A Class of Rough Generalized Marcinkiewicz Integrals on Product Domains

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Abstract: In this article, suitable estimates for a class of rough generalized Marcinkiewicz integrals on product spaces are established. By these estimates, together with employing Yano’s extrapolation technique, we obtain the boundedness of the aforementioned integral operators under weak conditions on singular kernels. A number of known previous results on Marcinkiewicz as well as generalized Marcinkiewicz operators over a symmetric space are essentially improved or extended.

Keywords: rough integral operators; Marcinkiewicz integrals; product domains; extrapolation

1. Introduction

Throughout this article, we let \(d \geq 2\) (\(d = n\ or\ m\)) and \(\mathbb{R}^d\) be a Euclidean space of dimensions \(d\). Furthermore, we let \(\mathbb{S}^{d-1}\) be the unit sphere in \(\mathbb{R}^d\) equipped with the normalized Lebesgue surface measure \(d\mu_{\mathbb{S}}(\cdot) \equiv d\mu\).

For \(\lambda_1 = \tau_1 + iv_1, \lambda_2 = \tau_2 + iv_2\ (\tau_1, \tau_2, v_1, v_2 \in \mathbb{R}\ with\ \tau_1, \tau_2 > 0)\), we assume that

\[
K_{\lambda_1, \lambda_2}(\omega, v) = \frac{\Omega(\omega, v)h(|\omega|, |v|)}{|\omega|^{\alpha_1} |v|^{\alpha_2}},
\]

where \(h\) is a measurable function defined on \(\mathbb{R}_+ \times \mathbb{R}_+\) and \(\Omega\) is a measurable function defined on \(\mathbb{R}^n \times \mathbb{R}^m\) which satisfies the following properties:

\[
\Omega(\omega, v) = \Omega(\omega, v),\ \forall r, s > 0, \quad (1)
\]

\[
\int_{\mathbb{S}^{n-1}} \Omega(\omega, v) d\mu(\omega) = \int_{\mathbb{S}^{m-1}} \Omega(\omega, v) d\mu(\omega) = 0, \quad (2)
\]

and

\[
\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}). \quad (3)
\]

For \(\alpha > 1\) and \(f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)\), we consider the generalized parametric Marcinkiewicz integral over the symmetric space \(\mathbb{R}^n \times \mathbb{R}^m\)

\[
\mathcal{M}_{\lambda_1, \lambda_2}(\alpha)(f)(x, y) = \left( \int_{\mathbb{R}_+ \times \mathbb{R}_+} |F_{\lambda_1, \lambda_2}(f)(x, y)|^\alpha \frac{drds}{rs} \right)^{1/\alpha}, \quad (4)
\]

where

\[
F_{\lambda_1, \lambda_2}(f)(x, y) = \int_{|\omega| \leq r} \int_{|v| \leq s} K_{\lambda_1, \lambda_2}(\omega, v)f(x - \omega, y - v) d\omega dv.
\]
When \( p = 2, h \equiv 1, \) and \( \lambda_1 = 1 = \lambda_2, \) we denote the operator \( \mathcal{M}_{\Omega,h}^{(a)} \) by \( \mathcal{M}_\Omega. \) In this case, \( \mathcal{M}_\Omega \) is essentially the classical Marcinkiewicz integral on product spaces. The study of the \( L^p \) boundedness of the operator \( \mathcal{M}_\Omega \) was started by Ding in [1], in which he established the \( L^2 \) boundedness of \( \mathcal{M}_\Omega \) whenever \( \Omega \) lies in the space \( L(\log L)^2(S^{n-1} \times S^{m-1}). \) Thereafter, the boundedness of \( \mathcal{M}_\Omega \) has been studied by many researchers. For example, Choi in [2] proved the \( L^2 \) boundedness of \( \mathcal{M}_\Omega \) if \( \Omega \) satisfies the weaker condition \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}). \) In [3], the authors proved that \( \mathcal{M}_\Omega \) is bounded on \( L^p(\mathbb{R}^n \times \mathbb{R}^m) \) for all \( p \in (1, \infty) \) if \( \Omega \in L(\log L)^2(S^{n-1} \times S^{m-1}). \) Later on, the authors of [4] improved and extended the above results. In fact, they showed that the operator \( \mathcal{M}_\Omega \) is of type \( (p, p) \) for all \( 1 < p < \infty, \) provided that \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}). \) Furthermore, they found that by adapting the technique employed in [5] to the product space setting, the condition \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}) \) is optimal in the sense that it cannot be replaced by a weaker condition \( \Omega \in L(\log L)^{1-\epsilon}(S^{n-1} \times S^{m-1}) \) for some \( \epsilon \in (0, 1). \) On the other hand, Al-Qassem in [6] showed that \( \mathcal{M}_\Omega \) is bounded on \( L^p(\mathbb{R}^n \times \mathbb{R}^m) \) for all \( 1 < p < \infty \) if \( \Omega \in B_q^{(0, 0)}(S^{n-1} \times S^{m-1}) \) with \( q > 1. \) Moreover, he showed that the condition \( \Omega \in B_q^{(0, 0)}(S^{n-1} \times S^{m-1}) \) is optimal in the sense that we cannot replace it by \( \Omega \in B_q^{(0, \nu)}(S^{n-1} \times S^{m-1}) \) for any \( \nu \in (-1, 0). \) Here, \( B_q^{(0, \nu)}(S^{n-1} \times S^{m-1}) \) is a special class of block spaces introduced in [7].

By using an extrapolation argument, the authors of [8] proved that the \( L^p \) boundedness of \( \mathcal{M}_{\Omega,h}^{(2)} \) for all \( |1/2 - 1/p| < \min\{1/\gamma, 1/2\} \) whenever \( h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+) \) for some \( \gamma > 1 \) and \( \Omega \) lies in either the space \( L(\log L)(S^{n-1} \times S^{m-1}) \) or in the space \( B_q^{(0,0)}(S^{m-1} \times S^{n-1}) \) with \( q > 1. \) Here, \( \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+) \) for \( \gamma > 1 \) indicates the class of measurable functions \( h \) which are defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \) and satisfy

\[
\|h\|_{\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)} = \sup_{j,k \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |h(r,s)|^\gamma \frac{dr ds}{rs} \right)^{1/\gamma} < \infty.
\]

Recently, the authors of [9] established that if \( h \equiv 1 \) and \( \Omega \in L(\log L)^{2/a}(S^{n-1} \times S^{m-1}) \) or \( \Omega \in B_q^{(0, \frac{2}{a}-1)}(S^{m-1} \times S^{n-1}), \) then

\[
\left\| \mathcal{M}_{\Omega,h}^{(a)}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_p \|f\|_{F_p^{\mathcal{M}_{\Omega,h}^{(a)}}(\mathbb{R}^n \times \mathbb{R}^m)} \tag{5}
\]

for all \( p \in (1, \infty). \)

It is well known that the Marcinkiewicz integral, \( \mathcal{M}_\Omega, \) on product spaces naturally generalizes the Marcinkiewicz integral in one parameter setting which was introduced by E. Stein in [10]. The singularity of \( \mathcal{M}_\Omega \) is along the diagonals \( \{x = y\} \) and \( \{y = y\}. \) The study of singular integrals on product spaces and the study of \( \mathcal{M}_\Omega \) as well as its generalizations, which may have singularities along subvarieties, has attracted the attention of many authors in recent years. One of the principal motivations for the study of such operators is the requirements of several complex variables and large classes of “subelliptic” equations. For more background information, readers may refer to Stein’s survey articles [11, 12].

Let us recall the definition of Triebel–Lizorkin spaces, \( F_p^{\mathcal{M}_{\Omega,h}}(\mathbb{R}^n \times \mathbb{R}^m). \) Assume that \( \vec{\alpha} = (\beta, \epsilon) \in \mathbb{R} \times \mathbb{R} \) and \( a, p \in (1, \infty). \) The homogeneous Triebel–Lizorkin space \( F_p^{\mathcal{M}_{\Omega,h}}(\mathbb{R}^n \times \mathbb{R}^m) \) is defined to be the class of all tempered distributions \( f \) on \( \mathbb{R}^n \times \mathbb{R}^m \) such that

\[
\|f\|_{F_p^{\mathcal{M}_{\Omega,h}}(\mathbb{R}^n \times \mathbb{R}^m)} = \left\| \left( \sum_{\lambda \in \mathbb{Z}} 2^{k\beta \lambda} 2^{j\epsilon \lambda} \|\langle \phi_j \otimes \psi_j \rangle * f\|_{a} \right)^{\frac{1}{a}} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} < \infty,
\]
where \( \tilde{\phi}_k(x) = 2^{-kn}E(2^{-k}x) \) for \( k \in \mathbb{Z} \), \( \tilde{\psi}_j(y) = 2^{-jm}J(2^{-j}y) \) for \( j \in \mathbb{Z} \), and the functions \( E \in C_0^\infty(\mathbb{R}^n) \) and \( J \in C_0^\infty(\mathbb{R}^m) \) are radial functions satisfying the following properties:

(i) \( E, J \in [0, 1] \);

(ii) \( \text{supp}(E) \subset \{ x : |x| \in [\frac{1}{2}, 2] \} \), \( \text{supp}(J) \subset \{ y : |y| \in [\frac{1}{2}, 2] \} \);

(iii) \( E(x), J(y) \geq T > 0 \) if \( |x|, |y| \in [\frac{3}{2}, \frac{5}{2}] \) for some constant \( T \);

(iv) \( \sum_{k \in \mathbb{Z}} E(2^{-k}x) = \sum_{j \in \mathbb{Z}} J(2^{-j}y) = 1 \) with \( x \neq 0 \neq y \).

It was shown in [13] that the space \( F_{p}^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m) \) satisfies the following:

(a) The Schwartz space \( S(\mathbb{R}^n \times \mathbb{R}^m) \) is dense in \( F_{p}^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m) \);

(b) \( F_{p}^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m) \subseteq L^p(\mathbb{R}^n \times \mathbb{R}^m) \) for \( 1 < p < \infty \);

(c) \( F_{p}^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m) \subseteq L^p(\mathbb{R}^n \times \mathbb{R}^m) \) if \( \alpha_1 \leq \alpha_2 \);

(d) \( F_{p}^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m) = F_{p}^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m) \),

where \( p' \) denotes the exponent conjugate to \( p \), that is, \( 1/p + 1/p' = 1 \) whenever \( 1 < p < \infty \) and \( p' := 1 \) or \( p' := +\infty \) for \( p := +\infty \) or \( p := 1 \), respectively.

In light of the results in [8] concerning the boundedness of the operator \( M_{\Omega_h}^{(2)} \) and of the results in [9] concerning the boundedness of the generalized operator \( M_{\Omega_h}^{(a)} \), a natural question arises in the following:

**Question:** Is the operator \( M_{\Omega_h}^{(a)} \) bounded under the same assumptions in [8] with replacing \( a = 2 \) by \( a > 1 \)?

The main purpose of this work is to answer the above question affirmatively. Precisely, we have the following:

**Theorem 1.** Let \( h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+) \) for some \( \gamma \in (1, 2] \) and \( \Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \) for some \( q \in (1, 2] \). Then, there is a constant \( C_{p,\Omega_h} \) such that

\[
||M_{\Omega_h}^{(a)}(f)||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{p,\Omega_h} \left( \frac{1}{\gamma - (q - 1)} \right)^{2/a} \|f\|_{F_p^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)}
\]

for all \( p \in (\frac{\gamma}{\gamma - (q - 1)}, \frac{\gamma}{\gamma - (q - 1)}) \) if \( \alpha \leq \gamma' \), and

\[
||M_{\Omega_h}^{(a)}(f)||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{p,\Omega_h} \left( \frac{1}{\gamma - (q - 1)} \right)^{2/a} \|f\|_{F_p^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)}
\]

for all \( \gamma' < p < \infty \) if \( \alpha \geq \gamma' \), where \( C_{p,\Omega_h} = C_p \|h\|_{\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)} \|\Omega\|_{L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \).

**Theorem 2.** Assume that \( h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+) \) with \( \gamma \in (2, \infty) \) and that \( \Omega \) lies in \( L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \) with \( q \in (1, 2] \). Then, we have

\[
||M_{\Omega_h}^{(a)}(f)||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{p,\Omega_h} \left( \frac{\gamma}{q - 1} \right)^{2/a} \|f\|_{F_p^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)}
\]

for all \( p \in (1, \alpha) \) if \( \alpha \leq \gamma' \), and

\[
||M_{\Omega_h}^{(a)}(f)||_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{p,\Omega_h} \left( \frac{\gamma}{q - 1} \right)^{2/a} \|f\|_{F_p^{\tilde{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)}
\]

for all \( p \in (\gamma', \infty) \) if \( \alpha \geq \gamma' \).
By employing the estimates in Theorems 1 and 2 and employing an extrapolation argument as in [14] (see also [15,16]), we obtain the following:

**Theorem 3.** Let \( h \) be given as in Theorem 1.

(i) If \( \Omega \in L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \), then the inequality

\[
\left\| \mathcal{M}^{(\alpha)}_{\Omega,h}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_p \left( 1 + \| \Omega \|_{L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \right) \left\| h \right\|_{\Delta_{\alpha}(\mathbb{R}^n \times \mathbb{R}^m)} \left\| f \right\|_{F_p^\alpha}(\mathbb{R}^n \times \mathbb{R}^m)
\]

holds for \( p \in \left( \frac{\alpha\gamma'}{\alpha+\gamma'-\gamma}, \frac{\alpha\gamma'}{\alpha-\gamma} \right] \) if \( \alpha \leq \gamma' \), and for \( \gamma' < p < \infty \) if \( \alpha \geq \gamma' \).

(ii) If \( \Omega \in B_q^{(0,\frac{2}{\alpha} - 1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \) for some \( q > 1 \), then the inequality

\[
\left\| \mathcal{M}^{(\alpha)}_{\Omega,h}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_p \left( 1 + \left\| \Omega \right\|_{B_q^{(0,\frac{2}{\alpha} - 1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \right) \left\| h \right\|_{\Delta_{\alpha}(\mathbb{R}^n \times \mathbb{R}^m)} \left\| f \right\|_{F_p^\alpha}(\mathbb{R}^n \times \mathbb{R}^m)
\]

holds for \( p \in \left( \frac{\alpha\gamma'}{\alpha+\gamma'-\gamma}, \frac{\alpha\gamma'}{\alpha-\gamma} \right] \) if \( \alpha \leq \gamma' \), and for \( \gamma' < p < \infty \) if \( \alpha \geq \gamma' \).

**Theorem 4.** Let \( h \in \Delta_{\alpha}(\mathbb{R}^n \times \mathbb{R}^m) \) for some \( \gamma \in (2, \infty) \) and \( \Omega \in L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cup B_q^{(0,\frac{2}{\alpha} - 1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \) for some \( q > 1 \). Then, the operator \( \mathcal{M}^{(\alpha)}_{\Omega,h} \) is bounded on \( L^p(\mathbb{R}^n \times \mathbb{R}^m) \) for \( p \in (1,\alpha) \) if \( \alpha \leq \gamma' \), and for \( p \in (\gamma', \infty) \) if \( \alpha \geq \gamma' \).

**Remark 1.**

1. The conditions assumed for \( \Omega \) in Theorems 3 and 4 are the weakest conditions in their respective classes for the case \( \alpha = 2 \) and \( h = 1 \) (see [4,6]).

2. For the special case \( h = 1 \), Theorem 4 gives that \( \mathcal{M}^{(\alpha)}_{\Omega,1} \) is bounded on \( L^p(\mathbb{R}^n \times \mathbb{R}^m) \) for all \( p \in (1,\infty) \), provided that \( \Omega \) belongs to \( L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \) or to \( B_q^{(0,\frac{2}{\alpha} - 1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \), which is Theorem 2.7 in [9].

3. The result in Theorem 3 in the case \( \alpha = 2 \) and \( 1 < \gamma \leq 2 \) essentially improves Theorem 2 in [8], in which the authors proved the \( L^p \) boundedness of \( \mathcal{M}^{(2)}_{\Omega,1} \) for \( p \in \left( \frac{2\gamma}{\gamma+2}, \frac{2\gamma}{\gamma-2} \right] \). Hence, the range of \( p \) in Theorem 3 is better than the range of that obtained in [8].

4. The authors of [17] proved the \( L^p(\gamma' < p < \infty) \) boundedness of \( \mathcal{M}^{(\alpha)}_{\Omega,h} \) only for the special case \( 1 < \gamma \leq 2 \) and \( \alpha = \gamma' \). Therefore, the results in Theorem 3 essentially improve the main results in [17].

5. For the special case \( \alpha = \gamma' \) with \( 2 < \gamma < \infty \), Theorem 4 leads to the boundedness of \( \mathcal{M}^{(\alpha)}_{\Omega,h} \) for all \( p \in (1,\infty) \).

Henceforward, the constant \( C \) signifies a positive real number that could be different at each occurrence but is independent of all essential variables.

### 2. Auxiliary Lemmas

This section is devoted to introducing some notation and establishing some lemmas that will be needed to prove the main results of this paper. For \( \theta \geq 2 \), consider the family of measures \( \{ \mu_{\Omega,h,r,s} := \mu_{r,s} : r, s \in \mathbb{R}^+ \} \) and its corresponding maximal operators \( \mu_{h}^{\theta} \) and \( S_{h,\theta} \) on \( \mathbb{R}^n \times \mathbb{R}^m \) by

\[
\int_{\mathbb{R}^n \times \mathbb{R}^m} f d\mu_{r,s} = \frac{1}{r^{1/\theta} s^{1/\theta}} \int_{1/2r \leq |\omega| \leq r} \int_{1/2s \leq |\omega| \leq s} f(\omega, v) K_{\Omega,h}(\omega, v) d\omega dv,
\]

\[
\mu_{h}^{\theta}(f)(\omega, v) = \sup_{r, s \in \mathbb{R}^+} ||| \mu_{r,s} * f(\omega, v) |||
\]
where $|\mu_{r,s}|$ is defined in the same way as $\mu_{r,s}$ but with $\Omega h$ replaced by $|\Omega h|$. 

**Lemma 1.** Assume that $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$, with $\gamma > 1$ and $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Then, for any $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$ with $p \in (\gamma', \infty)$, we have

$$\|\mu^*_h(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \tilde{C}_{p,h,\Omega} \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}$$

and

$$\|S_{h,\delta}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \tilde{C}_{p,h,\Omega} \ln^2(\theta) \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)},$$

where $\tilde{C}_{p,h,\Omega} = \|\Omega\|_{L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)}$.

**Proof.** Thanks to Hölder’s inequality, we obtain that

$$|\mu_{r,s}| * f(x,y) \leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)} \left\{ \frac{1}{rs} \int_0^r \int_0^s |\Omega(\omega,\nu)| d\omega d\nu \right\}^{1/\gamma'}$$

$$\times \frac{1}{r} \int_0^r \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} |\Omega(\omega,\nu)| |\mu^*(\{f\}^{1/\gamma})_{L^{(\gamma'/\gamma)(\mathbb{R}^n \times \mathbb{R}^m)}}| d\mu(\omega) d\mu(v) dr ds$$

$$\leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)} \|\mu^*(\{f\}^{1/\gamma})\|_{L^{(\gamma'/\gamma)(\mathbb{R}^n \times \mathbb{R}^m)}}$$

$$\leq C\tilde{C}_{p,h,\Omega} \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)},$$

where

$$\mu^*(f)(x,y) = \sup_{r,s > 0} \frac{1}{rs} \int_0^r \int_0^s |f(x-r\omega, y-s\nu)| dr ds.$$

Inequality (7) is easily deduced from Inequality (6). □

The next lemma is found in [8] with very minor modifications. We omit the proof.

**Lemma 2.** Let $\theta \geq 2$, $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$. Then, the following estimates hold:

$$\|\mu_{r,s}\| \leq C_{\Omega,\gamma'}$$

$$\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{m-1}} |\mu_{r,s}(\xi, \xi')|^2 \frac{d\omega d\nu}{s^2} \leq C_{\Omega,\gamma} \ln^2(\theta) \|\theta^{\frac{\gamma}{2}}\|_{L^\frac{\gamma}{\gamma'}} \|\mu_{r,s}\| \leq C_{\Omega,\gamma} \ln^2(\theta) \|\theta^{\frac{\gamma}{2}}\|_{L^\frac{\gamma}{\gamma'}},$$

where $2\delta' < 1$ and $\|\mu_{r,s}\|$ is the total variation of $\mu_{r,s}$.

In order to prove our main results, we need to prove the following lemmas.
Lemma 3. Suppose that $\theta \geq 2, h \in \Delta_\gamma(\mathbb{R}^n \times \mathbb{R}^m)$ with $1 < q \leq 2$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $1 < q \leq 2$. Let $\alpha \in (1, \gamma')$ and $\{G_j, \gamma, j, k \in \mathbb{Z}\}$ be arbitrary functions defined on $\mathbb{R}^n \times \mathbb{R}^m$. Then, there exists a positive constant $c_{\Omega, h}$ such that the inequality
\begin{equation}
\left\| \sum_{j,k \in \mathbb{Z}} \int_{\theta}^{\theta+1} \int_{\theta}^{\theta+1} |h(r,s) \ast G_{j,k}|^{\alpha} \frac{drds}{rs} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^{1/a} \leq c_{\Omega, h} \ln^{2/a} \left( \theta \right) \left\| \sum_{j,k \in \mathbb{Z}} \left| G_{j,k} \right|^{\alpha} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^{1/a}
\end{equation}
holds for all $p \in \left( \frac{\alpha}{\alpha' + \frac{\alpha}{\alpha'} - 1}, \frac{\alpha}{\alpha'} \right)$.

**Proof.** We employ a similar argument used in [19]. First, let us consider the case $p \in (\alpha', \frac{\alpha'}{\alpha'})$. By duality, there is a non-negative function $\vartheta \in L^{(\alpha')'}(\mathbb{R}^n \times \mathbb{R}^m)$ such that $\left\| \vartheta \right\|_{L^{(\alpha')'}(\mathbb{R}^n \times \mathbb{R}^m)} \leq 1$ and
\begin{equation}
\left\| \sum_{j,k \in \mathbb{Z}} \int_{\theta}^{\theta+1} \int_{\theta}^{\theta+1} |h(r,s) \ast G_{j,k}|^{\alpha} \frac{drds}{rs} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^{1/a} \leq \int_{\mathbb{R}^n \times \mathbb{R}^m} \left( \sum_{j,k \in \mathbb{Z}} \int_{\theta}^{\theta+1} \int_{\theta}^{\theta+1} |h(r,s) \ast G_{j,k}|^{\alpha} \frac{drds}{rs} \right)^{1/a} \vartheta(r,s) drd\sigma.
\end{equation}

By Hölder’s inequality, it is easy to obtain that
\begin{equation}
\left\| \int_{\mathbb{R}^n \times \mathbb{R}^m} \left( \sum_{j,k \in \mathbb{Z}} \int_{\theta}^{\theta+1} \int_{\theta}^{\theta+1} |h(r,s) \ast G_{j,k}|^{\alpha} \frac{drds}{rs} \right)^{1/a} \vartheta(r,s) drd\sigma \leq C \left\| \Omega \right\|_{L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \left\| h \right\|_{\Delta_\gamma(\mathbb{R}^n \times \mathbb{R}^m)}.
\end{equation}

Again, by using Hölder’s inequality and Inequalities (11) and (12), we have
\begin{equation}
\left\| \sum_{j,k \in \mathbb{Z}} \int_{\theta}^{\theta+1} \int_{\theta}^{\theta+1} |h(r,s) \ast G_{j,k}|^{\alpha} \frac{drds}{rs} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^{1/a} \leq C \left\| \Omega \right\|_{L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \left\| h \right\|_{\Delta_\gamma(\mathbb{R}^n \times \mathbb{R}^m)}
\end{equation}

where $\vartheta(\omega, v) = \vartheta(-\omega, -v)$. Therefore, since $\left\| h \right\|_{\Delta_\gamma(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{\Omega, h} \ln^{2/a} \left( \theta \right)$, we use (12) and Hölder’s inequality to obtain that
Finally, we consider the case \( p \in \left( \frac{\alpha'}{\alpha'}, 1 \right] \). Define the linear operator \( T \) on any function \( G = G_{j,k}(x,y) \) by \( T(G) = H_{\rho r, \theta/s} \ast G_{j,k}(x,y) \). Then, we have

\[
\left\| \left\| T(G) \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \ln^2(\theta) \left\| \left( \sum_{j,k \in \mathbb{Z}} |G_{j,k}| \right) \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)}.
\]  

On the other hand, by using (6), we obtain

\[
\left\| \sup_{j,k \in \mathbb{Z}} \sup_{r,s} \left\| H_{\rho r, \theta/s} \ast G_{j,k} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left\| \left( \sum_{j,k \in \mathbb{Z}} |G_{j,k}| \right) \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{\Omega,k} \left\| \sup_{j,k \in \mathbb{Z}} |G_{j,k}| \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}
\]

for all \( \gamma' < p < \infty \), which in turn implies

\[
\left\| \left\| \left\| H_{\rho r, \theta/s} \ast G_{j,k} \right\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^m)} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{\Omega,k} \left\| \left\| \sup_{j,k \in \mathbb{Z}} |G_{j,k}| \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}.
\]  

Therefore, by interpolating between (15) and (16) we get (10) for any \( p \in \left( \frac{\alpha'}{\alpha'}, 1 \right] \). \( \square \)

**Lemma 4.** Assume that \( \theta \geq 2, h \in \Delta_\eta(\mathbb{R}^n \times \mathbb{R}^m) \) for some \( 2 < \gamma < \infty \) and \( \Omega \in L^q \left( \mathbb{S}^{n-1} \times \mathbb{S}^{m-1} \right) \) for some \( 1 < q < 2 \). Let \( \alpha \leq \gamma' \) and \( \{ G_{j,k}(\cdot, \cdot), j,k \in \mathbb{Z} \} \) be arbitrary functions defined on \( \mathbb{R}^n \times \mathbb{R}^m \). Then, there exists a positive constant \( C_{\Omega,k} \) such that

\[
\left\| \left( \sum_{j,k \in \mathbb{Z}} |G_{j,k}| \right)^{\alpha/a} \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{\Omega,k} \ln^2(\theta) \left\| \left( \sum_{j,k \in \mathbb{Z}} |G_{j,k}| \right)^{\alpha/a} \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)}
\]  

for all \( p \in (1, \alpha) \).

**Proof.** By duality, there is a set of functions \( \{ M_{j,k}(\omega, v, r, s) \} \) defined on \( \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \) with

\[
\left\| \left\| M_{j,k} \right\|_{L^\infty(\mathbb{R}^n, [\alpha^1 \mu^{\kappa+1} \times [\alpha^1 \mu^{\kappa+1}] / \alpha \mu^{\kappa}]} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq 1
\]
\[ \sum_{j,k} \int_{B^1} \int_{B^k} |\mu_{r,s} \ast G_{j,k}| \frac{a \ drds}{rs} \right)^{1/a} \right\rangle_{L^p(R^n \times R^m)} \\
= \int_{R^n \times R^m} \sum_{j,k} \int_{B^1} \int_{B^k} \left( \mu_{r,s} \ast G_{j,k}(\omega, v) \right) M_{j,k}(\omega, v, r, s) \frac{drds}{rs} \omega \, dv \\
\leq C(\ln \theta)^{2/a} \left\| \left( M(\omega, v) \right)^{1/a} \right\|_{L^p(R^n \times R^m)} \left\langle \left( \sum_{j,k} |G_{j,k}|^a \right)^{1/a} \right\rangle_{L^p(R^n \times R^m)} 
\]

where 
\[ N(\omega, v) = \sum_{j,k} \int_{B^1} \int_{B^k} |\mu_{r,s} \ast M_{j,k}(\omega, v, r, s)| \frac{a' \ drds}{rs}. \]

Since \( \gamma \geq 2 \geq \gamma' \geq a \), then by Hölder’s inequality we obtain

\[
\left| \mu_{r,s} \ast M_{j,k}(\omega, r) \right|^{a'} \leq C \left\| \Omega \right\|_{L^1(S^{n-1} \times S^{n-1})} \left\| \mu \right\|_{\Delta_1(R^+ \times R^+)} \times \int_{B^1} \int_{B^k} \int_{S^{n-1} \times S^{n-1}} |M_{j,k}(\omega - \kappa x, v - \eta y, r, s)|^{a'} |\Omega(x, y)| |d\mu(x)| |d\mu(y)| \frac{dxdy}{\kappa \eta}. (19)
\]

Since \( p' > a' \), there exists a function \( \rho \in L^{(p'/a')'}(R^n \times R^m) \) such that

\[
\left\| N(\omega, v) \right\|_{L^{(p'/a')'}(R^n \times R^m)} = \sum_{j,k} \int_{R^n \times R^m} \int_{B^1} \int_{B^k} \left| \mu_{r,s} \ast M_{j,k}(\omega, v, r, s) \right| \frac{a' \ drds}{rs} \rho(\omega, v) \omega \, dv.
\]

Therefore, a simple change in variable together with Lemmas 1 and (19) give

\[
\left\| N(\omega, v) \right\|_{L^{(p'/a')'}(R^n \times R^m)} \leq C \left\| \Omega \right\|_{L^1(S^{n-1} \times S^{n-1})} \left\| \mu \right\|_{\Delta_1(R^+ \times R^+)} \left\| \rho \right\|_{L^{(p'/a')'}(R^n \times R^m)} \times \left\| \left( \sum_{j,k} \int_{B^1} \int_{B^k} |M_{j,k}(\omega, v, r, s)| \frac{a' \ drds}{rs} \right) \right\|_{L^{(p'/a')'}(R^n \times R^m)} \\
\leq C \left\| \Omega \right\|_{L^1(S^{n-1} \times S^{n-1})} \left\| \mu \right\|_{\Delta_1(R^+ \times R^+)} \left\| \rho \right\|_{L^{(p'/a')'}(R^n \times R^m)}. (20)
\]

Therefore, by (18) and (20), Inequality (17) is proved. Consequently, the proof of Lemma 4 is complete. □

**Lemma 5.** Assume that \( \theta \), \( \Omega \), and \( \{G_{j,k}(\cdot, \cdot) : j, k \in \mathbb{Z} \} \) are given as in Lemma 3. Suppose that \( h \in \Delta_1(R^+ \times R^+) \) with \( 1 < \gamma < \infty \) and \( a \geq \gamma' \). Then, there exists a constant \( C_{\Omega h} > 0 \) such that

\[
\left\| \sum_{j,k} \int_{B^1} \int_{B^k} |\mu_{r,s} \ast G_{j,k}| \frac{a \ drds}{rs} \right\|_{L^p(R^n \times R^m)} \leq C_{\Omega h} \gamma^{2/a}(\theta) \left\| \sum_{j,k} |G_{j,k}|^a \right\|_{L^p(R^n \times R^m)} \leq C_{\Omega h} \ln^{2/a}(\theta) \left\| \sum_{j,k} |G_{j,k}|^a \right\|_{L^p(R^n \times R^m)} 
\]

for all \( \gamma' < p < \infty \).
3. Proof of the Main Results

Proof of Theorem 1. We employ similar arguments as those in [19,20]. Assume that $\alpha > 1$, $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ with $1 < \gamma \leq 2$ and $\Omega \in L^q(S^{n-1} \times S^{m-1})$ with $q \in (1,2]$. By Minkowski’s inequality, we obtain
$$\mathcal{M}^{(a)}_{\Omega_{\delta}}(f)(x,y) = \left( \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left| \sum_{j,k=0}^{\infty} \frac{1}{r^{1+\frac{1}{2}}} \int_{2^{-j-1}r < |\omega| \leq 2^{-j}r} \int_{2^{-k-1}r < |v| \leq 2^{-k}r} K_{\Omega_{\delta}}(\omega,v) \right|^a \frac{drds}{rs} \right)^{1/a}$$

\[\leq \sum_{j,k=0}^{\infty} \left( \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left| \frac{1}{r^{1+\frac{1}{2}}} \int_{2^{-j-1}r < |\omega| \leq 2^{-j}r} \int_{2^{-k-1}r < |v| \leq 2^{-k}r} K_{\Omega_{\delta}}(\omega,v) f(x-\omega,y-v) \frac{drds}{rs} \right|^a \frac{drds}{rs} \right)^{1/a} \]

\[\leq \frac{2^{\gamma_1+\gamma_2}}{(2^{\gamma_1}-1)(2^{\gamma_2}-1)} \left( \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left| \mu_{r,s} \ast f(x,y) \right|^a \frac{drds}{rs} \right)^{1/a} . \] (25)

Take $\theta = 2^{1/\theta'}$, then $\ln(\theta) \leq \frac{C}{(\gamma_1-1)(\gamma_1-1)}$. Choose a set of functions $\{\varphi_k\}_{-\infty}^{\infty}$ defined on $(0,\infty)$ with the following properties:

$$\varphi_k \in C^\infty, \quad 0 \leq \varphi_k \leq 1, \quad \sum_{k \in \mathbb{Z}} \varphi_k(r) = 1,$$

$$\text{supp} \ (\varphi_k) \subseteq I_k \equiv [\theta^{-1-k}, \theta^{1-k}] \text{ and } \left| \frac{d^\theta \varphi_k(r)}{dr^\theta} \right| \leq \frac{C_{\beta}}{r^\beta},$$

where $C_{\beta}$ is independent of $\theta$. Define the operators $(\Omega_k(\zeta)) = \varphi_k(|\zeta|)$ and $(\Omega_j(\xi)) = \varphi_j(|\xi|)$ for $(\zeta, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$. Therefore, we obtain that for any $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left( \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left| \mu_{r,s} \ast f(x,y) \right|^a \frac{drds}{rs} \right)^{1/a} \leq C \sum_{l,j \in \mathbb{Z}} A_{l,j}(f)(x,y),$$ (26)

where

$$A_{l,j}(f)(x,y) = \left( \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left| B_{l,j}(f)(x,y,r,s) \right|^a \frac{drds}{rs} \right)^{1/a}$$

and

$$B_{l,j}(f)(x,y,r,s) = \sum_{j,k \in \mathbb{Z}} \mu_{r,s} \ast (\Omega_k(\zeta) \otimes \Omega_j(\xi)) \ast f(x,y) \chi_{\{j \leq k+1\} \times \{j \leq l+1\}}(r,s).$$

Therefore, to prove Theorem 1, it is sufficient to prove that there exists a positive constant $\epsilon$ such that

$$\|A_{l,j}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{p} C_{\Omega_{\delta}} 2^{-\gamma'|l|} (\ln \theta)^{2/a} \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}$$ (27)

for all $p \in \left(\frac{\alpha' \gamma'}{\alpha' \gamma' - \gamma}, \frac{\alpha' \gamma'}{\alpha' \gamma' - \gamma} \right)$ with $\gamma' \geq \alpha$, and also for all $\gamma' < p < \infty$ with $\gamma' \leq \alpha$.

Let us first estimate the norm of $A_{l,j}(f)$ for the case $p = \alpha = 2$. Indeed, by Plancherel’s theorem, Fubini’s theorem, and Lemma 2, we obtain
The authors would like to express their gratitude to the referees for their valuable comments and suggestions in improving writing this paper. In addition, they are grateful to the editor for handling the full submission of the manuscript.

All authors have read and agreed to the published version of the manuscript.

Formal analysis and writing-original draft preparation: M.A. and H.A.-Q.

Author Contributions: Theorem 1.

Conflicts of Interest: This research received no external funding.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors would like to express their gratitude to the referees for their valuable comments and suggestions in improving writing this paper. In addition, they are grateful to the editor for handling the full submission of the manuscript.

Conflicts of Interest: The authors declare that they have no conflict of interest.

\[ \| A_{\beta,i}(f) \|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \]
\[ \leq \sum_{j,k \in \mathbb{Z}} \int_{E_{j,k}} \left( \int_{\theta^j}^{\theta^{j+1}} \left( \int_{\theta^k}^{\theta^{k+1}} |\mu_{r,s}(\xi,\eta)|^2 \frac{dr ds}{r s} \right)^{\frac{1}{2}} f(\xi,\eta) \right) d\xi d\eta \]
\[ \leq C_p \ln^2(\theta) C_{\Omega,h} \sum_{j,k \in \mathbb{Z}} \int_{E_{j,k}} \left| f(\xi,\eta) \right|^2 d\xi d\eta \]
\[ \leq C_p \ln^2(\theta) C_{\Omega,h} \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \]
\[ \leq C_p \ln^2(\theta) \left( \int_{\mathbb{R}^n} \left| f(\xi,\eta) \right|^2 d\xi d\eta \right)^{\frac{1}{2}} \]
\[ \leq C_p \ln^2(\theta) C_{\Omega,h} \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \]

where \( E_{j,k} = \{ (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m : (|\xi|, |\eta|) \in I_k \times I_j \} \) and \( \epsilon \in (0, 1) \).

However, we estimate the \( L^p \)-norm of \( A_{\beta,i}(f) \) in the following. By Lemmas 3 and 5, together with the Littlewood–Paley theory and invoking Lemma 2.3 in [9], we obtain

\[ \| A_{\beta,i}(f) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \]
\[ \leq C \left( \sum_{j,k \in \mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \left( \int_{\theta^k}^{\theta^{k+1}} \left| \mu_{r,s} * ((\hat{U}_{k,i} \otimes \hat{U}_{j,i+1}) * f) \right|^2 \frac{dr ds}{r s} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \]
\[ \leq C_{\Omega,h} \ln^2(\theta) \left( \sum_{j,k \in \mathbb{Z}} \left| ((\hat{U}_{k,i} \otimes \hat{U}_{j,i+1}) * f) \right|^2 \right)^{\frac{1}{p}} \]
\[ \leq C_p \ln^{2/\alpha}(\theta) \| f \|_{F^\infty,\alpha(\mathbb{R}^n \times \mathbb{R}^m)} \]

for all \( p \in \left( \frac{\alpha' \gamma}{\alpha' \gamma + 1} \right), \) with \( \alpha \leq \gamma \), and also for all \( \gamma' < p < \infty \) with \( \alpha \geq \gamma' \). Therefore, by interpolating (28) with (29), we immediately obtain (27). This ends the proof of Theorem 1.

Proof of Theorem 2. To prove this theorem, we follow the exact procedure that was used in the proof of Theorem 1, employing Lemma 4 instead of Lemma 3.

4. Conclusions

In this article, we established appropriate \( L^p \) bounds for the generalized parameetric Marcinkiewicz integral operator \( M_{\Omega,h}^{(\alpha)} \) under the assumption that \( \Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \) for some \( q > 1 \). Then, we used these bounds, along with Yano’s extrapolation argument, to prove the boundedness of the operator \( M_{\Omega,h}^{(\alpha)} \) under very weak conditions on the kernel function \( \Omega \). Such conditions on \( \Omega \) are considered to be the best possible among their respective classes. The results in this article improve and extend several known results in the field of Marcinkiewicz and generalized Marcinkiewicz operators. In fact, our results improve and extend the results in [1–4,6,8,9,17].

Author Contributions: Formal analysis and writing-original draft preparation: M.A. and H.A.-Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors would like to express their gratitude to the referees for their valuable comments and suggestions in improving writing this paper. In addition, they are grateful to the editor for handling the full submission of the manuscript.

Conflicts of Interest: The authors declare that they have no conflict of interest.
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