The Existence Theorems of Fractional Differential Equation and Fractional Differential Inclusion with Affine Periodic Boundary Value Conditions

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Abstract: This paper is devoted to investigating the existence of solutions for the fractional differential equation and fractional differential inclusion of order \( \alpha \in (2, 3] \) with affine periodic boundary value conditions. Applying the Leray–Schauder fixed point theorem, the existence of the solutions for the fractional differential equation is established. Furthermore, for the fractional differential inclusion, we consider two cases: (i) the set-valued function has convex value and (ii) the set-valued function has nonconvex value. The main tools of our research are the Leray–Schauder alternative theorem, Covita and Nadler’s fixed point theorem and some set-valued analysis theories.

Keywords: fractional differential equation; fractional differential inclusion; affine periodic; boundary value problem; fixed point theorem

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1. Introduction

It is well known that many problems in real life can be solved by establishing mathematical models of differential equations. However, classical integer order differential equations have some difficulties when describing some complex phenomena or systems, such as certain materials and processes with memory and heritability. These problems can be solved by establishing mathematical models of fractional differential equations, which makes fractional differential equations have a wide range of applications in many fields, such as astrophysics, physics, biology, medicine, control science, image and signal processing, random diffusion, anomalous diffusion, etc. For relevant research on this aspect, we refer the interested readers to [1–4].

In recent years, boundary value problems of fractional differential equations have aroused the enthusiasm of scholars. There are many kinds of boundary value problems, including integral boundary value, multi-point boundary value, periodic and anti-periodic boundary value, affine periodic boundary value and so on. In [5], under integral boundary conditions, Rezapour et al. studied the existence of solutions to a Caputo fractional differential inclusion. Ahmad et al. used nonlinear alternative of Leray–Schauder type and some fixed point theorems to research a fractional differential inclusion of order \( q \in (1, 2] \), with four-point nonlocal boundary conditions involving convex and nonconvex multivalued maps in [6]. Agarwal et al. [7] investigated the existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions of the order \( q \in (3, 4] \). In [8], Gao et al. considered a sequential fractional differential equation with affine periodic boundary value conditions. Using Leray–Schauder and Krasnoselskii fixed point theorem, the existence theorem of the solution for the fractional differential equation was investigated, and via the Banach contraction mapping principle, the uniqueness theorem of the solution was also studied. For more research results, we refer to the readers to [9–13].
Affine period describes a physical phenomenon which is periodic in time and symmetric in space. The concept was proposed by Li [14] in 2013. It is widely used in electromagnetic, acoustic and other physical phenomena. However, most of the studies on affine period involve integer order differential systems, and there are few research results on fractional differential systems. Stimulated by [8], in this paper we study the existence results for fractional differential equation with (T, b) affine periodic boundary value conditions:

$$\begin{align*}
\left\{ \begin{array}{ll}
C D^a y(t) = f(t,y) & \text{for a.e. } t \in [0,T], \\
y(T) = by(0), y'(T) = by'(0), y''(T) = by''(0),
\end{array} \right.
\end{align*}$$

where $C D^a$ denotes the Caputo fractional derivative of order $a \in (2,3]$, $f(t,y) : [0,T] \times C([0,T];R) \rightarrow R$ is a continuous function, $b \in R \setminus \{0\}$ and $b \neq 1$.

The first contribution of this paper is to study the existence of solution for Equation (1) by using the Leray–Schauder alternative theorem. In a wide range of mathematical, economical, engineering and computational problems, the existence of solution for a theoretical or practical problem is equivalent to the existence of a fixed point for a suitable operator. Therefore, fixed points are crucial in many fields, such as mathematics and science. The research results of the fixed point theory can be found in [15–20].

The second contribution of this paper is to consider the following fractional differential inclusion:

$$\begin{align*}
\left\{ \begin{array}{ll}
C D^a y(t) \in F(t,y) & \text{for a.e. } t \in [0,T], \\
y(T) = by(0), y'(T) = by'(0), y''(T) = by''(0),
\end{array} \right.
\end{align*}$$

where $F(t,y) : [0,T] \times R \rightarrow P(R)$ is a set-valued map and $P(R)$ is the family of all nonempty subsets of $R$.

Precisely, we consider two cases: (i) when the set value function $F(\cdot)$ has convex value, we use the Leray–Schauder alternative theorem to verify that the problem (2) has at least one solution and (ii) when the set-valued function $F(\cdot)$ has a nonconvex value, the existence of the solution for the problem (2) has been researched based on the fixed point theorem of Covita and Nadler.

The organization of this paper is as follows. Some definitions and lemmas are presented in Section 2. The existence of solutions for the fractional differential equation and the fractional differential inclusion are given in Section 3 and Section 4, respectively.

2. Preliminaries

This section provides some basic definitions and properties on fractional calculus and some set-valued analysis theories which will be needed in our analysis. For more details on fractional calculus, we refer readers to [21,22], and for more set-valued analysis theories, we refer the interested readers to [23–25].

**Definition 1.** The Riemann–Liouville fractional integral of order $q > 0$ for a function $g$ is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} g(\tau) d\tau, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

**Definition 2.** The Caputo fractional derivative of order $q > 0$ for a function $g$ can be written as

$$C D^q g(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n-q)} \int_0^t (t - \tau)^{n-q-1} g^{(n)}(\tau) d\tau, & n - 1 < q < n \ (n \in \mathbb{N}^+), \\
\frac{1}{\Gamma(n)} \int_0^t (t - \tau)^{n-1} \frac{d^n}{d \tau^n} g(\tau) d\tau, & q = n \ (n \in \mathbb{N}),
\end{array} \right.$$

for $t > 0$.

The following proposition of the Caputo fractional derivative is important:
Proposition 1 ([22]). With the given notations, the following equality holds:

\[ I^q(C^qD^q g(t)) = g(t) - c_0 - c_1 t - \cdots - c_{n-1} t^{n-1}, \quad t > 0, \quad n - 1 < q < n, \quad (n \in \mathbb{N}^+) \]

where \( c_i (i = 0, 1, \ldots, n - 1) \) are arbitrary constants.

Let \( C([0, T]; R) \) denote a Banach space of continuous functions from \([0, T]\) into \( R \) with the norm \( \| y \| = \sup_{t \in [0, T]} |y(t)|. \) \( L^1([0, T]; R) \) is a Banach space of measurable functions \( y : [0, T] \rightarrow R, \) which are Lebesgue integrable and normed by \( \| y \|_{L^1} = \int_0^T |y(t)| dt. \)

Let \( P(X) = \{ A \subseteq X : A \text{ is nonempty} \}, \) \( P_b(X) = \{ A \in P(X) : A \text{ is bounded} \}, \) \( P_{cl}(X) = \{ A \in P(X) : A \text{ is closed} \}, \) and \( P_{cp,b,cl}(X) = \{ A \in P(X) : A \text{ is compact(bounded) and convex} \}. \) A set-valued map \( G : X \rightarrow P(X) \) is convex (closed) valued if \( G(x) \) is convex (closed) for all \( x \in X. \) The map \( G \) is bounded on bounded sets if \( G(B) = \bigcup_{x \in B} G(x) \) is bounded in \( X \) for all \( B \in P_b(X). \)

Definition 3. A set-valued map \( G : X \rightarrow P(X) \) is called completely continuous if \( G(B) \) is relatively compact for all \( B \in P_b(X). \)

Definition 4. A set-valued map \( G : X \rightarrow P(X) \) is called upper semicontinuous (u.s.c.) if for every open subset \( N \subseteq P(X), \) the set \( G^{-1}(N) = \{ x \in X : G(x) \subseteq N \} \) is open in \( X. \)

Proposition 2. If the set-valued map \( G \) is completely continuous with nonempty compact values, then \( G \) is u.s.c. if and only if \( G \) has a closed graph.

The set-valued map \( G \) has a fixed point if there is \( x \in X \) such that \( x \in G(x). \) In this paper, we use the following important fixed point theorems to study the existence of the solutions for the affine periodic boundary value problems.

Lemma 1. (Leray–Schauder fixed point theorem [26])

Let \( X \) be a Banach space, \( Q \in P_{cp,b,cl}(X), \) and \( \Omega \) is an open subset of \( Q \) with \( 0 \in \Omega. \) Let \( T : \overline{\Omega} \rightarrow Q \) be a continuous, compact map. Then, either there exists \( x \in \partial\Omega \) and \( \varepsilon \in (0, 1) \) with \( x = \varepsilon T(x) \) or \( T \) has a fixed point \( x \in \Omega \) such that \( x = T(x). \)

Lemma 2. (Leray–Schauder alternative theorem [27])

Let \( X \) be a Banach space, \( Q \in P_{cp,b,cl}(X) \) with \( 0 \in Q, \) and \( G : Q \rightarrow Q \) is an upper semicontinuous multifunction with compact, convex value which maps bounded sets into relatively compact sets, then one of the following statements is valid:

(i) The set \( \Gamma = \{ x \in Q : x \in \varepsilon G(x), \varepsilon \in (0, 1) \} \) is unbounded;

(ii) \( G(\cdot) \) has a fixed point, i.e., there exists \( x \in Q \) such that \( x \in G(x). \)

Lemma 3. (Covita and Nadler’s fixed point theorem [23])

Let \( (X, d) \) be a complete metric space. If \( G : X \rightarrow P_{cl}(X) \) is a contraction, then \( G \) has a fixed point \( x \in X \) such that \( x \in G(x). \)

A set-valued map \( G : [0, T] \rightarrow P_{cl}(R) \) is said to be measurable if for every \( x \in R, \) the function

\[ t \mapsto d(x, G(t)) = \inf\{ |x - y| : y \in G(t) \} \]

is measurable.

Definition 5. A set-valued map \( G : [0, T] \times R \rightarrow P(R) \) is called Carathéodory if

(i) \( t \rightarrow G(t, x) \) is measurable for each \( x \in R; \)

(ii) \( x \rightarrow G(t, x) \) is upper semicontinuous for almost every \( t \in [0, T]. \)
Moreover, a Carathéodory set-valued map $G : [0, T] \times R \to \mathcal{P}(R)$ is called $L^1$-Carathéodory if for each $\kappa > 0$ there exists $\phi_\kappa \in L^1([0, T]; R^+)$ such that
\[
\|G(t, x)\| = \sup \{ |u| : u \in G(t, x) \} \leq \phi_\kappa(t),
\]
for all $\|x\| \leq \kappa$ and for almost every $t \in [0, T]$.

For each $x \in C([0, T]; R)$, the set of selections of $G$ is defined by
\[
S_{G, x} := \{ g \in L^1([0, T]; R) : g(t) \in G(t, x(t)) \text{ for a.e. } t \in [0, T] \}.
\]

Using Aumann’s selection theorem ([28]), it is easy to check that for a measurable set-valued map $G : X \to \mathcal{P}(X)$, the set $S_{G, x}$ is nonempty if and only if
\[
t \to \inf \{ \|x\| : x \in G(t) \} \in L^1.
\]

Let $(X, d)$ be a metric space induced from the normed space $(X; \| \cdot \|)$. For $A, B \subset X$, the Hausdorff metric is obtained by
\[
d_H(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \},
\]
where $d(a, B) = \inf_{b \in B} d(a, b)$ and $d(A, b) = \inf_{b \in B} d(a, b)$.

**Definition 6.** A set-valued map $G : X \to \mathcal{P}_c(X)$ is called
(i) $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that
\[
d_H(G(x), G(y)) \leq \gamma d(x, y),
\]
for each $x, y \in X$;
(ii) A contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$.

**Lemma 4 ([24]).** Let $X$ be a Banach space. Let $G : [0, T] \times X \to \mathcal{P}_{cp,cv}(X)$ be an $L^1$-Carathéodory set-valued map and let $\Theta : L^1([0, T]; X) \to C([0, T]; X)$ be a linear continuous map. Then, the operator
\[
\Theta \circ S_{G, x} : C([0, T]; X) \to \mathcal{P}_{cp,cv}(C([0, T]; X))
\]
\[
x \mapsto (\Theta \circ S_{G, x})(x) = \Theta(S_{G, x})
\]
is a closed graph operator in $C([0, T]; X) \times C([0, T]; X)$.

3. Existence Theory of Fractional Differential Equation

In this section, we will consider the existence of solutions for the fractional differential Equation (1). For this purpose, we first give the following lemma.

**Lemma 5.** For any $\sigma(t) \in C([0, T])$, the $(T, b)$ affine periodic boundary value problem
\[
\begin{cases}
C^{\alpha}y(t) = \sigma(t) & \text{for a.e. } t \in [0, T], \\
y(T) = by(0), y'(T) = b'y(0), y''(T) = b''y(0)
\end{cases}
\]
has a unique solution expressed by
\[
y(t) = \int_0^t (t - s)^{-\alpha} \sigma(s) ds - \frac{1}{1-b} \int_0^T (T-s)^{\alpha-1} \sigma(s) ds + \mu_1(t) \int_0^T (T-s)^{\alpha-2} \sigma(s) ds + \mu_2(t) \int_0^T (T-s)^{\alpha-3} \sigma(s) ds
\]
\[
\text{where } \mu_1(t) = \frac{T}{(1-b)\Gamma} - \frac{t}{1-b} \text{ and } \mu_2(t) = -\frac{(1+b)T^2}{2(1-b)^3} + \frac{Tt}{(1-b)^2} - \frac{t^2}{2(1-b)}.
\]
**Proof.** Invoking Proposition 1, we take $I^a$ from (3) and obtain

$$y(t) = I^a \sigma(t) - c_0 - c_1 t - c_2 t^2$$

$$= \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} \sigma(s) ds - c_0 - c_1 t - c_2 t^2,$$

(5)

where $c_0, c_1$ and $c_2$ are arbitrary constants. We are now able to differentiate (5), obtaining

$$y'(t) = \int_0^t \frac{(t-s)^{a-2}}{\Gamma(a-1)} \sigma(s) ds - c_1 - 2c_2 t$$

and

$$y''(t) = \int_0^t \frac{(t-s)^{a-3}}{\Gamma(a-2)} \sigma(s) ds - 2c_2.$$

It results from the boundary conditions (3) that

$$c_0 = \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} \sigma(s) ds - \frac{T}{(1-b)^2} \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} \sigma(s) ds - \frac{(1+b)T^2}{2(1-b)^3} \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} \sigma(s) ds,$$

$$c_1 = \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} \sigma(s) ds - \frac{T}{(1-b)^2} \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} \sigma(s) ds,$$

$$c_2 = \frac{1}{2(1-b)} \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} \sigma(s) ds.$$

The substitution of the values of $c_0, c_1$ and $c_2$ into (5) gives the solution expressed as (4). This finishes the proof. □

The following estimate involving the integral inequalities will be used several times in the proof in our main results. For $\sigma(t) \in C([0, T])$, we yield

$$\left| \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} \sigma(s) ds \right| \leq \left| \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} ds \right| |\sigma(s)| \leq \frac{T^a}{\Gamma(a + 1)} \|\sigma\|. \quad (6)$$

In a similar fashion, one has

$$\left| \int_0^T \frac{(T-s)^{a-i}}{\Gamma(a + 1 - i)} \sigma(s) ds \right| \leq \left| \int_0^T \frac{(T-s)^{a-i}}{\Gamma(a + 1 - i)} ds \right| |\sigma(s)| \leq \frac{T^{a+1-i}}{\Gamma(a + 2 - i)} \|\sigma\|, \quad (7)$$

where $i = 1, 2, 3$.

For brevity, we let

$$M = \frac{T^{a-2}}{\Gamma(a - 1)} \left( \frac{(2-b)T^2}{(1-b)a(a-1)} + \overline{\mu}_1 \frac{T}{a-1} \right),$$

(8)

where $\overline{\mu}_1 = \sup_{t \in [0, T]} |\mu_1(t)|$ and $\overline{\mu}_2 = \sup_{t \in [0, T]} |\mu_2(t)|$.

Next, we will use Leray–Schauder fixed point theorem to research the $(T, b)$ affine periodic boundary value problem (1).

**Theorem 1.** Let $f(t, y) : [0, T] \times C([0, T]; R) \rightarrow R$ be a continuous function, which satisfies the following hypotheses:
\((H1)\) For all \(t \in [0, T]\) and \(y \in C([0, T]; R)\), there exists a positive continuous function \(\delta(t)\) and a nondecreasing continuous function \(\psi : [0, \infty) \to (0, \infty)\) such that 
\[ |f(t, y)| \leq \delta(t)\psi(\|y\|). \]

\((H2)\) There exists a positive constant \(\rho\) such that 
\[ \frac{\rho}{\|\delta\|\psi(\rho)} > M, \]
where \(M\) is the constant given in (8). Then, problem (1) admits at least one solution in \([0, T]\).

**Proof.** Let \(\Omega_p = \{ y \in C([0, T]; R) : \|y\| < \rho \}\), where \(\rho\) is given in (H2). It is easy to see that \(\Omega_p\) is a bounded open subset of \(C([0, T]; R)\).

In the meaning of Lemma 5, we introduce an operator \(H : C([0, T]; R) \to C([0, T]; R)\), which is expressed by 
\[ H(y)(t) = \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} f(s, y(s)) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} f(s, y(s)) ds \]
\[ + \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} f(s, y(s)) ds + \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} f(s, y(s)) ds, \]
where \(\mu_1(t)\) and \(\mu_2(t)\) are given in (4). Then, we can transform problem (1) into a fixed point problem, i.e., \(y = H(y)\).

The following uses Lemma 1 to prove the fixed point problem; the proof is divided into several steps:

**Step 1.** The operator \(H : C([0, T]; R) \to C([0, T]; R)\) is continuous.

Let \(\{y_n\}\) be a sequence such that \(y_n \to y\) in \(C([0, T]; R)\). Then, it holds that 
\[ |H(y_n)(t) - H(y)(t)| \leq \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} |f(s, y_n(s)) - f(s, y(s))| ds \]
\[ + \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} |f(s, y_n(s)) - f(s, y(s))| ds \]
\[ + |\mu_1(t)| \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} |f(s, y_n(s)) - f(s, y(s))| ds \]
\[ + |\mu_2(t)| \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} |f(s, y_n(s)) - f(s, y(s))| ds. \]

According to the continuity of \(f(t, y)\), one can conclude that \(|f(t, y_n) - f(t, y)| \to 0\) as \(n \to \infty\), which implies that 
\[ \|H(y_n)(t) - H(y)(t)\| = \sup_{t \in [0, T]} |H(y_n)(t) - H(y)(t)| \to 0 \quad \text{as} \quad n \to \infty. \]

**Step 2.** The operator \(H : C([0, T]; R) \to C([0, T]; R)\) is equicontinuous.
Let \( y \in \overline{\Omega}_\rho \) for any \( 0 \leq t_1 < t_2 \leq T \). From (H1), we infer that

\[
\begin{align*}
|H(y)(t_2) - H(y)(t_1)| & \leq \left| \int_0^{t_2} \frac{(t_2 - s)^{a-1}}{\Gamma(a)} f(s, y(s))ds \right| + |\mu_1(t_2) - \mu_1(t_1)| \left| \int_0^T \frac{(T - s)^{a-2}}{\Gamma(a-1)} f(s, y(s))ds \right| \\
& \quad + |\mu_2(t_2) - \mu_2(t_1)| \left| \int_0^T \frac{(T - s)^{a-3}}{\Gamma(a-2)} f(s, y(s))ds \right| \\
& \leq \frac{T^a}{\Gamma(a+1)} + \frac{T^a}{1 - b \Gamma(a+1)} + \frac{T^{a-1}}{\Gamma(a)} + \frac{T^{a-2}}{\Gamma(a-1)} |\delta(t)| |\psi(\|y\|)| \\
& \quad + \frac{T^{a-2}}{\Gamma(a-1)} \left( (2 - b)T^2 + \frac{T}{(1 - b)\alpha(a-1)} \right) \|\delta\| \|\psi(\rho)\| \\
& < \rho,
\end{align*}
\]

(11)

as \( t_1 \to t_2 \) for any \( y \in \overline{\Omega}_\rho \). This means \( H \) is equicontinuous.

**Step 3.** The operator \( H : C([0, T]; R) \to C([0, T]; R) \) is compact.

For each \( y \in \overline{\Omega}_\rho \) and \( t \in [0, T] \), owing to (H1) and (H2), one obtains

\[
\begin{align*}
|H(y)(t)| & \leq \left| \int_0^t \frac{(T - s)^{a-1}}{\Gamma(a)} f(s, y(s))ds \right| + \frac{1}{1 - b} \left| \int_0^T \frac{(T - s)^{a-1}}{\Gamma(a)} f(s, y(s))ds \right| \\
& \quad + |\mu_1(t)| \left| \int_0^T \frac{(T - s)^{a-2}}{\Gamma(a-1)} f(s, y(s))ds \right| \\
& \quad + |\mu_2(t)| \left| \int_0^T \frac{(T - s)^{a-3}}{\Gamma(a-2)} f(s, y(s))ds \right| \\
& \leq \frac{T^a}{\Gamma(a+1)} + \frac{T^a}{1 - b \Gamma(a+1)} + \frac{T}{\Gamma(\alpha)} + \frac{T}{\Gamma(\alpha-1)} |\delta(t)| |\psi(\|y\|)| \\
& \quad + \frac{T^{a-2}}{\Gamma(a-1)} \left( (2 - b)T^2 + \frac{T}{(1 - b)\alpha(a-1)} \right) \|\delta\| \|\psi(\rho)\| \\
& < \rho,
\end{align*}
\]

(12)

which yields \( \|H(y)(t)\| = \sup_{t \in [0, T]} |H(y)(t)| < \rho \). That is, \( H(y) \in \overline{\Omega}_\rho \). Therefore, due to the Arzela–Ascoli theorem, the operator \( H \) is compact.

**Step 4.** The operator \( H : C([0, T]; R) \to C([0, T]; R) \) has a fixed point.

Suppose \( y \in \partial \Omega_\rho \), there exists \( \varepsilon \in (0, 1) \) such that \( y = \varepsilon H(y) \). It then follows from (12) that

\[
\rho = \|y\| = \varepsilon \|H(y)(t)\| < \varepsilon M \|\delta\| \|\psi(\rho)\| < \varepsilon \rho < \rho.
\]

Obviously, this leads to a contradiction. Invoking Lemma 1, the operator \( H \) has a fixed point, i.e., \( y = H(y), y \in \overline{\Omega}_\rho \), which means the \((T, b)\) affine periodic boundary value problem (1) has at least one solution in \([0, T]\).

\( \square \)
Remark 1. If \( \alpha = 2 \), problem (1) is a second order differential equation whose affine periodic solutions have been studied in [29]. This paper mainly studies the existence of solutions for fractional differential equations.

Remark 2. For Theorem 1, we apply the Leray–Schauder fixed point theorem. Comparing Krasnosel’skii fixed point theorem and Banach fixed point theorem, they both require the function \( f \) to satisfy the Lipschitz condition, while the Leray–Schauder fixed point theorem does not. This gives problem (1) a wide range of applications.

Let us provide an example to verify Theorem 1:

Example 1. Let us consider the \((1, 2)\) affine periodic problem:

\[
\begin{align*}
\left\{\begin{array}{ll}
C^D y(t) = \frac{e^{-2t}}{\sqrt{9 + t^2}} \left( \cos y + \frac{y^2}{4 + y^2} \right) & \text{for a.e. } t \in [0,1], \\
y(1) &= 2y(0), \quad y'(1) = 2y'(0), \quad y''(1) = 2y''(0),
\end{array}\right.
\]

where here, \( \alpha = \frac{5}{2}, b = 2, T = 1 \) and \( f(t, y) = \frac{e^{-2t}}{\sqrt{9 + t^2}} \left( \cos y + \frac{y^2}{4 + y^2} \right) \). Clearly, we have

\[
|f(t, y)| = \left| \frac{e^{-2t}}{\sqrt{9 + t^2}} \left( \cos y + \frac{y^2}{4 + y^2} \right) \right| \leq \delta(t)\psi(\|y\|),
\]

where \( \delta(t) = \frac{e^{-2t}}{\sqrt{9 + t^2}} \) and \( \psi(\|y\|) = 1 + \frac{\|y\|}{4} \). With the above assumptions, we can obtain \( \Gamma(\frac{1}{2}) = \sqrt{3}, \quad \mathcal{F}_1 = \sup_{t \in [0,1]} |1 + t| = 2, \quad \mathcal{F}_2 = \sup_{t \in [0,1]} \left| \frac{3}{2} + t + \frac{t^2}{2} \right| = 3, \quad M \approx 4.891 \) and \( \|\delta\| = \frac{3}{2} \).

Then, using the condition (H2), we can find \( \rho > 2.751 \). It follows from Theorem 1 that problem (1) has a solution.

4. Existence Theories of Fractional Differential Inclusion

This section is devoted to research the following differential inclusion with \((T, b)\) affine periodic boundary value conditions:

\[
\begin{align*}
\left\{\begin{array}{ll}
C^D^\alpha y(t) \in F(t, y) & \text{for a.e. } t \in [0, T], \\
y(T) &= by(0), \quad y'(T) = by'(0), \quad y''(T) = by''(0),
\end{array}\right.
\]

where \( F(t, y) : [0, T] \times R \rightarrow P(R) \) is a set-valued map satisfying some hypotheses listed below. The existence results for problem (14) are provided for two cases when the set-valued map \( F(t, y) \) has a convex value and a nonconvex value.

Now, we first consider the convex case.

Theorem 2. Suppose that

(H3) \( F : [0, T] \times R \rightarrow P(R) \) has nonempty compact convex values and is Carathéodory.

(H4) For all \( t \in [0, T] \) and \( y \in C([0, T]; R) \), there exists a function \( \tilde{\delta}(t) \in L^1([0, T]; R^+) \) and a nondecreasing continuous function \( \tilde{\psi} : [0, \infty) \rightarrow (0, \infty) \) such that

\[
\|F(t, y)\| := \sup_{z \in F(t, y)} \{z\} \leq \tilde{\delta}(t)\tilde{\psi}(\|y\|).
\]

(H5) There exists a positive constant \( r \) such that

\[
\frac{r}{\|\delta\|\psi(r)} > M,
\]

where \( M \) is the constant given in (8). Then, the inclusion problem (14) admits at least one solution in \([0, T]\).
Proof. Let the operator $\Phi : C([0, T]; R) \rightarrow \mathcal{P}(C([0, T]; R))$ be defined by

$$
\Phi(y) = \left\{ z \in C([0, T]; R) : z(t) = \begin{cases} 
\int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} f(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} f(s) ds
+ \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} f(s) ds
+ \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} f(s) ds
\end{cases} \quad f \in S_{F,y} \right\}. 
$$

(15)

We claim that $\Phi$ satisfies the Leray–Schauder alternative theorem, i.e., the fixed point problem $y \in \Phi(y)$ has at least one fixed point. We divide the process of the proof into four steps:

**Step 1.** The operator $\Phi : C([0, T]; R) \rightarrow \mathcal{P}(C([0, T]; R))$ is convex.

Let $z_1, z_2 \in \Phi(y)$. For each $t \in [0, T]$, there exists $f_1, f_2 \in S_{F,y}$, so that

$$
z_j(t) = \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} f_j(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} f_j(s) ds
+ \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} f_j(s) ds
+ \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} f_j(s) ds, \quad (j = 1, 2).
$$

Let $0 \leq \theta \leq 1$. For any $t \in [0, T]$, one obtains

$$
[\theta z_1 + (1-\theta)z_2](t)
= \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} (\theta f_1 + (1-\theta)f_2)(s) ds
- \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} (\theta f_1 + (1-\theta)f_2)(s) ds
+ \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} (\theta f_1 + (1-\theta)f_2)(s) ds
+ \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} (\theta f_1 + (1-\theta)f_2)(s) ds.
$$

By virtue of that fact $F$ is convex, $S_{F,y}$ is convex, thus it follows that $\theta z_1 + (1-\theta)z_2 \in \Phi(y)$, which means that $\Phi$ is convex.

**Step 2.** The operator $\Phi : C([0, T]; R) \rightarrow \mathcal{P}(C([0, T]; R))$ is completely continuous.

First, we show that $\Phi$ maps the bounded sets into bounded sets in $C([0, T]; R)$. Let $\Omega_r = \{ y \in C([0, T]; R) : \|y\| < r \}$, where $r$ is given in (H5). Thus, for every $z \in \Phi(y), y \in \Omega_r$, there exists $f \in S_{F,y}$, satisfying

$$
z(t) = \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} f(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)} f(s) ds
+ \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)} f(s) ds
+ \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)} f(s) ds, \quad (16)
$$
and

\[ |z(t)| \leq \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds \right| + \left| \frac{1}{1-b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds \right| \\
+ |\mu_1(t)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s) ds + |\mu_2(t)| T \frac{T^{\alpha-3}}{\Gamma(\alpha-2)} f(s) ds \]

\[ \leq \left( \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^\alpha}{(1-b)\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \right) |\delta(t)| \tilde{\psi}(\|y\|). \]

As a result,

\[ \|z(t)\| \leq \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \left( \frac{(2-b)T^2}{(1-b)a(\alpha-1)} + \frac{T}{\alpha-1} \right) \|\delta\| \tilde{\psi}(r) = M\|\delta\| \tilde{\psi}(r) < r, \]

where M is given in (8).

Secondly, we show that \( \Phi \) maps the bounded sets into equicontinuous sets in \( C([0,T]; \mathbb{R}) \).

Let \( 0 \leq t_1 < t_2 \leq T \) and \( y \in \Omega_r \). For each \( z \in \Phi(y) \), one can deduce that

\[ |z(t_2) - z(t_1)| \leq \int_{t_1}^{t_2} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right| f(s) ds \\
+ |\mu_1(t_2) - \mu_1(t_1)| \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s) ds \\
+ |\mu_2(t_2) - \mu_2(t_1)| \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} f(s) ds \]

\[ \rightarrow 0, \quad (17) \]

as \( t_1 \to t_2 \). Owing to the Arzelà–Ascoli theorem, the operator \( \Phi : C([0,T]; \mathbb{R}) \to \mathcal{P}(C([0,T]; \mathbb{R})) \) is completely continuous.

**Step 3.** The operator \( \Phi : C([0,T]; \mathbb{R}) \to \mathcal{P}(C([0,T]; \mathbb{R})) \) has a closed graph.

Let \( \{y_n\} \) be a sequence such that \( y_n \to y \), and \( \forall n \in N, z_n \in \Phi(y_n) \) such that \( z_n \to z \). What follows is to show that \( z \in \Phi(y) \). For each \( n \), choose \( f_n \in S_{F,y_n} \) such that

\[ z_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s) ds \\
+ \mu_1(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f_n(s) ds + \mu_2(t) \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} f_n(s) ds. \]

Consider the continuous linear operator \( \Theta : L^1([0,T]; \mathbb{R}) \to C([0,T]; \mathbb{R}) \), defined by

\[ \Theta(f)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds \\
+ \mu_1(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s) ds + \mu_2(t) \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} f(s) ds. \]

In light of Lemma 4, \( \Theta \circ S_{F,y} \) is a closed graph operator. According to \( y_n \to y \), and \( z_n(t) \in \Theta(S_{F,y_n}) \), for all \( n \) there exists \( f_s \in S_{F,y} \), such that

\[ z_s(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_s(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f_s(s) ds \\
+ \mu_1(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f_s(s) ds + \mu_2(t) \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} f_s(s) ds. \]
Step 4. The operator $\Phi$ has a fixed point.

We claim a priori boundness of the solution. Let $y$ be a solution for problem (14). Then, for $t \in [0, T]$, there exists $f \in L^1([0, T]; R)$ such that

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds$$

$$+ \mu_1(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s) ds + \mu_2(t) \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} f(s) ds.$$ 

Taking into account (H4), for each $t \in [0, T]$, we derive that

$$|y(t)| \leq \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \left( \frac{(2-b)T^2}{1-b} + \bar{\theta}_1 \frac{T}{\alpha-1} + \bar{\theta}_2 \right) |\tilde{\phi}(t)||\tilde{\psi}(y||)$$

$$\leq M|\tilde{\phi}(||y||)|.$$ 

Therefore, we gain $\frac{|y|}{|\tilde{\phi}(||y||)|} \leq M$. Invoking (H5), there exists $r$ such that $|y| \neq r$. Let

$$V = \{ y \in C([0, T]; R) : |y| < r + 1 \}.$$ 

Note that the operator $\Phi : \nabla \rightarrow \mathcal{P}(C([0, T]; R))$ is u.s.c. and completely continuous. From the choice of $V$, there is no $y \in \partial V$ such that $y \in s\Phi(y)$ for some $s \in (0, 1)$. According to Lemma 2, the operator $\Phi$ has a fixed point, $y \in \overline{V}$, which is a solution for the $(T, b)$ affine periodic boundary value problem (14). This completes the proof. 

Next, we consider the existence of solutions for the affine periodic boundary value problem (14) with a nonconvex set-valued map by Covitz and Nadler’s fixed point theorem.

Theorem 3. Suppose that

(H6) $F : [0, T] \times R \rightarrow \mathcal{P}(R)$ is an integrable bounded set-valued map, it has nonempty compact values and $F(\cdot, y)$ is measurable for each $y \in R$.

(H7) For almost every $t \in [0, T]$ and $y_1, y_2 \in C([0, T]; R)$, there exists a function $l(t) \in L^1([0, T]; R^+)$ such that

$$d_H(F(t, y_1), F(t, y_2)) \leq l(t) |y_1 - y_2|.$$ 

Then, the $(T, b)$ affine periodic boundary value inclusion problem (14) admits at least one solution in $[0, T]$ if $M|l||l|_{L^1} < 1$, where $M$ is the constant given in (8).

Proof. From (H6), $F(\cdot, y)$ is measurable, which means that for each $y \in C([0, T]; R)$, $S_{F, y}$ is nonempty; therefore, $F$ has a measurable selection.

Now, we claim the operator $\Phi$ is closed for each $y \in C([0, T]; R)$. Let $\{y_n\}_{n \geq 0}$ be a sequence in $C([0, T]; R)$ with $y_n \rightarrow y$. Then, $y_n \in C([0, T]; R)$ and there exists $u_n \in S_{F, y}$ such that, for each $t \in [0, T]$,

$$y_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_n(s) ds - \frac{1}{1-b} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u_n(s) ds$$

$$+ \mu_1(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} u_n(s) ds + \mu_2(t) \int_0^T \frac{(T-s)^{\alpha-3}}{\Gamma(\alpha-2)} u_n(s) ds.$$ 

Therefore, we gain $\frac{|y_n|}{|\tilde{\phi}(||y_n||)|} \leq M$. Invoking (H5), there exists $r$ such that $|y_n| \neq r$. Let

$$V = \{ y_n \in C([0, T]; R) : |y_n| < r + 1 \}.$$ 

Note that the operator $\Phi : \nabla \rightarrow \mathcal{P}(C([0, T]; R))$ is u.s.c. and completely continuous. From the choice of $V$, there is no $y \in \partial V$ such that $y \in s\Phi(y)$ for some $s \in (0, 1)$. According to Lemma 2, the operator $\Phi$ has a fixed point, $y \in \overline{V}$, which is a solution for the $(T, b)$ affine periodic boundary value problem (14). This completes the proof. 

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In view of the fact that \( F \) has compact values, we may pass to a subsequence to obtain that \( u_n \) converges to \( u_* \in L^1([0,T]; R) \). It is easy to check that \( u_* \in S_{F,y} \) and for any \( t \in [0,T] \),

\[
y_u(t) \to y_* = \int_0^t (t-s)^{a-1}u_*(s)ds - \frac{1}{1-b} \int_0^T (T-s)^{a-1}u_*(s)ds + \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)}u_*(s)ds + \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)}u_*(s)ds.
\]

Namely, \( y_* \in \Phi(y) \), which implies \( \Phi \) is closed.

Next, we show that \( \Phi \) is a contractive set-valued map with constant \( \gamma := M\|l\|_{L^1} \). Let \( y_1, y_2 \in C([0,T]; R) \) and \( \tilde{z}_1 \in \Phi(y) \). Then, there exists \( m_1(t) \in F(t, y_1(t)) \) such that for every \( t \in [0,T] \),

\[
\tilde{z}_1(t) = \int_0^t (t-s)^{a-1}m_1(s)ds - \frac{1}{1-b} \int_0^T (T-s)^{a-1}m_1(s)ds + \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)}m_1(s)ds + \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)}m_1(s)ds.
\]

Applying the inequality of (H7), there exists \( \omega \in F(t, y_2) \) such that

\[
|m_1(t) - \omega| \leq l(t)|y_1 - y_2|, \quad t \in [0,T].
\]

Let us define an operator \( N : [0,T] \to \mathcal{P}(R) \) by

\[
N(t) = \{ \omega \in R : |m_1(t) - \omega| \leq l(t)|y_1 - y_2| \}.
\]

As the set-valued operator \( N(t) \cap F(t, y_2(t)) \) is measurable, there exists a function \( m_2(t) \in F(t, y_2(t)) \) and for every \( t \in [0,T] \),

\[
|m_1(t) - m_2(t)| \leq l(t)|y_1 - y_2|.
\]

For each \( t \in [0,T] \), let us define

\[
\tilde{z}_2(t) = \int_0^t (t-s)^{a-1}m_2(s)ds - \frac{1}{1-b} \int_0^T (T-s)^{a-1}m_2(s)ds + \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)}m_2(s)ds + \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)}m_2(s)ds.
\]

As a consequence,

\[
|\tilde{z}_1(t) - \tilde{z}_2(t)| \leq \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)}|m_1(s) - m_2(s)|ds + \left| \frac{1}{1-b} \right| \int_0^T \frac{(T-s)^{a-1}}{\Gamma(a)}|m_1(s) - m_2(s)|ds + \mu_1(t) \int_0^T \frac{(T-s)^{a-2}}{\Gamma(a-1)}|m_1(s) - m_2(s)|ds + \mu_2(t) \int_0^T \frac{(T-s)^{a-3}}{\Gamma(a-2)}|m_1(s) - m_2(s)|ds \leq M \int_0^T l(t)|y_1 - y_2|ds.
\]
Thus,
\[ \| \hat{z}_1(t) - \hat{z}_2(t) \| \leq M \| l \|_1 \| y_1 - y_2 \|. \]

Analogously, it follows that
\[ d_H(\Phi(y_1) - \Phi(y_2)) \leq M \| l \|_1 \| y_1 - y_2 \|. \]

Therefore, \( \Phi \) is a contraction. According to Lemma 3, \( \Phi \) has a fixed point \( y \) which is a solution for the \((T, b)\) affine periodic boundary value problem (14). The proof is complete. \( \square \)

5. Conclusions

In this paper, we use classical fixed point theory to research the existence of a solution to the fractional differential equation and the fractional differential inclusion with \((T, b)\) affine periodic boundary value conditions. Fixed point theory plays an important role in dynamic systems, nonlinear programming and other fields, and is widely used to study solutions of nonlinear differential, integral and functional equations. Thanks to the Leray–Schauder fixed point theorem, we translate the existence of a solution to the fractional differential equation into a fixed point problem and verify that the problem has at least one fixed point. The importance of Leray–Schauder fixed point theorem is that it is also true for quasilinear operators in infinite dimensional space, which gives Leray–Schauder fixed point theorem great advantages in studying the existence and uniqueness of solutions. In brief, it is a very simple and ingenious way to prove the existence of solutions of differential equations by using a fixed point principle.

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