma B_0^2 u}{\rho})ds. \end{aligned}$$

Taking Laplace on both sides

$$u(\lambda) = \frac{a}{\sqrt{2}\lambda^2} + \frac{1}{\lambda^{\beta}}[-Au(\lambda)] + \frac{1}{\lambda^{\beta}}F(u, u(\lambda)) + \frac{1}{\lambda^{\beta}}(P\frac{-\sigma B_0^2}{\rho}u(\lambda)).$$

Multiplying both sides by λ^{β}

$$\begin{aligned} \lambda^{\beta}u(\lambda) &= \frac{a}{\sqrt{2}}\lambda^{\beta-2} + [-Au(\lambda)] + F(u, u(\lambda)) + (P\frac{-\sigma B_0^2}{\rho}u(\lambda)) \\ (\lambda^{\beta} + A)u(\lambda) &= \frac{a}{\sqrt{2}}\lambda^{\beta-2} + F(u, u(\lambda)) + (P\frac{-\sigma B_0^2}{\rho}u(\lambda)) \\ u(\lambda) &= (\lambda^{\beta} + A)^{-1}\frac{a}{\sqrt{2}}\lambda^{\beta-2} + (\lambda^{\beta} + A)^{-1}F(u, u(\lambda)) \\ &+ (\lambda^{\beta} + A)^{-1}(P\frac{-\sigma B_0^2}{\rho}u(\lambda)). \end{aligned}$$

Taking Laplace inverse for both sides

$$\begin{aligned} u(\varsigma) &= \frac{a}{\sqrt{2}}\int_0^{\varsigma}E_{\beta}(-(\varsigma - s)^{\beta}A)ds + \int_0^{\varsigma}(\varsigma - s)^{\beta-1}E_{\beta,\beta}(-(\varsigma - s)^{\beta}A)F(u(s), u(s))ds \\ &+ \int_0^{\varsigma}(\varsigma - s)^{\beta-1}E_{\beta,\beta}(-(\varsigma - s)^{\beta}A)\big(P\frac{-\sigma B_0^2}{\rho}u(s)\big)ds. \end{aligned}$$

Definition 1. *A function $u : [0, \infty) \to H^{\delta,r} or (J_r)$ is termed the global mild solution of problem (2) in $H^{\delta,r}$, if $u \in C([0, \infty), H^{\delta,r})$ and $\varsigma \in [0, \infty)$,*

$$\begin{aligned} u(\varsigma) &= \frac{a}{\sqrt{2}}\int_0^{\varsigma}E_{\beta}(-(\varsigma - s)^{\beta}A)ds + \int_0^{\varsigma}(\varsigma - s)^{\beta-1}E_{\beta,\beta}(-(\varsigma - s)^{\beta}A)F(u(s), u(s))ds \\ &+ \int_0^{\varsigma}(\varsigma - s)^{\beta-1}E_{\beta,\beta}(-(\varsigma - s)^{\beta}A)\big(P\frac{-\sigma B_0^2}{\rho}u(s)\big)ds. \end{aligned} \tag{4}$$

Definition 2. *Let $0 < \tilde{\Im} < \infty$. If $u \in C([0, \tilde{\Im}], H^{\delta,r})$ or $C([0, \tilde{\Im}], H^{\delta,r})$ and u and satisfy (4) for $\varsigma \in [0, \tilde{\Im}]$, then the function $u : [0, \tilde{\Im}] \to H^{\beta,r}$ or (J_r) is called a local mild solution of problem (2) in $H^{\beta,r}$ or (J_r).*

Consider the three operators for case ξ, η, ϕ,

$$\begin{aligned} \xi(\varsigma) &= \frac{a}{\sqrt{2}}\int_0^{\varsigma}E_{\beta}(-(\varsigma - s)^{\beta}A)ds, \\ \eta(\varsigma) &= \int_0^{\varsigma}\varsigma - s)^{\beta-1}E_{\beta,\beta}(-(\varsigma - s)^{\beta}A)P(\frac{-\sigma B_0^2}{\rho}u(\varsigma))ds, \\ \phi(u, v)(\varsigma) &= \int_0^{\varsigma}(\varsigma - s)^{\beta-1}E_{\beta,\beta}(-(\varsigma - s)^{\beta}A)F(u(s), v(s)))ds. \end{aligned}$$

Definition 3. *A non-negative measurable function f defined on a measurable set $\mathcal{E}$ is integrable if,* $\int_{\mathcal{E}} f < \infty$.

Definition 4. *Let g be integrable over $\mathcal{E}$, and consider $< f_n >$ be a sequence of a measurable function with $|f_n| \leq g$ on $\mathcal{E}$ and $\lim_{n\to\infty} f_n = f$, i.e,*

$$\lim_{n\to\infty} \int_{\mathcal{E}} f_n = \int_{\mathcal{E}} f.$$

It is clear that every function f_n is integrable on $\mathcal{E}$ and, furthermore, it follows that $\lim_{n\to\infty} f_n = f$ on $\mathcal{E}$ and $|f_n| \leq g$ on $\mathcal{E}$ that $|f| \leq g$ and, hence, f is integrable on $\mathcal{E}$.

Lemma 4. *Let $(X, ||.||_X)$ denote the Banach space, and let a bilinear operator be defined as $G : X \times X \to X$ and a positive real number L such that*

$$||G(u,v)||_X \leq L||u||_X||v||_X,\ \forall\, u,v \in X.$$

Then, for any $u_0 \in X$ with $||u_0||_X < \dfrac{1}{4L}$, there is just a unique solution $u \in X$ to the equation $u = u_0 + G(u,u)$.

3. Global and Local Existence in $H^{\delta,r}$

For the existence and unique property of a mild solution to the situation (2) in $H^{\delta,r}$, we provide adequate conditions. For this, we suppose that:

(e) For $\varsigma > 0$, $(P\frac{-\sigma B_0^2}{\rho}u)$ is continuous and $|P(\dfrac{-\sigma B_0^2}{\rho}u(s))|_r = 0(\varsigma^{-\beta(1-\delta})$ for $0 < \delta < 1$ as $\varsigma \to 0$.

Lemma 5. *Let $1 < r < \infty$ and $\delta_1 \leq \delta_2$. The existence of constant is $\mathcal{C} = \mathcal{C}(\delta_1,\delta_2)$ such that*

$$|e^{-\varsigma A}v|_{H^{\delta_2,r}} \leq \mathcal{C}\varsigma^{-(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}},\ \ \varsigma > 0,$$

for $v \in H^{\delta_1,r}$. Furthermore,

$$\lim_{t\to 0} \varsigma^{(\delta_2-\delta_1)}|e^{-\varsigma A}v|_{H^{\delta_2,r}} = 0.$$

We now examine a fundamental lemma that enables us to demonstrate the final major theorems of this section.

Lemma 6. *Let $1 < r < \infty$ and also $\delta_1 \leq \delta_2$. Then, for any $\tilde{\Im} > 0$, there is constant $\mathcal{C}_1 = \mathcal{C}_1(\delta_1,\delta_2) > 0$ such that*

$$|E_\beta(-\varsigma^\beta A)|_{H^{\delta_2,r}} \leq \mathcal{C}_1\varsigma^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}} \text{ and } |E_{\beta,\beta}(-\varsigma^\beta A)|_{H^{\delta_2,r}} \leq \mathcal{C}_1\varsigma^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}},$$

for all $v \in H^{\delta_1,r}$ and $\varsigma \in (0,\tilde{\Im}]$. Furthermore, $\lim_{\varsigma\to 0} \varsigma^{\beta(\delta_2-\delta_1)}|E_\beta(-\varsigma^\beta A)v|_{H^{\delta_2,r}} = 0$.

Proof. Let $v \in H^{\delta_1,r}$. By the previous Lemma 5, we find that

$$\begin{aligned}
|E_\beta(-\varsigma^\beta A)v|_{H^{\delta_2,r}} &\leq \int_0^\infty \bar{M}_\beta(s)|e^{-s\varsigma^\beta A}v|_{H^{\delta_2,r}}ds \\
&\leq \left(\mathcal{C}\int_0^\infty \bar{M}_\beta(s)s^{-(\delta_2-\delta_1)}ds\right)\varsigma^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}} \\
&\leq \mathcal{C}_1\varsigma^{-\beta(\delta_2-\delta_1)}|v|_{H^{\delta_1,r}}.
\end{aligned}$$

Additionally, the dominated convergence theorem of Lebesgue demonstrates that

$$\lim_{\varsigma\to 0}\varsigma^{\beta(\delta_2-\delta_1)}|E_\beta(-\varsigma^\beta A)\nu|_{H^{\delta_2,r}} \leq \int_0^\infty \bar{M}_\beta(s)\lim_{\varsigma\to 0}\varsigma^{\beta(\delta_2-\delta_1)}|e^{-s\varsigma^\beta A}\nu|_{H^{\beta_2,r}}ds = 0.$$

Similarly,

$$\begin{aligned} |E_{\beta,\beta}(-\varsigma^\beta A)\nu|_{H^{\delta_2,r}} &\leq \int_0^\infty \beta s\bar{M}_\beta(s)|e^{-s\varsigma^\beta A}\nu|_{H^{\delta_2,r}}ds \\ &\leq \left(\beta\mathcal{C}\int_0^\infty \bar{M}_\beta(s)s^{1-(\delta_2-\delta_1)}ds\right)\varsigma^{-\beta(\delta_2-\delta_1)}|\nu|_{H^{\delta_1,r}} \\ &\leq \mathcal{C}_1\varsigma^{-\beta(\delta_2-\delta_1)}|\nu|_{H^{\delta_1,r}}, \end{aligned}$$

where constant $\mathcal{C}_1 = \mathcal{C}_1(\beta,\delta_1,\delta_2)$ is

$$\mathcal{C}_1 \geq \mathcal{C}\ max\left\{\frac{\Gamma(1-\delta_2+\delta_1)}{\Gamma(1+\beta(\delta_1-\delta_2))}, \frac{\beta\Gamma(2-\delta_2+\delta_1)}{\Gamma(1+\beta(1+\delta_1-\delta_2))}\right\}.$$

□

4. Global Existence in $H^{\delta,r}$

Now, a portion of the above section of this article deals with existence of a global mild solution of problem (2) in $H^{\delta,r}$, and we let

$$\begin{aligned} \mathcal{M}(\varsigma) &= \sup_{s\in(0,\varsigma]}\{s^{\beta(1-\delta)}|P(\frac{-\sigma B_0^2}{\rho}u(s))|_r\}, \\ \mathcal{B}_1 &= \mathcal{C}_1\ max\{\mathcal{B}(\beta(1-\delta)), 1-(\beta(1-\delta)), \mathcal{B}(\beta(1-\alpha), 1-\beta(1-\delta))\}, \\ L &\geq \mathcal{M}\mathcal{C}_1\ max\Big\{\mathcal{B}(\beta(1-\delta)), 1-2\beta(\alpha-\delta)), \mathcal{B}(\beta(1-\alpha), 1-2\beta(\alpha-\delta))\Big\}, \end{aligned}$$

assuming $\mathcal{M}$ is provided afterward.

Theorem 1. *Let $1 < r < \infty$, $0 < \delta < 1$ and (e) hold. For all $\frac{a\varsigma}{\sqrt{2}} \in H^{\delta,r}$, suppose that*

$$\mathcal{C}_1\left|\frac{a\varsigma}{\sqrt{2}}\right|_{H^{\delta,r}} + \mathcal{B}_1\mathcal{M}_\infty < \frac{1}{4L}. \tag{5}$$

The above-mentioned $\mathcal{M}_\infty$ is defined as $\mathcal{M}_\infty := \sup_{s\in(0,\infty)}\{s^{\beta(1-\delta)}P(\frac{-\sigma B_0^2 u(s)}{\rho})\}$. If $\frac{n}{2r} - \frac{1}{2} < \delta$, then there is a $\alpha > max\{\delta, \frac{1}{2}\}$ and a function which is unique $u : [0,\infty) \to H^{\delta,r}$ helps to satisfy:

(a) *The function $u : [0,\infty) \to H^{\delta,r}$ is continuous and $u(0) = \frac{a\varsigma}{\sqrt{2}}$;*
(b) *The function $u : (0,\infty) \to H^{\alpha,r}$ is continuous and $lim_{\varsigma\to 0}\varsigma^{\beta(\alpha-\delta)}|u(\varsigma)|_{H^{\alpha,r}} = 0$;*
(c) *u satisfy (4) for $\varsigma \in [0,\infty)$.*

Proof. Take $\alpha = \frac{1+\delta}{2}$. Here, X_∞ is a space containing all the well-defined curves $u : (0,\infty) \to H^{\delta,r}$, also $X_\infty = X[\infty]$, and the X_∞ term is a complete metric space, and it is nonempty:

(i) The continuous and bounded function is $u : [0,\infty) \to H^{\delta,r}$;
(ii) Additionally, the function is continuous and bounded $u : (0,\infty) \to H^{\alpha,r}$; moreover,

$$\lim_{\varsigma\to 0}\varsigma^{\beta(\alpha-\delta)}|u(\varsigma)|_{H^{\alpha,r}} = 0;$$

having a fundamental norm

$$||u||_{X_\infty} = max\left\{ \sup_{\varsigma\geq 0} |u(\varsigma)|_{H^{\delta,r}}, \sup_{\varsigma\geq 0} \varsigma^{\beta(\alpha-\delta)}|u(\varsigma)|_{H^{\alpha,r}} \right\}.$$

Since it is clear the mapping $\mathcal{F} : H^{\alpha,r} \times H^{\alpha,r} \to J_r$ is a well-defined, bounded as well as bilinear mapping because of a Weissler argument, $\exists\ \mathcal{M}$ in such a way that for $u, \nu \in H^{\alpha,r}$,

$$\begin{aligned} |\mathcal{F}(u,\nu)|_r &\leq \mathcal{M}|u|_{H^{\alpha,\varsigma}}|\nu|_{H^{\alpha,r}}, \\ |\mathcal{F}(u,u) - \mathcal{F}(\nu,\nu)|_r &\leq \mathcal{M}(|u|_{H^{\alpha,r}} + |\nu|_{H^{\alpha,r}})|u-\nu|_{H^{\alpha,r}}. \end{aligned} \tag{6}$$

Step I

Let us suppose that $u, \nu \in X_\infty$. Here, the operator $\phi(u(\varsigma), \nu(\varsigma)) \in \mathcal{C}([0,\infty), H^{\delta,r})$ and also the operator $\phi(u(\varsigma), \nu(\varsigma)) \in \mathcal{C}((0,\infty), H^{\alpha,r})$. Consider $\varsigma < \varsigma_0$ for completely arbitrary, $\varsigma_0 \geq 0$ is fixed and $\varepsilon > 0$ is very small (the following situation is related). There are

$$\begin{aligned} \left|\phi(u(\varsigma)\nu(\varsigma)) - \phi(u(\varsigma_0)\nu(\varsigma_0))\right|_{H^{\delta,r}} &\leq \int_{\varsigma_0}^{\varsigma} (\varsigma-s)^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^\beta A)F(u(s)\nu(s)|_{H^{\delta,r}}ds \\ &+ \int_0^{\varsigma_0} |((\varsigma-s)^{\beta-1} - (\varsigma_0-s)^{\beta-1})E_{\beta,\beta}(-(\varsigma-s)^\beta A)F(u(s)\nu(s)|_{H^{\delta,r}}ds \\ &+ \int_0^{\varsigma_0-\varepsilon} (\varsigma_0-s)^{\beta-1}|\big(E_{\beta,\beta}(-(\varsigma-s)^\beta A) - E_{\beta,\beta}(-(\varsigma_0-s)^\beta A)\big)F(u(s),\nu(s)|_{H^{\delta,r}}ds \\ &+ \int_{\varsigma_0-\varepsilon}^{\varsigma_0} (\varsigma_0-s)^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^\beta A) - E_{\beta,\beta}(-(\varsigma_0-s)^\beta A)F(u(s),\nu(s)|_{H^{\delta,r}}ds \\ &:= G_{11}(\varsigma) + G_{22}(\varsigma) + G_{33}(\varsigma) + G_{44}(\varsigma). \end{aligned}$$

Each of these four terms is estimated independently. For $G_{11}(\varsigma)$, in the light of the above Lemma 6, we find

$$\begin{aligned} G_{11}(\varsigma) &\leq \mathcal{C}_1 \int_{\varsigma_0}^{\varsigma} (\varsigma-s)^{\beta(1-\delta)-1}|F(u(s),\nu(s)|_r ds \\ &\leq \mathcal{M}\mathcal{C}_1 \int_{\varsigma_0}^{\varsigma} (\varsigma-s)^{\beta(1-\delta)-1}|u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}}ds \\ &\leq \mathcal{M}\mathcal{C}_1 \int_{\varsigma_0}^{\varsigma} (\varsigma-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in[0,\varsigma]} \left\{ s^{2\beta(\alpha-\delta)}|u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}} \right\} \\ &= \mathcal{M}\mathcal{C}_1 \int_{\varsigma_0/\varsigma}^{1} (1-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in[0,\varsigma]} \left\{ s^{2\beta(\alpha-\delta)}|u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}} \right\}. \end{aligned}$$

There exists $\tilde{\delta} > 0$, and $\tilde{\delta}$ is very small for $0 < \varsigma - \varsigma_0 < \tilde{\delta}$ from definition of the β function

$$\int_{\varsigma_0/\varsigma}^{1} (1-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \to 0.$$

Consequently, $G_{11}(\varsigma)$ approaches to 0 as $\varsigma - \varsigma_0$ approaches to 0. For $G_{12}(\varsigma)$,

$$\begin{aligned} G_{12}(\varsigma) &\leq \mathcal{C}_1 \int_0^{\varsigma_0} \big((\varsigma_0-s)^{\beta-1} - (\varsigma-s)^{\beta-1}\big)(\varsigma-s)^{-\beta\delta}|F(u(s),\nu(s)|_r ds \\ &\leq \mathcal{M}\mathcal{C}_1 \int_0^{\varsigma_0} \big((\varsigma_0-s)^{\beta-1} - (\varsigma-s)^{\beta-1}\big)(\varsigma-s)^{-\beta\delta}s^{-2\beta(\alpha-\delta)}ds \\ &\quad \sup_{s\in[0,\varsigma_0]} \left\{ s^{2\beta(\alpha-\delta)}|(u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}} \right\}. \end{aligned}$$

It is interesting to note

$$\begin{aligned}
&\int_0^{\varsigma_0} |(\varsigma_0-s)^{\beta-1}-(\varsigma-s)^{\beta-1}|(\varsigma-s)^{-\beta\delta}s^{-2\beta(\alpha-\delta)}ds \\
&\leq \int_0^{\varsigma_0}(\varsigma-s)^{\beta-1}(\varsigma-s)^{-\beta\delta}s^{-2\beta(\alpha-\delta)}ds+\int_0^{\varsigma_0}(\varsigma_0-s)^{\beta-1}(\varsigma-s)^{-\beta\delta}s^{-2\beta(\alpha-\delta)}ds \\
&\leq 2\int_0^{\varsigma_0}(\varsigma_0-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \\
&= 2\mathcal{B}(\beta(1-\delta),1-2\beta(\alpha-\delta)).
\end{aligned}$$

So, by theorem (LDC), there are

$$\int_0^{\varsigma_0}(\varsigma_0-s)^{\beta-1}-(\varsigma-s)^{\beta-1}(\varsigma-s)^{-\beta\delta}s^{-2\beta(\alpha-\delta)}ds \to 0 \; as \; \varsigma \to \varsigma_0.$$

We conclude the limiting value of $G_{12}(\varsigma)$ is equal to zero as $\varsigma \to \varsigma_0$. Now, we move towards $G_{13}(\varsigma)$,

$$\begin{aligned}
G_{13}(\varsigma) &\leq \int_0^{\varsigma_0-\varepsilon}(\varsigma_0-s)^{\beta-1}|\big(E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)+E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)F(u(s),\nu(s)\big)|_{H^{\delta,r}}ds \\
&\leq \int_0^{\varsigma_0-\varepsilon}(\varsigma_0-s)^{\beta-1}\big((\varsigma-s)^{-\alpha\beta}+(\varsigma_0-s)^{-\alpha\beta}\big)|F(u(s),\nu(s)|_{H^{\delta,r}}ds \\
&\leq 2\mathcal{M}\mathcal{C}_1\int_0^{\varsigma_0-\varepsilon}(\varsigma_0-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds \sup_{s\in[0,\varsigma_0]}\{s^{2\beta(\alpha-\delta)}|u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}}\},
\end{aligned}$$

by using the (LDC) theorem one more time, and the operator $E_{\beta,\beta}(-\varsigma^{\beta}A)$ is uniform continuous by Lemma 1, which shows

$$\begin{aligned}
\lim_{\varsigma\to\varsigma_0}G_{13}(\varsigma) &= \int_0^{\varsigma_0-\varepsilon}(\varsigma_0-s)^{\beta-1}\lim_{\varsigma\to\varsigma_0}|E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)-E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)F(u(s),\nu(s))|_{H^{\delta,r}}ds \\
&= 0.
\end{aligned}$$

For $G_{14}(\varsigma)$, from calculations, we find conclusions that

$$\begin{aligned}
G_{14}(\varsigma) &\leq \int_{\varsigma_0-\varepsilon}^{\varsigma_0}(\varsigma_0-s)^{\beta-1}\big((\varsigma-s)^{-\beta\delta}+(\varsigma_0-s)^{-\beta\delta}\big)|F(u(s),\nu(s)|_r ds \\
&\leq 2\mathcal{M}\mathcal{C}_1\int_{\varsigma_0-\varepsilon}^{\varsigma_0}(t_0-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}\sup_{s\in[\varsigma_0-\varepsilon,\varsigma_0]}\{s^{2\beta(\alpha-\delta)}|(u(s)|_{H^{\delta,r}}|\nu(s)|_{H^{\delta,r}}\}\to 0, \; as \; \varepsilon\to 0
\end{aligned}$$

From the characteristics of β-function, we find that

$$\Big|\phi(u(\varsigma),\nu(\varsigma))-\phi(u(\varsigma_0),\nu(\varsigma_0))\Big|_{H^{\delta,r}} \to 0 \; as \; \varsigma \to \varsigma_0.$$

The continuous operator $\phi(u,\nu)$ is calculated in $\mathcal{C}((0,\infty),H^{\alpha,r})$. The same conversation as before follows. So, we skip the explanation.

Step II

This must prove that $\phi : X_\infty \times X_\infty \to X_\infty$ is a bilinear, as well as continuous, operator. By Lemma 6, we have

$$\begin{aligned}
|\phi(u(\varsigma),\nu(\varsigma))|_{H^{\delta,r}} &\le \left|\int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)F(u(s),\nu(s)ds\right|_{H^{\alpha\varsigma,r}}\\
&\le \mathcal{C}_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}|F(u(s),\nu(s)|_r ds\\
&\le \mathcal{M}\mathcal{C}_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds\sup_{s\in[0,\varsigma]}\left\{s^{2\beta(\alpha-\delta)}|u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\delta,r}}\right\}\\
&= \mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\delta),1-2\beta(\alpha-\delta))||u||_{X_\infty}||\nu||_{X_\infty}.
\end{aligned}$$

In addition to

$$\begin{aligned}
|\phi(u(\varsigma),\nu(\varsigma))|_{H^{\alpha,r}} &\le \left|\int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)F(u(s),\nu(s))ds\right|_{H^{\alpha,r}}\\
&\le \mathcal{C}_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\alpha)-1}|F(u(s),\nu(s)|_r\, ds\\
&\le \mathcal{M}\mathcal{C}_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\alpha)-1}s^{-2\beta(\alpha-\delta)}ds\sup_{s\in[0,\varsigma]}\left\{s^{2\beta(\alpha-\delta)}|u(s)|_{H^{\alpha,r}}|\nu(s)|_{H^{\alpha,r}}\right\}\\
&= \mathcal{M}\mathcal{C}_1\varsigma^{-\beta(\alpha-\delta)}\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))||u||_{X_\infty}||\nu||_{X_\infty}.
\end{aligned}$$

Hence,

$$\sup_{\varsigma\in[0,\infty)}\varsigma^{\beta(\alpha-\delta)}|\phi(u(\varsigma),\nu(\varsigma))|_{H^{\alpha,r}}\le\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))||u||_{X_\infty}||\nu||_{X_\infty}.$$

To be more accurate,

$$\lim_{\varsigma\to\varsigma_0}\varsigma^{\beta(\alpha-\delta)}|\phi(u(\varsigma),\nu(\varsigma))|_{H^{\alpha,r}}=0.$$

Thus, $\phi(u,\nu)$ belongs to X_∞, and $||\phi(u(\varsigma),\nu(\varsigma))||_{X_\infty}\le L||u||_{X_\infty}||\nu||_{X_\infty}$.
Step III
Let $0\le\varsigma\le\varsigma_0$. Since

$$\begin{aligned}
&|\eta(\varsigma)-\eta(\varsigma_0)|_{H^{\delta,r}}\\
\le\ &\int_{\varsigma_0}^{\varsigma}(\varsigma-s)^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)P(\frac{-\sigma B_0^2}{\rho}u(s))|_{H^{\delta,r}}ds\\
+\ &\int_0^{\varsigma_0}((\varsigma_0-s)^{\beta-1}-(\varsigma-s))^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)P(\frac{-\sigma B_0^2}{\rho}u(s))|_{H^{\delta,r}}ds\\
+\ &\int_0^{\varsigma_0-\varepsilon}(\varsigma_0-s)^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)-E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)P(\frac{-\sigma B_0^2}{\rho}u(s))|_{H^{\delta,r}}ds\\
+\ &\int_{\varsigma_0-\varepsilon}^{\varsigma_0}(\varsigma_0-s)^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)-E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)P(\frac{-\sigma B_0^2}{\rho}u(s))|_{H^{\delta,r}}ds\\
\le\ &\mathcal{C}_1\int_{\varsigma_0}^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}|P(\frac{-\sigma B_0^2}{\rho}u(s))|_r ds\\
+\ &\mathcal{C}_1\int_0^{\varsigma_0}\left((\varsigma_0-s)^{\beta-1}-(\varsigma-s)^{\beta-1}\right)(\varsigma-s)^{-\beta\delta}|P(\frac{-\sigma B_0^2}{\rho}u(s))|_r ds\\
+\ &\mathcal{C}_1\int_0^{\varsigma_0-\varepsilon}(\varsigma_0-s)^{\beta-1}|E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)-E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)P(\frac{-\sigma B_0^2}{\rho}u(s))|_{H^{\delta,r}}ds
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\mathcal{C}_1 \int_{\varsigma_0-\varepsilon}^{\varsigma_0} (\varsigma_0 - s)^{\beta(1-\delta)-1} |P(\frac{-\sigma B_0^2}{\rho} u(s))|_r ds \\
&\leq \quad \mathcal{M}(\varsigma)\mathcal{C}_1 \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\beta(1-\delta)-1} s^{-\beta(1-\delta)} ds \\
&+ \quad \mathcal{M}(\varsigma)\mathcal{C}_1 \int_0^{\varsigma_0} ((\varsigma_0 - s)^{\beta-1} - (\varsigma - s)^{\beta-1})(\varsigma - s)^{-\beta\delta} s^{-\beta(1-\delta)} ds \\
&+ \quad \mathcal{M}(\varsigma)\mathcal{C}_1 \int_0^{\varsigma_0-\varepsilon} (\varsigma_0 - s)^{\beta-1} |E_{\beta,\beta}(-(\varsigma - s)^{\beta} A) - E_{\beta,\beta}(-(\varsigma_0 - s)^{\beta} A) s^{-\beta(1-\delta)} ds \\
&+ \quad 2\mathcal{M}(\varsigma)\mathcal{C}_1 \int_{\varsigma_0-\varepsilon}^{\varsigma_0} (\varsigma_0 - s)^{\beta(1-\delta)-1} s^{-\beta(1-\delta)} ds.
\end{aligned}
$$

From the result of Lemma 1 as well as the property of β function, the 1st, 2nd, 3rd and final integral approaches to 0 as $\varsigma \to \varsigma_0$ as ε tends to 0, which suggests

$$
|\eta(\varsigma) - \eta(\varsigma_0)|_{H^{\delta,r}} \to 0 \; as \; \varsigma \to \varsigma_0.
$$

We evaluated that $\eta(\varsigma)$ is continuous in $H^{\alpha,r}$, which is implied by the similar prior explanation.

$$
\begin{aligned}
|\eta(\varsigma)|_{H^{\delta,r}} &\leq \left| \int_0^{\varsigma} (\varsigma - s)^{\beta-1} E_{\beta,\beta}(-(\varsigma - s)^{\beta} A) P(\frac{-\sigma B_0^2}{\rho} u(s)) ds \right|_{H^{\delta,r}} \\
&\leq \mathcal{C}_1 \int_0^{\varsigma} (\varsigma - s)^{\beta(1-\delta)-1} \left| P(\frac{-\sigma B_0^2}{\rho} u(s)) \right|_r ds \\
&\leq \mathcal{M}(\varsigma)\mathcal{C}_1 \int_0^{\varsigma} (\varsigma - s)^{\beta(1-\delta)-1} s^{-\beta(1-\delta)} ds \\
&= \mathcal{M}(\varsigma)\mathcal{C}_1 \mathcal{B}(\beta(1-\delta), 1 - \beta(1-\delta)),
\end{aligned} \tag{7}
$$

and also

$$
\begin{aligned}
|\eta(\varsigma)|_{H^{\alpha,r}} &\leq \left| \int_0^{\varsigma} (\varsigma - s)^{\beta-1} E_{\beta,\beta}(-(\varsigma - s)^{\beta} A) P(\frac{-\sigma B_0^2}{\rho} u(s)) ds \right|_{H^{\alpha,r}} \\
&\leq \mathcal{C}_1 \int_0^{\varsigma} (\varsigma - s)^{\beta(1-\alpha)-1} \left| P(\frac{-\sigma B_0^2}{\rho} u(s)) \right|_r ds \\
&\leq \mathcal{M}(\varsigma)\mathcal{C}_1 \int_0^{\varsigma} (\varsigma - s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds \\
&= \varsigma^{-\beta(\alpha-\delta)} \mathcal{C}_1 \mathcal{M}(\varsigma) \mathcal{B}(\beta(1-\alpha), 1 - \beta(1-\delta)).
\end{aligned}
$$

Specifically,

$$
\varsigma^{\beta(\alpha-\delta)} |\eta(\varsigma)|_{H^{\alpha,r}} \leq \mathcal{M}(\varsigma)\mathcal{C}_1 \mathcal{B}(\beta(1-\alpha), 1 - \beta(1-\delta)) \to 0, \;\; as \;\; \varsigma \to 0.
$$

As we know, if $\varsigma \to 0$, then $\mathcal{M}(\varsigma) \to 0$, as a result of an assumption (e). This guarantees that $\eta(\varsigma) \in X_\infty$ and $||\eta(\varsigma)||_\infty \leq \mathcal{B}_1 \mathcal{M}_\infty$. For $\frac{a\varsigma}{\sqrt{2}} \in H^{\delta,r}$. From the statement of Lemma 1, it is simple to see this

$$
\begin{aligned}
E_\beta(-\varsigma^\beta A)\frac{a}{\sqrt{2}} &\in \mathcal{C}([0,\infty), H^{\delta,r}) \\
E_\beta(-\varsigma^\beta A)\frac{a}{\sqrt{2}} &\in \mathcal{C}((0,\infty), H^{\alpha,r}).
\end{aligned}
$$

Therefore, also

$$\begin{aligned} \int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in \mathcal{C}([0,\infty),H^{\delta,r}) \\ \int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in \mathcal{C}((0,\infty),H^{\alpha,r}). \end{aligned}$$

Lemma 6 suggests that $\forall\varsigma \in (0,\tilde{\Im}]$,

$$\begin{aligned} \int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in X_\infty, \\ \varsigma^{\beta(\alpha-\delta)}\int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds &\in \mathcal{C}((0,\infty),H^{\alpha,r}), \\ \left\|\int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds\right\|_{X_\infty} &\leq \int_0^{\varsigma}\left\|E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds\right\|_{X_\infty}, \\ &\leq \int_0^{t} C_1\left|\frac{a}{\sqrt{2}}\right|_{H^{\delta,r}}ds, \\ &= C_1\left|\frac{a\varsigma}{\sqrt{2}}\right|_{H^{\delta,r}}. \end{aligned}$$

With the help of Equation (5), the inequality gives

$$\begin{aligned} \left\|\int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds+\eta(\varsigma)\right\|_{X_\infty} &\leq \left\|\int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds\right\|_{X_\infty}+\left\|\eta(\varsigma)\right\|_{X_\infty} \\ &\leq C_1\left|\frac{a\varsigma}{\sqrt{2}}\right|_{H^{\delta,r}}+\mathcal{B}_1\mathcal{M}_\infty \leq \frac{1}{4L} \end{aligned}$$

which is showing the result that F has a unique and special fixed point.

Step IV

For the purpose of demonstrating that $u(\varsigma) \to \frac{a\varsigma}{\sqrt{2}}$ in $H^{\delta,r}$ by assigning $\varsigma \to 0$. We must demonstrate this as

$$\begin{aligned} \lim_{\varsigma\to 0}\xi(\varsigma) &= \lim_{\varsigma\to 0}\frac{a}{\sqrt{2}}\int_0^{\varsigma} E_\beta(-\varsigma^\beta A)ds = 0, \\ \lim_{\varsigma\to 0}\eta(\varsigma) &= \lim_{\varsigma\to 0}\int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)P\frac{-\sigma B_0^2}{\rho}u(s)ds = 0, \\ \lim_{\varsigma\to 0}\phi(u,v)(\varsigma) &= \lim_{\varsigma\to 0}\int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)F(u(s),u(s))ds = 0 \end{aligned}$$

in $H^{\delta,r}$. It is understood that $\lim_{\varsigma\to\varsigma_0}\eta(\varsigma) = 0$ and $\lim_{\varsigma\to\varsigma_0}\mathcal{M}(\varsigma) = 0$ with Equation (7). Additionally,

$$\begin{aligned} &\left|\int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)F(u(s),u(s))ds\right|_{H^{\delta,r}} \\ \leq\ & C_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}\left|F(u(s),u(s))\right|_r ds \\ \leq\ & \mathcal{M}C_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}|u(s)|^2_{H^{\alpha,r}}ds \\ \leq\ & \mathcal{M}C_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}|u(s)|^2_{H^{\alpha,r}}ds \\ \leq\ & \mathcal{M}C_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\delta)-1}s^{-2\beta(\alpha-\delta)}ds\sup_{s\in(0,\varsigma]}\left\{s^{2\beta(\alpha-\delta)}|u(s)|^2_{H^{\alpha,r}}\right\} \\ =\ & \mathcal{M}C_1\mathcal{B}(\beta(1-\delta),1-2\beta(\alpha-\delta))\sup_{s\in(0,\varsigma]}\left\{s^{2\beta(\alpha-\delta)}|u(s)|^2_{H^{\alpha,r}}\right\} \to 0 \ \ as \ \ \varsigma\to 0. \end{aligned}$$

□

5. Local Existence in $H^{\delta,r}$

This section is further divided into the $H^{\delta,r}$ local mild solution to problem (2).

Theorem 2. *Let $1 < r < \infty, 0 < \delta < 1$ and (e) exists. Let us suppose that*

$$\frac{n}{2r} - \frac{1}{2} < \delta. \tag{8}$$

The existence of a function $\alpha > max\{\delta, \frac{1}{2}\}$ knowing that for all $\frac{a\varsigma}{\sqrt{2}} \in H^{\delta,r} \exists \tilde{\Im}_* > 0$ and a distinctive continuous function $u : [0, \tilde{\Im}_*] \to H^{\delta,r}$ such that

(i) *A continuous mapping in $H^{\delta,r}$ is defined as $u : [0, \tilde{\Im}_*] \to H^{\delta,r}$ with $u(0) = \dfrac{at}{\sqrt{2}}$;*
(ii) *A continuous mapping in $H^{\alpha,r}$ is defined as $u : (0, \tilde{\Im}_*] \to H^{\alpha,r}$ with limiting value of function $\lim_{\varsigma\to 0} \varsigma^{\beta(\alpha-\delta)} |u(\varsigma)|_{H^{\alpha,r}} = 0$;*
(iii) *u holds (4) for $\varsigma \in [0, \tilde{\Im}_*]$.*

Proof. Take $\alpha = \dfrac{1+\delta}{2}$. Additionally, value $\dfrac{a\varsigma}{\sqrt{2}} \in H^{\delta,r}$. Let us define $X_{\tilde{\Im}}$ as the space of curves in such a way that $u : (0, \tilde{\Im}] \to H^{\delta,r}$, moreover $X_{\tilde{\Im}} = X[\tilde{\Im}]$:
(a^*) A continuous mapping is defined as $u : [0, \tilde{\Im}] \to H^{\delta,r}$;
(b^*) A continuous mapping with limiting value of function is defined as $u : (0, \tilde{\Im}] \to H^{\alpha,r}$, with $\lim_{\varsigma\to 0} \varsigma^{\beta(\alpha-\delta)} |u(\varsigma)|_{H^{\alpha,r}} = 0$,
with a norm defined by

$$||u||_X = \sup_{\varsigma\in[0,\tilde{\Im}]} \left\{ \varsigma^{\beta(\alpha-\delta)} |u(\varsigma)|_{H^{\alpha,r}} \right\}.$$

From the proof of Theorem 1, we notice that the operator ϕ is continuous and it is a linear map $\phi : X \times X \to X$, and $\eta(\varsigma) \in X$. From Lemma 1, we can claim $\forall\, \varsigma \in (0, \tilde{\Im}]$,

$$\begin{aligned}
E_\beta(-\varsigma^\beta A)\frac{a}{\sqrt{2}} &\in C([0, \tilde{\Im}], H^{\delta,r}), \\
E_\beta(-\varsigma^\beta A)\frac{a}{\sqrt{2}} &\in C([0, \tilde{\Im}], H^{\alpha,r}), \\
\int_0^\varsigma E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}} ds &\in C([0, \tilde{\Im}, H^{\delta,r}), \\
\int_0^\varsigma E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}} ds &\in C((0, \tilde{\Im}, H^{\alpha,r}).
\end{aligned}$$

Hence, from the previous Lemma 6, this yields that

$$\begin{aligned}
\int_0^\varsigma E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}} ds &\in X, \\
\varsigma^{\beta(\alpha-\delta)} \int_0^\varsigma E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}} ds &\in C([0, \tilde{\Im}], H^{\alpha,r}).
\end{aligned}$$

$$\begin{aligned}
\left|\left| \int_0^\varsigma E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}} ds \right|\right|_X &\leq \int_0^\varsigma \left|\left| E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}} ds \right|\right|_X, \\
&\leq \int_0^\varsigma C_1 |\frac{a}{\sqrt{2}}|_{H^{\delta,r}} ds, \\
&= C_1 |\frac{a\varsigma}{\sqrt{2}}|_{H^{\delta,r}}.
\end{aligned}$$

With the help of Equation (5), the inequality gives us

$$\begin{aligned}\left|\left|\int_0^{\varsigma} E_{\beta}(-(\varsigma-s)^{\beta}A)\frac{a}{\sqrt{2}}ds+\eta(\varsigma)\right|\right|_{X[\tilde{\Im}_*]} &\leq \left|\left|\int_0^{\varsigma} E_{\beta}(-(\varsigma-s)^{\beta}A)\frac{a}{\sqrt{2}}ds\right|\right|_{X[\tilde{\Im}_*]}+\left|\left|\eta(\varsigma)\right|\right|_{X[\tilde{\Im}_*]}\\ &\leq \mathcal{C}_1\left|\frac{a\varsigma}{\sqrt{2}}\right|_{H^{\delta,r}}+\mathcal{B}_1\mathcal{M}_{\infty}\leq\frac{1}{4L},\end{aligned}$$

which gives the result that $\mathcal{F}$ has a unique fixed point due to Lemma 4. □

6. Existence Locally in J_r

We are using the Iteration method in this part intended for thinking of the local existence of a mild solution to problem (2) in J_r. Let $\alpha=\frac{1+\delta}{2}$.

Theorem 3. *Suppose $1<r<\infty$, $0<\delta<1$ and (e) holds. Let us take value in $H^{\delta,r}$*

$$\frac{a\varsigma}{\sqrt{2}}\in H^{\delta,r}\ \text{with}\ \frac{n}{2r}-\frac{1}{2}<\delta.$$

So, the mild solution of (2) is a unique answer for $\frac{a\varsigma}{\sqrt{2}}\in H^{\delta,r}$ in J_r. Additionally, $\varsigma^{\beta(\alpha-\delta)}A^{\alpha}u(\varsigma)$ is bounded as $\varsigma\to 0$. Moreover, u and $A^{\alpha}u$ are both continuous function in $[0,\tilde{\Im}]$ and $(0,\tilde{\Im}]$, respectively.

Proof. Step I

$$\tilde{\kappa}(\varsigma):=\sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^{\alpha}u(s)|_r$$

also

$$\tilde{\psi}(\varsigma):=\phi(u,u)(\varsigma) = \int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)F(u(s),u(s))ds.$$

The instant results from Step II in Theorem 1 reveal that A is continuous functions $\tilde{\psi}(\varsigma)$, and $A^{\alpha}\tilde{\psi}(\varsigma)$ exists in $[0,\tilde{\Im}]$ and $(0,\tilde{\Im}]$, respectively, with value

$$\begin{aligned}|A^{\alpha}\tilde{\psi}(\varsigma)|_r &\leq \left|\int_0^{\varsigma}A^{\alpha}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)F(u(s),u(s))ds\right|_r\\ &\leq \mathcal{C}_1\int_0^{\varsigma}A^{\alpha}(\varsigma-s)^{\beta(1-\alpha)-1}|F(u(s),u(s))|_r ds\\ &\leq \mathcal{M}\mathcal{C}_1\int_0^{\varsigma}A^{\alpha}(\varsigma-s)^{\beta(1-\alpha)-1}s^{-2\beta(\alpha-\delta)}\sup_{s\in(0,\varsigma]}s^{2\beta(\alpha-\delta)}|u(s)|_r|u(s)|_r ds\\ &= \mathcal{M}\mathcal{C}_1\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\alpha)-1}s^{-2\beta(\alpha-\delta)}ds\{\sup_{s\in(0,\varsigma]}A^{\alpha}s^{2\beta(\alpha-\delta)}|(u(s)|^2\}\\ |A^{\alpha}\tilde{\psi}(\varsigma)|_r &\leq \mathcal{M}\mathcal{C}_1\varsigma^{-\beta(\alpha-\delta)}\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}^2(\varsigma).\end{aligned} \tag{9}$$

We also take into account the integral $\eta(\varsigma)$. Given that (e) is true,

$$\left|P(\frac{-\sigma B_0^2}{\rho}u(s))\right|_r\leq\mathcal{M}(\varsigma)s^{\beta(1-\delta)}$$

the above inequality with a continuous function $\mathcal{M}(\varsigma)$ holds. Theorem 1's third stage reveals that $A^{\alpha}\eta(\varsigma)$ is continuous in $(0, \tilde{\Im}]$, with results

$$
\begin{aligned}
|A^{\alpha}\eta(\varsigma)|_r &= \left| \int_0^{\varsigma} (\varsigma-s)^{\beta-1} A^{\alpha} E_{\beta,\beta}(-(\varsigma-s)^{\beta}A) P(\frac{-\sigma B_0^2}{\rho} u(s)) \right|_{H^{\alpha,r}} ds \\
&\leq \mathcal{C}_1 \int_0^{\varsigma} (\varsigma-s)^{\beta(1-\alpha)-1} \left|A^{\alpha} P(\frac{-\sigma B_0^2}{\rho} u(s))\right|_r ds \\
&\leq \mathcal{C}_1 \mathcal{M}(\varsigma) \int_0^{\varsigma} (\varsigma-s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds \\
&= \varsigma^{-\beta(\alpha-\delta)} \mathcal{C}_1 \mathcal{M}(\varsigma) \mathcal{B}(\beta(1-\delta), 1-\beta(1-\delta)) \\
|A^{\alpha}\eta(\varsigma)|_r &\leq \varsigma^{-\beta(\alpha-\delta)} \mathcal{C}_1 \mathcal{M}(\varsigma) \mathcal{B}(\beta(1-\delta), 1-\beta(1-\delta)).
\end{aligned} \tag{10}
$$

For $|P(\dfrac{-\sigma B_0^2}{\rho} u(\varsigma))|_r = 0(\varsigma^{-\beta(1-\delta)})$ as ς tends to zero, and $\mathcal{M}(\varsigma) = 0$. The above Equation (10) concludes that $|A^{\alpha}\eta(\varsigma)|_r = 0(\varsigma^{-\beta(\alpha-\delta)})$, ς approaches to 0. We establish the continuity of η in J_r. In actuality, we take $0 < \varsigma_0 < \varsigma < \tilde{\Im}$, resulting in

$$
\begin{aligned}
|\eta(\varsigma) - \eta(\varsigma_0)|_r \leq\ & \mathcal{C}_3 \int_{\varsigma_0}^{\varsigma} (\varsigma-s)^{\beta-1} \left|P \frac{(-\sigma B_0^2 u(s))}{\rho}\right|_r ds \\
+\ & \mathcal{C}_3 \int_0^{\varsigma_0} ((\varsigma_0-s)^{\beta-1} - (\varsigma-s))^{\beta-1} \left|P \frac{(-\sigma B_0^2 u(s))}{\rho}\right|_r ds \\
+\ & \mathcal{C}_3 \int_0^{\varsigma_0-\varepsilon} (\varsigma_0-s)^{\beta-1} ||E_{\beta,\beta}(-(\varsigma-s)^{\beta}A) - E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)||\, |P \frac{(-\sigma B_0^2 u(s))}{\rho}|_r ds \\
+\ & 2\mathcal{C}_3 \int_{\varsigma_0-\varepsilon}^{\varsigma_0} (\varsigma_0-s)^{\beta-1} |P \frac{(-\sigma B_0^2 u(s))}{\rho}|_r ds \\
\leq\ & \mathcal{C}_3 \mathcal{M}(\varsigma) \int_{\varsigma_0}^{\varsigma} (\varsigma-s)^{\beta-1} s^{-\beta(1-\delta)} ds \\
+\ & \mathcal{C}_3 \mathcal{M}(\varsigma) \int_0^{\varsigma_0} ((\varsigma-s)^{\beta-1} - (\varsigma_0-s)^{\beta-1})\, s^{-\beta(1-\delta)} ds \\
+\ & \mathcal{C}_3 \mathcal{M}(\varsigma) \int_0^{\varsigma_0-\varepsilon} (\varsigma_0-s)^{\beta-1} s^{-\beta(1-\delta)} ds \sup_{s\in[0,\varsigma-\varepsilon]} ||E_{\beta,\beta}(-(\varsigma-s)^{\beta}A) - E_{\beta,\beta}(-(\varsigma_0-s)^{\beta}A)|| \\
+\ & 2\mathcal{C}_3 \mathcal{M}(\varsigma) \int_{\varsigma_0-\varepsilon}^{\varsigma_0} (\varsigma_0-s)^{\beta-1} s^{-\beta(1-\delta)} ds \ \to\ 0,\ \ as\ \varsigma\ \to\ \varsigma_0.
\end{aligned}
$$

Additionally, we think about the function $\int_0^{\varsigma} E_{\beta}(-(\varsigma-s)^{\beta}A) \dfrac{a}{\sqrt{2}}\, ds$. It is clear from the Lemma 6 that

$$
\begin{aligned}
\left| \int_0^{\varsigma} A^{\alpha} E_{\beta}(-(\varsigma-s)^{\beta}A) \frac{a}{\sqrt{2}} ds \right|_r &\leq \int_0^{\varsigma} \left| A^{\alpha} E_{\beta}(-(\varsigma-s)^{\beta}A) \frac{a}{\sqrt{2}} \right|_r ds \\
&\leq \mathcal{C}_1 \varsigma^{-\beta(\alpha-\delta)} A^{\delta} \left| \int_0^{\varsigma} \frac{a}{\sqrt{2}} ds \right|_r \\
&= \mathcal{C}_1 \varsigma^{-\beta(\alpha-\delta)} \left| \frac{a}{\sqrt{2}} \varsigma \right|_{H^{\delta,r}} \\
& \lim_{\varsigma\to 0} \varsigma^{\beta(\alpha-\delta)} \left| \int_0^{\varsigma} A^{\alpha} E_{\beta}(-(\varsigma-s)^{\beta}A) \frac{a}{\sqrt{2}} ds \right|_r, \\
&= \lim_{\varsigma\to 0} \varsigma^{\beta(\alpha-\delta)} \left| \int_0^{\varsigma} E_{\beta}(-(\varsigma-s)^{\beta}A) \frac{a}{\sqrt{2}} ds \right|_{H^{\alpha,r}} \\
&= 0.
\end{aligned}
$$

Step II

Using the successive approximation method, we now arrive at the following solution:

$$\begin{aligned} u_0(\varsigma) &= \int_0^{\varsigma} E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}ds + \eta(\varsigma), \\ u_{n+1}(\varsigma) &= u_0(\varsigma) + \phi(u_n,u_n)(\varsigma),\ n = 0,1,2,.... \end{aligned} \tag{11}$$

Using the results above, we have that $\tilde{\kappa}_n(\varsigma) := \sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^\alpha u_n(s)|_r$ are continuous and increasing in $[0,\tilde{\Im}]$ with value $\tilde{\kappa}_n(0) = 0$. However, given (9) and (10), and that the inequality is satisfied by $\tilde{\kappa}_n(\varsigma)$,

$$\tilde{\kappa}_{n+1}(\varsigma) \le \tilde{\kappa}_0(\varsigma) + \mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_n^2(\varsigma). \tag{12}$$

For $\tilde{\kappa}_0(0) = 0$, select $\tilde{\Im} > 0$,

$$4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\tilde{\Im}) < 1. \tag{13}$$

Therefore, it is assured that the sequence $\tilde{\kappa}_n(\tilde{\Im})$ is constrained by the fundamental consideration (12), i.e.,

$$\tilde{\kappa}_n(\tilde{\Im}) \le \tilde{\rho}(\tilde{\Im}),\ n = 0,1,2,....$$

There are

$$\tilde{\rho}(\varsigma) = \frac{1-\sqrt{1-4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\varsigma)}}{2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))}.$$

It is true that $\tilde{\kappa}n(\varsigma) \le \tilde{\rho}(\varsigma)$ holds for any value of $\varsigma \in (0,\tilde{\Im}]$, similarly to how we say $\tilde{\rho}(\varsigma) \le 2\tilde{\kappa}_0(\varsigma)$.

Let us think about the equality

$$z_{n+1}(\varsigma) = \int_0^{\varsigma} (\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)[F(u_{n+1}(s),u_{n+1}(s) - F(u_n(s),u_n(s))]ds,$$

for $\varsigma \in (0,\tilde{\Im}]$ and $z_n = u_{n+1} - u_n,\ n = 0,1,2,...$ by writing

$$Z_n(\varsigma) := \sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^\alpha z_n(s)|_r.$$

In light of (6), there are

$$\begin{aligned} \big|F(u_{n+1}(s),u_{n+1}(s) - F(u_n(s),u_n(s))\big|_r &\le \mathcal{M}\big(|u_{n+1}|_{H^{\alpha,r}} + |u_n|_{H^{\alpha,r}}\big)\big|u_{n+1}-u_n\big| \\ &= \mathcal{M}\big(|u_{n+1}|_{H^{\alpha,r}} + |u_n|_{H^{\alpha,r}}\big)A^\alpha z_n \sup_{s\in(0,\varsigma]} s^{-\beta(\alpha-\delta)}s^{\beta(\alpha-\delta)} \\ &= \mathcal{M}(|A^\alpha u_{n+1}|_r + |A^\alpha u_n|_r)z_n \sup_{s\in(0,\varsigma]} s^{-\beta(\alpha-\delta)}s^{\beta(\alpha-\delta)} \\ &= \mathcal{M}\Bigg(\sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^\alpha u_{n+1}|_r \\ &+ \sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^\alpha u_n|_r\Bigg)z_n s^{-\beta(\alpha-\delta)} \\ &\le \mathcal{M}(\tilde{\kappa}_{n+1}+\tilde{\kappa}_n)z_n s^{-\beta(\alpha-\delta)} \sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}s^{-\beta(\alpha-\delta)}A^\alpha \\ &\le \mathcal{M}(\tilde{\kappa}_{n+1}+\tilde{\kappa}_n) \sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^\alpha z_n(s)|_r s^{-2\beta(\alpha-\delta)} \\ \big|F(u_{n+1}(s),u_{n+1}(s) - F(u_n(s),u_n(s))\big|_r &\le \mathcal{M}(\tilde{\kappa}_{n+1}+\tilde{\kappa}_n)Z_n(s)s^{-2\beta(\alpha-\delta)}. \end{aligned}$$

It is implied by step II in Theorem 1 that

$$\varsigma^{\beta(\alpha-\delta)}|A^{\alpha}z_{n+1}(\varsigma)|_r \leq 2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-\beta(1-\delta))\tilde{\rho}(\tilde{\Im})Z_n(\varsigma).$$

Such inequality results in

$$\begin{aligned} Z_{n+1}(\tilde{\Im}) &\leq 2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\rho}(\tilde{\Im})Z_n(\tilde{\Im}) \\ &\leq 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\tilde{\Im})Z_n(\tilde{\Im}). \end{aligned} \tag{14}$$

As per (13) with (14), it is indeed obvious that

$$\lim_{n\to 0}\frac{Z_{n+1}(\tilde{\Im})}{Z_n(\tilde{\Im})} < 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\delta))\tilde{\kappa}_0(\tilde{\Im}) < 1.$$

Thus, convergence of the series $\Sigma_{n=0}^{\infty}Z_n(\tilde{\Im})$ implies the uniform convergence of series $\Sigma_{n=0}^{\infty}\varsigma^{\beta(\alpha-\delta)}A^{\alpha}z_n(\varsigma)$ for $\varsigma \in (0,\tilde{\Im}]$; the uniform convergence of sequence $\{\varsigma^{\beta(\alpha-\delta)}A^{\alpha}u_n(\varsigma)\}$ holds in $(0,\tilde{\Im}]$. So,

$$\lim_{n\to\infty}u_n(\varsigma) = u(\varsigma) \in D(A^{\alpha})$$

and

$$\lim_{n\to\infty}\varsigma^{\beta(\alpha-\delta)}A^{\alpha}u_n(\varsigma) = \varsigma^{\beta(\alpha-\delta)}A^{\alpha}u(\varsigma)\ uniformly.$$

From the boundedness theorem, "A function f continuous on a bounded and closed interval is necessarily a bounded function". So, both $A^{-\alpha}$ and A^{α} are bound and closed, respectively. The function $\tilde{\kappa}(\varsigma) = \sup\limits_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^{\alpha}u(s)|_r$ satisfies

$$\tilde{\kappa}(\varsigma) \leq \tilde{\rho}(\varsigma) \leq 2\tilde{\kappa}_0(\varsigma),\ \varsigma \in (0,\varsigma], \tag{15}$$

as well as

$$\begin{aligned} \varsigma_n &= \sup_{s\in(0,\tilde{\Im}]} s^{2\beta(\alpha-\delta)}|F(u_n(s),u_n(s)-F(u(s),u(s))|_r \\ &\leq \mathcal{M}(\tilde{\kappa}_n(\tilde{\Im})+\tilde{\kappa}(\tilde{\Im}))\sup_{s\in(0,\tilde{\Im}]} s^{\beta(\alpha-\delta)}|A^{\alpha}(u_n(s)-u(s))|_r \ \to\ 0,\ as\ n\to\ \infty. \end{aligned}$$

To finalize this step, it is important to verify that u is a suitable solution to the issue (2) in the range of $[0,\tilde{\Im}]$.

$$|\phi(u_n,u_n)(\varsigma)-\phi(u,u)(\varsigma)|_r \leq \int_0^{\varsigma}(\varsigma-s)^{\beta-1}\varsigma_n s^{-2\beta(\alpha-\delta)}ds = \varsigma^{\beta\delta}\varsigma_n\ \to\ 0,\ (n\to\infty).$$

To put it another way, we get $\phi(u_n,u_n)(\varsigma)-\phi(u,u)(\varsigma)$. Limits are taken on both sides of (10), and we conclude

$$u(\varsigma) = u_0(\varsigma)+\phi(u,u)(\varsigma). \tag{16}$$

Suppose $u(0) = \frac{a\varsigma}{\sqrt{2}}$; what we learn is that (16) is true for both $\varsigma \in [0,\tilde{\Im}]$ and $u \in \mathcal{C}([0,\tilde{\Im}],J_r)$. Additionally, the continuity of $A^{\alpha}u(\varsigma)$ in $(0,\tilde{\Im}]$ is derived from the uniform convergence of $\varsigma^{\beta(\alpha-\delta)}A^{\alpha}u_n(\varsigma)$ to $\varsigma^{\beta(\alpha-\delta)}A^{\alpha}u(\varsigma)$. We conclude that $|A^{\alpha}u(\varsigma)|_r = 0(\varsigma^{-\beta(\alpha-\delta)})$ is clear from (15) and $\tilde{\kappa}_0(\varsigma) = 0$.

Step III

We illustrate the difference between "mild solutions." Assume that the issue has mild solutions in u and v (2). Suppose $z = u - v$; consider the inequality.

$$z(\varsigma) = \int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)[F(u(s),u(s)-F(v(s),v(s))]ds.$$

Expressing the solution

$$\tilde{\kappa}(\varsigma) := max\{\sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^{\alpha}(u(s))|_r, \sup_{s\in(0,\varsigma]} s^{\beta(\alpha-\delta)}|A^{\alpha}(\nu(s))|_r\}.$$

From (5) and Lemma 6, we have the inequality

$$|A^{\alpha}z(\varsigma)|_r \leq \mathcal{M}\mathcal{C}_1\tilde{\kappa}(\varsigma)\int_0^{\varsigma}(\varsigma-s)^{\beta(1-\alpha)-1}s^{-\beta(\alpha-\delta)}|A^{\alpha}z(s)|_r ds.$$

The Gronwall inequality shows that $\varsigma \in (0,\tilde{\Im}]$, $A^{\alpha}z(\varsigma) = 0$, which indicates it for $\varsigma \in [0,\tilde{\Im}]$, $z(\varsigma) = u(\varsigma) - \nu(\varsigma) \equiv 0$. As a result, the mild solution is unique. □

7. Regularity

In this Section (2), the regularity of "the solution u that" resolves the issue is studied. In this essay, we'll assume: (e_1) with an exponent $\vartheta \in (0, \beta(1-\alpha))$, $P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})$ is Hölder-continuous, that is,

$$\begin{aligned} |f(x)-f(y)| &\leq ||x-y||^{\beta}, \\ \left|P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - P(\frac{-\sigma B_0^2 u(s)}{\rho})\right|_r &\leq L|\varsigma - s|^{\vartheta}, \forall 0<\varsigma, s \leq \tilde{\Im}. \end{aligned}$$

Definition 5. *An expression $u : [0,\tilde{\Im}] \to J_r$. If $u \in \mathcal{C}([0,\tilde{\Im}], J_r)$ with ${}^C D_{\varsigma}^{\beta} u(\varsigma) \in \mathcal{C}((0,\tilde{\Im}], Jr)$, which accepts values in $D(A)$ and solves (2) for every $\varsigma \in (0,\tilde{\Im}]$, then J_r is referred to as the classical solution of (2).*

Lemma 7. *Assume (e_1) be satisfied. If*

$$\eta_1(\varsigma) := \int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)\left(P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})\right)ds, \quad \forall\, \varsigma \in (0,\tilde{\Im}],$$

so, $\eta_1(\varsigma) \in D(A)$ and, also, $A\eta_1(\varsigma) \in \mathcal{C}^{\vartheta}([0,\tilde{\Im}], J_r)$.

Proof. To be fixed $\varsigma \in (0,\tilde{\Im}]$. Let us think about

$$(\varsigma-s)^{\beta-1}\left|AE_{\beta,\beta}(-(\varsigma-s)^{\beta}A)\left(P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})\right)\right|_r.$$

Lemma 6 with (e_1), give us

$$\begin{aligned} &(\varsigma-s)^{\beta-1}\left|AE_{\beta,\beta}(-(\varsigma-s)^{\beta}A)\left(P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})\right)\right|_r \\ \leq\ & \mathcal{C}_1(\varsigma-s)^{\beta-1}(\varsigma-s)^{-\beta}\left|P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})\right|_r \\ =\ & \mathcal{C}_1(\varsigma-s)^{-1}\left|P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})\right|_r \\ \leq\ & \mathcal{C}_1 L(\varsigma-s)^{\vartheta-1} \in L^1([0,\tilde{\Im}], J_r). \end{aligned} \tag{17}$$

Then,

$$\begin{aligned} |A\eta_1(\varsigma)|_r &\leq \int_0^{\varsigma}(\varsigma-s)^{\beta-1}\left|AE_{\beta,\beta}(-(\varsigma-s)^{\beta}A)\left(P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})\right)\right|_r ds \\ &\leq \mathcal{C}_1 L\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}ds \leq \frac{\mathcal{C}_1 L\varsigma^{\vartheta}}{\vartheta} < \infty. \end{aligned}$$

Since A is closed, we can write $\eta_1(\varsigma) \in D(A)$.
It is necessary to demonstrate that $A\eta_1(\varsigma)$ is Hölder-continuous because

$$\frac{d}{d\varsigma}\left(\varsigma^{\beta-1}E_{\beta,\beta}(-\mu\varsigma^{\beta})\right) = \varsigma^{\beta-2}E_{\beta,\beta-1}(-\mu\varsigma^{\beta}).$$

Then,

$$\begin{aligned}
&\frac{d}{d\varsigma}\left(\varsigma^{\beta-1}AE_{\beta,\beta}(-\varsigma^{\beta}A)\right)\\
=\;& \frac{1}{2\pi i}\int_{\Gamma_\theta}\varsigma^{\beta-2}E_{\beta,\beta-1}(-\mu\varsigma^{\beta})A(\mu I+A)^{-1}d\mu\\
=\;& \frac{1}{2\pi i}\int_{\Gamma_\theta}\varsigma^{\beta-2}E_{\beta,\beta-1}(-\mu\varsigma^{\beta})d\mu - \frac{1}{2\pi i}\int_{\Gamma_\theta}\varsigma^{\beta-2}\mu E_{\beta,\beta-1}(-\mu\varsigma^{\beta})(\mu I+A)^{-1}d\mu.
\end{aligned}$$

Put

$$\begin{aligned}
-\mu\varsigma^{\beta} &= \psi\\
-\varsigma^{\beta}d\mu &= d\psi\\
d\mu &= -\frac{d\psi}{\varsigma^{\beta}}\\
&= \frac{1}{2\pi i}\int_{\Gamma'_\theta}-\varsigma^{\beta-2}E_{\beta,\beta-1}(\psi)\frac{1}{\varsigma^{\beta}}d\psi - \frac{1}{2\pi i}\int_{\Gamma'_\theta}\varsigma^{\beta-2}E_{\beta,\beta-1}(\psi)\frac{\psi}{\varsigma^{\beta}}\left(\frac{-\psi}{\varsigma^{\beta}}I+A\right)^{-1}\frac{1}{\varsigma^{\beta}}d\psi,
\end{aligned}$$

in the light of

$$||(\mu I+A)^{-1}|| \;\leq\; \frac{\mathcal{C}}{|\mu|},$$

we derive that

$$\left|\left|\frac{d}{d\varsigma}\left(\varsigma^{\beta-1}AE_{\beta,\beta}(-\varsigma^{\beta}A)\right)\right|\right| \;\leq\; \mathcal{C}_\beta\varsigma^{-2},\ 0<\varsigma<\tilde{\Im}.$$

From MVT, we derive that $\forall\ 0<s<\varsigma\leq\tilde{\Im}$, and we obtain

$$\begin{aligned}
\left|\left|\varsigma^{\beta-1}AE_{\beta,\beta}(-\varsigma^{\beta}A)-s^{\beta-1}AE_{\beta,\beta}(-s^{\beta}A)\right|\right| &= \left|\left|\int_s^{\varsigma}\frac{d}{d\tau}\left(\tau^{\beta-1}AE_{\beta,\beta}(-\tau^{\beta}A)\right)d\tau\right|\right|\\
&\leq \int_s^{\varsigma}\left|\left|\frac{d}{d\tau}\left(\tau^{\beta-1}AE_{\beta,\beta}(-\tau^{\beta}A)\right)\right|\right|d\tau\\
&\leq \mathcal{C}_\beta\int_s^{\varsigma}\tau^{-2}d\tau = \mathcal{C}_\beta(s^{-1}-\varsigma^{-1}).
\end{aligned} \tag{18}$$

Take $h>0$ such that $0<\varsigma<\varsigma+h\leq\tilde{\Im}$, then

$$
\begin{aligned}
& A\eta_1(\varsigma+h) - A\eta_1(\varsigma) \\
= & \int_0^{\varsigma} \Big((\varsigma+h-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) \\
- & (\varsigma-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma-s)^{\beta} A) \Big) \Big(P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) \Big) ds \\
+ & \int_0^{\varsigma} (\varsigma+h-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) \Big(P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma+h)}{\rho}) \Big) ds \\
+ & \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) \Big(P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma+h)}{\rho}) \Big) ds \\
:= & I_1(\varsigma) + I_2(\varsigma) + I_3(\varsigma).
\end{aligned} \tag{19}
$$

The three major terms are discussed here one by one. We have Equation (18) with (e_1) for $I_1(\varsigma)$

$$
\begin{aligned}
|I_1(\varsigma)|_r \leq & \int_0^{\varsigma} \Big\| (\varsigma+h-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) - (\varsigma-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma-s)^{\beta} A) \Big\| \\
& \big| P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) \big|_r ds \\
\leq & \, \mathcal{C}_{\beta} L h \int_0^{\varsigma} (\varsigma+h-s)^{-1} (\varsigma-s)^{\vartheta-1} ds \\
\leq & \, \mathcal{C}_{\beta} L h \int_0^{\varsigma} (h+s)^{-1} (\varsigma-s)^{\vartheta-1} ds \\
\leq & \, \mathcal{C}_{\beta} L \int_0^{\varsigma} \frac{h}{s+h} s^{\vartheta-1} ds + \mathcal{C}_{\beta} L h \int_h^{\infty} \frac{s}{s+h} s^{\vartheta-1} ds \\
\leq & \, \mathcal{C}_{\beta} L h^{\vartheta}.
\end{aligned} \tag{20}
$$

For solving $I_2(\varsigma)$, Lemma 6 and (e_1) are used here, so we have

$$
\begin{aligned}
|I_2(\varsigma)|_r \leq & \int_0^{\varsigma} (\varsigma+h-s)^{\beta-1} \Big| A E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) \Big(P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma+h)}{\rho}) \Big) \Big|_r ds \\
\leq & \, \mathcal{C}_1 \int_0^{\varsigma} (\varsigma+h-s)^{-1} \Big| \Big(P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma+h)}{\rho}) \Big) \Big|_r ds \\
\leq & \, \mathcal{C}_1 L h^{\vartheta} \int_0^{\varsigma} (\varsigma+h-s)^{-1} ds \\
= & \, \mathcal{C}_1 L [ln(h) - ln(\varsigma+h)] h^{\vartheta}.
\end{aligned} \tag{21}
$$

Moreover, for solving $I_3(\varsigma)$, Lemma 6 and (e_1) are used here, so we have

$$
|I_3(\varsigma)|_r \leq \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\beta-1} \Big| A E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) (P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma+h)}{\rho})) \Big|_r ds \tag{22}
$$

$$
\begin{aligned}
\leq & \, \mathcal{C}_1 \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{-1} \Big| P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma+h)}{\rho})) \Big|_r ds \\
\leq & \, \mathcal{C}_1 L \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\vartheta-1} ds = \mathcal{C}_1 L \frac{h^{\vartheta}}{\vartheta}.
\end{aligned} \tag{23}
$$

By combining all of the above (20–22), we deduce that Hölder's continuity of $A\eta_1(\varsigma)$ exists. □

Theorem 4. *Assume Theorem 3's assumptions are satisfied. The mild solution of Equation (2) is the classical one applicable to any* $\frac{a\varsigma}{\sqrt{2}} \in D(A)$ *if* (e_1) *is true.*

Proof. From Lemma 2(ii), it is guaranteed that function $u(\varsigma) = \int_0^\varsigma E_\beta(-(\varsigma-s)^\beta A)\frac{a}{\sqrt{2}}d\varsigma$ ($\varsigma > 0$) is a classical solution for the given problem for value $\frac{at}{\sqrt{2}} \in D(A)$,

$$\begin{cases} {}^C D_\varsigma^\beta u = -Au, \quad \varsigma > 0, \\ u(0) = \dfrac{ax}{\sqrt{2}}. \end{cases}$$

Step I
By verifying that

$$\eta(\varsigma) = \int_0^\varsigma (\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)P(\frac{-\sigma B_0^2}{\rho}u(s))ds$$

is the classical solution to this following problem

$$\begin{cases} {}^C D_\varsigma^\beta u = -Au + P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}), \; \varsigma > 0, \\ u(0) = 0. \end{cases}$$

Thus, it follows from Theorem 3 that $\eta \in \mathcal{C}([0,\tilde{\Im}], J_r)$. By rewriting $\eta(\varsigma) = \eta_1(\varsigma)_+\eta_2(\varsigma)$, then

$$\begin{aligned} \eta_1(\varsigma) &= \int_0^\varsigma (\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)(P(\frac{-\sigma B_0^2 u(s)}{\rho}) - P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}))ds, \\ \eta_2(\varsigma) &= \int_0^\varsigma (\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})ds. \end{aligned}$$

From Lemma 7 , we can write $\eta_1(\varsigma) \in D(A)$ to demonstrate the similar results for $\eta_2(\varsigma)$. By Lemma 2(iii), we find that $A\eta_2(\varsigma) = P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho}) - E_\beta(-\varsigma^\beta A)P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})$. Since (e_1) exists, $|A\eta_2(\varsigma)|_r \leq (1+\mathcal{C}_1)|P(\frac{-\sigma B_0^2 u(\varsigma)}{\rho})|_r$; thus,

$$\eta_2(\varsigma) \in D(A) \; for \; \varsigma \in (0,\tilde{\Im}] \; and \; \eta_2(\varsigma) \in \mathcal{C}^\vartheta((0,\tilde{\Im}], J_r). \tag{24}$$

Additionally, we verify for ${}^C D_\varsigma^\beta \eta \in \mathcal{C}((0,\tilde{\Im}], J_r)$. In light of the Lemma 2(iv) with the condition $\eta(0) = 0$, there are

$${}^C D_\varsigma^\beta \eta(\varsigma) = \frac{d}{d\varsigma}(I_\varsigma^{1-\beta}\eta(\varsigma)) = \frac{d}{d\varsigma}\left(E_\beta(-\varsigma^\beta A) * P(\frac{-\sigma B_0^2 u}{\rho})\right).$$

The continuous differentiability of $E_\beta(\varsigma^\beta A) * P(\frac{-\sigma B_0^2 u}{\rho})$ in J_r remains to be proven. If we assume that $0 < h \leq \tilde{\Im} - \varsigma$, we can derive:

$$\begin{aligned} &\frac{1}{h}\left(E_\beta(-(\varsigma+h)^\beta A) * P(\frac{-\sigma B_0^2 u}{\rho}) - E_\beta(-\varsigma^\beta A) * P(\frac{-\sigma B_0^2 u}{\rho})\right) \\ &= \int_0^\varsigma \frac{1}{h}\left(E_\beta(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho}) - E_\beta(-(\varsigma-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho})\right)ds \\ &\quad + \frac{1}{h}\int_t^{\varsigma+h} E_\beta(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho})ds. \end{aligned}$$

Consider

$$
\begin{aligned}
& \int_0^\varsigma \frac{1}{h}\Big| E_\beta(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho}) - E_\beta(-(\varsigma-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho})\Big|_r ds \\
\leq \;& \mathcal{C}_1 \frac{1}{h}\int_0^\varsigma \Big| E_\beta(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho})\Big|_r ds \\
+ \;& \mathcal{C}_1 \frac{1}{h}\int_0^\varsigma \Big| E_\beta(-(\varsigma-s))^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho})\Big|_r ds \\
\leq \;& \mathcal{C}_1 \mathcal{M}(\varsigma)\frac{1}{h}\int_0^\varsigma (\varsigma+h-s)^\beta s^{-\beta(1-\delta)} ds \\
+ \;& \mathcal{C}_1 \mathcal{M}(\varsigma)\frac{1}{h}\int_0^\varsigma (\varsigma-s)^{-\beta} s^{-\beta(1-\delta)} ds \\
\leq \;& \mathcal{C}_1 \mathcal{M}(\varsigma)\frac{1}{h}\Big((\varsigma+h)^{1-\beta} + \varsigma^{1-\beta}\Big)\mathcal{B}(1-\beta, 1-\beta(1-\delta)).
\end{aligned}
$$

Using Lebesgue's LDC theorem, we arrive at the following conclusion:

$$
\begin{aligned}
& \lim_{h\to 0}\int_0^\varsigma \frac{1}{h}\Big(E_\beta(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho}) - E_\beta(-(\varsigma-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho})\Big) ds \\
&= -\int_0^\varsigma (\varsigma-s)^{\beta-1} A E_{\beta,\beta}(-(\varsigma-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho}) ds \\
&= A\eta(\varsigma).
\end{aligned}
$$

In contrast with

$$
\begin{aligned}
& \frac{1}{h}\int_\varsigma^{\varsigma+h} E_\beta(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2 u(s)}{\rho}) ds \\
&= \frac{1}{h}\int_0^h E_\beta(-s^\beta A)P(\frac{-\sigma B_0^2}{\rho} u(\varsigma+h-s) ds \\
&= \frac{1}{h}\int_0^h E_\beta(-s^\beta A)\Big(P(\frac{-\sigma B_0^2}{\rho} u(\varsigma+h-s)) - P(\frac{-\sigma B_0^2}{\rho} u(\varsigma-s))\Big) ds \\
&+ \frac{1}{h}\int_0^h E_\beta(-s^\beta A)P(\frac{-\sigma B_0^2}{\rho} u(\varsigma-s)) - P(\frac{-\sigma B_0^2}{\rho} u(\varsigma)) ds \\
&+ \frac{1}{h}\int_0^h E_\beta(-s^\beta A)P(\frac{-\sigma B_0^2}{\rho} u(\varsigma)) ds.
\end{aligned}
$$

Lemmas 1 and 6 with property (e_1) give us

$$
\begin{aligned}
& \Big| \frac{1}{h}\int_0^\varsigma E_\beta(-s^\beta A)P(\frac{-\sigma B_0^2}{\rho} u(\varsigma+h-s)) - P(\frac{-\sigma B_0^2}{\rho} u(\varsigma-s)) ds\Big|_r \leq \mathcal{C}_1 L h^\vartheta, \\
& \Big| \frac{1}{h}\int_0^t E_\beta(-s^\beta A)P(\frac{-\sigma B_0^2}{\rho} u(\varsigma-s)) - P(\frac{-\sigma B_0^2}{\rho} u(\varsigma)) ds\Big|_r \leq \mathcal{C}_1 L \frac{h^\vartheta}{\vartheta+1}.
\end{aligned}
$$

Consequently, Lemma 2(i) offers

$$
\lim_{h\to 0}\frac{1}{h}\int_0^h E_\beta\big((-s)^\beta A\big)P(\frac{-\sigma B_0^2}{\rho} u(s)) ds = P(\frac{-\sigma B_0^2}{\rho} u(\varsigma)).
$$

Hence,

$$
\lim_{h\to 0}\frac{1}{h}\int_\varsigma^{\varsigma+h} E_\beta\big((\varsigma+h-s)^\beta A\big)P(\frac{-\sigma B_0^2}{\rho} u(s)) ds = P(\frac{-\sigma B_0^2}{\rho} u(\varsigma)).
$$

Our conclusion is that $E_\beta(\varsigma^\beta A) * P(\frac{-\sigma B_0^2}{\rho}u)$ is differentiable at ς_+ and

$$\frac{d}{d\varsigma}\big(E_\beta(\varsigma^\beta A) * P(\frac{-\sigma B_0^2}{\rho}u)\big)_+ = A\eta(\varsigma) + P(\frac{-\sigma B_0^2}{\rho}u(\varsigma)).$$

Similarly, $E_\beta(\varsigma^\beta A) * P(\frac{-\sigma B_0^2}{\rho}u)$ is differential at ς_- and

$$\frac{d}{d\varsigma}\big(E_\beta(\varsigma^\beta A) * P(\frac{-\sigma B_0^2}{\rho}u)\big)_- = A\eta(\varsigma) + P(\frac{-\sigma B_0^2}{\rho}u(\varsigma)).$$

We show that $A\eta = A\eta_1 + A\eta_2 \in \mathcal{C}((0,\tilde{\Im}], J_r)$. It is obviously clear that given function

$$\eta_2(\varsigma) = P(\frac{-\sigma B_0^2}{\rho}u(\varsigma)) - E_\beta(\varsigma^\beta A)P(\frac{-\sigma B_0^2}{\rho}u(\varsigma)).$$

In consideration of Lemma 1, it is continuous because of Lemma 2(iii). Additionally, Lemma 7 tells us that $A\eta_1(\varsigma)$ is also continuous. Accordingly, ${}^C D_\varsigma^\beta \eta \in \mathcal{C}((0,\tilde{\Im}], J_r)$.

Step II

Suppose u is a mild solution of (2). To demonstrate $F(u,u) \in \mathcal{C}^\vartheta((0,\tilde{\Im}], J_r)$ from (5), we must verify that $A^\alpha u$ is Hölder-continuous in J_r. Apply $h > 0$ in a way that $0 < \varsigma < \varsigma + h$. Indicate $\tilde{\phi}(\varsigma) := E_\beta(-\varsigma^\beta A)\dfrac{a}{\sqrt{2}}$ through Lemma 2(iv) and (6), then

$$\begin{aligned}
|A^\alpha\tilde{\phi}(\varsigma+h) - A^\alpha\tilde{\phi}(\varsigma)|_r &= \left|\int_\varsigma^{\varsigma+h} -s^{\beta-1}A^\alpha E_{\beta,\beta}(-s^\beta A)\frac{a}{\sqrt{2}}ds\right|_r \\
&\le \int_\varsigma^{\varsigma+h} s^{\beta-1}\left|A^{\alpha-\delta}E_{\beta,\beta}(-s^\beta A)A^\delta\frac{a}{\sqrt{2}}\right|_r ds \\
&\le \mathcal{C}_1\int_\varsigma^{\varsigma+h} s^{\beta-1}s^{-\beta(\alpha-\delta)}\left|A^\delta\frac{a}{\sqrt{2}}\right|_r ds \\
&= \mathcal{C}_1\int_\varsigma^{\varsigma+h} s^{\beta(\delta-\alpha)+\beta-1}\left|A^\delta\frac{a}{\sqrt{2}}\right|_r ds \\
&= \mathcal{C}_1\int_\varsigma^{\varsigma+h} s^{\beta(1+\delta-\alpha)-1}ds|A^\delta\frac{a}{\sqrt{2}}|_r \\
&= \frac{\mathcal{C}_1|a|_{H^{\delta,r}}}{\beta(1+\delta-\alpha)}\big((\varsigma+h)^{\beta(1+\delta-\alpha)} - \varsigma^{\beta(1+\delta-\alpha)}\big) \\
&\le \frac{\mathcal{C}_1|\frac{a}{\sqrt{2}}|_{H^{\delta,r}}}{\beta(1+\delta-\alpha)}h^{\beta(1+\delta-\alpha)}.
\end{aligned}$$

Thus, $A^\alpha\tilde{\phi} \in \mathcal{C}^\vartheta((0,\tilde{\Im}], J_r)$. Apply h in a way that $\varepsilon \le \varsigma < \varsigma + h \le \tilde{\Im}$ for all small $\varepsilon > 0$,

$$\begin{aligned}
&|A^\alpha\eta(\varsigma+h) - A^\alpha\eta(\varsigma)|_r \\
\le\ &\left|\int_\varsigma^{\varsigma+h}(\varsigma+h-s)^{\beta-1}A^\alpha E_{\beta,\beta}(-(\varsigma+h-s)^\beta A)P(\frac{-\sigma B_0^2}{\rho}u(s))ds\right|_r \\
+\ &\left|\int_0^\varsigma A^\alpha\big((\varsigma+h-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma+h-s)^\beta A) - (\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^\beta A)\big)P(\frac{-\sigma B_0^2}{\rho}u(s))ds\right|_r \\
=\ &\eta_1(\varsigma) + \eta_2(\varsigma).
\end{aligned}$$

Lemma 6 and (e) are applied, and the outcome is

$$\begin{aligned}
\eta_1(\varsigma) &\leq \mathcal{C}_1 \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\beta(1-\alpha)-1} |P(\frac{-\sigma B_0^2}{\rho} u(s))|_r ds \\
&\leq \mathcal{C}_1 \mathcal{M}(\varsigma) \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds \\
&\leq \mathcal{M}(\varsigma) \frac{\mathcal{C}_1}{\beta(1-\alpha)} h^{\beta(1-\alpha)} \varsigma^{-\beta(1-\delta)} \\
&\leq \mathcal{M}(\varsigma) \frac{\mathcal{C}_1}{\beta(1-\alpha)} h^{\beta(1-\alpha)} \varepsilon^{-\beta(1-\delta)}.
\end{aligned}$$

Finding the inequality, we may calculate $\eta_2(\varsigma)$,

$$\begin{aligned}
\frac{d}{d\varsigma}(\varsigma^{\beta-1} A^{\alpha} E_{\beta,\beta}(-\varsigma^{\beta} A)) &= \frac{1}{2\pi i} \int_{\Gamma} \mu^{\alpha} \varsigma^{\beta-2} E_{\beta,\beta-1}(-\mu\varsigma^{\beta})(\mu I + A)^{-1} d\mu \\
&= \frac{1}{2\pi i} \int_{\Gamma'} -\Big(-\frac{\psi}{\varsigma^{\beta}}\Big)^{\alpha} \varsigma^{\beta-2} E_{\beta,\beta-1}(\psi) \Big(-\frac{\psi}{\varsigma^{\beta}} I + A\Big)^{-1} \frac{1}{\varsigma^{\beta}} d\psi,
\end{aligned}$$

The above equation gives the results that

$$\Big|\Big|\frac{d}{d\varsigma}(\varsigma^{\beta-1} A^{\alpha} E_{\beta,\beta}(-\varsigma^{\beta} A))\Big|\Big| \leq \mathcal{C}_{\beta} \varsigma^{\beta(1-\alpha)-2}.$$

The mean value theorem yields

$$\begin{aligned}
\Big|\Big|\varsigma^{\beta-1} A^{\alpha} E_{\beta,\beta}(-\varsigma^{\beta} A) - s^{\beta-1} A^{\alpha} E_{\beta,\beta}(-s^{\beta} A)\Big|\Big| &\leq \int_s^{\varsigma} \Big|\Big|\frac{d}{d\tau}(\tau^{\beta-1} A^{\alpha} E_{\beta,\beta}(-\tau^{\beta} A))\Big|\Big| d\tau \\
&\leq \mathcal{C}_{\beta} \int_s^{\varsigma} \tau^{\beta(1-\alpha)-2} d\tau = \mathcal{C}_{\beta}\big(s^{\beta(1-\alpha)-1} - \varsigma^{\beta(1-\alpha)-1}\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\eta_2(\varsigma) &\leq \int_0^{\varsigma} \Big| A^{\alpha}\Big((\varsigma+h-s)^{\beta-1} E_{\beta,\beta}(-(\varsigma+h-s)^{\beta} A) - (\varsigma-s)^{\beta-1} E_{\beta,\beta}(-(\varsigma-s)^{\beta} A\Big) P(\frac{-\sigma B_0^2}{\rho} u(s))\Big|_r ds \\
&\leq \int_0^{\varsigma} \Big((\varsigma-s)^{\beta(1-\alpha)-1} - (\varsigma+h-s)^{\beta(1-\alpha)-1}\Big) \Big|P(\frac{-\sigma B_0^2}{\rho} u(s))\Big|_r ds \\
&\leq \mathcal{C}_{\beta} M(\varsigma)\Big(\int_0^{\varsigma} (\varsigma-s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds - \int_0^{\varsigma+h} (\varsigma+h-s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds\Big) \\
&+ \mathcal{C}_{\beta} \mathcal{M}(\varsigma) \int_{\varsigma}^{\varsigma+h} (\varsigma+h-s)^{\beta(1-\alpha)-1} s^{-\beta(1-\delta)} ds \\
&\leq \mathcal{C}_{\beta} \mathcal{M}(\varsigma)\big(\varsigma^{\beta(\delta-\alpha)} - (\varsigma+h)^{\beta(\delta-\alpha)}\big) \mathcal{B}(\beta(1-\alpha), 1-\beta(1-\delta)) + \mathcal{C}_{\beta} M(\varsigma) h^{\beta(1-\alpha)} \varsigma^{-\beta(1-\delta)} \\
&\leq \mathcal{C}_{\beta} \mathcal{M}(\varsigma) h^{\beta(\alpha-\delta)} [\varepsilon(\varepsilon+h)]^{\beta(\delta-\alpha)} + \mathcal{C}_{\beta} \mathcal{M}(\varsigma) h^{\beta(1-\alpha)} \varepsilon^{-\beta(1-\delta)}.
\end{aligned}$$

This guarantees that $A^{\alpha}\eta \in \mathcal{C}^{\vartheta}([\varepsilon, \tilde{\Im}], J_r)$. Owing to random ε, $A^{\alpha}\eta \in \mathcal{C}^{\vartheta}((0, \tilde{\Im}], J_r)$,

$$\tilde{\psi}(\varsigma) = \int_0^{\varsigma} (\varsigma-s)^{\beta-1} E_{\beta,\beta}(-(\varsigma-s)^{\beta} A) F(u(s), u(s)) ds.$$

Here, we know that $|F(u(s), u(s))|_r \leq \mathcal{M}\tilde{\kappa}^2(\varsigma) s^{-2\beta(\alpha-\delta)}$, in which the supplied function's continuity exists but is also bounded in $(0, \tilde{\Im}]$, and $\tilde{\kappa}(\varsigma) := \sup\limits_{s \in (0,\varsigma]} s^{\beta(\alpha-\delta)} |u(s)|_{H^{\alpha,r}}$.

We are able to provide the Hölder continuity of $A^{\alpha}\tilde{\psi}$ in the same fashion in $\mathcal{C}^{\vartheta}((0,\tilde{\Im}],J_r)$. Therefore,

$$A^{\alpha}u(\varsigma) = A^{\alpha}\tilde{\varphi}(\varsigma) + A^{\alpha}\eta(\varsigma) + A^{\alpha}\tilde{\psi}(\varsigma) \in \mathcal{C}^{\vartheta}((0,\tilde{\Im}],J_r).$$

Seeing as $F(u,u) \in \mathcal{C}^{\vartheta}((0,\tilde{\Im}],J_r)$ is demonstrated from the previous step II, this gives the results that ${}^{C}D^{\beta}_{\varsigma}\psi \in \mathcal{C}^{\vartheta}((0,\tilde{\Im}],J_r)$, $A\tilde{\psi} \in \mathcal{C}((0,\tilde{\Im}],J_r)$ and ${}^{C}D^{\beta}_{\varsigma}\tilde{\psi} = -A\tilde{\psi} + F(u,u)$. Similarly, we obtained that ${}^{C}D^{\beta}_{\varsigma}u \in \mathcal{C}((0,\tilde{\Im}],J_r)$, $Au \in \mathcal{C}((0,\tilde{\Im}],J_r)$ and ${}^{C}D^{\beta}_{\varsigma}u = -Au + F(u,u) + P(\frac{-\sigma B_0^2}{\rho}u)$. Consequently, the conclusion is that u is a classic solution. □

Theorem 5. *Suppose (e_1) is true. If u is showing a classical solution of (2), then $Au \in \mathcal{C}^{\nu}((0,\tilde{\Im}],J_r)$ and, also, ${}^{C}D^{\beta}_{\varsigma}u \in \mathcal{C}^{\nu}((0,\tilde{\Im}],J_r)$.*

Proof. We may put $u(\varsigma) = \tilde{\varphi}(\varsigma) + \eta(\varsigma) + \tilde{\psi}(\varsigma)$ if u exhibits the classical solution of (2). The evidence is sufficient to demonstrate $A\tilde{\varphi} \in \mathcal{C}^{\beta(1-\delta)}((0,\tilde{\Im}],J_r)$. It is necessary to demonstrate that $A\tilde{\phi} \in \mathcal{C}^{\beta(1-\delta)}([\varepsilon,\tilde{\Im}],J_r)$ is true for any $\varepsilon > 0$. In fact, by choosing h such that $\varepsilon \leq \varsigma < \varsigma + h \leq \tilde{\Im}$, using Lemma 2(iii),

$$\begin{aligned}
|A\tilde{\phi}(\varsigma+h) - A\tilde{\phi}(\varsigma)|_r &= \left| \int_{\varsigma}^{\varsigma+h} -s^{\beta-1}A^2E_{\beta,\beta}(-s^{\beta}A)\frac{a}{\sqrt{2}}ds \right|_r \\
&\leq \mathcal{C}_1 \int_{\varsigma}^{\varsigma+h} s^{-\beta(1-\delta)-1}ds|\frac{a}{\sqrt{2}}|_{H^{\delta,r}} \\
&= \frac{\mathcal{C}_1|\frac{a}{\sqrt{2}}|_{H^{\delta,r}}}{\beta}\left(\varsigma^{-\beta(1-\delta)} - (\varsigma+h)^{-\beta(1-\delta)}\right) \\
&\leq \frac{\mathcal{C}_1|\frac{a}{\sqrt{2}}|_{H^{\delta,r}}}{\beta}\frac{h^{\beta(1-\delta)}}{[\varepsilon(\varepsilon+h)]^{\beta(1-\delta)}}.
\end{aligned}$$

As from Lemma 7, we can write $\eta(\varsigma)$ as

$$\begin{aligned}
\eta(\varsigma) &= \eta_1(\varsigma) + \eta_2(\varsigma) \\
&= \int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)\left(P(\frac{-\sigma B_0^2}{\rho}u(s)) - P(\frac{-\sigma B_0^2}{\rho}u(\varsigma))\right)ds \\
&+ \int_0^{\varsigma}(\varsigma-s)^{\beta-1}E_{\beta,\beta}(-(\varsigma-s)^{\beta}A)(P(\frac{-\sigma B_0^2}{\rho}u(s))ds,
\end{aligned}$$

in the domain of $\varsigma \in (0,\tilde{\Im}]$ from Lemma 7 and (24) it follows that $A\eta_1(\varsigma) \in \mathcal{C}^{\nu}([0,\tilde{\Im}],J_r)$ and $A\eta_2(\varsigma) \in \mathcal{C}^{\tilde{\phi}}((0,\tilde{\Im}],J_r)$, accordingly. □

8. Conclusions

The Helmholtz-Leray projection is used in this work to show the existence and uniqueness of fractional-order Navier-Stokes equations for the Cauchy problem solution. In the interim, we provide a workable local solution in $\mathbf{S}_{\wp}$. To model phenomenon diffusion in fractal media, Navier-Stokes equations (NSEs) with time-fractional derivatives of order $\gamma \in (0,1)$ are utilised. We use $\mathbf{S}_{\wp}$ to show that these equations have regular classical solutions. Additional study may build on the idea presented in this article by incorporating validity and generalising other activities. Numerous studies are being done in this interesting field, which could result in a variety of ideas and uses.

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References

1. Gorenflo, R.; Mainardi, F. Fractional calculus. In *Fractals and Fractional Calculus in Continuum Mechanics*; Springer: Vienna, Austria, 1997; pp. 223–276.
2. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]
3. Machado, T.; Kiryakova, V.; Mainardi, F. A poster about the recent history of fractional calculus. *Fract. Calc. Appl. Anal.* **2010**, *13*, 329–334.
4. El-Shahed, M.; Salem, A. On the generalized Navier-Stokes equations. *Appl. Math. Comput.* **2004**, *156*, 287–293. [CrossRef]
5. Doering, C.R.; Gibbon, J.D. *Applied Analysis of the Navier-Stokes Equations (No. 12)*; Cambridge University Press: Cambridge, UK, 1995.
6. Gresho, P.M.; Sani, R.L. On pressure boundary conditions for the incompressible Navier-Stokes equations. *Int. J. Numer. Methods Fluids* **1987**, *7*, 1111–1145. [CrossRef]
7. Lemarié-Rieusset, P.G. *Recent Developments in the Navier-Stokes Problem*; CRC: Boca Raton, FL, USA, 2002.
8. Von Wahl, W. *The Equations of Navier-Stokes and Abstract Parabolic Equations*; Vieweg+Teubner Verlag Wiesbaden: Braunschweig, Germany, 1985.
9. Iwabuchi, T.; Takada, R. Global solutions for the Navier-Stokes equations in the rotational framework. *Math. Ann.* **2013**, *357*, 727–741. [CrossRef]
10. Davidson, P.A. An Introduction to magnetohydrodynamics. *Am. J. Phys.* **2002**, *70*, 781. [CrossRef]
11. Goedbloed, J.H.; Goedbloed, J.P.; Poedts, S. *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas*; Cambridge University Press: Cambridge, UK, 2004.
12. Desjardins, B.; Le Bris, C. Remarks on a nonhomogeneous model of magnetohydrodynamics. *Differ. Integral Equ.* **1998**, *11*, 377–394. [CrossRef]
13. Qayyum, M.; Abbas, T.; Afzal, S.; Saeed, S.T.; Akgül, A.; Inc, M.; Mahmoud, K.H.; Alsubaie, A.S. Heat Transfer Analysis of Unsteady MHD Carreau Fluid Flow over a Stretching/Shrinking Sheet. *Coatings* **2022**, *12*, 1661. [CrossRef]
14. Rehman, A.U.; Riaz, M.B.; Saeed, S.T.; Jarad, F.; Jasim, H.N.; Enver, A. An exact and comparative analysis of MHD free convection flow of water-based nanoparticles via CF derivative. *Math. Probl. Eng.* **2022**, *2022*, 9977188. [CrossRef]
15. Saeed, S.T.; Riaz, M.B.; Awrejcewicz, J.; Ahmad, H. Exact Symmetric Solutions of MHD Casson Fluid Using Chemically Reactive Flow with Generalized Boundary Conditions. *Energies* **2021**, *14*, 6243. [CrossRef]
16. Niazi, A.U.K.; He, J.; Shafqat, R.; Ahmed, B. Existence, Uniqueness, and E q–Ulam-Type Stability of Fuzzy Fractional Differential Equation. *Fractal Fract.* **2021**, *5*, 66. [CrossRef]
17. Shafqat, R.; Niazi, A.U.K.; Jeelani, M.B.; Alharthi, N.H. Existence and Uniqueness of Mild Solution Where $\alpha \in (1,2)$ for Fuzzy Fractional Evolution Equations with Uncertainty. *Fractal Fract.* **2022**, *6*, 65. [CrossRef]
18. Alnahdi, A.S.; Shafqat, R.; Niazi, A.U.K.; Jeelani, M.B. Pattern formation induced by fuzzy fractional-order model of COVID-19. *Axioms* **2022**, *11*, 313. [CrossRef]
19. Khan, A.; Shafqat, R.; Niazi, A.U.K. Existence Results of Fuzzy Delay Impulsive Fractional Differential Equation by Fixed Point Theory Approach. *J. Funct. Spaces* **2022**, *2022*, 4123949. [CrossRef]
20. Abuasbeh, K.; Shafqat, R.; Niazi, A.U.K.; Awadalla, M. Local and global existence and uniqueness of solution for time-fractional fuzzy Navier-Stokes equations. *Fractal Fract.* **2022**, *6*, 330. [CrossRef]
21. Abbas, A.; Shafqat, R.; Jeelani, M.B.; Alharthi, N.H. Significance of Chemical Reaction and Lorentz Force on Third-Grade Fluid Flow and Heat Transfer with Darcy–Forchheimer Law over an Inclined Exponentially Stretching Sheet Embedded in a Porous Medium. *Symmetry* **2022**, *14*, 779. [CrossRef]
22. Phong, T.T.; Long, L.D. Well-posed results for nonlocal fractional parabolic equation involving Caputo-Fabrizio operator. *J. Math. Comput. Sci.* **2022**, *26*, 357–367. [CrossRef]
23. Shafqat, R.; Niazi, A.U.K.; Yavuz, M.; Jeelani, M.B.; Saleem, K. Mild Solution for the Time-Fractional Navier-Stokes Equation Incorporating MHD Effects. *Fractal Fract.* **2022**, *6*, 580. [CrossRef]
24. Mehmood, Y.; Shafqat, R.; Sarris, I.E.; Bilal, M.; Sajid, T.; Akhtar, T. Numerical Investigation of MWCNT and SWCNT Fluid Flow along with the Activation Energy Effects over Quartic Auto Catalytic Endothermic and Exothermic Chemical Reactions. *Mathematics* **2022**, *10*, 4636. [CrossRef]
25. Boulares, H.; Benchaabane, A.; Pakkaranang, N.; Shafqat, R.; Panyanak, B. Qualitative Properties of Positive Solutions of a Kind for Fractional Pantograph Problems using Technique Fixed Point Theory. *Fractal Fract.* **2022**, *6*, 593. [CrossRef]

26. Zhou, Y.; Vijayakumar, V.; Murugesu, R. Controllability for fractional evolution inclusions without compactness. *Evol. Equ. Control. Theory* **2015**, *4*, 507–524. [CrossRef]
27. Wang, R.N.; Chen, D.H.; Xiao, T.J. Abstract fractional Cauchy problems with almost sectorial operators. *J. Differ. Equ.* **2012**, *252*, 202–235. [CrossRef]
28. Podlubny, I.; Chechkin, A.; Skovranek, T.; Chen, Y.; Jara, B.M.V. Matrix approach to discrete fractional calculus II: Partial fractional differential equations. *J. Comput. Phys.* **2009**, *228*, 3137–3153. [CrossRef]