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Sharp Coefficient Bounds for a New Subclass of q-Starlike Functions Associated with q-Analogue of the Hyperbolic Tangent Function

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Abstract: In this study, by making the use of q-analogous of the hyperbolic tangent function and a Sălăgean q-differential operator, a new class of q-starlike functions is introduced. The prime contribution of this study covers the derivation of sharp coefficient bounds in open unit disk U, especially the first three coefficient bounds, Fekete–Szegő type functional, and upper bounds of second- and third-order Hankel determinant for the functions to this class. We also use Zalcman and generalized Zalcman conjectures to investigate the coefficient bounds of a newly defined class of functions. Furthermore, some known corollaries are highlighted based on the unique choices of the involved parameters l and q.

Keywords: analytic functions; univalent functions; q-starlike functions; Hankel determinants; q-derivative operator; Zalcman conjecture; Sălăgean q-differential operator; subordination

MSC: 30C45; 30C50; 30C80

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1. Introduction

Let an analytic function η in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

assume to satisfy the conditions $\eta(0) = 0$ and $\eta'(0) = 1$, and all such types of functions contained in class $\mathcal{A}$ and every $\eta \in \mathcal{A}$ have the following series of the form

$$\eta(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The set of all such types of univalent functions, which are normalized by the conditions $\eta(0) = 0$ and $\eta'(0) = 1$, is denoted by $\mathcal{S}$. A function η is called a starlike function if η maps U onto the starshape domain. Moreover, the set of all univalent functions which satisfy the condition

$$\text{Re}\left(\frac{z\eta'(z)}{\eta(z)}\right) > 0,\ z \in U$$

is denoted by the class $\mathcal{S}^*$ of starlike functions.

An analytic function w along the following conditions

$$w(0) = 0,$$

and

$$|w(z)| < 1,$$

is called Schwarz function. For two functions, η and g, which are analytic in U, we say that η is subordinate to g (denoted by $\eta \prec g$), if there exists a Schwarz function w, such that

$$\eta(z) = g(w(z)), \tag{2}$$

In particular, if g is univalent in U, then

$$\eta \prec g \Leftrightarrow \eta(0) = g(0),$$

and

$$\eta(U) \subset g(U).$$

The class $\mathcal{P}$ represents the class of Caratheodory functions [1], which satisfy the conditions

$$p(0) = 1 \text{ and } \mathrm{Re}(p(z) > 0, \ z \in U.$$

For every $p \in \mathcal{P}$, there is a Taylor series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{3}$$

Although function theory was initiated in 1851, in 1916, Bieberbach [2] conjecture unfolded this field and gave a new direction for research. In 1985, De-Branges [3] proved Bieberbach conjecture. A number of outstanding scholars found some new sub-families of the class $\mathcal{S}$ of normalized univalent functions associated with different image domains, for example, starlike $(\mathcal{S}^*)$ and convex $(\mathcal{K})$ functions, respectively. These classes are primary and remarkable subclasses of the univalent class $\mathcal{S}$. In 1992, Ma and Minda [4] made a very good contribution and defined the general form of the family of univalent functions as follows:

$$\mathcal{S}^*(\varphi) = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec \varphi(z) \right\}, \tag{4}$$

where φ is an analytic function along the conditions $\varphi(0) > 0$ and $\mathrm{Re}(\varphi(z)) > 0$ in U. If we choose $\varphi(z) = \frac{1+z}{1-z}$ in (4), then we have a class of starlike functions which is given below:

$$\mathcal{S}^* = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec \frac{1+z}{1-z} \right\}.$$

Janowski [5] investigated the class of Janowski starlike functions and defined it as follows:

$$\mathcal{S}^*(L, M) = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec \varphi(z) = \frac{1+Lz}{1+Mz}, \ (-1 \le M < L \le 1) \right\}.$$

Sokól and Stankiewicz [6] chose $\varphi(z) = \sqrt{1+z}$ and defined the family of class $\mathcal{S}^*_{\mathcal{L}}$ as

$$S^*_{\mathcal{L}} = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec \sqrt{1+z} \right\}.$$

The function $\varphi(z) = \sqrt{1+z}$ maps the region U onto the image domain, which is bounded by

$$\left| w^2 - 1 \right| < 1.$$

Recently, Cho et al. [7] chose $\varphi(z) = 1 + \sin z$ and defined a class $(\mathcal{S}^*_{\sin})$ of starlike functions:

$$S^*_{\sin} = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec 1 + \sin z \right\}.$$

Recently, Bano and Raza [8] chose $\varphi(z) = \cos z$, and Alotaibi et al. [9] considered $\varphi(z) = \cosh z$; they defined a family of new classes as follows:

$$S^*_{\cos z} = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec \cos z \right\}$$

and

$$S^*_{\cosh z} = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec \cosh z \right\}.$$

Kumar and Arora [10] considered a function $\varphi(z) = 1 + \sinh^{-1} z$ and used a petal-shaped domain and defined a class of starlike functions:

$$S^*_{\sinh^{-1}(z)} = \left\{ \eta \in \mathcal{A} : \frac{z\eta'(z)}{\eta(z)} \prec 1 + \sinh^{-1}(z) \right\}.$$

In 2021, Barukab et al. [11] investigated a class $\mathcal{R}_s$ and found a third Hankel determinant.

For $\eta \in \mathcal{A}$, the jth Hankel determinant is defined by

$$\mathcal{H}_{j,n}(\eta) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix},$$

where $n, j \in \mathbb{N}$, and $a_1 = 1$.

For different values of j and n, the $\mathcal{H}_{j,n}(\eta)$ has the following form:

(i) For $j = 2$ and $n = 1$, we obtain the Fekete–Szegő functional, that is,

$$\mathcal{H}_{2,1}(\eta) = \left| a_3 - a_2^2 \right|$$

and the generalized form of this functional is

$$\left| a_3 - \mu a_2^2 \right|,$$

where μ is a real or complex number, (see [12]).

(ii) Janteng [13] gave the following form of a second Hankel determinant and then a number of researchers studied it for some new classes of analytic functions

$$\mathcal{H}_{2,2}(\eta) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left| a_2 a_4 - a_3^2 \right|.$$

(iii) For $j = 3$ and $n = 1$, we have the following form of the third Hankel determinant:

$$\mathcal{H}_{3,1}(\eta) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2 a_4 - a_3^2\right) - a_4(a_4 - a_2 a_3) + a_5\left(a_3 - a_2^2\right).$$

In 1966, Pommerenke [14] investigated the Hankel determinants for univalent starlike functions. In 1983, Noor [15] defined a class of close-to-convex functions of higher order and investigated Hankel determinants while Ehrenborg [16] investigated the Hankel determinants associated with exponential function.

Recently, numerous researchers have started to pay more attention to finding the sharp bounds of Hankel determinants for a particular family of functions. For example, Janteng et al. [13] calculated the sharp bounds of the second Hankel determinant of the subfamily $(\mathcal{K}, \mathcal{S}^*, \mathcal{R})$ of univalent functions class $\mathcal{S}$. Cho et al. [17] investigated classes of starlike functions of order β and strongly starlike functions of order β, and established a Hankel determinant and showed that $|\mathcal{H}_{2,2}(\eta)|$ is bounded by $(1-\beta)^2$ and β^2. Zaprawa [18] used the new methodology and investigated the third Hankal determinant for the class of starlike $(\mathcal{S}^*)$ and convex $(\mathcal{K})$ functions:

$$|\mathcal{H}_{3,1}(\eta)| \leq \left\{ \begin{array}{cc} 1 & \text{if } \eta \in \mathcal{S}^* \\ \frac{49}{540} & \text{if } \eta \in \mathcal{K} \end{array} \right\}.$$

In 2018, Kwon et al. [19] improved the Zaprawa's result and proved that

$$|\mathcal{H}_{3,1}(\eta)| \leq \frac{8}{9}, \ \ \eta \in \mathcal{S}^*.$$

Again in 2021, Zaprawa et al. [20] improved the result of Kwon et al. and proved that

$$|\mathcal{H}_{3,1}(\eta)| \leq \frac{5}{9}, \ \ \eta \in \mathcal{S}^*.$$

Jackson [21] used the idea of basic q-calculus and defined the q-analogues of derivatives (D_q), and further, Ismail et al. [22] used this operator and defined a q-analogous of starlike functions. Recently, number of researchers started to use the D_q and defined many new subclasses of starlike and convex functions and found sharp bounds of second- and third-order Hankal determinants. For example, in 2019, Mahmood et al. [23] defined the family of q-starlike functions and investigated the third-order Hankel determinant, and for close-to-convex functions the same work was carried out by Srivastava in [24]. Arif et al. [25] used the technique of subordination and defined a new subclass of starlike functions associated with sine function [26] and then determined the third Hankel determinant for this class. Meanwhile, Srivastava et al. [27] defined another new subclass of q-starlike functions and investigated Hankel and Toeplitz determinants related with the generalized conic domain. In 2022, Raza et al. [28] defined a new class of q-starlike functions associated with symmetric Booth Lemniscate and studied Hankel determinants. For further research, see [29,30].

The calculus without limits is called q-calculus and it can be dealt with just like classical calculus. However, in q-calculus we do not use limits. Here, we define the q-derivative operator of a function $\eta \in \mathcal{A}$.

Definition 1 ([21])**.** *For $\eta \in \mathcal{A}$, the q-derivative operator or q-difference operator is defined by*

$$\begin{aligned} D_q\eta(z) &= \frac{\eta(z)-\eta(qz)}{(1-q)z}, \quad z \neq 0, q \neq 1, \\ &= 1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}, \end{aligned} \tag{5}$$

where

$$[t]_q = \frac{1-q^t}{1-q}, \ \ (t \in \mathbb{C}).$$

In particular, $t = n \in \mathbb{N}$,

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q} \\ &= \sum_{v=0}^{n-1} q^v. \end{aligned}$$

Definition 2 ([31])**.** *The Sălăgean q-differential operator for η is defined by*

$$\begin{aligned} \mathcal{S}_q^0\eta(z) &= \eta(z),\ \mathcal{S}_q^1\eta(z) = zD_q\eta(z) = \frac{\eta(qz)-\eta(z)}{(q-1)},\cdots, \\ \mathcal{S}_q^l\eta(z) &= zD_q\left(\mathcal{S}_q^{l-1}\eta(z)\right) = \eta(z)*\left(z+\textstyle\sum_{n=2}^{\infty}\left([n]_q\right)^l z^n\right), \\ &= z+\textstyle\sum_{n=2}^{\infty}\left([n]_q\right)^l a_n z^n, \end{aligned} \tag{6}$$

where $l \in \mathbb{N}$.

Remark 1. *For $q \to 1-$ in (6), then we have a familiar Sălăgean derivative introduced in [32].*

By using the definition of subordination, we introduce a new class $\mathcal{S}_s^*(l,q)$ of q-starlike functions associated with the Sălăgean q-differential operator and connected with the q-analogue of the hyperbolic tangent function.

Definition 3. *A function $\eta \in \mathcal{S}$ is said to be in the class $\mathcal{S}_s^*(l,q)$ if it satisfies the following condition*

$$\frac{\mathcal{S}_q^l\eta(z)}{\eta(z)} \prec \varphi(z),\ z \in U. \tag{7}$$

That is,

$$\mathcal{S}_s^*(l,q) = \left\{\eta \in \mathcal{S} : \frac{\mathcal{S}_q^l\eta(z)}{\eta(z)} \prec \varphi(z)\right\},$$

where

$$\varphi(z) = 1+\tanh(qz).$$

Remark 2. *For $l = 1$ and $\varphi(z) = 1+\sin(qz)$, then $\mathcal{S}_s^*(l,q)$ reduces to the class $\mathcal{S}_{qs}^*$ studied by Taj et al. [33].*

Remark 3. *For $l = 1$ and $q \to 1-$, then $\mathcal{S}_s^*(l,q)$ reduces to the class $\mathcal{S}_{qs}^*$ studied by Ullah et al. [34].*

2. A Set of Lemmas

We need the following lemmas to investigate the sharp coefficient problems for the class $\mathcal{S}_s^*(l,q)$.

Lemma 1 ([35])**.** *Let the function $p(z)$ be of the form (3), then*

$$|b_n| \le 2,\ n \in \mathbb{N}. \tag{8}$$

Additionally,

$$|b_n - \mu b_i b_{n-i}| \le 2,\ n > i,\ \mu \in [0,1]. \tag{9}$$

The equality holds for

$$\eta(z) = (1+z)(1-z)^{-1}.$$

Lemma 2 ([25])**.** *Let the function $p \in \mathcal{P}$ be given by (3), then*

$$\left|b_3 - 2Bb_1b_2 + Db_1^3\right| \le 2,$$

if

$$0 \le B \le 1,\ and\ B(2B-1) \le D \le B.$$

Lemma 3 ([35]). *Let an analytic function $p(z)$ be of the form (3), then*

$$2b_2 = b_1^2 + x(4 - b_1^2)$$

and

$$4b_3 = b_1^3 + 2(4 - b_1^2)b_1 x - (4 - b_1^2)b_1 x^2 + 2(4 - b_1^2)(1 - \left|x^2\right|)z,$$

where $x, z \in \mathbb{C}$, with $|z| \le 1$ and $|x| \le 1$.

Lemma 4 ([36]). *Consider the function $p \in \mathcal{P}$ of the form (3), $0 < a < 1$, $0 < \alpha < 1$ and*

$$\begin{aligned} & 8a(1-a)\left\{(\alpha\beta - 2\lambda)^2 + (\alpha(a+\alpha) - \beta)^2\right\} + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \\ \le \quad & 4\alpha^2 a(1-\alpha)^2(1-a). \end{aligned} \tag{10}$$

Then,

$$\left|\lambda b_1^4 + a b_2^2 + 2\alpha b_1 b_3 - \frac{3}{2}\beta b_1^2 b_2 - b_4\right| \le 2. \tag{11}$$

3. Main Results

Bound of $|H_{3,1}(\eta)|$ for the class $\mathcal{S}_s^*(l, q)$.

Theorem 1. *If $\eta \in \mathcal{S}_s^*(q, l)$ is of the form (1), then*

$$\begin{aligned} |a_2| &\le \frac{q}{(1+q)^l - 1}, \quad 0 < q < 1, \\ |a_3| &\le \frac{q}{(1+q+q^2)^l - 1}, \quad 0 < q < 1, \\ |a_4| &\le \frac{q}{(1+q+q^2+q^3)^l - 1}, \quad 0 < q < 1, \\ |a_5| &\le \frac{q}{(1+q+q^2+q^3+q^4)^l - 1}, \quad 0 < q < 0.8651682397. \end{aligned}$$

All bounds of Theorem 1 are sharp for the functions given in (29)–(32).

Proof. Let $\eta \in \mathcal{S}_s^*(q, l)$, then η satisfies (7), we have

$$\frac{\mathcal{S}_q^l \eta(z)}{\eta(z)} \prec 1 + \tanh(qz). \tag{12}$$

By using (2), we have

$$\frac{\mathcal{S}_q^l \eta(z)}{\eta(z)} = 1 + \tanh(q(w(z)). \tag{13}$$

Let

$$\begin{aligned} w(z) &= \frac{p(z) - 1}{p(z) + 1} \\ &= \frac{b_1 z + b_2 z^2 + b_3 z^3 + \dots}{2 + b_1 z + b_2 z^2 + \dots} \\ &= \frac{1}{2}b_1 z + \frac{1}{2}\left(b_2 - \frac{1}{2}b_1^2\right)z^2 + \frac{1}{2}\left(b_3 - b_1 b_2 + \frac{1}{4}b_1^3\right)z^3 + \cdots. \end{aligned} \tag{14}$$

In view of (13) and (14), we have

$$
\begin{aligned}
& 1+\tanh(q(w(z))) \\
= {} & 1+\frac{q}{2}b_1 z+q\left(\frac{1}{2}b_2-\frac{1}{4}b_1^2\right)z^2 \\
& +q\left(\frac{1}{2}b_3-\frac{1}{2}b_1b_2+\frac{(3-2q^2)}{24}b_1^3\right)z^3+ \\
& +24q\left(\frac{1}{2}b_4-\frac{1}{2}b_1b_3-\frac{q}{4}b_1^2+B(q)b_1^2b_2+B(q)b_1^4\right)z^4+\dots,
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
B(q) &= \left(\frac{3q}{8}-\frac{q^2}{256\times 24}+\frac{8q^3}{256\times 3}\right), \\
C(q) &= \left(\frac{q}{16}+\frac{q^2}{256\times 24}+\frac{q^3}{256}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \frac{\mathcal{S}_q^l \eta(z)}{\eta(z)} \\
= {} & 1+\left(\left([2]_q\right)^l-1\right)a_2 z+\left\{\left(\left([3]_q\right)^l-1\right)a_3-\left(\left([2]_q\right)^l-1\right)a_2^2\right\}z^2 \\
& +\left\{\left(\left([4]_q\right)^l-1\right)a_4-\left\{\left([2]_q\right)^l+\left([3]_q\right)^l-2\right\}a_2a_3+\left(\left([2]_q\right)^l-1\right)a_2^3\right\}z^3+ \\
& +\left\{\begin{array}{c} \left(\left([5]_q\right)^l-1\right)a_4-\left\{\left([2]_q\right)^l+\left([4]_q\right)^l-2\right\}a_2a_4-\left(\left([3]_q\right)^l-1\right)a_3^2 \\ +\left\{2\left([2]_q\right)^l+\left([3]_q\right)^l-3\right\}a_3a_2^2-\left(\left([2]_q\right)^l-1\right)a_2^4 \end{array}\right\}z^4.
\end{aligned} \tag{16}
$$

Equating the corresponding coefficients of (15) and (16), we have

$$
a_2 = \frac{qb_1}{2\left(\left([2]_q\right)^l-1\right)}, \tag{17}
$$

$$
a_3 = \frac{q}{2\left(\left([3]_q\right)^l-1\right)}\left\{b_2-\frac{1}{2}\left(\frac{\left([2]_q\right)^l-q-1}{\left([2]_q\right)^l-1}\right)b_1^2\right\}, \tag{18}
$$

$$
a_4 = \frac{q}{2\left(\left([4]_q\right)^l-1\right)}\left\{b_3-B(l,q)b_1b_2-D(l,q)b_1^3\right\}, \tag{19}
$$

$$
a_5 = \frac{q}{2\left(\left([5]_q\right)^l-1\right)}\left(\lambda b_1^4+ab_2^2+2\alpha b_1b_3-\frac{3}{2}\beta b_1^2b_2-b_4\right), \tag{20}
$$

where

$$B(l,q) = 1-\frac{\left(\left([2]_q\right)^l+\left([3]_q\right)^l-2\right)q}{2\left(\left([2]_q\right)^l-1\right)\left(\left([3]_q\right)^l-1\right)}, \tag{21}$$

$$D(l,q) = \left\{\frac{q\left(\left([2]_q\right)^l+\left([3]_q\right)^l-2\right)\left(\left([2]_q\right)^l-q-1\right)}{4\left(\left([2]_q\right)^l-1\right)^2\left(\left([3]_q\right)^l-1\right)}+\frac{q^2}{4\left(\left([2]_q\right)^l-1\right)^2}-\frac{(3-2q^3)}{12}\right\}, \tag{22}$$

$$\begin{aligned}\lambda &= \frac{2}{q}C(q)-\frac{q^3}{8\left(\left([2]_q\right)^l-1\right)^3}-\frac{2}{q}D(l,q)-\frac{q\left([2]_q\right)^l-q-1}{8\left(\left([2]_q\right)^l-1\right)\left(\left([3]_q\right)^l-1\right)} \\ &\quad -\frac{q^2\left(2\left([2]_q\right)^l+\left([3]_q\right)^l-3\right)\left(\left([2]_q\right)^l-q-1\right)}{8\left(\left([2]_q\right)^l-1\right)^3\left(\left([3]_q\right)^l-1\right)},\end{aligned} \tag{23}$$

$$\alpha = \frac{1}{2}-\frac{q\left(\left([2]_q\right)^l+\left([4]_q\right)^l-2\right)}{4\left(\left([2]_q\right)^l-1\right)\left(\left([4]_q\right)^l-1\right)}, \tag{24}$$

$$\begin{aligned}\beta &= \frac{4}{3q}B(q)-\frac{4}{3q}B(l,q)+\frac{q\left(\left([2]_q\right)^l-q-1\right)}{3\left(\left([2]_q\right)^l-1\right)\left(\left([2]_q\right)^l-1\right)} \\ &\quad -\frac{q^2\left(2\left([2]_q\right)^l+\left([3]_q\right)^l-3\right)}{6\left(\left([2]_q\right)^l-1\right)^2\left(\left([3]_q\right)^l-1\right)},\end{aligned} \tag{25}$$

$$a = \frac{1}{2}+\frac{q}{2\left(\left([3]_q\right)^l-1\right)}. \tag{26}$$

Applying the Lemma 1 on (17), we have

$$\begin{aligned}|a_2| &\le \frac{q}{\left([2]_q\right)^l-1} \\ &= \frac{q}{(1+q)^l-1}.\end{aligned}$$

Applying the Lemma 1 on (18), we have

$$\begin{aligned}|a_3| &\le \frac{q}{\left([3]_q\right)^l-1}, \\ &= \frac{q}{(1+q+q^2)^l-1}.\end{aligned}$$

Now consider (19), we obtain

$$|a_4| = \frac{q}{2\left(\left([4]_q\right)^l - 1\right)}\left|b_3 - B(l,q)b_1b_2 + D(l,q)b_1^3\right|,$$

where $B(l,q)$ and $D(l,q)$ are given by (21) and (22). Assuming the value, $B = B(l,q)$ and $D = -D(l,q)$, which satisfy $B(2B-1) \leq D \leq B$, when $q \in (0,1)$, by the applications of Lemma 2, we obtain

$$|a_4| \leq \frac{q}{(1+q+q^2+q^3)^l - 1}.$$

Now, from (20), we consider

$$a_5 = \frac{q}{2\left(\left([5]_q\right)^l - 1\right)}\left(\lambda b_1^4 + ab_2^2 + 2\alpha b_1b_3 - \frac{3}{2}\beta b_1^2b_2 - b_4\right),$$

we see that $0 < a < 1, 0 < \alpha < 1, q \in (0,1)$. Now, we take

$$\begin{aligned} & 8a(1-a)\left\{(\alpha\beta - 2\lambda)^2 + (\alpha(a+\alpha) - \beta)^2\right\} + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \\ \leq \; & 4\alpha^2 a(1-\alpha)^2(1-a). \end{aligned} \tag{27}$$

Using (23)–(26) in (27), we obtain

$$\begin{aligned} & 8a(1-a)\left\{(\alpha\beta - 2\lambda)^2 + (\alpha(a+\alpha) - \beta)^2\right\} \\ & +\alpha(1-\alpha)(\beta - 2a\alpha)^2 - 4\alpha^2 a(1-\alpha)^2(1-a) \\ = \; & 2\left(1 - \frac{q^2}{\left(\left([3]_q\right)^l - 1\right)^2}\right)\left\{\Phi_1(l,q) + \Phi_2(l,q) + \Phi_3(l,q) - \Phi_4(l,q)^2\right\} \\ = \; & \Psi(q,a,\alpha,\beta), \end{aligned}$$

where

$$\begin{aligned} \Phi_1(l,q) = \; & \Bigg\{\left(\frac{1}{2} - qT_1(q)\right)\left(\frac{4}{3q}(C(q) - B(l,q)) + qT_2(q) - q^2T_3(q)\right) \\ & -\frac{4}{q}(C(q) - D(l,q)) + \frac{q^3}{4\left(\left([3]_q\right)^l - 1\right)^3} + \frac{3q}{8}T_2(q) \\ & +\frac{3}{2}q^2\left(\frac{\left([2]_q\right)^l - (q+1)}{\left([2]_q\right)^l - 1}\right)T_3(q)\Bigg\}^2, \end{aligned}$$

$$\begin{aligned} \Phi_2(l,q) = \; & \left\{\left(\frac{1}{2} - qT_1(q)\right)\left(1 + \frac{q}{2\left(\left([3]_q\right)^l - 1\right)} - qT_1(q)\right)\right. \\ & \left. -\frac{4}{3q}(C(q) - B(l,q)) - qT_2(q) + q^2T_3(q)\right\}^2, \end{aligned}$$

$$\Phi_3(l,q) = \left(\frac{1}{4} - q^2(T_1(q))^2\right)\left\{\left(\frac{4}{3q}(C(q) - B(l,q)) + qT_2(q) - q^2T_3(q)\right) - \left(1 + \frac{q}{\left([3]_q\right)^l - 1}\right)\left(\frac{1}{2} - qT_1(q)\right)\right\}^2,$$

$$\Phi_4(l,q) = \left(\frac{1}{4} - q^2(T_1(q))^2\right)\left(1 - \frac{q^2}{\left(\left([3]_q\right)^l - 1\right)^2}\right),$$

and

$$T_1(q) = \frac{\left([2]_q\right)^l + \left([4]_q\right)^l - 2}{4\left(\left([2]_q\right)^l - 1\right)\left(\left([4]_q\right)^l - 1\right)},$$

$$T_2(q) = \frac{\left([2]_q\right)^l - (q+1)}{3\left(\left([2]_q\right)^l - 1\right)\left(\left([3]_q\right)^l - 1\right)},$$

$$T_3(q) = \frac{2\left([2]_q\right)^l + \left([3]_q\right)^l - 3}{6\left(\left([2]_q\right)^l - 1\right)^2\left(\left([3]_q\right)^l - 1\right)}.$$

Therefore, we have $\Psi(q,a,\alpha,\beta) \leq 0$, when $0 < q < 0.8651682397$.

Now, by using Lemma 4, we have

$$|a_5| \leq \frac{q}{(1+q+q^2+q^3+q^4)^l - 1}.$$

For sharpness, consider the function $\eta_n : U \longrightarrow \mathbb{C}$, defined by

$$\frac{\mathcal{S}_q^l \eta_n(z)}{\eta_n(z)} = 1 + \tanh(qz^n),\ n = 2,3,4,5. \tag{28}$$

The following functions show that the result is sharp.

$$\eta_2(z) = z + \frac{q}{\left([2]_q\right)^l - 1} z^2 + \ldots, \tag{29}$$

$$\eta_3(z) = z + \frac{q}{\left([3]_q\right)^l - 1} z^3 + \frac{q^2}{\left(\left([3]_q\right)^l - 1\right)\left(\left([5]_q\right)^l - 1\right)} z^5 + \ldots, \tag{30}$$

$$\eta_4(z) = z + \frac{q}{\left([4]_q\right)^l - 1} z^4 + \ldots, \tag{31}$$

$$\eta_5(z) = z + \frac{q}{\left([5]_q\right)^l - 1} z^5 + \ldots. \tag{32}$$

□

Considering $l = 1$, in Theorem 2, we obtain the result which is proved in [33].

Corollary 1. *If $\eta \in \mathcal{S}_s^*(q)$ is of the form (1), then*

$$
\begin{aligned}
|a_2| &\leq 1, \ 0 < q < 1, \\
|a_3| &\leq \frac{1}{1+q}, \ 0 < q < 1, \\
|a_4| &\leq \frac{1}{1+q+q^2}, \ 0 < q < 1, \\
|a_5| &\leq \frac{1}{1+q+q^2+q^3}, \ 0 < q < 0.8651682397.
\end{aligned}
$$

Let $l = 1$ and $q \to 1-$, in Theorem 1, we obtain the following corollary proved in [37]:

Corollary 2. *If $\eta \in \mathcal{S}_s^*$ has the series as given in (1), then*

$$
\begin{aligned}
|a_2| &\leq 1, \\
|a_3| &\leq \frac{1}{2}, \\
|a_4| &\leq \frac{1}{3}.
\end{aligned}
$$

These bounds are sharp for the following functions:

$$
\begin{aligned}
\eta_2(z) &= z + z^2 + \ldots, \\
\eta_3(z) &= z + \frac{1}{2}z^3 + \ldots, \\
\eta_4(z) &= z + \frac{1}{3}z^4 + \ldots.
\end{aligned}
$$

Let $l = 1$ and $q \to 1-$ in Theorem 1, we obtain the following corollary proved in [30]:

Corollary 3. *If $\eta \in \mathcal{S}_s^*$ has the series as given in (1), then*

$$
|a_n| \leq \frac{1}{n-1}, \ n = 2, 3, 4.
$$

Zalcman and Generalized Zalcman Conjecture

In 1960, Zalcman defined the conjecture for univalent functions. He stated that every $\eta \in \mathcal{S}$ of the form (1) satisfies the following inequality:

$$
\left| a_n^2 - a_{2n-1} \right| \leq (n-1)^2, \ \ n \geq 2. \tag{33}
$$

In 1999, Ma [38] proved a generalized version of Zalcman conjecture and stated that every univalent function $\eta \in \mathcal{S}$ satisfies the following inequality:

$$
|a_n a_i - a_{n+i-1}| \leq (n-1)(i-1), \ \ \forall\, i, n \in \mathbb{N}, \ n \geq 2, \ i \geq 2. \tag{34}
$$

In [39,40], authors discussed the Fekete–Szegő functional, the second-order Hankel determinant,and Zalcman conjecture, and these results are shown to be sharp. Furthermore, we have estimated the bounds of the third-order Hankel determinant for this class $\mathcal{S}_s^*(l, q)$ for different values of n and i. For $n = 2$, the inequality (33) has the form

$$
\left| a_2^2 - a_3 \right| \leq 1.
$$

Theorem 2. *If $\eta \in \mathcal{S}_s^*(l,q)$ is of the form (1), then*

$$\left|a_2^2 - a_3\right| \leq \frac{q}{(1+q+q^2)^l - 1}. \tag{35}$$

The inequality (35) is sharp for the function η_3 given in (29).

Proof. From (17) and (18), consider

$$\begin{aligned}
\left|a_3 - a_2^2\right| &= \left|\frac{q}{2\left(\left([3]_q\right)^l - 1\right)}\left\{b_2 - \frac{1}{2}\left(\frac{\left([2]_q\right)^l - q - 1}{\left([2]_q\right)^l - 1}\right)b_1^2\right\} - \frac{q^2 b_1^2}{4\left(\left([2]_q\right)^l - 1\right)^2}\right| \\
&= \frac{q}{2\left(\left([3]_q\right)^l - 1\right)}\left|b_2 - \frac{1}{2}\left\{\frac{\left([2]_q\right)^l - q - 1}{\left([2]_q\right)^l - 1} + \frac{q\left(\left([3]_q\right)^l - 1\right)}{\left(\left([2]_q\right)^l - 1\right)^2}\right\}b_1^2\right| \\
&= \frac{q}{2\left((1+q+q^2)^l - 1\right)}\left|b_2 - v b_1^2\right|,
\end{aligned}$$

where

$$v = \frac{1}{2}\left\{\frac{\left([2]_q\right)^l - q - 1}{\left([2]_q\right)^l - 1} + \frac{q\left([3]_q^l - 1\right)}{\left(\left([2]_q\right)^l - 1\right)^2}\right\}.$$

Since, $0 < q < 1$, therefore $v \in (0,1)$. Now, using Lemma 9, for $n = 2,\ i = 1$, we obtain (35).

For sharpness, consider the function η_3 such that

$$\eta_3(z) = z + \frac{q}{\left([3]_q\right)^l - 1}z^3 + \dots.$$

Now,

$$a_2 = 0, \text{ and } a_3 = \frac{q}{(1+q+q^2)^l - 1}.$$

□

Considering $l = 1$ in Theorem 2, we obtain the result which is proved in [33].

Corollary 4. *If $\eta \in \mathcal{S}_s^*(q)$ is of the form (1), then*

$$\left|a_2^2 - a_3\right| \leq \frac{1}{1+q}.$$

Considering $l = 1$ and $q \to 1-$ in Theorem 2, we have the following known corollary proved in [37]:

Corollary 5. *If $\eta \in \mathcal{S}_s^*$ is of the form (1), then*

$$\left|a_2^2 - a_3\right| \leq \frac{1}{2}.$$

Take $n = 3, i = 2$, in the inequality (34), then we have $|a_4 - a_2 a_3| \leq 2$. Now we discuss it as follows:

Theorem 3. *If $\eta \in \mathcal{S}_s^*(l,q)$ is the form (1), then*

$$|a_4 - a_2a_3| \leq \frac{q}{(1+q+q^2+q^3)^l - 1}, \quad 0 < q < 0.8651682.$$

The inequality is sharp for the function η_4 given in (31).

Proof. From (17)–(19), consider

$$|a_4 - a_2a_3| = \frac{q}{2\left(\left([4]_q\right)^l - 1\right)} \left| \begin{array}{c} b_3 - \left(B(l,q) + \frac{q\left(\left([4]_q\right)^l - 1\right)}{2\left(\left([2]_q\right)^l - 1\right)\left(\left([3]_q\right)^l - 1\right)}\right) b_1b_2 \\ - \left(D(l,q) - \frac{q\left(\left([4]_q\right)^l - 1\right)\left(\left([2]_q\right)^l - q - 1\right)}{4\left(\left([2]_q\right)^l - 1\right)^2\left(\left([3]_q\right)^l - 1\right)}\right) b_1^3 \end{array} \right|.$$

Assuming the values

$$\begin{aligned} B &= 1 - \frac{q\left((1+q)^l - 1\right)\left(\left((1+q)^l - (1+q+q^2)^l\right) - 2\right) + q\left(\left((1+q+q^2+q^3)^l\right) - 1\right)}{2\left((1+q)^l - 1\right)\left((1+q+q^2)^l - 1\right)}, \\ D &= -\left(D(l,q) - \frac{q\left(\left([4]_q\right)^l - 1\right)\left(\left([2]_q\right)^l - q - 1\right)}{4\left(\left([2]_q\right)^l - 1\right)^2\left(\left([3]_q\right)^l - 1\right)}\right), \\ &= -\left(\begin{array}{c} q(1+q)^l \left(\frac{\left\{(1+q)^l + (1+q+q^2)^l - 2\right\} - \left\{(1+q+q^2+q^3)^l - 1\right\}}{4\left((1+q)^l - 1\right)^2\left((1+q+q^2)^l - 1\right)} \right) \\ + \frac{q^2}{4\left((1+q)^l - 1\right)^2} + \left(\frac{3-2q^2}{12}\right) \end{array} \right). \end{aligned}$$

We see that,

$$B(2B-1) - D < 0, \text{ when } 0 < q < 0.8651682,$$

which shows that

$$B(2B-1) \leq D \leq B.$$

Thus, using Lemma 2, we obtain

$$|a_4 - a_2a_3| = \frac{q}{(1+q+q^2+q^3)^l - 1}.$$

The equality holds for the extremal function

$$\eta_4(z) = z + \frac{q}{\left([4]_q\right)^l - 1} z^4 + \dots.$$

□

Considering $l = 1$ in Theorem 3, we obtain the following known corollary, proved in [33]:

Corollary 6. *If $\eta \in \mathcal{S}_s^*(q)$, has the series of the form as given in (1), then*

$$|a_4 - a_2 a_3| \leq \frac{1}{1+q+q^2}.$$

Considering $l = 1$ and $q \to 1-$ in Theorem 3, we obtain the following known corollary, proved in [37]:

Corollary 7. *If $\eta \in \mathcal{S}_s^*$ has the series of the form as given in (1), then*

$$|a_4 - a_2 a_3| \leq \frac{1}{3}.$$

In the following result, we prove the second Hankel determinant $H_{2,2}(\eta)$.

Theorem 4. *If $\eta \in \mathcal{S}_s^*(l,q)$ is of the form (1), then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{q^2}{\left((1+q+q^2)^l - 1\right)^2}.$$

Proof. Making use of (17)–(19), we obtain

$$\left|a_2 a_4 - a_3^2\right| = Q_1 \left| -b_1^4 - \frac{Q_2}{Q_1} b_1^2 b_2 + \frac{Q_3}{Q_1} b_1 b_3 - \frac{Q_4}{Q_1} b_2^2 \right|,$$

where

$$Q_1 = Q_3 D(l,q) - \left(\frac{q\left(([2]_q)^l - q - 1\right)}{4\left(\left([3]_q\right)^l - 1\right)\left(\left([2]_q\right)^l - 1\right)} \right)^2,$$

$$Q_2 = Q_3 B(l,q) - \frac{q^2\left(([2]_q)^l - q - 1\right)}{4\left(\left([3]_q\right)^l - 1\right)^2\left(\left([2]_q\right)^l - 1\right)},$$

$$Q_3 = \frac{q^2}{4\left(\left([2]_q\right)^l - 1\right)\left(\left([4]_q\right)^l - 1\right)},$$

$$Q_4 = \frac{q^2}{4\left(\left([3]_q\right)^l - 1\right)^2}.$$

By using Lemma 3, we assume that $b = b_1$, $(0 \leq b \leq 2)$, so that

$$\begin{aligned} &\left|a_2 a_4 - a_3^2\right| \\ = \ & Q_1 \left| \begin{array}{c} \alpha_1 b^4 - \alpha_2 (4-b^2) b^2 x^2 - \alpha_3 (4-b^2)^2 x^2 \\ + \alpha_4 (4-b^2) b \delta \left(1 - |x|^2\right) + \alpha_5 (4-b^2) b^2 x \end{array} \right|, \end{aligned} \tag{36}$$

where

$$\begin{aligned}\alpha_1 &= 1-\frac{Q_2}{2Q_1}+\alpha_2-\alpha_3,\\ \alpha_2 &= \frac{Q_3}{4Q_1},\ \alpha_3=\frac{Q_4}{4Q_1},\ \alpha_4=2\alpha_2,\\ \alpha_5 &= 2\alpha_2-\frac{Q_2}{2Q_1}-\frac{2Q_4}{Q_1}.\end{aligned}$$

Using $|\delta| \leq 1$ and $|x| \leq 1$, and applying the triangle inequality, if we take $b \in [0,2]$, we obtain

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\ \leq\ & Q_1\left(\begin{array}{c}\alpha_1b^4+\alpha_2(4-b^2)b^2x^2+\alpha_3(4-b^2)^2x^2\\ +\alpha_4(4-b^2)b(1-x^2)+\alpha_5(4-b^2)b^2x\end{array}\right)\\ =\ & \mathcal{Q}(b,x).\end{aligned}$$

Now, trivially we have $\mathcal{Q}'(b,x) > 0$ on $[0,1]$. Therefore, $\mathcal{Q}(b,x)$ is an increasing function in the interval $[0,1]$. The maximum value occurs at $x=1$,

$$\max \mathcal{Q}(b,1) = \mathcal{Q}(b).$$

Hence,

$$\mathcal{Q}(b) = Q_1\left(\alpha_1b^4+\alpha_2\left(4-b^2\right)b^2+\alpha_3\left(4-b^2\right)^2+\alpha_5\left(4-b^2\right)b^2\right).$$

As

$$\mathcal{Q}'(b) \leq 0,$$

then $\mathcal{Q}(b)$ is a decreasing function of b, so that it gives the maximum value at $b=0:$

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\leq 16Q_1\alpha_3\\ &= \frac{q^2}{\left((1+q+q^2)^l-1\right)^2}.\end{aligned}$$

The result is sharp for the function η_3 given in (28). □

Considering $l=1$ in Theorem 4, we obtain the result which is proved in [33].

Corollary 8. *If $\eta \in \mathcal{S}_s^*(q)$ has the series of the form as given in (1), then*

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{(1+q)^2}.$$

Considering, $l=1$ and $q \to 1-$, in Theorem 4, we get the result which is proved in [37].

Corollary 9. *If $\eta \in \mathcal{S}_s^*$ has the series of the form as given in (1), then*

$$\left|a_2a_4-a_3^2\right| \leq \frac{1}{4}.$$

Theorem 5. *If $\eta \in \mathcal{S}_s^*(l,q)$ is of the form (1), and $0 < q < 0.8651682$, then*

$$\begin{aligned} |H_{3,1}(\eta)| \leq & \left(\frac{q}{(1+q+q^2+q^3+q^4)^l - 1}\right)\left(\frac{q}{(1+q+q^2)^l - 1}\right) \\ & + \left(\frac{q}{(1+q+q^2+q^3)^l - 1}\right)\left(\frac{q}{(1+q+q^2+q^3)^l - 1}\right) \\ & + \left(\frac{q}{(1+q+q^2)^l - 1}\right)\left(\frac{q^2}{\left((1+q+q^2+q^3)^l - 1\right)^2}\right). \end{aligned}$$

Proof. We know that

$$|H_{3,1}(\eta)| \leq |a_5|\left|a_3 - a_2^2\right| + |a_4||a_4 - a_2a_3| + |a_3|\left|a_2a_4 - a_3^2\right|.$$

Using Theorems 1–4, we have the required result when $0 < q < 0.8651682$. □

Considering $l = 1$ in Theorem 5, we obtain the result which is proved in [33].

Corollary 10. *If $\eta \in \mathcal{S}_s^*(l,q)$ is of the form (1), and $0 < q < 0.8651682$, then*

$$\begin{aligned} |H_{3,1}(\eta)| \leq & \left(\frac{1}{1+q+q^2+q^3}\right)\left(\frac{1}{1+q}\right) \\ & + \left(\frac{1}{1+q+q^2}\right)\left(\frac{1}{1+q+q^2}\right) \\ & + \left(\frac{1}{1+q}\right)\left(\frac{1}{(1+q+q^2)^2}\right). \end{aligned}$$

4. Conclusions

In this work, we introduced a new subclass $\mathcal{S}(l,q)$ of q-starlike functions related to the q-analogue of the hyperbolic tangent function through subordination relation. This class generalized a number of known subclasses of starlike functions. For this class, we have investigated the Fekete–Szegő type functional, and estimates of the second- and third-order Hankel determinant. We also considered Zalcman and generalized Zalcman inequalities and found sharp estimates. We have shown that all the results of this article are sharp.

Moreover, for future work, the class $\mathcal{S}(l,q)$ can be further investigated for finding the upper bounds of higher-order Hankel and Toeplitz determinants.

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References

1. Miller, S.S. Differential inequalities and Carathéodory functions. *Bull. Am. Math. Soc.* **1975**, *81*, 79–81. [CrossRef]
2. Bieberbach, L. Über die koeffizienten derjenigen potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitz. Ber. Preuss. Akad. Wiss.* **1916**, *138*, 940–955.
3. De Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
4. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–22 June 1992; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; Conference Proceedings and Lecture Notes in Analysis; International Press: Cambridge, UK, 1994; Volume I, pp. 157–169.
5. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1970**, *23*, 159–177. [CrossRef]
6. Sokół, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzesz. Mat. Fiz.* **1996**, *19*, 101–105.

7. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
8. Bano, K.; Raza, M. Starlike functions associated with cosine function. *Bull. Iran. Math. Soc.* **2021**, *47*, 1513–1532. [CrossRef]
9. Alotaibi, A.; Arif, M.; Alghamdi, M.A.; Hussain, S. Starlikness associated with cosine hyperbolic function. *Mathematics* **2020**, *8*, 1118. [CrossRef]
10. Kumar, S.S.; Arora, K. Starlike functions associated with a petal shaped domain. *Bull. Korean Math. Soc*. **2022**, *59*, 993–1010.
11. Barukab, O.; Arif, M.; Abbas, M.; Khan, S.K. Sharp bounds of the coefficient results for the family of bounded turning functions associated with petal shaped domain. *J. Funct. Spaces* **2021**, *2021* , 5535629. [CrossRef]
12. Karthikeyan, K.R.; Murugusundaramoorthy, G.; Purohit, S.D.; Suthar, D.L. Certain class of analytic functions with respect to symmetric points defined by q-calculus. *J. Math.* **2021**, *2021*, 8298848. [CrossRef]
13. Janteng, A.; Halim, A.S.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math.* **2007**, *1*, 619–625.
14. Pommerenke, C. On starlike and close-to-convex functions. *Proc. Lond. Math. Soc.* **1963**, *3*, 290–304. [CrossRef]
15. Noor, K.I. On subclasses of close-to-convex functionsof higher order. *Int. J. Math. Math. Sci.* **1983**, *6*, 327–334. [CrossRef]
16. Ehrenborg, R. The Hankel determinant of exponential polynomials. *Am. Math. Mon.* **2000**, *107*, 557–560. [CrossRef]
17. Cho, N.E.; Kowalczyk, B.; Kwon, O.S.; Lecko, A.; Sim, Y.J. The bounds of some determinants for starlike functions of order alpha. *Bull. Malays. Math. Sci. Soc.* **2018**, *41*, 523–535. [CrossRef]
18. Zaprawa, P. Third Hankel determinants for subclasses of univalent functions. *Mediterr. J. Math.* **2017**, *14*, 19. [CrossRef]
19. Kwon, O.S.; Lecko, A.; Sim, Y.J. The bound of the Hankel determinant of the third kind for starlike functions. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 767–780. [CrossRef]
20. Zaprawa, P.; Obradovic, M.; Tuneski, N. Third Hankel determinant for univalent starlike functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2021**, *115*, 9. [CrossRef]
21. Jackson, F.H. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1908**, *46*, 253–281. [CrossRef]
22. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]
23. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper bound of the third Hankel determinant for a subclass of q -starlike functions. *Symmetry* **2019**, *11*, 347. [CrossRef]
24. Srivastava, H.M.V.; Ahmad, Q.Z.; Darus, M.; Khan, N.; Khan, B.; Zaman, N.; Shah, H.H. Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the Lemniscate of Bernoulli. *Mathematics* **2019**, *7*, 848. [CrossRef]
25. Arif, M.; Raza, M.; Tang, H.; Hussain, S.; Khan, H. Hankel determinant of order three for familiar subsets of analytic functions related with sine function. *Open Math.* **2019**, *17*, 1615–1630. [CrossRef]
26. Zhang, H.Y.; Tang, H. A study of fourth-order Hankel determinants for starlike functions connected with the sine function. *J. Funct. Spaces* **2021**, *2021*, 9991460. [CrossRef]
27. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a generalized conic Domain. *Mathematics* **2019**, *7*, 181. [CrossRef]
28. Raza, M.; Riaz, A.; Xin, Q.; Malik, S.N. Hankel determinants and coefficient estimates for starlike functions related to symmetric booth lemniscate. *Symmetry* **2022**, *14*, 1366. [CrossRef]
29. Shi, L.; Srivastava, H.M.; Arif, M.; Hussain, S.; Khan, H. An investigation of the third Hankel determinant problem forcertain subfamilies of univalent functions involving the exponential function. *Symmetry* **2019**, *11*, 598. [CrossRef]
30. Zhang, H.Y.; Srivastava, R.; Tang, H. Third-order Hankel and Toeplitz determinants for starlike functions connected with the sine function. *Mathematics* **2019**, *7*, 404. [CrossRef]
31. Govindaraj, M.; Sivasubramanian, S. On a class of analytic functions related to conic domains involving q-calculus. *Anal. Math.* **2017**, *43*, 475–487. [CrossRef]
32. Sãlaãgean, G.S. Subclasses of univalent functions. In *Complex Analysis, Fifth Romanian—Finnish Seminar, Part 1 (Bucharest, 1981)*; Lecture Notes in Mathematics, 1013; Springer: Berlin, Germany, 1983; pp. 362–372.
33. Taj, Y.; Zainab, S.; Xin, Q.; Tawfiq, M.O.; Raza, M.; Malik, S.N. Certain coefficient problems for q-starlike functions associated with q-analogue of sine function. *Symmetry* **2022**, *14*, 2200. [CrossRef]
34. Ullah, K.; Zainab, S.; Arif, M.; Darus, M.; Shutaywi, M. Radius problems for starlike functions associated with the tan hyperbolic function. *J. Funct. Spaces* **2021**, *2021*, 9967640. [CrossRef]
35. Duren, P.L. Univalent Functions. In *Grundlehren der Mathematischen Wissenschaften*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Iapan, 1983; Volume 259.
36. Ravichandran, V.; Verma, S. Bound for the fifth coefficient of certain starlike functions. *Comptes Rendus Math. Acad. Sci.* **2015**, *353*, 505–510. [CrossRef]
37. Ullah, K.; Srivastava, H.M.; Rafiq, A.; Arif, M.; Arjika, S. A study of sharp coefficient bounds for a new subfamily of starlike functions. *J. Inequalities Appl.* **2021**, *2021*, 194. [CrossRef]
38. Ma, W. Generalized Zalcman conjecture for starlike and typically real functions. *J. Math. Anal. Appl.* **1999**, *234*, 328–339. [CrossRef]

39. Murugusundaramoorthy, G. Fekete–Szegő inequalities for certain subclasses of analytic functions related with nephroid domain. *J. Contemp. Math. Anal.* **2022**, *57*, 56–69. [CrossRef]
40. Murugusundaramoorthy, G.; Khan, M.G.; Ahmad, B.; Ahmad, B.; Abdeljawad, T.; Salleh, Z. Coefficient functionals for a class of bounded turning functions connected to three leaf function. *J. Math. Comput. Sci.* **2022**, *28*, 213–223. [CrossRef]