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# Multiplicatively Simpson Type Inequalities via Fractional Integral

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**Abstract:** Multiplicative calculus, also called non-Newtonian calculus, represents an alternative approach to the usual calculus of Newton (1643–1727) and Leibniz (1646–1716). This type of calculus was first introduced by Grossman and Katz and it provides a defined calculation, from the start, for positive real numbers only. In this investigation, we propose to study symmetrical fractional multiplicative inequalities of the Simpson type. For this, we first establish a new fractional identity for multiplicatively differentiable functions. Based on that identity, we derive new Simpson-type inequalities for multiplicatively convex functions via fractional integral operators. We finish the study by providing some applications to analytic inequalities.

**Keywords:** non-Newtonian calculus; Simpson inequality; multiplicatively convex functions

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**Citation:** Moumen, A.; Boulares, H.; Meftah, B.; Shafqat, R.; Alraqad, T.; Ali, E.E.; Khaled, Z. Multiplicatively Simpson Type Inequalities via Fractional Integral. *Symmetry* **2023**, *15*, 460. https://doi.org/10.3390/sym15020460

Academic Editors: Hassen Fourati, Abdellatif Ben Makhlouf and Omar Naifar

Received: 6 January 2023
Revised: 31 January 2023
Accepted: 5 February 2023
Published: 9 February 2023



## 1. Introduction

Between 1967 and 1970, Grossman and Katz created the first non-Newtonian calculation system, called geometric calculation. Over the next few years they created an infinite family of non-Newtonian calculi, thus modifying the classical calculus introduced by Newton and Leibniz in the 17th century each of which differed markedly from the classical calculus of Newton and Leibniz known today as the non-Newtonian calculus or the multiplicative calculus, where the ordinary product and ratio are used, respectively, as sum and exponential difference over the domain of positive real numbers see [1]. This calculation is useful for dealing with exponentially varying functions. It is worth noting that the complete mathematical description of multiplicative calculus was given by Bashirov et al. [2]. We recall that the multiplicative derivatives $\rho^*$ of positive function $\rho$ is defined as follows:

$$\rho^*(t) = \lim_{h\to 0}\left(\frac{\rho(t+h)}{\rho(t)}\right)^{\frac{1}{h}}.$$

The relation between $\rho^*$ and the ordinary derivative $\rho'$ is as follows:

$$\rho^*(t) = e^{(\ln \rho(t))'} = e^{\frac{\rho'(t)}{\rho(t)}}.$$

**Theorem 1.** *Let $\rho : [l,k] \subset \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable function on $(l,k)$. Then we have*

$$\left|\frac{1}{6}\left(\rho(l) + 4\rho\left(\frac{l+k}{2}\right) + \rho(k)\right) - \frac{1}{k-l}\int_l^k \rho(u)du\right| \le \frac{(k-l)^4}{2880}\left\|\rho^{(4)}\right\|_\infty, \quad (1)$$

*where* $\left\|\rho^{(4)}\right\|_\infty = \sup_{u\in[l,k]}\left|\rho^{(4)}(u)\right| < \infty.$

The above inequality is well known in the literature as Simpson's integral inequality. Regarding some results connected with inequality (1) and related inequalities, we refer readers to [3–17]. Shafqat et al. [18] investigated the existence and uniqueness of the Fuzzy fractional evolution equations. Boulares et al. [19,20] studied the existence and uniqueness of solutions for non-linear fractional differential equations.

The multiplicative derivative admits the following properties:

**Theorem 2** ([2])**.** *Let $\rho$ and $\vartheta$ be two multiplicatively differentiable functions, and c is an arbitrary constant. Then functions $c\rho, \rho\vartheta, \rho+\vartheta, \rho/\vartheta$ and $\rho^{\vartheta}$ are $^*$ differentiable and*

- $(c\rho)^*(t) = \rho^*(t)$,
- $(\rho\vartheta)^*(t) = \rho^*(t)\vartheta^*(t)$,
- $(\rho+\vartheta)^*(t) = \rho^*(t)^{\frac{\rho(t)}{\rho(t)+\vartheta(t)}}\vartheta^*(t)^{\frac{\vartheta(t)}{\rho(t)+\vartheta(t)}}$,
- $\left(\frac{\rho}{\vartheta}\right)^*(t) = \frac{\rho^*(t)}{\vartheta^*(t)}$,
- $\left(\rho^{\vartheta}\right)^*(t) = \rho^*(t)^{\vartheta(t)}\rho(t)^{\vartheta'(t)}$.

The multiplicative integral noted $^*$ integral $\int_l^k (\rho(t))^{dt}$ has the following relationship with the Riemann integral

$$\int_l^k (\rho(t))^{dt} = \exp\left\{\int_l^k \ln(\rho(t))dt\right\}.$$

The multiplicative integral enjoy the following properties:

**Theorem 3** ([2])**.** *Let $\rho$ be a positive and Riemann integrable on $[l,k]$, then $\rho$ is multiplicative integrable on $[l,k]$ and*

- $\int_l^k ((\rho(t))^p)^{dt} = \left(\int_l^k (\rho(t))^{dt}\right)^p$,
- $\int_l^k (\rho(t)\vartheta(t))^{dt} = \int_l^k (\rho(t))^{dt}\int_l^k (\vartheta(t))^{dt}$,
- $\int_l^k \left(\frac{\rho(t)}{\vartheta(t)}\right)^{dt} = \frac{\int_l^k (\rho(t))^{dt}}{\int_l^k (\vartheta(t))^{dt}}$,
- $\int_l^k (\rho(t))^{dt} = \int_l^c (\rho(t))^{dt}\int_c^k (\rho(t))^{dt}, l < c < k$,
- $\int_l^l (\rho(t))^{dt} = 1$ and $\int_l^k (\rho(t))^{dt} = \left(\int_k^l (\rho(t))^{dt}\right)^{-1}$.

The multiplicative integration by parts is given by the following Theorem:

**Theorem 4** ([2])**.** *Let $\rho : [l,k] \to \mathbb{R}$ be multiplicative differentiable, let $\vartheta : [l,k] \to \mathbb{R}$ be differentiable so the function $\rho^{\vartheta}$ is multiplicative integrable, and*

$$\int_l^k \left(\rho^*(t)^{\vartheta(t)}\right)^{dt} = \frac{\rho(k)^{\vartheta(k)}}{\rho(l)^{\vartheta(l)}} \times \frac{1}{\int_l^k \left(\rho(t)^{\vartheta'(t)}\right)^{dt}}.$$

Using the above result and the properties of multiplicative derivatives and integrals, Ali et al. [21], established an interesting identity given by the following lemma.

**Lemma 1** ([21])**.** *Let $\rho : [l,k] \to \mathbb{R}$ be multiplicative differentiable, let $h : [l,k] \to \mathbb{R}$ and let $\vartheta : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then we have*

$$\int_l^k \left(\rho^*(h(t))^{h'(t)\vartheta(t)}\right)^{dt} = \frac{\rho(h(k))^{\vartheta(k)}}{\rho(h(l))^{\vartheta(l)}} \times \frac{1}{\int_l^k \left(\rho(h(t))^{\vartheta'(t)}\right)^{dt}}.$$

In recent years, much interest has been given to the development of the theory and applications of multiplicative calculus. Aniszewska [22] presented the multiplicative version of the Runge–Kutta method and used it to solve multiplicative differential equations. Rıza et al. [23], gave the numerical solutions of multiplicative differential equations by introducing the multiplicative finite difference methods. Misirli and Gurefe [24] presented the multiplicative Adams Bashforth-Moulton methods. Bashirov and Norozpour [25] extended the multiplicative integral to complex valued functions. Bhat et al. defined multiplicative Fourier transform in [26] and multiplicative Sumudu transform [27]. Bashirov [28] studied double integrals in the sense of multiplicative calculus. In [29], Ali et al. introduced the multiplicative Hermite–Hadamard inequality for multiplicative integral as follows:

**Theorem 5.** *Let $f$ be a positive and multiplicatively convex function on interval $[u_1, u_2]$, then following inequalities hold*

$$f\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2} (f(t))^{dt}\right)^{\frac{1}{u_2-u_1}} \leq G(f(u_1), f(u_2)),$$

*where $G(\cdot,\cdot)$ is a geometric mean.*

In [30], Ali et al. generalized the obtained results in [29], via $\phi$-convexity. In [31], Özcan generalized the results in [29] under the $h$-convexity. In [32], Özcan established the analogue preinvex of the Hermite–Hadamard inequality. In [33], Özcan generalized the results of [32] for $h$-preinvex functions.

In [34], Meftah studied the so-called Maclaurin type inequalities.

**Theorem 6.** *Let $f : [u_1, u_2] \to \mathbb{R}^+$ be a multiplicative differentiable mapping on $[u_1, u_2]$ with $u_1 < u_2$. If $f^*$ is multiplicative convex on $[u_1, u_2]$, then we have*

$$\begin{aligned} &\left| \left( f\left(\frac{5u_1+u_2}{6}\right)^3 f\left(\frac{u_1+u_2}{2}\right)^2 f\left(\frac{u_1+5u_2}{6}\right)^3 \right)^{\frac{1}{8}} \left(\int_{u_1}^{u_2} f(t)^{dt}\right)^{\frac{1}{u_1-u_2}} \right| \\ \leq\ &\left( (f^*(u_1))^{64} \left(f^*\left(\frac{5u_1+u_2}{6}\right)\right)^{379} \left(f^*\left(\frac{u_1+u_2}{2}\right)\right)^{314} \left(f^*\left(\frac{u_1+5u_2}{6}\right)\right)^{379} \right. \\ &\left. \times (f^*(u_2))^{64} \right)^{\frac{u_2-u_1}{13824}}. \end{aligned}$$

In [21], Ali et al. gave some Ostrowski and Simpson type inequalities for multiplicative integrals as follow:

**Theorem 7.** *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}^{+}$ be a multiplicative differentiable mapping on $I^{\circ}$, $u_1, u_2 \in I^{\circ}$ with $u_1 < u_2$. If $f$ is increasing on $[u_1, u_2]$ and $f^*$is multiplicatively convex on $[u_1, u_2]$, then the following Ostrowski type inequality for multiplicative integrals holds for all $x \in [u_1, u_2]$*

$$\left| f(x) \left( \int_{u_1}^{u_2} (f(t))^{dt} \right)^{\frac{1}{u_1 - u_2}} \right|$$
$$\leq \quad (f^*(a))^{\frac{x-u_1}{2(u_2-u_1)} + \frac{(u_2-x)^3+(x-u_1)^3}{8(u_2-u_1)^2}} (f^*(b))^{\frac{u_2-x}{2(u_2-u_1)} + \frac{(u_2-x)^3+(x-u_1)^3}{8(u_2-u_1)^2}}.$$

**Theorem 8.** *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}^{+}$ be a multiplicative differentiable mapping on $I^{\circ}$, $u_1, u_2 \in I^{\circ}$ with $u_1 < u_2$. If $f$ is increasing on $[u_1, u_2]$ and $f^*$ is multiplicatively convex on $[u_1, u_2]$, then we the following Ostrowski type inequality for multiplicative integrals holds for all $x \in [u_1, u_2]$*

$$\left| \left( f(u_1) f^2 \left( \frac{u_1 + u_2}{2} \right) f(u_2) \right) \left( \int_{u_1}^{u_2} (f(t))^{dt} \right)^{\frac{1}{u_1 - u_2}} \right| \leq ((f^*(a))(f^*(b)))^{\frac{5(u_2-u_1)}{72}}.$$

Recently, Abdeljawad and Grossman [35] introduced the multiplicative Riemann–Liouville fractional integrals as follows:

**Definition 1.** *The of order $\alpha \in \mathbb{C}$ ,$Re(\alpha) > 0$, respectively, are defined by*

$$({}_l I_*^{\alpha} \rho)(x) = e^{\left(J_{l+}^{\alpha} (\ln \circ \rho)\right)(x)}$$

*and*

$$({}_* I_k^{\alpha} \rho)(x) = e^{\left(J_{k-}^{\alpha} (\ln \circ \rho)\right)(x)},$$

*where $J_{l+}^{\alpha}$and $J_{k-}^{\alpha}$ denote the left and right Riemann–Liouville fractional integral, defined by*

$$\left(J_{l+}^{\alpha} \rho\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{l}^{x} (x-t)^{\alpha-1} \rho(t) dt, l < x$$

*and*

$$\left(J_{k-}^{\alpha} \rho\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{k} (t-x)^{\alpha-1} \rho(t) dt, x < k.$$

Budak and Özçelik [36], used the above operator and presented some Hermite–Hadamard type inequalities for multiplicatively fractional integrals.

Hoping to stimulate future research in this direction and motivated by paper [21] and some of the existing results in the literature, in this study, we prove a new integral identity. Based on this, we establish some symmetrical fractional multiplicatively Simpson type inequalities for convex functions. Some applications to special means are proposed to demonstrate the effectiveness of our finding.

## 2. Main Results

We first recall that a positive function $\rho$ is said to be multiplicatively convex, if the following inequality holds

$$\rho(tx + (1-t)y) \leq [\rho(x)]^t [\rho(y)]^{1-t}.$$

**Lemma 2.** *Let $\rho : [l, k] \to \mathbb{R}^{+}$ be a multiplicative differentiable mapping on $[l, k]$ with $l < k$. If $\rho^*$ is multiplicative integrable on $[l, k]$, then we have the following identity for multiplicative integrals*

$$\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\left({}_{l}I_{*}^{\alpha}\rho\right)\left(\frac{l+k}{2}\right)\left({}_{*}I_{\frac{l+k}{2}}^{\alpha}\rho\right)(k)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^{\alpha}}}$$
$$=\left(\int_0^1\left(\rho^*\left((1-t)l+t\frac{l+k}{2}\right)^{t^{\alpha}-\frac{1}{3}}\right)^{dt}\right)^{\frac{k-l}{4}}\left(\int_0^1\left(\rho^*\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{1}{3}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{k-l}{4}}.$$

**Proof.** Let
$$I_1=\left(\int_0^1\left(\rho^*\left((1-t)l+t\frac{l+k}{2}\right)^{t^{\alpha}-\frac{1}{3}}\right)^{dt}\right)^{\frac{k-l}{4}}$$
and
$$I_2=\left(\int_0^1\left(\rho^*\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{1}{3}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{k-l}{4}}.$$
Using the integration by parts for multiplicative integrals, $I_1$ gives

$$\begin{aligned}
I_1 &= \left(\int_0^1\left(\rho^*\left((1-t)l+t\frac{l+k}{2}\right)^{t^{\alpha}-\frac{1}{3}}\right)^{dt}\right)^{\frac{k-l}{4}}\\
&= \left(\int_0^1\left(\rho^*\left((1-t)l+t\frac{l+k}{2}\right)^{\frac{k-l}{2}\left(\frac{1}{2}t^{\alpha}-\frac{1}{6}\right)}\right)^{dt}\right)\\
&= \frac{\left(\rho\left(\frac{l+k}{2}\right)\right)^{\frac{1}{3}}}{(\rho(l))^{-\frac{1}{6}}}\cdot\frac{1}{\int_0^1\left(\rho\left(\left((1-t)l+t\frac{l+k}{2}\right)\right)^{\frac{\alpha}{2}t^{\alpha-1}}\right)^{dt}}\\
&= (\rho(l))^{\frac{1}{6}}\left(\rho\left(\frac{l+k}{2}\right)\right)^{\frac{1}{3}}\frac{1}{\exp\left\{\int_0^1\frac{\alpha}{2}(t)^{\alpha-1}\ln\left(\rho\left((1-t)l+t\frac{l+k}{2}\right)\right)dt\right\}}\\
&= (\rho(l))^{\frac{1}{6}}\left(\rho\left(\frac{l+k}{2}\right)\right)^{\frac{1}{3}}\frac{1}{\exp\left\{\frac{2^{\alpha-1}\alpha}{(k-l)^{\alpha}}\int_0^1(u-l)^{\alpha-1}\ln(\rho(u))du\right\}}\\
&= \frac{(\rho(l))^{\frac{1}{6}}\left(\rho\left(\frac{l+k}{2}\right)\right)^{\frac{1}{3}}}{\left(\exp\left\{\left(\frac{1}{\Gamma(\alpha)}\int_l^{\frac{l+k}{2}}(u-l)^{\alpha-1}\ln(\rho(u))du\right)\right\}\right)^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^{\alpha}}}}\\
&= (\rho(l))^{\frac{1}{6}}\left(\rho\left(\frac{l+k}{2}\right)\right)^{\frac{1}{3}}\left(\left({}_{l}I_{*}^{\alpha}\rho\right)\left(\frac{l+k}{2}\right)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^{\alpha}}}.
\end{aligned}$$

Similarly, we obtain

$$I_2 = \left(\int_0^1\left(\rho^*\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{1}{3}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{k-l}{4}}$$

$$
\begin{aligned}
&= \left( \int_0^1 \left( \rho^* \left( (1-t)\frac{l+k}{2} + tk \right)^{\frac{k-l}{2}\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)} \right)^{dt} \right) \\
&= \frac{(\rho(k))^{\frac{1}{6}}}{\left(\rho\left(\frac{l+k}{2}\right)\right)^{-\frac{1}{3}}} \cdot \frac{1}{\int_0^1 \left( \rho\left( (1-t)\frac{l+k}{2} + tk \right)^{\frac{\alpha}{2}(1-t)^{\alpha-1}} \right)^{dt}} \\
&= \rho\left(\frac{l+k}{2}\right)^{\frac{1}{3}} \rho(k)^{\frac{1}{6}} \cdot \frac{1}{\exp\left\{ \frac{\alpha}{2}\int_0^1 (1-t)^{\alpha-1} \ln\rho\left( (1-t)\frac{l+k}{2} + tk \right) dt \right\}} \\
&= \rho\left(\frac{l+k}{2}\right)^{\frac{1}{3}} \rho(k)^{\frac{1}{6}} \cdot \frac{1}{\exp\left\{ \frac{2^{\alpha-1}}{(k-l)^\alpha} \alpha \int_{\frac{l+k}{2}}^{k} (k-u)^{\alpha-1} \ln\rho(u) dt \right\}} \\
&= \left( \rho\left(\frac{l+k}{2}\right) \right)^{\frac{n}{2(n+2)}} (\rho(k))^{\frac{1}{n+2}} \cdot \frac{1}{\exp\left\{ \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha} \left( \frac{1}{\Gamma(\alpha)} \int_{\frac{l+k}{2}}^{k} (k-u)^{\alpha-1} \ln f(u) du \right) \right\}} \\
&= \frac{\rho\left(\frac{l+k}{2}\right)^{\frac{1}{3}} \rho(k)^{\frac{1}{6}}}{\left( \exp\left\{ \left( \frac{1}{\Gamma(\alpha)} \int_{\frac{l+k}{2}}^{k} (k-u)^{\alpha-1} \ln\rho(u) du \right) \right\} \right)^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}}} . \\
&= \rho\left(\frac{l+k}{2}\right)^{\frac{1}{3}} \rho(k)^{\frac{1}{6}} \cdot \left( \left( {}_*I^\alpha_{\frac{l+k}{2}} \rho \right)(k) \right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}} .
\end{aligned}
$$

Multiplying the above equalities, we obtain

$$
\begin{aligned}
I_1 \times I_2 &= (\rho(l))^{\frac{1}{6}} \left( \rho\left(\frac{l+k}{2}\right) \right)^{\frac{1}{3}} \left( \left({}_lI^\alpha_* \rho\right)\left(\frac{l+k}{2}\right) \right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}} \\
&\times \left( \rho\left(\frac{l+k}{2}\right) \right)^{\frac{1}{3}} (\rho(k))^{\frac{1}{6}} \cdot \left( \left( {}_*I^\alpha_{\frac{l+k}{2}} \rho \right)(k) \right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}} \\
&= \left( (\rho(l)) \left( \rho\left(\frac{l+k}{2}\right) \right)^4 (\rho(k)) \right)^{\frac{1}{6}} \left( \left({}_lI^\alpha_* \rho\right)\left(\frac{l+k}{2}\right) \left( {}_*I^\alpha_{\frac{l+k}{2}} \rho \right)(k) \right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}} .
\end{aligned}
$$

which is the result. The proof is completed. □

**Theorem 9.** *Let $\rho : [l,k] \to \mathbb{R}^+$ be a multiplicatively differentiable mapping on $[l,k]$ with $l < k$. If $|\ln \rho^*| \leq \ln \mathcal{M}$ on $[l,k]$, then we have*

$$
\begin{aligned}
&\left| \left( (\rho(l)) \left( \rho\left(\frac{l+k}{2}\right) \right)^4 (\rho(k)) \right)^{\frac{1}{6}} \left( \left({}_lI^\alpha_* \rho\right)\left(\frac{l+k}{2}\right) \left( {}_*I^\alpha_{\frac{l+k}{2}} \rho \right)(k) \right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}} \right| \\
&\leq \mathcal{M}^{\frac{k-l}{2}\left( \frac{\alpha+2}{3(\alpha+1)} + \frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}+1} \right)} .
\end{aligned}
$$

**Proof.** From Lemma 2, properties of multiplicative integral and using the fact that $|\ln f^*| \leq \ln \mathcal{M}$, we have

$$
\begin{aligned}
&\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^{4}(\rho(k))\right)^{\frac{1}{6}}\left(\left({}_{l}I^{\alpha}_{*}\rho\right)\left(\frac{l+k}{2}\right)\left({}_{*}I^{\alpha}_{\frac{l+k}{2}}\rho\right)(k)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^{\alpha}}}\right| \\
= &\left|\left(\int_{0}^{1}\left(\rho^{*}\left((1-t)l+t\frac{l+k}{2}\right)^{t^{\alpha}-\frac{1}{3}}\right)^{dt}\right)^{\frac{k-l}{4}}\right| \\
&\times\left|\left(\int_{0}^{1}\left(\rho^{*}\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{1}{3}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{k-l}{4}}\right| \\
= &\left|\left(\int_{0}^{1}\left(\rho^{*}\left((1-t)l+t\frac{l+k}{2}\right)^{\frac{k-l}{4}\left(t^{\alpha}-\frac{1}{3}\right)}\right)^{dt}\right)\right| \\
&\times\left|\left(\int_{0}^{1}\left(\rho^{*}\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{k-l}{4}\left(\frac{1}{3}-(1-t)^{\alpha}\right)}\right)^{dt}\right)\right| \\
\leq &\left(\exp\left\{\int_{0}^{1}\left|\frac{k-l}{4}\left(t^{\alpha}-\frac{1}{3}\right)\ln\left(\rho^{*}\left((1-t)l+t\frac{l+k}{2}\right)\right)\right|dt\right\}\right) \\
&\times\left(\exp\left\{\int_{0}^{1}\left|\frac{k-l}{4}\left(\frac{1}{3}-(1-t)^{\alpha}\right)\ln\left(\rho^{*}\left((1-t)\frac{l+k}{2}+tk\right)\right)\right|dt\right\}\right) \\
= &\left(\exp\left\{\frac{k-l}{4}\int_{0}^{1}\left|t^{\alpha}-\frac{1}{3}\right|\left|\ln\left(\rho^{*}\left((1-t)l+t\frac{l+k}{2}\right)\right)\right|dt\right\}\right) \\
&\times\left(\exp\left\{\frac{k-l}{4}\int_{0}^{1}\left|\frac{1}{3}-(1-t)^{\alpha}\right|\left|\ln\left(\rho^{*}\left((1-t)\frac{l+k}{2}+tk\right)\right)\right|dt\right\}\right) \\
\leq &\left(\exp\left\{\frac{k-l}{4}\ln\mathcal{M}\int_{0}^{1}\left|t^{\alpha}-\frac{1}{3}\right|dt\right\}\right)\left(\exp\left\{\frac{k-l}{4}\ln\mathcal{M}\int_{0}^{1}\left|\frac{1}{3}-(1-t)^{\alpha}\right|dt\right\}\right) \\
= &\left(\exp\left\{\frac{k-l}{4}\ln\mathcal{M}\left(\int_{0}^{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{3}-t^{\alpha}\right)dt+\int_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(t^{\alpha}-\frac{1}{3}\right)dt\right)\right\}\right) \\
&\times\left(\exp\left\{\frac{k-l}{4}\ln\mathcal{M}\left(\int_{0}^{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\frac{1}{3}\right)dt+\int_{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{3}-(1-t)^{\alpha}\right)dt\right)\right\}\right) \\
= &\left(\exp\left\{\frac{k-l}{4}\left(\frac{\alpha+2}{3(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}+1}\right)\ln\mathcal{M}\right\}\right) \\
&\times\left(\exp\left\{\frac{k-l}{4}\left(\frac{\alpha+2}{3(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}\right)\ln\mathcal{M}\right\}\right) \\
= &\ \mathcal{M}^{\frac{k-l}{2}\left(\frac{\alpha+2}{3(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}+1}\right)}.
\end{aligned}
$$

The proof is completed. □

**Corollary 1.** *In Theorem 9, if we take $\alpha = 1$, we obtain*

$$\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\int_l^k \rho(u)^{du}\right)^{\frac{1}{l-k}}\right| \leq \mathcal{M}^{\frac{11}{36}(k-l)}.$$

**Theorem 10.** *Let $\rho : [l,k] \to \mathbb{R}^+$ be a multiplicative differentiable mapping on $[l,k]$ with $l < k$. If $\rho^*$ is multiplicatively convex function on $[l,k]$, then we have*

$$\begin{aligned}
&\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\left({}_lI_*^{\alpha}\rho\right)\left(\frac{l+k}{2}\right)\left({}_*I^{\alpha}_{\frac{l+k}{2}}\rho\right)(k)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^{\alpha}}}\right| \\
\leq\ & [(\rho^*(l))(f^*(k))]^{\frac{k-l}{2}\left(\frac{4-\alpha^2-3\alpha}{12(\alpha+1)(\alpha+2)}+\frac{\alpha}{(\alpha+1)}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)} \\
&\times\left(f^*\left(\frac{l+k}{2}\right)\right)^{\frac{k-l}{2}\left(\frac{4-\alpha}{6(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)}.
\end{aligned}$$

**Proof.** From Lemma 2, modulus and properties of multiplicative integral, we have

$$\begin{aligned}
&\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\left({}_lI_*^{\alpha}\rho\right)\left(\frac{l+k}{2}\right)\left({}_*I^{\alpha}_{\frac{l+k}{2}}\rho\right)(k)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^{\alpha}}}\right| \\
=\ & \left|\left(\int_0^1\left(\rho^*\left((1-t)l+t\frac{l+k}{2}\right)^{t^{\alpha}-\frac{1}{3}}\right)^{dt}\right)^{\frac{k-l}{4}}\right| \\
&\times\left|\left(\int_0^1\left(\rho^*\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{1}{3}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{k-l}{4}}\right| \\
=\ & \int_0^1\left|\rho^*\left((1-t)l+t\frac{l+k}{2}\right)^{\frac{k-l}{2}\left(\frac{1}{2}t^{\alpha}-\frac{1}{6}\right)}\right|^{dt} \\
&\times\int_0^1\left|\rho^*\left((1-t)\frac{l+k}{2}+tk\right)^{\frac{k-l}{2}\left(\frac{1}{6}-\frac{1}{2}(1-t)^{\alpha}\right)}\right|^{dt} \\
\leq\ & \left(\exp\left\{\frac{k-l}{2}\int_0^1\left|\left(\frac{1}{2}t^{\alpha}-\frac{1}{6}\right)\right|\left|\ln\rho^*\left((1-t)l+t\frac{l+k}{2}\right)\right|dt\right\}\right) \\
&\times\left(\exp\left\{\frac{k-l}{2}\int_0^1\left|\left(\frac{1}{6}-\frac{1}{2}(1-t)^{\alpha}\right)\right|\left|\ln\rho^*\left((1-t)\frac{l+k}{2}+tk\right)\right|dt\right\}\right).
\end{aligned} \tag{2}$$

From the multiplicative convexity of $\rho^*$ and properties on ln, we have

$$\begin{aligned}
\left|\ln\rho^*\left((1-t)l+t\frac{l+k}{2}\right)\right| &\leq \left|\ln(\rho^*(l))^{(1-t)}\left(f^*\left(\frac{l+k}{2}\right)\right)^t\right| \\
&= \left((1-t)\ln(\rho^*(l))+t\ln\left(f^*\left(\frac{l+k}{2}\right)\right)\right)
\end{aligned} \tag{3}$$

and

$$\left|\ln\rho^*\left((1-t)\frac{l+k}{2}+tk\right)\right| \leq \left|\ln(\rho^*(l))^{(1-t)}\left(f^*\left(\frac{l+k}{2}\right)\right)^t\right| \tag{4}$$

$$= \left((1-t)\ln\left(\rho^*\left(\frac{l+k}{2}\right)\right) + t\ln(f^*(k))\right).$$

Combining (2)–(4), we obtain

$$\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\left({}_lI_*^\alpha\rho\right)\left(\frac{l+k}{2}\right)\left({}_*I_{\frac{l+k}{2}}^\alpha\rho\right)(k)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}}\right|$$

$$\leq \exp\left\{\frac{k-l}{2}\left(\int_0^{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{6}-\frac{1}{2}t^\alpha\right)\left((1-t)\ln(\rho^*(l)) + t\ln\left(f^*\left(\frac{l+k}{2}\right)\right)\right)dt\right.\right.$$

$$\left.\left.+\int_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{2}t^\alpha-\frac{1}{6}\right)\left((1-t)\ln(\rho^*(l)) + t\ln\left(f^*\left(\frac{l+k}{2}\right)\right)\right)dt\right)\right\}$$

$$\times\exp\left\{\frac{k-l}{2}\left(\int_0^{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{2}(1-t)^\alpha-\frac{1}{6}\right)\left((1-t)\ln\left(\rho^*\left(\frac{l+k}{2}\right)\right) + t\ln(f^*(k))\right)dt\right.\right.$$

$$\left.\left.+\int_{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^\alpha\right)\left((1-t)\ln\left(\rho^*\left(\frac{l+k}{2}\right)\right) + t\ln(f^*(k))\right)dt\right)\right\}$$

$$= \exp\left\{\frac{k-l}{2}\left(\left(\int_0^{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{6}-\frac{1}{2}t^\alpha\right)(1-t)dt + \int_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{2}t^\alpha-\frac{1}{6}\right)(1-t)dt\right)\ln(\rho^*(l))\right.\right.$$

$$\left.\left.+\left(\int_0^{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{6}-\frac{1}{2}t^\alpha\right)tdt + \int_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{2}t^\alpha-\frac{1}{6}\right)tdt\right)\ln\left(f^*\left(\frac{l+k}{2}\right)\right)\right)\right\}$$

$$\times\exp\left\{\ln\left(\rho^*\left(\frac{l+k}{2}\right)\right)\frac{k-l}{2}\left(\left(\int_0^{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{2}(1-t)^\alpha-\frac{1}{6}\right)(1-t)dt\right.\right.\right.$$

$$\left.+\int_{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^\alpha\right)(1-t)dt\right)$$

$$\left.\left.+\left(\int_0^{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{2}(1-t)^\alpha-\frac{1}{n+2}\right)tdt + \int_{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^\alpha\right)tdt\right)\ln(f^*(k))\right)\right\}$$

$$= \exp\left\{\frac{k-l}{2}\left(\ln(\rho^*(l))^{\left(\frac{4-\alpha^2-3\alpha}{12(\alpha+1)(\alpha+2)}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)}\right.\right.$$

$$
\begin{aligned}
& + \ln\left(f^*\left(\frac{l+k}{2}\right)\right)^{\frac{1}{2}\left(\frac{4-\alpha}{(n+2)(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)}\Bigg)\Bigg\} \\
& \times \exp\left\{\frac{k-l}{2}\left(\ln\left(\rho^*\left(\frac{l+k}{2}\right)\right)^{\frac{1}{2}\left(\frac{4-\alpha}{(n+2)(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)}\right.\right. \\
& \left.\left. + \ln(f^*(k))^{\left(\left(\frac{4-\alpha^2-3\alpha}{12(\alpha+1)(\alpha+2)}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)\right)}\right)\right\} \\
= \; & [(\rho^*(l))(f^*(k))]^{\frac{k-l}{2}\left(\frac{4-\alpha^2-3\alpha}{12(\alpha+1)(\alpha+2)}+\frac{\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)} \\
& \left(f^*\left(\frac{l+k}{2}\right)\right)^{\frac{k-l}{2}\left(\frac{4-\alpha}{6(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}\right)},
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int_0^{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{6}-\frac{1}{2}t^\alpha\right)(1-t)dt &= \int_{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{6}-\frac{1}{2}(1-t)^\alpha\right)tdt \\
&= \left(\frac{\alpha}{6(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{12(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{n+2}-\frac{1}{2}t^\alpha\right)tdt &= \int_{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{6}-\frac{1}{2}(1-t)^\alpha\right)(1-t)dt \\
&= \frac{\alpha}{12(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
& \int_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{2}t^\alpha-\frac{1}{6}\right)(1-t)dt = \int_0^{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{2}(1-t)^\alpha-\frac{1}{6}\right)tdt \\
= \; & \frac{4-\alpha^2-3\alpha}{12(\alpha+1)(\alpha+2)}+\frac{\alpha}{6(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{12(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{1}{2}t^\alpha-\frac{1}{6}\right)tdt &= \int_0^{1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{2}(1-t)^\alpha-\frac{1}{6}\right)(1-t)dt \\
&= \frac{4-\alpha}{12(\alpha+2)}+\frac{\alpha}{12(\alpha+2)}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}.
\end{aligned}
$$

The proof is completed. □

**Corollary 2.** *In Theorem 10, using the multiplicative convexity of $\rho^*$, i.e., $f^*\left(\frac{l+k}{2}\right) \leq \sqrt{\rho^*(l)\rho^*(k)}$, we obtain*

$$\begin{aligned}&\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\left({}_lI_*^\alpha\rho\right)\left(\frac{l+k}{2}\right)\left({}_*I_{\frac{l+k}{2}}^\alpha\rho\right)(k)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-l)^\alpha}}\right|\\ \leq\ & [(\rho^*(l))(f^*(k))]^{\frac{k-l}{2}\left(\frac{4-2\alpha}{12(\alpha+1)}+\frac{\alpha}{(\alpha+1)}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}\right)}.\end{aligned}$$

**Corollary 3.** *In Theorem 10, if we take $\alpha = 1$, we obtain*

$$\begin{aligned}&\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\int_l^k \rho(u)^{du}\right)^{\frac{1}{l-k}}\right|\\ \leq\ & [(\rho^*(l))(f^*(k))]^{\frac{2}{81}(k-l)}\left(f^*\left(\frac{l+k}{2}\right)\right)^{\frac{29}{324}(k-l)}.\end{aligned}$$

**Corollary 4.** *In Corollary 3, using the multiplicative convexity of $f^*$, we obtain*

$$\left|\left((\rho(l))\left(\rho\left(\frac{l+k}{2}\right)\right)^4(\rho(k))\right)^{\frac{1}{6}}\left(\int_l^k \rho(u)^{du}\right)^{\frac{1}{l-k}}\right| \leq [(\rho^*(l))(f^*(k))]^{\frac{5}{72}(k-l)}.$$

## 3. Applications to Special Means

We shall consider the means for arbitrary real numbers $l, k$.
The Arithmetic mean: $A(l,k) = \frac{l+k}{2}$.
The Harmonic mean: $H(l,k) = \frac{2lk}{l+k}$.
The logarithmic means: $L(l,k) = \frac{k-l}{\ln k - \ln l}$, $l,k > 0$ , and $l \neq k$.
The $p$-Logarithmic mean: $L_p(l,k) = \left(\frac{k^{p+1}-l^{p+1}}{(p+1)(k-l)}\right)^{\frac{1}{p}}$, $l,k > 0, l \neq k$ and $p \in \mathbb{R}\setminus\{-1,0\}$.

**Proposition 1.** *Let $l,k \in \mathbb{R}$ with $0 < l < k$, then we have*

$$\left|e^{\frac{1}{6}\left(2H^{-1}(l,k)-A^4(l,k)\right)-L^{-1}(l,k)}\right| \leq e^{-\frac{11}{36l^2}(k-l)}.$$

**Proof.** The assertion follows from Corollary 1 applied to the function $\rho(t) = e^{\frac{1}{t}}$ whose $\rho^*(t) = e^{-\frac{1}{t^2}}$, $\mathcal{M} = e^{-\frac{1}{l^2}}$ and $\left(\int_l^k \rho(u)^{du}\right)^{\frac{1}{l-k}} = \exp\{-L^{-1}(l,k)\}$. □

**Proposition 2.** *Let $l,k \in \mathbb{R}$ with $0 < l < k$, then we have*

$$\left|e^{\frac{1}{6}\left(2A(l^p,k^p)+A^{4p}(l,k)\right)-L_p^p(l,k)}\right| \leq e^{p\frac{5(k-l)\left(l^{p-1}+k^{p-1}\right)}{72}}.$$

**Proof.** The assertion follows from Corollary 4, applied to the function $\rho(t) = e^{t^p}$ with $p \geq 2$ whose $\rho^*(t) = e^{pt^{p-1}}$ and $\left(\int_l^k \rho(u)^{du}\right)^{\frac{1}{l-k}} = \exp\left\{-L_p^p(l,k)\right\}$. □

## 4. Conclusions

Multiplicative calculus is an alternative to Newtonian calculus. Since its inception as one of the non-Newtonian calculus, a number of works have been devoted to different applications of multiplicative calculus. In this study, we discussed Simpson-type fractional integral inequalities for multiplicatively differentiable functions based on a new identity. Some special cases are derived and applications of our findings are provided. We hope that the new strategy formulated in this paper will inspire and stimulated further research in this promising field of multiplicative fractional inequalities.

**Author Contributions:** Conceptualization, A.M., H.B. and B.M.; methodology, A.M. and H.B.; writing—original draft preparation, R.S.; writing—review and editing, T.A., E.E.A. and Z.K. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research has been funded by Scientific Research Deanship at University of Ha'il—Saudi Arabia through project number RG-21021.

**Data Availability Statement:** Not available.

**Conflicts of Interest:** The authors declare no conflict of interest.

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