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Existence of Global and Local Mild Solution for the Fractional Navier–Stokes Equations

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Abstract: Navier–Stokes equations (NS-equations) are applied extensively for the study of various waves phenomena where the symmetries are involved. In this paper, we discuss the NS-equations with the time-fractional derivative of order $\beta \in (0,1)$. In fractional media, these equations can be utilized to recreate anomalous diffusion equations which can be used to construct symmetries. We examine the initial value problem involving the symmetric Stokes operator and gravitational force utilizing the Caputo fractional derivative. Additionally, we demonstrate the global and local mild solutions in $H^{\alpha,p}$. We also demonstrate the regularity of classical solutions in such circumstances. An example is presented to demonstrate the reliability of our findings.

Keywords: Navier–Stokes equations; Caputo fractional derivatives; mild solutions; regularity

MSC: 34A08; 34A12

1. Introduction

Because of their importance in fluid mechanics, the Navier–Stokes equations have been extensively studied by various researchers. NS-equations are partial differential equations that describe the flow of incompressible fluid. These equations are generalization of the equations devised by Swiss mathematician Leonhard Euler in the eighteen century to describe the flow of incompressible and frictionless fluids. The NS-equations are useful because they describe the physics of many scientific and engineering phenomena. These can be used to simulate weather, ocean currents, water flow in a pipe, and airflow around a wing etc. The difference between the NS-equations and the Euler equations is that the NS-equations account for viscosity, whereas the Euler equations exclusively simulate inviscid flow.

As a result, the NS-equations are parabolic equations, which have exceptional analytic features. In a purely mathematical sense, the NS-equations are extremely interesting. Despite its extensive range of applications, it is still unknown if smooth solutions always exist in three dimensions, that is, whether these are infinite and differentiable at all points in the domain. The existence and smoothness problem is known as the Navier–Stokes problem.

Different scholars focus on mass and momentum conservation and describe useful phenomena concerning the motion of the incompressible fluid flow, ranging from large-scale atmospheric motions to the lubricant in ball bearings; see, Varnhorn [1], as well as Cannone [2]. Similarly, Rieusset [3] discussed the existence, uniqueness and regularity of NS-equations.

Jean Leray was a French mathematician who work on both PDEs and algebraic topology and explained a fascinating phenomenon. The Leray projection is a linear operator that is useful in the theory of partial differential equations, particularly in the subject of

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fluid dynamics. It can be considered as a projection on a vector field with no divergence. In the Stokes equations and NS-equations, it is applied to eliminate both the pressure term and the divergence-free term; see [4].

Aljandro Rangel-Huerts and Blanca Bermudez solved NS-equations using two unique formulations with moderate and high Reynolds numbers. They used two numerical solutions of lid-driven cavity and Taylor vortex problems. These problems can be solved by using stream function vorticity in two dimensions of NS-equations; see [5]. Moreover, Gallgher [6], Giga [7], Rejaiba [8], Kozono [9], Sell [10] and Choe [11] found unique results on the regularity of weak and strong solutions. Emilia Bazhlekova et al. [12] analyzed the Rayleigh Stokes' problems. Rayleigh problem is also known as Stokes' first problem which is a problem of determining the flow created by a sudden movement of an infinitely long plate from rest named after Lord Rayleigh and Sir George Stokes. The authors studied the Reyleigh problems involving RL-fractional derivative. They worked on smooth and non-smoothness initial data for Sobolev regularity of homogeneous problems.

On the contrary, fractional calculus has received a lot of attention in recent years. Many of the fundamental piece of calculus are related to fluid mechanics like total derivative, gradients, divergence and rotation. Fractional calculus proved that the topic indeed is very promising like in control theory of dynamical system, porous structure, viscoelasticity and among others; see, e.g., Hilfer [13], Herrmann [14], and Zhou [15–17]. Such models are important not just in Physics but also in pure mathematics. Recently, experimental data and theoretical analysis have shown that the diffusion equation fails to describes the diffusion phenomena in porous media. Basically, the diffusion equation is a parabolic PDE. In Physics, it describe the microscopic behavior of many microparticles in Brownain motion.

Do NS-equations describe all the motion of the fluid? Serkan Solmaz gave an interesting fact that the NS-equations encompass all types of fluid motion in case they are combined with a related mathematical model such as multi-phase flow, chemical reaction and turbulent etc. It is significant to specify the degree of error throughout the analysis in which the NS-equations enable a reasonable range of error. Thereby, these are the most famous equations that examine the motion of fluid reliably. Different authors talked about the time fractional NS-equations; see [18–20]. Moreover, to the best of our insight there are not many results on the existence, uniqueness and regularity of mild solution for time fractional NS-equations.

Keeping this in view, we discuss the time fractional NS-equations in an open set $\Omega \subset R^m (m \geq 3)$:

$$\begin{cases} \partial_t^\beta v - \mu \Delta v + (v \cdot \nabla) v = -\nabla p + \rho g + \mu \nabla^2 \vec{v}, 0 < t, \\ \nabla \cdot v = 0, \\ \frac{v}{\partial \Omega} = 0, \\ v(0, y) = ax + b, \end{cases} \tag{1}$$

where $\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v\right) = \rho \frac{Dv}{Dt}, g$ is a gravitational force or body force, $-\nabla p$ is a pressure gradient, $\mu \nabla^2 \overrightarrow{v}$ is viscous term or diffusion term, $\rho \frac{Dv}{Dt}$ is local acceleration and ∂_t^β be the Caputo fractional derivative with order $\beta \in (0, 1), y \in \Omega$ and the time $0 < t$. By applying a well-known Helmholtz projector P on (1) for getting rid of the pressure term, one has

$$\begin{cases} \partial_t^\beta v - \mu P \Delta v + P(v \cdot \nabla) v = P g, 0 < t, \\ \nabla \cdot v = 0, \\ \frac{v}{\partial \Omega} = 0, \\ v(0, y) = b. \end{cases}$$

B is the Stokes operator under consideration, where b is the initial velocity and $-\mu P\Delta$ is the Dirichlet boundary condition. The abstract form of (1) is

$$\begin{cases} {}^{C}D_{\mathfrak{t}}^{\beta}v = -Bv + F(v,w) + Pg, 0 < \mathfrak{t}, \\ v(0) = b, \end{cases} \quad (2)$$

where $-P(v \cdot \nabla)w = F(v,w)$.

The arrangement of the paper is as: In Section 2, we review some helpful preliminaries. In Section 3, study of the global and local existence of mild solutions of problem (2) in $H^{\beta,p}$ is conducted. In Section 4, the regularity of classic solutions in Q_p will be discussed. At last, an example will be presented.

2. Preliminaries

In this section, we discuss some known definitions, notations and results.
Suppose that, $\omega = \{(y_1, ..., y_m) : y_m > 0\}$ be an open subset of R^m where $m \geq 3$ and $1 < p < \infty$. Then there exists a bounded projection

$$C_{\varrho}^{\infty}(\omega) = \{v \in (C^{\infty}(\omega))^m : \nabla \cdot v = 0, vhascompactin\omega\},$$

as well as the null space is the closure of

$$\{v \in (C^{\infty}(\omega))^m : v = \nabla\varphi, \varphi \in C^{\infty}(\omega)\}.$$

Suppose that, $Q_p = \overline{C_{\varrho}^{\infty}(\omega)}^{|\cdot|}$, be the closed subspace of $(L^p(\omega))^m$. $(M^{n,p}(\omega))^m$ be a Sobolev space along the norm $|\cdot|_{n,p}$.

$B = -\mu P\Delta$ is said to be the Stokes operator in Q_p whose domain is $D_p(B) = D_p(\Delta) \bigcap h_p$. Here

$$D_p(\Delta) = \{v \in (M^{2,p}(\omega))^m : \frac{v}{\partial\omega} = 0\}.$$

It is noted that $-B$ is a closed linear operator as well as generates the bounded analytic semi-group $\{e^{-\mathfrak{t}B}\}$ on Q_p.

We present new fractional power space definitions that are connected to $-B$. For $\alpha > 0$ as well as $v \in Q_p$, define

$$B^{-\alpha}v = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \mathfrak{t}^{\alpha-1} e^{-\mathfrak{t}B} v d\mathfrak{t}.$$

$B^{-\alpha}$ is bounded and one-to-one operator on Q_p. Suppose that B^{α} is the inverse of $B^{-\alpha}$. For $\alpha > 0$, indicate the space $H^{\alpha,p}$ according to the range $B^{-\alpha}$ along the norm

$$|v|_{H^{\alpha,p}} = |B^{\alpha}v|_p.$$

It is not difficult to see that $e^{-\mathfrak{t}B}$ restrict to be a bounded analytic semi-group on $H^{\alpha,p}$, for further details; see [21].

Suppose that Y is a Banach space as well as Q is the interval of $\mathbb{R}$. All continuous Y valued functions are represented by $C(Q,Y)$. So for $0 < \zeta < 1, C^{\zeta}(Q,Y)$ indicates for the set of all functions is Holder continuous along the exponent ζ.

Assume that $\beta \in (0,1)$ as well as $w : [0,\infty) \to Y$, the fractional integral with the order β along the lower limit zero for the function w is defined as

$$I_{\mathfrak{t}}^{\beta}w(\mathfrak{t}) = \int_0^{\infty} h_{\beta}(\mathfrak{t}-s)w(s)ds, 0 < \mathfrak{t},$$

the R.H.S is point-wise defined on the interval $[0,\infty)$, where h_{β} is said to be the Riemann-Liouville kernel

$$h_{\beta}(\mathfrak{t}) = \frac{\mathfrak{t}^{\beta-1}}{\Gamma(\beta)}, 0 < \mathfrak{t}.$$

${}^{C}D_{\mathfrak{t}}^{\beta}$ indicates the Caputo fractional derivative operator with order β. It can be describe as

$$ {}^{C}D_{\mathfrak{t}}^{\beta}w(\mathfrak{t}) = \frac{d}{d\mathfrak{t}}[I_{\mathfrak{t}}^{1-\beta}(w(\mathfrak{t}) - w(0))] = \frac{d}{d\mathfrak{t}}\bigg(\int_{0}^{\mathfrak{t}} h_{1-\beta}(\mathfrak{t}-s)(w(\mathfrak{t}) - w(0))ds\bigg), 0 < \mathfrak{t}. $$

Generally, for $w = [0,\infty) \times R^{m} \to R^{m}$, Caputo fractional derivative w.r.t time for the function w can be defined as

$$ \partial_{\mathfrak{t}}^{\beta}v(\mathfrak{t},y) = \partial_{\mathfrak{t}}\bigg(\int_{0}^{\mathfrak{t}} h_{1-\beta}(\mathfrak{t}-s)(v(\mathfrak{t},y) - v(\mathfrak{t},0))ds\bigg), 0 < \mathfrak{t}, $$

for further details; see [22]. Now, we define generalized Mittag-Leffler functions:

$$ E_{\beta}(-\mathfrak{t}^{\beta}B) = \int_{0}^{\infty} \mathcal{M}_{\beta}(s)e^{-s\mathfrak{t}^{\beta}B}ds, E_{\beta,\beta}(-\mathfrak{t}^{\beta}B) = \int_{0}^{\infty} \beta s\mathcal{M}_{\beta}(s)e^{-s\mathfrak{t}^{\beta}B}ds, $$

where $\mathcal{M}(\theta)$ is Mainardi's Wright Type function defined as

$$ \mathcal{M}_{\beta}(\theta) = \sum_{g=0}^{\infty} \frac{\theta^{m}}{m!\Gamma(1-\beta(1+m))}. $$

Lemma 1. *In uniform operator topology,* $0 < \mathfrak{t}, E_{\beta}(-\mathfrak{t}^{\beta}B)$ *and* $E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)$ *are continuous. On the interval* $[r,\infty]$*, the continuity is uniform for every* $0 < r$.

Lemma 2. *Let* $0 < \beta < 1$. *At that point the following properties holds:*
(i) for every $v \in Y, \lim_{\mathfrak{t}\to 0^{+}} E_{\beta}(-\mathfrak{t}^{\beta}B)v = v$;
(ii) for every $v \in D(B)$ *and* $0 < \mathfrak{t}, {}^{C}D_{\beta}^{\mathfrak{t}}E_{\beta}(-\mathfrak{t}^{\beta}B)v = -BE_{\beta}(-\mathfrak{t}^{\beta}B)v$;
(iii) for every $v \in Y, E_{\beta}'(-\mathfrak{t}^{\beta}B)v = -\mathfrak{t}^{\beta-1}BE_{\beta,\beta}(-\mathfrak{t}^{\beta}B)v$;
(iv) for $0 < \mathfrak{t}, E_{\beta}(-\mathfrak{t}^{\beta}B)v = I_{\mathfrak{t}}^{1-\beta}(\mathfrak{t}^{\beta-1}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)v)$.

Definition 1. *A function* $v : [0,\infty) \to H^{\alpha,p}$ *is said to be the global mild solution of (2) in* $H^{\alpha,p}$*, if* $v \in C([0,\infty), H^{\alpha,p})$ *and for* $\mathfrak{t} \in [0,\infty)$

$$ \begin{aligned} v(\mathfrak{t}) &= E_{\beta}(-\mathfrak{t}^{\beta}B)b + \int_{0}^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)F(v(s),w(s))ds \\ &+ \int_{0}^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(s)ds. \end{aligned} \tag{3} $$

Definition 2. *Suppose that* $0 < \mathfrak{T} < \infty$. *A local mild solution of problem (2) in* $H^{\alpha,p}$ *or in* Q_{p}*, is a function* $v : [0,\mathfrak{T}] \to H^{\alpha,p}$ (Q_{p})*, if* $v \in C([0,\mathfrak{T}], H^{\alpha.p})$ *as well as* v *fulfils (3) for interval* $\mathfrak{t} \in [0,\mathfrak{T}]$.

$$ \begin{aligned} \varphi(\mathfrak{t}) &= \int_{0}^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)g(s)ds \\ \mathcal{U}(v,w) &= \int_{0}^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)F(v(s),w(s))ds. \end{aligned} $$

Lemma 3. *Suppose that* $(Y, \|\cdot\|_{Y})$ *is a Banach space,* $O : Y \times Y \to Y$ *be a bi-linear operator as well as* K *be a non-negative real number in such a way that*

$$ \|O(v,w)\|_{Y} \leq K\|v\|_{Y}\|w\|_{Y}, forall v,w \in Y. $$

Then, for some $v_{0} \in Y$ *with* $\|v_{0}\|_{Y} < \frac{1}{4K}$*, the relation* $v = v_{0} + O(v,w)$ *must have a unique solution* $v \in Y$.

The system (2) is equal to the following integral:

$$v(\mathfrak{t}) = b + \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\beta-1} \Big(Bv(s) + F(v(s), w(s)) + Pg(s) \Big) ds, 0 \leq \mathfrak{t}, \tag{4}$$

provided the integral (4) exist.

Theorem 1. *If (4) holds, then*

$$\begin{aligned} v(\mathfrak{t}) &= E_\beta(-\mathfrak{t}^\beta B)b + \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\beta-1} E_{\beta,\beta}(-(\mathfrak{t} - s)^\beta B) F(v(s), w(s)) ds \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\beta-1} E_{\beta,\beta}(-(\mathfrak{t} - s)^\beta B) Pg(s) ds, \end{aligned}$$

where

$$E_\beta(-\mathfrak{t}^\beta B) = \int_0^\infty M_\beta(\theta) T(\mathfrak{t}^\beta \theta) d\theta, E_{\beta,\beta}(-\mathfrak{t}^\beta B) = \int_0^\infty \beta\theta M_\beta(\theta) T(\mathfrak{t}^\beta \theta) d\theta.$$

Proof. Let $\lambda > 0$

$$\nu(\lambda) = \int_0^\infty e^{-\lambda s} v(s) ds, \mu(\lambda) = \int_0^\infty e^{-\lambda s} g(s) ds.$$

Apply Laplace Transformation on (4)

$$\nu(\lambda) = \lambda^{\beta-1}(\lambda^\beta I - B)^{-1} b + (\lambda^\beta I - B)^{-1} \mu(\lambda),$$

for $\mathfrak{t} \geq 0$

$$\nu(\lambda) = \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} T(s) b ds + \int_0^\infty e^{-\lambda^\beta s} T(s) \mu(\lambda) ds.$$

Let

$$\phi_\beta(\theta) = \frac{\beta}{\theta^{\beta+1}} M_\beta(\theta^{-\beta}), \beta \in (0, 1),$$

and its Laplace Transform is given by

$$\int_0^\infty e^{-\lambda\theta} \phi_\beta(\theta) d\theta = e^{-\lambda^\beta}, \tag{5}$$

using (4), so

$$\begin{aligned} \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} T(s) b ds &= \int_0^\infty \beta(\lambda\mathfrak{t})^{\beta-1} e^{-(\lambda\mathfrak{t})^\beta} T(\mathfrak{t}^\beta) b d\mathfrak{t} \\ &= \int_0^\infty -\frac{1}{\lambda} \frac{d}{d\mathfrak{t}} \Big(\int_0^\infty e^{(-\lambda\mathfrak{t})^\beta} \phi_\beta(\theta) d\theta \Big) T(\mathfrak{t}^\beta) b d\mathfrak{t} \\ &= \int_0^\infty \int_0^\infty \frac{-\lambda\theta}{-\lambda} e^{-\lambda\mathfrak{t}\theta} \phi_\beta(\theta) T(\mathfrak{t}^\beta) b d\mathfrak{t} \\ &= \int_0^\infty \int_0^\infty \theta \phi_\beta(\theta) e^{-\lambda\mathfrak{t}\theta} T(\mathfrak{t}^\beta) b d\mathfrak{t} d\theta \\ &= \int_0^\infty \int_0^\infty \phi_\beta(\theta) e^{-\lambda\mathfrak{t}} T(\frac{\mathfrak{t}^\beta}{\theta^\beta}) b d\theta d\mathfrak{t} \\ &= \int_0^\infty e^{-\lambda\mathfrak{t}} \Big[\int_0^\infty \phi_\beta(\theta) T(\frac{\mathfrak{t}^\beta}{\theta^\beta}) b \Big] d\theta d\mathfrak{t} \\ &= \mathcal{L}\Big[\int_0^\infty M_\beta(\theta) T(\mathfrak{t}^\beta \theta) b d\theta \Big](\lambda) \\ &= \mathcal{L}[E_\beta(-\mathfrak{t}^\beta B) b](\lambda). \end{aligned} \tag{6}$$

Similarly

$$\begin{aligned}
\int_0^\infty e^{-\lambda^\beta s}T(s)\mu(\lambda)ds &= \int_0^\infty\int_0^\infty \beta \mathrm{t}^{\beta-1}e^{(-\lambda \mathrm{t})^\beta}T(\mathrm{t}^\beta)e^{-\lambda s}[F(v(s),w(s))+Pg(s)]dsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta \mathrm{t}^{\beta-1}\phi_\beta(\theta)e^{-\lambda \mathrm{t}\theta}T(\mathrm{t}^\beta)e^{-\lambda s}[F(v(s),w(s))+Pg(s)]d\theta dsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta\frac{\mathrm{t}^{\beta-1}}{\theta^\beta}\phi_\beta(\theta)T(\frac{\mathrm{t}^\beta}{\theta^\beta})e^{-\lambda(\mathrm{t}+s)}[F(v(s),w(s))+Pg(s)]d\theta dsd\mathrm{t} \\
&= \int_0^\infty e^{-\lambda \mathrm{t}}\bigg[\beta\int_0^{\mathrm{t}}\int_0^\infty \phi_\beta(\theta)T\bigg(\frac{(\mathrm{t}-s)^\beta}{\theta^\beta}\bigg)\frac{(\mathrm{t}-s)^{\beta-1}}{\theta^\beta} \\
&\quad [F(v(s),w(s))+Pg(s)]d\theta ds\bigg]d\mathrm{t}. \qquad (7)
\end{aligned}$$

Combining Equations (5)–(7), one has

$$\begin{aligned}
\nu(\lambda) &= \int_0^\infty e^{-\lambda \mathrm{t}}\bigg[\int_0^\infty \phi_\beta(\theta)T(\frac{\mathrm{t}^\beta}{\theta^\beta})bd\theta+\beta\int_0^{\mathrm{t}}\int_0^\infty \phi_\beta(\theta)T\bigg(\frac{(\mathrm{t}-s)^\beta}{\theta^\beta}\bigg)\frac{(\mathrm{t}-s)^{\beta-1}}{\theta^\beta} \\
&\quad [F(v(s),w(s))+Pg(s)]d\theta ds\bigg].
\end{aligned}$$

By applying the Laplace Transform,

$$\begin{aligned}
& v(\mathrm{t}) \\
=& \int_0^\infty \phi_\beta(\theta)T(\frac{\mathrm{t}^\beta}{\theta^\beta})bd\theta+\beta\int_0^{\mathrm{t}}\int_0^\infty \phi_\beta(\theta)T\bigg(\frac{(\mathrm{t}-s)^\beta}{\theta^\beta}\bigg)\frac{(\mathrm{t}-s)^{\beta-1}}{\theta^\beta}[F(v(s),w(s))+Pg(s)]d\theta ds \\
=& \int_0^\infty M_\beta(\theta)T(\mathrm{t}^\beta\theta)bd\theta+\beta\int_0^{\mathrm{t}}\int_0^\infty \theta(\mathrm{t}-s)^{\beta-1}M_\beta(\theta)T((\mathrm{t}-s)^\beta\theta)[F(v(s),w(s))+Pg(s)]d\theta ds \\
=& E_\beta(-\mathrm{t}^\beta B)b+\int_0^{\mathrm{t}}(\mathrm{t}-s)^{\beta-1}E_{\beta,\beta}(-\mathrm{t}^\beta B)[F(v(s),w(s))+Pg(s)].
\end{aligned}$$

We rewrite the above equation

$$v(\mathrm{t}) = b+\frac{1}{\Gamma(\beta)}\int_0^{\mathrm{t}}(\mathrm{t}-s)^{\beta-1}\bigg(Bv(s)+F(v(s),w(s))+Pg(s)\bigg)ds.$$

Thus, the proof is complete. □

Proposition 1. *Prove that*

(i) $E_{\beta,\beta}(-\mathrm{t}^\beta B) = \frac{1}{2\pi i}\int_{\Gamma\theta}E_{\beta,\beta}(-\nu \mathrm{t}^\beta)(\nu I+B)^{-1}d\nu$;

(ii) $B^\gamma E_{\beta,\beta}(-\mathrm{t}^\beta B) = \frac{1}{2\pi i}\int_{\Gamma\theta}\nu^\gamma E_{\beta,\beta}(-\nu \mathrm{t}^\beta)(\nu I+B)^{-1}d\nu$

Proof. (i) Since $\int_0^\infty \beta s M_\beta(s)e^{-s\mathrm{t}^\beta B}ds = E_{\beta,\beta}(-\mathrm{t})$, by using Fabini's Theorem, we get

$$\begin{aligned}
E_{\beta,\beta}(-\mathrm{t}) &= \int_0^\infty \beta s M_\beta(s)e^{-s\mathrm{t}^\beta B}ds \\
&= \frac{1}{2\pi i}\int_0^\infty \beta s M_\beta(s)\int_{\Gamma\theta}e^{-\nu s\mathrm{t}^\beta}(\nu I+B)^{-1}d\nu ds \\
&= \frac{1}{2\pi i}\int_0^\infty \beta s M_\beta(s)e^{-\nu s\mathrm{t}^\beta}ds\int_{\Gamma\theta}(\nu I+B)^{-1}d\nu \\
&= \frac{1}{2\pi i}\int_{\Gamma\theta}E_{\beta,\beta}(-\nu \mathrm{t}^\beta)(\nu I+B)^{-1}d\nu.
\end{aligned}$$

(ii) We follow the same steps

$$\begin{aligned} B^{\gamma}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B) &= \int_0^{\infty} \beta s M_{\beta}(s) B^{\gamma} e^{-s\mathfrak{t}^{\beta}B} ds \\ &= \frac{1}{2\pi i}\int_0^{\infty} \beta s M_{\beta}(s) \int_{\Gamma\theta} \nu^{\gamma} e^{-\nu s \mathfrak{t}^{\beta}} (\nu I + B)^{-1} d\nu ds \end{aligned}$$

$$\begin{aligned} B^{\gamma}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B) &= \frac{1}{2\pi i}\int_0^{\infty} \nu^{\gamma} \beta s M_{\beta}(s) e^{-\nu s \mathfrak{t}^{\beta}} ds \int_{\Gamma\theta} (\nu I + B)^{-1} d\nu \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} \nu^{\gamma} E_{\beta,\beta}(-\nu \mathfrak{t}^{\beta})(\nu I + B)^{-1} d\nu. \end{aligned}$$

□

3. Global and Local Existence in $H^{\alpha,p}$

In this section, our main purpose is to build up sufficient conditions for the existence and uniqueness of the mild solution of problem (2) in $H^{\alpha,p}$. We suppose that

Hypothesis 1 (H1). *Pg is said to be continuous for $0 < \mathfrak{t}$ and $|Pg(\mathfrak{t})|_p = s(\mathfrak{t}^{-\beta(1-\alpha)})$ as $\mathfrak{t} \to 0$ for $1 > \alpha > 0$.*

Lemma 4. *See ([23]). Suppose that $1 < p < \infty$ and $\alpha_1 \leq \alpha_2$. Then, at that point there exist a constant $\mathfrak{C} = \mathfrak{C}(\alpha_1, \alpha_2)$ in such a way that*

$$|e^{-\mathfrak{t}B} w|_{H^{\alpha_2,p}} \leq \mathfrak{C}\mathfrak{t}^{-(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}}, 0 < \mathfrak{t},$$

for $w \in H^{\alpha_1,p}$. Moreover, $\lim_{\mathfrak{t}\to 0} \mathfrak{t}^{(\alpha_2-\alpha_1)}|e^{-\mathfrak{t}B} w|_{H^{\alpha_2,p}} = 0$.

Lemma 5. *Suppose that $1 < p < \infty$ and $\alpha_1 \leq \alpha_2$. For any $R > 0$ there is a constant $\mathfrak{C}_1 = \mathfrak{C}_1(\alpha_1, \alpha_2) > 0$ in such a way that*

$$|E_{\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} \leq \mathfrak{C}_1 \mathfrak{t}^{-\beta(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}} and |E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} \leq \mathfrak{C}_1 \mathfrak{t}^{-\beta(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}}$$

for all $w \in H^{\alpha_1,p}$ as well as $\mathfrak{t} \in (0, R]$. Moreover,

$$\lim_{\mathfrak{t}\to 0} \mathfrak{t}^{\beta(\alpha_2-\alpha_1)}|E_{\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} = 0.$$

Proof. Let $w \in H^{\alpha_1,p}$. According to Lemma 4, we consider

$$\begin{aligned} |E_{\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} &\leq \int_0^{\infty} \mathcal{M}_{\beta}(s)|e^{-s\mathfrak{t}^{\beta}B} w|_{H^{\alpha_2,p}} ds \\ &\leq \left(\mathfrak{C}\int_0^{\infty} \mathcal{M}_{\beta}(s) s^{-(\alpha_2-\alpha_1)} ds\right)\mathfrak{t}^{-\beta(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}} \\ &\leq \mathfrak{C}_1 \mathfrak{t}^{-\beta(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}}. \end{aligned}$$

A well-known theorem, $\mathcal{LebesgueDominatedConvergence}$ theorem shows that

$$\lim_{\mathfrak{t}\to 0} \mathfrak{t}^{\beta(\alpha_2-\alpha_1)}|E_{\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} \leq \int_0^{\infty} \mathcal{M}(s) \lim_{\mathfrak{t}\to 0} \mathfrak{t}^{\beta(\alpha_2-\alpha_1)}|E_{\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} = 0.$$

Similarly

$$|E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} \leq \int_0^{\infty} \beta s \mathcal{M}_{\beta}(s)|e^{-s\mathfrak{t}^{\beta}B} w|_{H^{\alpha_2,p}} ds$$

$$
\begin{aligned}
|E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)w|_{H^{\alpha_2,p}} &\leq \left(\beta\mathfrak{C}\int_0^{\infty}\mathcal{M}_{\beta}(s)s^{1-(\alpha_2-\alpha_1)}ds\right)\mathfrak{t}^{-\beta(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}} \\
&\leq \mathfrak{C}_1\mathfrak{t}^{-\beta(\alpha_2-\alpha_1)}|w|_{H^{\alpha_1,p}},
\end{aligned}
$$

where the constant term is $\mathfrak{C}_1 = \mathfrak{C}_1(\beta, \alpha_1, \alpha_2)$, such that

$$
\mathfrak{C}_1 \geq \mathfrak{C}\max\left\{\frac{\Gamma(1-\alpha_2+\alpha_1)}{\Gamma(1+\beta(\alpha_1-\alpha_2))}, \frac{\beta\Gamma(2-\alpha_2+\alpha_1)}{\Gamma(1+\beta(\alpha_1-\alpha_2))}\right\}.
$$

□

3.1. Global Existence in $H^{\alpha,p}$

The global mild solution of (2) in $H^{\alpha,p}$ is investigated in this subsection. For comfort, we signify

$$
\begin{aligned}
\mathcal{N}(\mathfrak{t}) &= \sup_{s\in(0,\mathfrak{t}]}\{s^{\beta(1-\alpha)}|Pg(s)|_p\}, \\
V_1 &= \mathfrak{C}_1\max\{V(\beta(1-\alpha), 1-\beta(1-\alpha)), V(\beta(1-\xi), 1-\beta(1-\alpha))\}, \\
K &\geq \mathcal{M}\mathfrak{C}_1\max\left\{V(\beta(1-\alpha), 1-2\beta(\xi-\alpha)), V(\beta(1-\xi), 1-2\beta(\xi-\alpha))\right\}.
\end{aligned}
$$

Theorem 2. *Suppose that $1 < p < \infty, 0 < \alpha < 1$ and condition (H_1) holds. For each $\beta \in H^{\alpha,p}$. Let*

$$
\mathfrak{C}_1|b|_{H^{\alpha,p}} + V_1\mathcal{N}_{\infty} < \frac{1}{4K}, \tag{8}
$$

where $\mathcal{N}_{\infty} = \sup_{s\in(0,\infty)}\{s^{\beta(1-\alpha)}|Pg(s)|_p\}$. If $\frac{m}{2p} - \frac{1}{2} < \alpha$, then at that point there is $b\xi > \max\{\alpha, \frac{1}{2}\}$ and a unique function $v : [0,\infty) \to H^{\alpha,p}$ fulfils the conditions given below:

(i) $v : [0,\infty) \to H^{\alpha,p}$ is continuous as well as $v(0) = b$;

(ii) $v : [0,\infty) \to H^{\xi,p}$ is continuous as well as $\lim_{\mathfrak{t}\to 0}\mathfrak{t}^{\beta(\xi-\alpha)}|v(\mathfrak{t})|_{H^{\xi,p}} = 0$;

(iii) v fulfils (3) for $t \in [0,\infty)$.

Proof. The proof of this theorem is similar to that in [24] with a slight change according to our problem. □

3.2. Local Existence in $H^{\alpha,p}$

The local mild solution of (2) in $H^{\alpha,p}$ is discussed in this section.

Theorem 3. *Let $1 < p < \infty, 0 < \alpha < 1$ and $(\mathbf{H1})$ (the supposition is given in the beginning of Section 3) holds. Assume that*

$$
\frac{m}{2p} - \frac{1}{2} < \alpha.
$$

Then, there is $\xi > \max\{\alpha, \frac{1}{2}\}$ in such a way that for each $b \in H^{\alpha,p}$ there exist $\mathfrak{T}_* > 0$ as well as $v : [0,\mathfrak{T}_*] \to H^{\alpha,p}$ is a unique function that fulfils the following properties:

(i) $v : [0,\mathfrak{T}_*] \to H^{\alpha,p}$ is continuous and $v(0) = b$;

(ii) $v : [0,\mathfrak{T}_*] \to H^{\xi,p}$ is continuous and $\lim_{\mathfrak{t}\to 0}\mathfrak{t}^{\beta(\xi-\alpha)}|v(\mathfrak{t})|_{H^{\xi,p}} = 0$;

(iii) For $\mathfrak{t} \in [0,\mathfrak{T}_*]$, v satisfy (3).

Proof. Suppose that $\xi = \frac{1+\alpha}{2}$ and the space of all curves is $Y = Y[\mathfrak{T}]\ v : (0, \mathfrak{T}] \to H^{\alpha,p}$ in such a way that:

(i) $v : [0, \mathfrak{T}_*] \to H^{\alpha,p}$ is continuous and $v(0) = b$;

(ii) $v : [0, \mathfrak{T}_*] \to H^{\xi,p}$ is continuous and $\lim_{t\to 0} t^{\beta(\xi-\alpha)}|v(t)|_{H^{\xi,p}} = 0$; with its neutral form

$$\|v\|_Y = \sup_{t\in[0,\mathfrak{T}]} \{t^{\beta(\xi-\alpha)}|v(t)|_{H^{\xi,p}}\}.$$

Alike the proof of Theorem 2, it is not difficult to claim that $\mathcal{U} : Y \times Y \to Y$ be continuous linear mapping as well as $\varphi(t) \in Y$.

$$\begin{aligned} E_\beta(-t^\beta B)b &\in \mathfrak{C}([0, \mathfrak{T}], H^{\alpha,p}), \\ E_\beta(-t^\beta B)b &\in \mathfrak{C}([0, \mathfrak{T}], H^{\xi,p}). \end{aligned}$$

By Lemma 5, it can easily be seen that

$$\begin{aligned} E_\beta(-t^\beta B)b &\in Y, \\ t^{\beta(\xi-\alpha)}E_\beta(-t^\beta B)b &\in \mathfrak{C}([0, \mathfrak{T}], H^{\xi,p}). \end{aligned}$$

Therefore, let $\mathfrak{T}_* > 0$ be small in such a way that

$$\|E_\beta(-t^\beta B)b + \varphi(t)\|_{Y[\mathfrak{T}_*]} \le \|E_\beta(-t^\beta B)b\|_{Y[\mathfrak{T}_*]} + \|\phi(t)\|_{Y[\mathfrak{T}_*]} < \frac{1}{4K}.$$

As a result of Lemma 3, $\mathcal{F}$ has a fixed point that is unique. $\square$

4. Local Existence in Q_p

In this section, we discuss the local mild solution of (2) by using iteration method. Suppose that $\xi = \frac{1+\alpha}{2}$:

Theorem 4. *Suppose that* $1 < p < \infty, 0 < \alpha < 1$ *and* (***H1***)*(the supposition is given in the beginning of Section 3) holds. Assume that*

$$b \in H^{\alpha,p} with \frac{m}{2p} - \frac{1}{2} < \alpha.$$

Then, the problem (2) has mild solution v *by* Q_p *for* $b \in H^{\alpha,p}$*. Furthermore,* v *must be continuous on* $(0, \mathfrak{T}]$*,* $B^\xi v$*, be continuous on* $(0, \mathfrak{T}]$ *and* $t^{\beta(\xi-\alpha)}B^\xi v(t)$ *is bounded as* $t \to 0$.

Proof. Step 1: Describe

$$\mathfrak{R}(t) := \sup_{s\in(0,t]} s^{\beta(\xi-\alpha)}|B^\xi v(s)|_p,$$

and

$$\psi(t) := \mathcal{U}(v, w)(t) = \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta B)F(v(s) - w(s))ds.$$

$$|B^\xi \psi(t)|_p \le \mathcal{N}\mathfrak{C}_1 V(\beta(1-\xi), 1 - 2\beta(\xi - \alpha))\mathfrak{R}^2(t)t^{-\beta(\xi-\alpha)},$$

considering the integral $\varphi(t)$. Thus

$$|Pg(s)|_p \le \mathcal{N}(t)s^{\beta(1-\alpha)},$$

where $\mathcal{N}$ is a continuous function. Using Theorem 2, we show that $B^{\xi}(\mathrm{t})$ is continuous in the interval $(0,\mathfrak{T}]$ by using

$$|B^{\xi}\varphi(\mathrm{t})|_p \leq \mathfrak{C}_1\mathcal{N}(\mathrm{t})V(\beta(1-\xi),1-\beta(1-\alpha))\mathrm{t}^{-\beta(\xi-\alpha)}. \tag{9}$$

For $|Pg(\mathrm{t})|_p = s(\mathrm{t}^{-\beta(1-\alpha)})$ as $\mathrm{t}\to 0$, $\mathcal{N}(\mathrm{t}) = 0$ is the solution. Here, (9) denotes, $|B^{\xi}\varphi(\mathrm{t})|_p s(\mathrm{t}^{-\beta(1-\alpha)})$ as $\mathrm{t}\to 0$. In Q_p, we show that φ is continuous. In fact, if we take $0\leq \mathrm{t}_0 < \mathrm{t} < \mathfrak{T}$, we get

$$\begin{aligned}
& |\varphi(\mathrm{t})-\varphi(\mathrm{t}_0)|_p \\
\leq\ & \mathfrak{C}_3\int_{\mathrm{t}_0}^{\mathrm{t}}(\mathrm{t}-s)^{\beta-1}|Pg(s)|_p ds + \mathfrak{C}_3\int_0^{\mathrm{t}_0}\left((\mathrm{t}_0-s)^{\beta-1}-(\mathrm{t}-s)^{\beta-1}\right)|Pg(s)|_p ds \\
+\ & \mathfrak{C}_3\int_0^{\mathrm{t}_0-\epsilon}(\mathrm{t}_0-s)^{\beta-1}\|E_{\beta,\beta}(-(\mathrm{t}-s)^{\beta}B)-E_{\beta,\beta}(-(\mathrm{t}_0-s)^{\beta}B\||Pg(s)|_p ds \\
+\ & 2\mathfrak{C}_3\int_{\mathrm{t}_0-\epsilon}^{\mathrm{t}_0}(\mathrm{t}_0-s)^{\beta-1}|Pg(s)|_p ds \\
\leq\ & \mathfrak{C}_3\mathcal{N}(\mathrm{t})\int_{\mathrm{t}_0}^{\mathrm{t}}(\mathrm{t}-s)^{\beta-1}s^{-\beta(1-\alpha)}ds + \mathfrak{C}_3\mathcal{N}(\mathrm{t})\int_0^{\mathrm{t}}\left((\mathrm{t}-s)^{\beta-1}-(\mathrm{t}_0-s)^{\beta-1}\right)s^{-\beta(1-\alpha)}ds \\
+\ & \mathfrak{C}_3\mathcal{N}(\mathrm{t})\int_0^{\mathrm{t}_0-\epsilon}(\mathrm{t}_0-s)^{\beta-1}s^{-\beta(1-\alpha)}ds \sup_{s\in[0,\mathrm{t}-\epsilon]}\|E_{\beta,\beta}(-(\mathrm{t}-s)^{\beta}B)-E_{\beta,\beta}(-(\mathrm{t}_0-s)^{\beta}B\| \\
+\ & 2\mathfrak{C}_3\mathcal{N}(\mathrm{t})\int_{\mathrm{t}_0-\epsilon}^{\mathrm{t}_0}(\mathrm{t}_0-s)^{\beta-1}s^{-\beta(1-\alpha)}ds \to 0, as \mathrm{t}\to \mathrm{t}_0,
\end{aligned}$$

as a result of previous conversations.

We also consider the function $E_{\beta}(-\mathrm{t}^{\beta}B)b$. It is clear by Lemma 5 that

$$\begin{aligned}
|B^{\xi}E_{\beta}(-\mathrm{t}^{\beta}B)b|_p \leq \mathfrak{C}_1\mathrm{t}^{-\beta(1-\alpha)}|B^{\alpha}b|_p = \mathfrak{C}_1\mathrm{t}^{-\beta(1-\alpha)}|b|_{H^{\alpha,p}}, \\
\lim_{\mathrm{t}\to 0}\mathrm{t}^{\beta(\xi-\alpha)}|B^{\xi}E_{\beta}(-\mathrm{t}^{\beta}B)b|_p = \lim_{\mathrm{t}\to 0}\mathrm{t}^{\beta(\xi-\alpha)}|E_{\beta}(-\mathrm{t}^{\beta}B)b|_{H^{\alpha,p}} = 0.
\end{aligned}$$

Step 2: Now, we derive the result using successive approximations:

$$\begin{aligned}
v_0(\mathrm{t}) &= E_{\beta}(-\mathrm{t}^{\beta}B)b+\varphi(\mathrm{t}), \\
v_{m+1} &= v_0(\mathrm{t})+\mathcal{U}(v_m,w_m)(\mathrm{t}), m=0,1,2\cdots.
\end{aligned} \tag{10}$$

Using the information presented above, we can deduce that

$$\mathfrak{R}_m(\mathrm{t}) := \sup_{s\in(0,\mathrm{t}]} s^{\beta(\xi-\alpha)}|B^{\xi}v_m(s)|_p$$

are increasing and continuous functions on $[0,\mathfrak{T}]$ with $\mathfrak{R}_m(0)=0$. Furthermore, $\mathfrak{R}_m(t)$ fulfils the following inequality as a result of (9) and (10):

$$\mathfrak{R}_{m+1}(\mathrm{t}) \leq \mathfrak{R}_0(\mathrm{t})+\mathcal{N}\mathfrak{C}_1V(\beta(1-\xi),1-2\beta(\xi-\alpha))\mathfrak{R}_m^2(\mathrm{t}). \tag{11}$$

We choose $\mathfrak{T}>0$ such that $\mathfrak{R}_0(0)=0$,

$$4\mathcal{N}\mathfrak{C}_1V(\beta(1-\xi),1-2\beta(\xi-\alpha))\mathfrak{R}_0(\mathfrak{T}) < 1. \tag{12}$$

The sequence $\mathfrak{R}_m(\mathfrak{T})$ is thus bounded, according to a fundamental consideration of (11).

$$\mathfrak{R}_m(\mathfrak{T}) \leq \varrho(\mathfrak{T}), m=0,1,2...,$$

where

$$\varrho(\mathfrak{t}) = \frac{1 - \sqrt{1 - 4\mathcal{N}\mathfrak{C}_1 V(\beta(1-\xi), 1 - 2\beta(\xi - \alpha))\mathfrak{K}_0(\mathfrak{t})}}{2\mathcal{N}V\mathfrak{C}_1(\beta(1-\xi), 1 - 2\beta(\xi - \alpha))}.$$

In the same way, $\mathfrak{R}_m(\mathfrak{t}) \leq \varrho(\mathfrak{t})$ holds for any $\mathfrak{t} \in (0, \mathfrak{T})$. Similarly, we may see that

$$\varrho(\mathfrak{t}) \leq 2\mathfrak{R}_0(\mathfrak{t}).$$

Suppose that the equality

$$k_{m+1}(\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\beta - 1} E_{\beta,\beta}(-(\mathfrak{t} - s)^{\beta} B)[F(v_{m+1}(s), w_{m+1}(s)) - F(v_m(s), w_m(s))]ds,$$

where $k_m = v_{m+1} - v_n, m = 0, 1, ...,$ as well as $\mathfrak{t} \in (0, \mathfrak{T}]$. Writing

$$\mathcal{W}_m(\mathfrak{t}) := \sup_{s \in (0,\mathfrak{t}]} s^{\beta(\xi - \alpha)} |B^{\xi} k_m(s)|_p.$$

By Equation (8), we get

$$|J(v_{m+1}(s), w_{m+1}(s)) - J(v_m(s), w_m(s))|_p \leq \mathcal{N}(\mathfrak{R}_{m+1}(s) + \mathfrak{R}_m(t))J_m(s)s^{-2\beta(\xi - \alpha)},$$

by Theorem 2, we have

$$\mathfrak{t}^{\beta(\xi - \alpha)} |B^{\xi} k_{m+1}(\mathfrak{t})| \leq 2\mathcal{N}\mathfrak{C}_1 V(\beta(1-\xi), 1 - \beta(1-\alpha))\varrho(\mathfrak{t})\mathcal{W}_m(\mathfrak{t}).$$

The above inequality gives

$$\begin{aligned} \mathcal{W}_{m+1}(\mathfrak{T}) &\leq 2\mathcal{N}\mathfrak{C}_1 V(\beta(1-\xi), 1 - \beta(1-\alpha))\varrho(\mathfrak{t})\mathcal{W}_m(\mathfrak{t}) \\ &\leq 4\mathcal{N}\mathfrak{C}_1 V(\beta(1-\xi), 1 - \beta(1-\alpha))\varrho(\mathfrak{t})\mathfrak{R}_0(\mathfrak{t})\mathcal{W}_m(\mathfrak{t}). \end{aligned} \tag{13}$$

By Equations (12) and (13), it is not difficult to show that

$$\lim_{m \to 0} \frac{J_{m+1}(\mathfrak{T})}{J_m(\mathfrak{T})} < 4\mathcal{N}\mathfrak{C}_1 V(\beta(1-\xi), 1 - \beta(1-\alpha)) < 1,$$

as a result, the series $\Sigma_{m=0}^{\infty} J_m(\mathfrak{T})$ converge. It prove that for $\mathfrak{t} \in (0, \mathfrak{T}]$ the series

$$\Sigma_{m=0}^{\infty} \mathfrak{t}^{\beta(\xi - \alpha)} B^{\xi} k_m(t)$$

converge uniformly. As a result, the sequence $\mathfrak{t}^{\beta(\xi - \alpha)} B^{\xi} v_m(\mathfrak{t})$ converge uniformly in $(0, \mathfrak{T}]$. This suggest that

$$\lim_{m \to 0} v_m(\mathfrak{t}) = v(\mathfrak{t}) \in D(B^{\xi})$$

as well as

$$\lim_{m \to 0} \mathfrak{t}^{\beta(\xi - \alpha)} B^{\xi} v_m(\mathfrak{t}) = \mathfrak{t}^{\beta(\xi - \alpha)} B^{\xi} v(\mathfrak{t}) uniformly,$$

since B^{ξ} is both bounded and $B^{-\xi}$ is closed. As a result, the function

$$\mathfrak{R}(\mathfrak{t}) = \sup_{s \in (0,\mathfrak{T}]} \mathfrak{t}^{\beta(\xi - \alpha)} |B^{\xi} v(s)|_p$$

also meets the condition

$$\mathfrak{R}(\mathfrak{t}) \leq \varrho(\mathfrak{t}) \leq 2\mathfrak{R}_0(\mathfrak{t}), \mathfrak{t} \in (0, \mathfrak{t}]. \tag{14}$$

as well as

$$\begin{aligned} S_m &:= \sup_{s\in(0,\mathfrak{T}]} s^{2\beta(\xi-\alpha)}|F(v_m(s),w_m(s)) - F(v(s),w(s))|_p \\ &\le \mathcal{N}(\mathfrak{R}_m(\mathfrak{T}) + \mathfrak{R}(\mathfrak{T})) \sup_{s\in(0,\mathfrak{T}]} s^{\beta(\xi-\alpha)}|B^{\xi}(v_m(s) - v(s))|_p \to 0, asm \to \infty. \end{aligned}$$

Finally, make sure that v in $[0,\mathfrak{T}]$ is a mild solution to problem (2). Since

$$|\mathcal{U}(v_n,w_n)(\mathfrak{t}) - \mathcal{U}(v,w)|_p \le \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta-1} S_m s^{-2\beta(\xi-\alpha)} ds = \mathfrak{t}^{\beta\alpha} S_m \to 0, (m \to \infty),$$

we have $\mathcal{U}(v_m,w_m)(\mathfrak{t}) \to \mathcal{U}(v,w)(\mathfrak{t})$. We get (9) by taking the limits on both sides

$$v(\mathfrak{t}) = v_0(\mathfrak{t}) + \mathcal{U}(v,w)(\mathfrak{t}). \tag{15}$$

If we set $v(0) = b$, we get (15) for $\mathfrak{t} \in [0,\mathfrak{T}]$ and $v \in \mathfrak{C}([0,\mathfrak{T}],Q_p)$. Furthermore, the consistent convergence of $\mathfrak{t}^{\beta(\xi-\alpha)}B^{\xi}v_m(\mathfrak{t})to\mathfrak{t}^{\beta(\xi-\alpha)}B^{\xi}v(\mathfrak{t})$ drive the continuity of $B^{\xi}v(\mathfrak{t})on(0,\mathfrak{T}]$. According to (14) and $\mathfrak{R}_0(0) = 0$, we have $|B^{\xi}v(t)|_p = s(\mathfrak{t}^{-\beta(\xi-\alpha)})$ is obvious.

Step 3: We show that the mild solution is unique. Assume that v and w are the mild solutions of problem (2). We consider the equality $k = v - w$

$$k(\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta-1} E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)[F(v(s),v(s)) - F(w(s),w(s))]ds.$$

Introducing the function

$$\tilde{\mathfrak{R}}(\mathfrak{t}) := \max\{\sup_{s\in(0,\mathfrak{t}]} s^{\beta(\xi-\alpha)}|B^{\xi}v(s)|_p, \sup_{s\in(0,\mathfrak{t}]} s^{\beta(\xi-\alpha)}|B^{\xi}w(s)|_p\}.$$

By (8) and Lemma 5, we get

$$|B^{\xi}k(\mathfrak{t})|_p \le \mathcal{N}\mathfrak{C}_1\tilde{\mathfrak{R}}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta(1-\xi)-1} s^{-\beta(\xi-\alpha)}|B^{\xi}k(s)|_p ds.$$

For $\mathfrak{t} \in (0,\mathfrak{T})$, the Gronwall inequality demonstrates that $B^{\xi}k(\mathfrak{t}) = 0$. Since $\mathfrak{t} \in [0,\mathfrak{T}]$, this means that $k(\mathfrak{t}) = v(\mathfrak{t}) - w(\mathfrak{t}) = 0$. As a result, the mild solution is unique. □

5. Regularity

Considering the regularity of v which satisfy (2), overall in this section, we suppose that:

Hypothesis 2 (H2). *$Pg(\mathfrak{t})$ be the Holdër continuous along the exponent $\theta \in (0,\beta(1-\xi))$, i.e,*

$$|Pg(\mathfrak{t}) - Pg(s)|_p \le K|\mathfrak{t}-s|^{\theta}, \forall \mathfrak{t} > 0, s \le \mathfrak{T}.$$

Definition 3. *The function $v : [0,\mathfrak{T}] \to Q_p$ is said to be the classical solution of (2), if $v \in \mathfrak{C}([0,\mathfrak{T}],Q_p)$ with ${}^{C}D_{\mathfrak{t}}^{\mathfrak{t}}v(\mathfrak{t}) \in \mathfrak{C}([0,\mathfrak{T}],Q_p)$, which takes the value of $D(B)$ and satisfy (2) for every $\mathfrak{t} \in (0,\mathfrak{T}]$.*

Lemma 6. *Let (**H2**) (the supposition is given in the beginning of Sec. 5) be fulfilled. If*

$$\varphi_1(\mathfrak{t}) := \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta-1} E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)(Pg(s) - Pg(\mathfrak{t}))ds, \mathfrak{t} \in (0,\mathfrak{T}],$$

then $\varphi_1(\mathfrak{t}) \in D(B)$ and $B\varphi_1(\mathfrak{t})\mathfrak{C}^{\theta}([0,\mathfrak{T}],Q_p)$.

Proof. As

$$
\begin{aligned}
(\mathfrak{t}-s)^{\beta-1}|BE_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)(Pg(s)-Pg(\mathfrak{t}))|_p &\leq (\mathfrak{t}-s)^{-1}|(Pg(s)-Pg(\mathfrak{t}))|_p \\
&\leq \mathfrak{C}_1 K(\mathfrak{t}-s)^{\theta-1} \in \mathcal{L}^1([0,\mathfrak{T}],Q_p), \quad (16)
\end{aligned}
$$

then

$$
\begin{aligned}
|B\varphi_1(\mathfrak{t})|_p &\leq \int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}|BE_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)(Pg(s)-Pg(\mathfrak{t}))|_p ds \\
&\leq \mathfrak{C}_1 K\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\theta-1} \leq \frac{\mathfrak{C}_1 R}{\theta}\mathfrak{t}^{\theta} < \infty.
\end{aligned}
$$

We must show that $B\varphi_1(\mathfrak{t})$ is $H\dot{o}$lder continuous.

$$
\frac{d}{d\mathfrak{t}}\big(\mathfrak{t}^{\beta-1}E_{\beta,\beta}(-\nu\mathfrak{t}^{\beta})\big) = \mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(-\nu\mathfrak{t}^{\beta}),
$$

then

$$
\begin{aligned}
\frac{d}{d\mathfrak{t}}\big(\mathfrak{t}^{\beta-1}E_{\beta,\beta}(-\nu\mathfrak{t}^{\beta})\big) &= \frac{1}{2\pi i}\int_{\Gamma\theta}\mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(-\nu\mathfrak{t}^{\beta})B(\nu I+B)^{-1}d\nu \\
&= \frac{1}{2\pi i}\int_{\Gamma\theta}\mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(-\nu\mathfrak{t}^{\beta})d\nu - \frac{1}{2\pi i}\int_{\Gamma\theta}\mathfrak{t}^{\beta-2}\nu E_{\beta,\beta-1}(-\nu\mathfrak{t}^{\beta})(\nu I+B)^{-1}d\nu \\
&= \frac{1}{2\pi i}\int_{\Gamma\theta}-\mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(\zeta)\frac{1}{\mathfrak{t}^{\beta}}d\zeta \\
&- \frac{1}{2\pi i}\int_{\Gamma\theta}-\mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(\zeta)\frac{\zeta}{\mathfrak{t}^{\beta}}\left(-\frac{\zeta}{\mathfrak{t}^{\beta}}I+B\right)^{-1}\frac{1}{\mathfrak{t}^{\beta}}d\zeta.
\end{aligned}
$$

In view of $\|\nu I+B\|^{\leq}\frac{\mathfrak{C}}{|\nu|}$, we derive that

$$
\left\|\frac{d}{d\mathfrak{t}}\big(\mathfrak{t}^{\beta-1}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)\big)\right\| \leq \mathfrak{C}_{\beta}\mathfrak{t}^{-2}, 0<\mathfrak{t}<\mathfrak{T}.
$$

By the Mean Value Theorem, for each $\mathfrak{T}\geq\mathfrak{t}>s>0$, we get

$$
\begin{aligned}
\|\mathfrak{t}^{\beta-1}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)-s^{\beta-1}BE_{\beta,\beta}(-s^{\beta}B)\| &= \left\|\int_s^{\mathfrak{t}}\big(\tau^{\beta-1}BE_{\beta,\beta}(\tau^{\beta}B)\big)d\tau\right\| \\
&\leq \left\|\int_s^{\mathfrak{t}}\big(\tau^{\beta-1}BE_{\beta,\beta}(\tau^{\beta}B)\big)\right\|d\tau \\
&\leq \mathfrak{C}_{\beta}\int_s^{\mathfrak{t}}\tau^{-2}d\tau = \mathfrak{C}+\beta(s^{-1}-\mathfrak{t}^{-1}). \quad (17)
\end{aligned}
$$

Let $k>0$ in such a way that $0<\mathfrak{t}<\mathfrak{t}+k\leq\mathfrak{T}$, then

$$
\begin{aligned}
B\varphi_1(\mathfrak{t}+k)-B\varphi_1(\mathfrak{t}) &= \int_0^{\mathfrak{t}}\big((\mathfrak{t}+k-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)\big) \\
&- (\mathfrak{t}-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)(Pg(s)-Pg(\mathfrak{t}))ds \\
&+ \int_0^{\mathfrak{t}}(\mathfrak{t}+k-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)(Pg(\mathfrak{t})-Pg(\mathfrak{t}+k))ds \\
&+ \int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)(Pg(\mathfrak{t})-Pg(\mathfrak{t}+k))ds \\
&:= k_1(\mathfrak{t})+k_2(\mathfrak{t})+k_3(\mathfrak{t}).
\end{aligned}
$$

We discuss these terms step by step. For $k_1(\mathfrak{t})$, by (16) and (**H1**), we get

$$\begin{aligned}|k_1(\mathfrak{t})|_p &\leq \int_0^{\mathfrak{t}} \|(\mathfrak{t}+k-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B) \\ &- (\mathfrak{t}-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)\||(Pg(s)-Pg(\mathfrak{t}))|_p ds \\ &\leq K\mathfrak{C}_\beta k\int_0^{\mathfrak{t}}(\mathfrak{t}+k-s)^{-1}(\mathfrak{t}-s)^{\theta-1}ds \\ &\leq K\mathfrak{C}_\beta k\int_0^{\mathfrak{t}}(s+k)^{-1}(\mathfrak{t}-s)^{\theta-1}ds \\ &\leq \mathfrak{C}_\beta K\int_0^{k}\frac{k}{s+k}s^{\theta-1}ds + KC_\beta k\int_h^{\infty}\frac{s}{s+k}s^{\theta-1}ds,\end{aligned}$$

so

$$|k_1(\mathfrak{t})|_p \leq K\mathfrak{C}_\beta k^{\theta}. \tag{18}$$

For $k_2(\mathfrak{t})$, by using Lemma 5 and (**H2**),

$$\begin{aligned}|k_2(\mathfrak{t})|_p &\leq \int_0^{\mathfrak{t}}(\mathfrak{t}+k-s)^{\beta-1}|BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)(Pg(\mathfrak{t})-Pg(\mathfrak{t}+k))|_p ds \\ &\leq \mathfrak{C}_1\int_0^{\mathfrak{t}}(\mathfrak{t}+k-s)^{-1}|(Pg(\mathfrak{t})-Pg(\mathfrak{t}+k))|_p ds \\ &\leq K\mathfrak{C}_1 k^{\theta}\int_0^{\mathfrak{t}}(\mathfrak{t}+k-s)^{-1}ds \\ &= K\mathfrak{C}_1[\ln k-\ln(\mathfrak{t}+k)]k^{\theta}.\end{aligned} \tag{19}$$

Moreover, for $k_3(\mathfrak{t})$, again we use (**H2**) and Lemma 5, we get

$$\begin{aligned}|k_3(\mathfrak{t})|_p &\leq \int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{\beta-1}|BE_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)(Pg(\mathfrak{t})-Pg(\mathfrak{t}+k))|_p ds \\ &\leq \mathfrak{C}_1\int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{-1}|(Pg(s)-Pg(\mathfrak{t}+k))|_p ds \\ &\leq \mathfrak{C}_1 K\int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{\theta-1}ds = \mathfrak{C}_1 K\frac{k^{\theta}}{\theta}.\end{aligned} \tag{20}$$

Combining Equations (18), (19) and (20), we conclude that $B\varphi_1(\mathfrak{t})$ is Hölder continuous. □

Theorem 5. *Assume that the suppositions of Theorem 4 are fulfilled. The mild solution of Theorem 4 is classic if for each $b \in D(B)$, (**H2**) holds.*

Proof. In the case of $b \in D(B)$, Part (ii) of Lemma 2 show that $v(\mathfrak{t}) = E_\beta(-\mathfrak{t}^{\beta}B)b(0<\mathfrak{t})$ the following problem has a classic solution:

$$\begin{cases} {}^{C}D_{\mathfrak{t}}^{\beta}v = -Bv, 0<\mathfrak{t}, \\ v(0) = b. \end{cases}$$

Step 1: We show that

$$\varphi(\mathfrak{t}) = \int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-((\mathfrak{t}-s)^{\beta}B)Pg(s)ds,$$

is classic solution of the problem

$$\begin{cases} {}^{C}D_{\mathfrak{t}}^{\beta}v = -Bv + Pg(\mathfrak{t}), 0<\mathfrak{t}, \\ v(0) = b. \end{cases}$$

From Theorem 4 $\varphi \in \mathfrak{C}([0,\mathfrak{T}], Q_p)$, we write $\varphi(\mathfrak{t}) = \varphi_1(\mathfrak{t}) + \varphi_2(\mathfrak{t})$, where

$$\varphi_1(\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta-1} E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)\big(Pg(\mathfrak{t}) - Pg(\mathfrak{t}+k)\big)ds$$

$$\varphi_2(\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta-1} E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(\mathfrak{t})ds.$$

$$B\varphi_2(\mathfrak{t}) = Pg(\mathfrak{t}) - E_{\beta}(-\mathfrak{t}^{\beta}B)Pg(\mathfrak{t}).$$

Since (**H2**) hold, it observes that

$$|B\varphi_2(\mathfrak{t})|_p \leq (1 + (\mathfrak{C}_1)|Pg(\mathfrak{t})|_p,$$

as a result

$$\varphi_2(\mathfrak{t}) \in D(B) as well as B\varphi_2(\mathfrak{t}) \in \mathfrak{C}^{\mu}((0,\mathfrak{T}], Q_p) for \mathfrak{t} \in (0,\mathfrak{T}].$$

We also explain that ${}^{C}D_{\mathfrak{t}}^{\beta}\varphi \in \mathfrak{C}((0,\mathfrak{T}], Q_p)$. By Lemma 2(iv), as well as $\varphi(0) = 0$, we get

$${}^{C}D_{\mathfrak{t}}^{\beta}\varphi(\mathfrak{t}) = \frac{d}{d\mathfrak{t}}\big(I_{\mathfrak{t}}^{1-\beta}\varphi(\mathfrak{t})\big) = \frac{d}{d\mathfrak{t}}(E_{\beta}(-\mathfrak{t}^{\beta}B) * Pg).$$

It remains to show that $E_{\beta}(-\mathfrak{t}^{\beta}B) * Pg$ is continuously differentiable in Q_p. Suppose that $\mathfrak{T} - \mathfrak{t} \geq k > 0$, we have

$$\begin{aligned}\frac{1}{k}(E_{\beta}(-(\mathfrak{t}+k)^{\beta}B) * Pg - E_{\beta}(-\mathfrak{t}^{\beta}B) * Pg) &= \int_0^{\mathfrak{t}} \frac{1}{k}\big(E_{\beta}(-(\mathfrak{t}+k-s)^{\beta}B)Pg(s) \\ &- E_{\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(s)\big)ds \\ &+ \frac{1}{k}\int_0^{\mathfrak{t}+k} E_{\beta}(-(\mathfrak{t}+k-s)^{\beta}B)Pg(s).\end{aligned}$$

Note that

$$\begin{aligned}&\int_0^{\mathfrak{t}} \frac{1}{k}|E_{\beta}(-(\mathfrak{t}+k-s)^{\beta}B)Pg(s) - E_{\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(s)|_p ds \\ \leq\ & \mathfrak{C}_1\frac{1}{k}\int_0^{\mathfrak{t}} |E_{\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(s)|_p \\ +\ & \mathfrak{C}_1\frac{1}{k}\int_0^{\mathfrak{t}} |E_{\beta}(-(\mathfrak{t}+k-s)^{\beta}B)Pg(s)|_p ds \\ \leq\ & \mathfrak{C}_1\mathcal{N}(\mathfrak{t})\frac{1}{k}\int_0^{\mathfrak{t}} (\mathfrak{t}+k-s)^{-\beta}s^{-\beta(1-\alpha)}ds \\ +\ & \mathfrak{C}_1\mathcal{N}(\mathfrak{t})\frac{1}{k}\int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{-\beta}s^{-\beta(1-\alpha)}ds \\ \leq\ & \mathfrak{C}_1\mathcal{N}(\mathfrak{t})\frac{1}{k}\big((\mathfrak{t}+k)^{1-\beta} + \mathfrak{t}^{1-\beta}\big) \\ & V(1-\beta, 1-\beta(1-\alpha)),\end{aligned}$$

according to Dominated Convergence Theorem, we note that

$$\begin{aligned}&\lim_{k\to 0}\int_0^{\mathfrak{t}} \frac{1}{k}\big(E_{\beta}(-(\mathfrak{t}+k-s)^{\beta}B)Pg(s) - E_{\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(s)\big)ds \\ =\ & \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\beta-1}BE_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)Pg(s)ds \\ =\ & B\varphi(\mathfrak{t}).\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{1}{k}\int_{\mathfrak{t}}^{\mathfrak{t}+k} E_\beta(-(\mathfrak{t}+k-s)^\beta B)Pg(s) &= \frac{1}{k}\int_0^k E_\beta(-s^\beta B)Pg(\mathfrak{t}+k-s)ds \\ &= \frac{1}{k}\int_0^k E_\beta(-s^\beta B)\big(Pg(\mathfrak{t}+k-s)ds - Pg(\mathfrak{t}-s)\big)ds \\ &+ \frac{1}{k}\int_0^k E_\beta(-s^\beta B)\big(Pg(\mathfrak{t}-s) - Pg(\mathfrak{t})\big)ds \\ &+ \frac{1}{k}\int_0^k E_\beta(-s^\beta B)Pf(s)ds.\end{aligned}$$

By Lemma 1 and 5 and (**H2**), we get

$$\begin{aligned}\left|\frac{1}{k}\int_0^k E_\beta(-s^\beta B)\big(Pg(\mathfrak{t}+k-s)ds - Pg(\mathfrak{t}-s)\big)ds\right|_p &\leq \mathfrak{C}_1 k^\theta, \\ \left|\frac{1}{k}\int_0^k E_\beta(-s^\beta B)\big(Pg(\mathfrak{t}-s) - Pg(\mathfrak{t})\big)ds\right|_p &\leq \mathfrak{C}_1 K\frac{k^\theta}{\theta+1}.\end{aligned}$$

We conclude that $E_\beta(\mathfrak{t}^\beta B) * Pg$ is differentiable at $\mathfrak{t}_+$ as well as $\frac{d}{d\mathfrak{t}}\big(E_\beta(\mathfrak{t}^\beta B) * Pg\big)_+ = B\varphi(\mathfrak{t}) + Pg(\mathfrak{t})$. Same as $E_\beta(\mathfrak{t}^\beta B) * Pg$ is differentiable at $\mathfrak{t}_-$ as well as $\frac{d}{d\mathfrak{t}}\big(E_\beta(\mathfrak{t}^\beta B) * Pg\big)_- = B\varphi(\mathfrak{t}) + Pg(\mathfrak{t})$.

We indicate $\varphi(\mathfrak{t}) := E_\beta(-\mathfrak{t}^\beta B)b$. According to Lemma 2(iv) and (5)

$$\begin{aligned}|B^\xi\varphi(\mathfrak{t}+k) - B^\xi\varphi(\mathfrak{t})|_p &= \left|\int_{\mathfrak{t}}^{\mathfrak{t}+k} -s^{\beta-1}B^\xi E_{\beta,\beta}(-s^{\beta-1}B)bds\right|_p \\ &\leq \int_{\mathfrak{t}}^{\mathfrak{t}+k} s^{\beta-1}|B^{\xi-\alpha}E_{\beta,\beta}(-s^{\beta-1}B)B^\beta b|_p ds \\ &\leq L_1\int_{\mathfrak{t}}^{\mathfrak{t}+k} s^{\beta(1+\alpha-\xi)-1}ds|B^\beta b|_p \\ &= \frac{L_1|b|_{H^{\alpha,p}}}{\beta(1+\alpha-\xi)}k^{\beta(1+\alpha-\xi)}.\end{aligned}$$

Thus, $B^\xi\varphi \in \mathfrak{C}^\theta((0,\mathfrak{T}], Q_p)$.

For each small $\epsilon > 0$, take k in such a way that $\epsilon \leq \mathfrak{t} < \mathfrak{t}+k \leq k$, since

$$\begin{aligned}|B^\xi\varphi(\mathfrak{t}+k) - B^\xi\varphi(\mathfrak{t})|_p &\leq \left|\int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{\beta-1}B^\xi E_{\beta,\beta}(-(\mathfrak{t}+k-s)^\beta B)Pg(s)ds\right|_p \\ &+ \Big|B^\xi\big((\mathfrak{t}+k-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}+k-s)^\beta B) \\ &- (\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^\beta B)\big)Pg(s)ds\Big|_p \\ &= \varphi_1(\mathfrak{t}) + \varphi_2(\mathfrak{t}).\end{aligned}$$

By applying (**H1**) and Lemma 5, we have

$$\begin{aligned}\varphi_1(\mathfrak{t}) &\leq \mathfrak{C}_1\int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{\beta(1-\xi)-1}|Pg(s)|_p ds\\ &\leq \mathfrak{C}_1\mathcal{N}(\mathfrak{t})\int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}+k-s)^{\beta(1-\xi)-1}s^{-\beta(1-\alpha)}ds\\ &\leq \mathcal{N}(\mathfrak{t})\frac{\mathfrak{C}_1}{\beta(1-\xi)}k^{\beta(1-\xi)}\mathfrak{t}^{-\beta(1-\alpha)}\\ &\leq \mathcal{N}(\mathfrak{t})\frac{\mathfrak{C}_1}{\beta(1-\xi)}k^{\beta(1-\xi)}\epsilon^{-\beta(1-\alpha)}.\end{aligned}$$

To prove $\varphi_2(\mathfrak{t})$, we consider the inequality

$$\begin{aligned}\frac{d}{d\mathfrak{t}}\big(\mathfrak{t}^{\beta-1}B^{\xi}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)\big) &= \frac{1}{2\pi\iota}\int_{\Gamma}\nu^{\xi}\mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(-\nu\mathfrak{t}^{\beta})(\nu I+B)^{-1}d\nu\\ &= \frac{1}{2\pi\iota}\int_{\Gamma'}-\Big(-\frac{\zeta}{\mathfrak{t}^{\beta}}\Big)^{\xi}\mathfrak{t}^{\beta-2}E_{\beta,\beta-1}(\zeta)\Big(-\frac{\zeta}{\mathfrak{t}^{\beta}}I+B\Big)^{-1}\frac{1}{\mathfrak{t}^{\beta}}d\zeta.\end{aligned}$$

This gives that $\|\frac{d}{d\mathfrak{t}}\big(\mathfrak{t}^{\beta-1}B^{\xi}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)\big)\| \leq \mathfrak{C}_{\beta}\mathfrak{t}^{\beta(1-\xi)-2}$. By Mean Value Theorem

$$\begin{aligned}\|\mathfrak{t}^{\beta-1}B^{\xi}E_{\beta,\beta}(-\mathfrak{t}^{\beta}B)-s^{\beta-1}B^{\xi}E_{\beta,\beta}(-s^{\beta}B)\| &\leq \int_s^{\mathfrak{t}}\Big\|\frac{d}{d\tau}\big(\tau^{\beta-1}B^{\xi}E_{\beta,\beta}(-\tau^{\beta}B)\big)\Big\|d\tau\\ &\leq \mathfrak{C}_{\beta}\int_s^{\mathfrak{t}}\tau^{\beta(1-\xi)-2}d\tau = \mathfrak{C}_{\beta}\big(s^{\beta(1-\xi)-1}-\mathfrak{t}^{\beta(1-\xi)-1}\big),\end{aligned}$$

thus

$$\begin{aligned}&\varphi_2(\mathfrak{t})\\ \leq\ &\int_0^{\mathfrak{t}}|B^{\xi}\big((\mathfrak{t}+k-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}+k-s)^{\beta}B)-(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)\big)Pg(s)ds|_p\\ \leq\ &\int_0^{\mathfrak{t}}\big((\mathfrak{t}-s)^{\beta(1-\xi)-1}-(\mathfrak{t}+k-s)^{\beta(1-\xi)-1}\big)|Pg(s)|_p ds\\ \leq\ &\mathfrak{C}_{\beta}\mathcal{N}(\mathfrak{t})\Big(\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta(1-\xi)-1}s^{-\beta(1-\alpha)}ds-\int_0^{\mathfrak{t}+k}(\mathfrak{t}-s+k)^{\beta(1-\xi)-1}s^{-\beta(1-\alpha)}ds\Big)\\ +\ &\mathfrak{C}_{\beta}\mathcal{N}(\mathfrak{t})\int_{\mathfrak{t}}^{\mathfrak{t}+k}(\mathfrak{t}-s+k)^{\beta(1-\xi)-1}s^{-\beta(1-\alpha)}ds\\ \leq\ &\mathfrak{C}_{\beta}\mathcal{N}(\mathfrak{t})\big(\mathfrak{t}^{\beta(\alpha-\xi)}-(\mathfrak{t}+k)^{\beta(\alpha-\xi)}\big)B(\beta(1-\xi),1-\beta(1-\alpha))+\mathfrak{C}_{\beta}\mathcal{N}(\mathfrak{t})k^{\beta(1-\xi)}\mathfrak{t}^{-\beta(1-\alpha)}\\ \leq\ &\mathfrak{C}_{\beta}\mathcal{N}(\mathfrak{t})k^{\beta(1-\xi)}[\epsilon(\epsilon+k)]^{\beta(\alpha-\xi)}+\mathfrak{C}_{\beta}\mathcal{N}(\mathfrak{t})k^{\beta(1-\xi)}\epsilon^{-\beta(1-\alpha)},\end{aligned}$$

which shows that $B^{\xi}\varphi \in \mathfrak{C}^{\theta}([\epsilon,\mathfrak{T}],Q_p)$. Therefore $B^{\xi}\varphi \in \mathfrak{C}^{\theta}([0,\mathfrak{T}],Q_p)$, because of arbitrary ϵ.

Recall

$$\psi(\mathfrak{t}) = \int 0^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}E_{\beta,\beta}(-(\mathfrak{t}-s)^{\beta}B)F(v(s),v(s))ds.$$

Since $|F(v(s),w(s))|_p \leq \mathcal{N}\mathfrak{R}^2(\mathfrak{t})s^{-2\beta(\xi-\alpha)}$, where $\mathfrak{R}(\mathfrak{t}) := \sup_{s\in[0,\mathfrak{t}]}s^{\beta(\xi-\alpha)}|v(s)|_{H^{\xi,p}}$ in $(0,\mathfrak{T}]$, is bounded and continuous. A similar conversation made it possible to provide the Holder continuity of $B^{\xi}\psi$ in $\mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$. Hence, we have $B^{\xi}v(\mathfrak{t}) = B^{\xi}\varphi(\mathfrak{t})+B^{\xi}\varphi(\mathfrak{t})+B^{\xi}\psi(\mathfrak{t}) \in \mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$.

Since $F(v,w) \in \mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$, by Step 2, this proves that ${}^{C}D_{\mathfrak{t}}^{\beta}\psi \in \mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$, $B\psi \in \mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$. and ${}^{C}D_{\mathfrak{t}}^{\beta}\psi = -B\psi+F(v,w)$. We obtain ${}^{C}D_{\mathfrak{t}}^{\beta}v \in \mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$, $Bv \in \mathfrak{C}^{\theta}((0,\mathfrak{T}],Q_p)$ and ${}^{C}D_{\mathfrak{t}}^{\beta}v = -Bv+F(v,w)+Pg$.

Hence, we prove that v is a classical solution. □

6. Example

In this section, we present an example to indicate the applicability of our results:

Example 1. *Suppose that* $Y \in L^2(0, 2\pi)$ *as well as* $\mathfrak{e}_m(y) = 3\sqrt{\frac{3}{2}\pi}\cos x, m = 1, 2, ...$ *At that point, we define infinitesimal dimensional space* $\mathcal{U} = Y$ *and consider a system*

$$\begin{cases} {}^C D_t^{\frac{4}{5}} \mathfrak{z}(t, y) = {}^C D_t^{\frac{2}{3}} \mathfrak{z}(t, y) + f(t, \mathfrak{z}(t, y)) + Qw(t, y), 0 < t < d, 0 < y < 2\pi, \\ \mathfrak{z}(0, y) = \mathfrak{z}_0(y), 0 \le y \le 2\pi, \\ \mathfrak{z}(t, 0) = \mathfrak{z}(t, 2\pi), 0 \le y \le d, \end{cases}$$

where (**H1**) *is satisfied by the nonlinear function* f *as an operator for every* $w \in L^2(0, d; \mathcal{U})$ *and* $\sum_{m=1}^{\infty} \hat{w}_m s(t)\mathfrak{e}_m$*. Consider*

$$Qw(t) = \sum_{m=1}^{\infty} \hat{w}_m s(t)\mathfrak{e}_m,$$

$$\hat{w}_m(t) = \begin{cases} 0, 0 \le t < d(1 - \frac{1}{m}), \\ w_m(t), d(1 - \frac{1}{m}) \le t \le d. \end{cases}$$

Because

$$\|Qw\|_{L^2(0,d;\mathcal{U})} \le \|w\|_{L^2(0,d;\mathcal{U})'},$$

from $\mathcal{U}$ *into* $L^2(\mathfrak{J}, Y)$*, the operator* Q *is bounded. However, it is not easy to see that* $\overline{Q\mathcal{U}} \neq L^2(\mathfrak{J}, Y)$*. Suppose that* φ *is an arbitrary element in* $L^2(0, d, Y)$ *and* $k \in Y$ *is defined as*

$$k = E_\beta(-d - s)^\beta \mathfrak{z}(0)y + \int_0^d (d - s)^{\beta - 1} T_{\frac{4}{5}}(d - s)\varphi(s)ds.$$

Suppose that

$$\varphi(t) = \sum_{m=1}^{\infty} f_m(t)\mathfrak{e}_m,$$

as well as

$$k = \sum_{m=1}^{\infty} k_m(t)\mathfrak{e}_m.$$

Hence, we declare that for each given $\varphi \in L^2(0, d, Y)$*, there exist* $w \in \mathcal{U}$ *in such a way that*

$$\begin{aligned} & E_\beta(-d - s)^\beta \mathfrak{z}(0)y + \int_0^t (d - s)^{\beta - 1} T_{\frac{4}{5}}(d - s)Qw(s)ds \\ = \; & E_\beta(-d - s)^\beta \mathfrak{z}(0)y + \int_0^d (d - s)^{\beta - 1} T_{\frac{4}{5}}(d - s)\varphi(s)ds, \end{aligned}$$

this indicates that (**H2**) *is fulfilled.*

7. Conclusions

The purpose of this paper is to study the time fractional NS-equations using initial value problem with the Caputo derivative. We proved the global and local existence of mild solution in $H^{\alpha,p}$. We established sufficient conditions for the existence and uniqueness of the mild solution for problem (2) in $H^{\alpha,p}$. Moreover, we showed that classical solutions that satisfy problem (2) are regular. Furthermore, we presented the regularity of mild solutions for time fractional NS-equations. In the end, we presented an example.

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