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A Novel Problem for Solving Permuted Cordial Labeling of Graphs

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Abstract: In this paper, we used the permutation group together with the concept of cordiality in graph theory to introduce a new method of labeling. This construed permuted cordial labeling can be applied to all paths, cycles, fans and wheel graphs. Moreover, some other properties are investigated and show that the union of any two paths and the union of any two cycles are permuted cordial graphs. In addition, we investigated the permuted cordiality for the union of any path with cycle.

Keywords: graph; cordial labeling; permuted labeling

1. Introduction

If the vertices of a graph are assigned values subject to certain conditions, then it is known as graph labeling. Most of the graph labeling problems have the following three characteristics in common: a set of numbers for the assignment of vertex labels, a rule that assigns a label to each edge and some condition(s) that these labels must satisfy. Cordial labeling were introduced by Cahit [1] who called a graph $G(V,E)$ cordial if there is a vertex labeling $h : V \rightarrow \{0,1\}$ such that the induced labeling $h^* : E \rightarrow \{0,1\}$, defined by $f^*(uv) = |h(u) - h(v)|$ for all edges $u, v \in V$ and with the following inequalities holding: $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| \leq 1$, where $v(i)$ (respectively, $e(i)$) is the number of vertices (respectively, edges) labeled with i. For a detailed survey on graph labeling one can refer to Gallian [2]. For more details about the cordial labeling, the reader can refer to [3–11]. Let $v(0)$ and $v(1)$, be the number of vertices of G with labels 0 and 1, respectively, under h. Let $e(0)$ and $e(1)$ be the number of edges having labels 0 and 1, respectively, under h^*. A binary vertex labeling of a graph G is called a cordial labeling if $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| \leq 1$. In the paper [12] they investigate the existence of the local super ant magic total chromatic number for some particular classes of graphs such as a trees, paths, and cycles. In the paper [13], they gave a characterization of the locating chromatic number of powers of paths. In addition, they find sharp upper and lower bounds for the locating chromatic number of powers of cycles. The main purpose of the paper [14] was to determine the crossing numbers of the join products of six symmetric graphs on six vertices with paths and cycles on n vertices. According to the paper [15], the Cartesian product of the directed path P_m and the directed cycle C_n for any positive integers m and n is the exact value of the signed.

In the paper [16], they obtain the g-extra connectivity of the strong product of two paths, the strong product of a path and a cycle, and the strong product of two cycles. In [17], they obtained an $S-(a,0)-EAMT$ and $S-(a',2)-EAMT$ labeling of symmetric classes of networks termed as hexagonal lattice $HTT_{m,n}$ and prismatic lattice $T_{m,n}$. In this

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papers [18,19] designs a feature selection algorithm based on fuzzy relative knowledge distances regards relative fuzzy knowledge distances as evaluation functions and conducts feature selections for information.

As it is known, every permutation can be expressed as a composite of disjoint cycles. Let k be a permutation of degree n defined on a set S having n distinct elements. Let it be possible to arrange m elements of the set S in a row in such a way that the k range of each element in the row is the element which follows it, the k image of the last element is the first element and the remaining $(n-m)$ elements of the set S are left unchanged by k. Then k is called a cyclic permutation or an m-cycle of length m. A cyclic of length two is called a transposition. A permutation is said to be even or odd according as it is expressible as the product of an even or an odd number of transpositions. Every permutation can be expressed as a composite of disjoint cycles. Theorem of the $n!$ Permutations of n cycles, $\frac{n!}{2}$ are even permutations and $\frac{n!}{2}$ are odd permutations. As a well know theorem states: The set A_n of all even permutations of degree n forms a finite group of order $\frac{n!}{2}$ with respect to permutation multiplications as the composition [20]. Therefore, for any set of distinct three elements, we have only three even permutations. Now we labelled the vertices of a graph $G=(V,E)$ by $h: V \to \{i,g,f\}$ and induced edge labeling $h^*: E(G) \to \{i,g,f\}$, defined as follows:

If $e = uv \in E$ then the following table determines the labeling of e considering that our permutation is $\{i,f,g\}$. We called this permuted cordial labeling.

This paper studies each path P_n, $n \geq 2$ admits permuted cordial labeling for all n. Each cycle C_n, $n \geq 3$ admits a permuted cordial labeling. Each Fan F_n, $n \geq 2$ admits permuted cordial labeling for all n. The Wheel graph $W_n, n \geq 3$ admits permuted cordial labeling except $n \equiv 2 \ mod \ 3$ and n even. The union of $P_n \cup P_m$, $n,m \geq 2$ admits a permuted cordial labeling for all n,m. The union of $C_n \cup C_m$, $n,m \geq 3$ admits a permuted cordial labeling for all n,m. The union of $P_n \cup C_m$, $n \geq 2, m \geq 3$ admits a permuted cordial labeling for all n,m.

The rest of this paper is structured as follows: Permuted cordial labeling of cycles and paths are presented in Section 3. Permuted cordial labeling of cycles and paths are presented in Section 4. Finally, in Section 5, conclusions are drawn.

2. Terminology and Notation

By A_{3r}, we mean the labeling $ifg \dots ifg$ (r-times) of the path P_{3r} or the cycle C_{3r}. D_{3r} and B_{3r} represent the labeling of $igf \dots igf$(r-times) and $gif \dots gif$(r-times) of either P_{3r} or C_{3r}. Sometimes, we modify A_{3r}, D_{3r}, and B_{3r} by adding symbols at one end or the other or both; for example, $A_{3r}gi$ means the labeling $igf \dots igf$(r-times)gi for P_{3r+2} or the cycle C_{3r+2}. Similarly, $D_{3r}f$ is the cycle C_{3r+1} labeling $igf \dots igf$(r-time)f (or the path $P(3r+1)$). $v(i)$, $v(g)$, and $v(f)$ represent the number of vertices labeled i, g, and f, respectively. Similarly, we denoted $e(i)$, $e(g)$, and $e(f)$ to be the number of edges labeled i, g, and f, respectively, for the graph G.

A vertex labeling of graph G of $h: V \to \{i,g,f\}$ with induced edge labeling $h^*: E(G) \to \{i,g,f\}$ defined by

.	$v(i)$	$v(f)$	$v(g)$
$u(i)$	i	f	g
$u(f)$	f	g	i
$u(g)$	g	i	f

is called permuted cordial labeling if $|v(x) - v(y)| \leq 1$ and $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x,y \in \{i,f,g\}$ where $v(x)$ (respectively, $e(x)$) is the number of vertices (respectively, edges) labeled with $x \in \{i,f,g\}$. A graph G is permuted cordial if it admits a permuted labeling.

3. Permuted Cordial Labeling of Some Graphs

In this section we shall prove that the permuted cordial labeling of paths, cycles, fan and wheel graphs.

Theorem 1. *Each path P_n, $n \geq 2$ admits permuted cordial labeling for all n.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of P_n and $E = \{v_r v_{r+1}; 1 \leq r \leq n-1\}$ its edges set. Define vertex labeling $h : V \to \{i, g, f\}$ as follows: suppose $n \equiv 0, 1, 2 (mod\, 3)$ then,

$$h(v_r) = \begin{cases} i & ;0 \ \ mod \ \ 3 \\ f & ;1 \ \ mod \ \ 3 \\ g & ;2 \ \ mod \ \ 3 \end{cases}$$

□

The induced edge labeling h^* is given in Section 2 where our permutation is $\{i, g, f\}$ Let us study the following three cases:

Case (1): $n \equiv 0(mod\, 3)$. The total number of vertices labeled i is denoted by $v(i)$ is the same as the total number of vertices labeled f and denoted by $v(f)$ and also equivalent to the total number of vertices labeled g denoted by $v(g)$ and this number is r, i.e., $v(i) = v(f) = v(g) = r$. Obviously, $|v(x) - v(y)| = 0,\ x \neq y$ and $x, y \in \{i, f, g\}$. Similarly, in this same one can see that the number of edges labeled i denoted by $e(i)$ the same as the number of edges labeled f denoted by $e(f)$ and this number is r, while the number of edges labeled g denoted by $e(g)$ and this number is $r - 1$. Consequently, $|e(x) - e(y)| \leq 1,\ x \neq y$ and $x, y \in \{i, f, g\}$.

Case (2): $n \equiv 1(mod\, 3)$. The total number of vertices labeled g is denoted by $v(g)$ is the same as the total number of vertices labeled f denoted by $v(f)$ and this number is r, while the number of vertices of labeled i denoted by $v(i)$ and this number is $r + 1$. Consequently, $|v(x) - v(y)| \leq 1,\ x \neq y$ and $x, y \in \{i, f, g\}$. Similarly, in this, one can see that the number of edges labeled i denoted by $e(i)$ is the same as the number of edges labeled f denoted by $e(f)$, the same as the number of edges labeled g denoted by $e(g)$ and this number is r. Therefore, $|e(x) - e(y)| = 0,\ x \neq y$ and $x, y \in \{i, f, g\}$.

Case (3): $n \equiv 2(mod\, 3)$. The total number of vertices labeled i is denoted by $v(i)$ is the same as the total number of vertices labeled f, is denoted by $v(f)$ and this number is $r + 1$, while the total number of vertices labeled g is denoted by $v(g)$ and this number is r. Therefore, $|v(x) - v(y)| \leq 1,\ x \neq y$ and $x, y \in \{i, f, g\}$. Similarly, in this same one can see that the total number of edges labeled i is is denoted by $e(i)$ is the same as the total number of edges labeled g denoted by $e(g)$ and this number is r, while the number of edges labeled f denoted by $e(f)$ and this number is $r + 1$. Therefore, $|e(x) - e(y)| \leq 1,\ x \neq y$ and $x, y \in \{i, f, g\}$.

Thus, we have seen in each case $|v(x) - v(y)| \leq 1$ and $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x, y \in \{i, f, g\}$. Hence the path P_n, $n \geq 2$ admits permuted cordial labeling.

Theorem 2. *Each cycle C_n, $n \geq 3$ admits a permuted cordial labeling.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n and $E = v_r v_{r+1}; 1 \leq r \leq n-1\} \cup v_1 v_n$ be the edge set of C_n. Define vertex labeling $h : V(G) \to \{i, g, f\}$ as given in Table 1.

Table 1. Labeling of Vertices of $\boldsymbol{C_n}$.

when $n \equiv 0, 1(mod\ \ 3)$	and when $n \equiv 2(mod\ \ 3)$
$h(v_r) = \begin{cases} i & ;0 \ \ mod \ \ 3 \\ f & ;1 \ \ mod \ \ 3 \\ g & ;2 \ \ mod \ \ 3 \end{cases}$	$h(v_r) = \begin{cases} g & ;r = 0 \ \ mod \ \ 3, r \neq n-1 \\ i & ;r = 1 \ \ mod \ \ 3, r \neq n \\ f & ;r = 2 \ \ mod \ \ 3 \\ f & ;r = n-1 \\ g & ;r = n \end{cases}$

Now, we defined the labeling of the edge set of C_n using the function as follows in Table 2.

Table 2. Labeling of Edge set.

	.	$v(i)$	$v(f)$	$v(g)$
$h^*(e)$	$v(i)$	i	f	g
	$v(f)$	f	g	i
	$v(g)$	g	i	f

It is follows that C_n, $n \geq 3$ admits permuted cordial labeling as required in Table 3.

Table 3. Vertex and edge conditions of a cycle graph.

n	$v(i)$	$v(f)$	$v(g)$	$\|v(x) - v(y)\| x \neq y$ and $x, y \in \{i, f, g\}$
$n \equiv 0 \ mod \ 3$ i.e., $n = 3r$,	r	r	r	$0, 0, 0$
$n \equiv 1 \ mod \ 3$ i.e., $n = 3r + 1$,	$r + 1$	r	r	$1, 1, 0$
$n \equiv 2 \ mod \ 3$ i.e., $n = 3r + 2$,	r	$r + 1$	$r + 1$	$1, 1, 0$
n	$e(i)$	$e(f)$	$e(g)$	$\|e(x) - e(y)\| x \neq y$ and $x, y \in \{i, f, g\}$
$n \equiv 0 \ mod \ 3$ i.e., $n = 3r$,	r	r	r	$0, 0, 0$
$n \equiv 1 \ mod \ 3$ i.e., $n = 3r + 1$,	$r + 1$	r	r	$1, 1, 0$
$n \equiv 2 \ mod \ 3$ i.e., $n = 3r + 2$,	r	$r + 1$	$r + 1$	$1, 1, 0$

Hence the cycle graph $C_n, n \geq 3$ admits permuted cordial labeling. □

Theorem 3. *Each Fan F_n, $n \geq 2$ admits permuted cordial labeling for all n.*

Proof. Let $V = \{u, v_1, v_2, \ldots, v_n\}$ be the vertex set of F_n and $E = \{v_r v_{r+1}; 1 \leq r \leq n - 1\} \cup \{uv_r; 1 \leq r \leq n\}$ its edges set. Define vertex labeling $h : V \to \{i, g, f\}$ as follows in Table 4.

Table 4. Labeling of Vertices of $\boldsymbol{F_n}$.

when $n \equiv 0(mod \ 3)$	when $n \equiv 1(mod \ 3)$	and when $n \equiv 2(mod \ 3)$, n odd
$h(u) = i$ $h(v_r) = \begin{cases} i & ; 0 \ mod \ 3 \\ f & ; 1 \ mod \ 3 \\ g & ; 2 \ mod \ 3 \end{cases}$	$h(u) = g$ $h(v_r) = \begin{cases} i & ; 0 \ mod \ 3 \\ f & ; 1 \ mod \ 3 \\ g & ; 2 \ mod \ 3 \end{cases}$	$h(u) = g$ $h(v_r) = \begin{cases} i & ; 0 \ mod \ 3 \\ f & ; 1 \ mod \ 3 \\ g & ; 2 \ mod \ 3 \end{cases}$

The induced edge labeling h^* is given in Section 2 where our permutation is $\{i, g, f\}$ Let us study the following three cases:

Case (1): $n \equiv 0(mod\, 3)$. The total number of vertices labeled i is denoted by $v(i)$ is the same as the total number of vertices labeled g and denoted by $v(g)$ and this number is r, also the total number of vertices labeled f denoted by $v(f)$ and this number is $r + 1$, i.e., $v(i) = v(g) = r$, $v(f) = r + 1$. Obviously, $|v(x) - v(y)| \leq 1$, $x \neq y$ and $x, y \in \{i, f, g\}$.

Similarly, in this same one can see that the number of edges labeled i denoted by $e(i)$ the same as the number of edges labeled f denoted by $e(f)$ and also equivalent to the number of edges labeled g denoted by $e(g)$ and this number is $2r$. Consequently, $|e(x) - e(y)| = 0$, $x \neq y$ and $x, y \in \{i, f, g\}$.

Case (2): $n \equiv 1(mod\,3)$. The total number of vertices labeled g is denoted by $v(g)$ is the same as the total number of vertices labeled f denoted by $v(i)$ and this number is $r+1$, while the number of vertices of labeled f denoted by $v(f)$ and this number is r. Consequently, $|v(x) - v(y)| \leq 1$, $x \neq y$ and $x, y \in \{i, f, g\}$. Similarly, in this same one can see that the number of edges labeled i denoted by $e(i)$ is the same as the number of edges labeled f denoted by $e(f)$ and this number is $2r$, while the number of edges labeled g denoted by $e(g)$ and this number is $2r+1$. Therefore, $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x, y \in \{i, f, g\}$.

Case (3): $n \equiv 2(mod\,3)$. The total number of vertices labeled i is denoted by $v(i)$ is the same as the total number of vertices labeled f is denoted by $v(f)$ and also equivalent to the number of vertices labeled g is denoted by $v(g)$ and this number is $r+1$. Therefore, $|v(x) - v(y)| = 0$, $x \neq y$ and $x, y \in \{i, f, g\}$. Similarly, in this same one can see that the number of edges labeled i denoted by $e(i)$ the same as the number of edges labeled f denoted by $e(f)$ and also equivalent to the number of edges labeled g denoted by $e(g)$ and this number is $2r+1$. Consequently, $|e(x) - e(y)| = 0$, $x \neq y$ and $x, y \in \{i, f, g\}$.

Thus, we have seen in each case $|v(x) - v(y)| \leq 1$ and $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x, y \in \{i, f, g\}$. Hence the fan F_n, $n \geq 2$ admits permuted cordial labeling. □

Theorem 4. *Each wheel W_n, $n \geq 3$ admits a permuted cordial labeling except $n \equiv 2$ mod 3 and n even.*

Proof. Let $V = \{u, v_1, v_2, \ldots, v_n\}$ be the vertex set of W_n and $E = \{v_r v_{r+1}; 1 \leq r \leq n-1\} \cup v_1 v_n \cup \{u v_r; 1 \leq r \leq n\}$ be the edge set of W_n. Define vertex labeling $h : V(G) \to \{i, g, f\}$ as given in Table 5.

Table 5. Labeling of Vertices of $\boldsymbol{W_n}$.

when $n \equiv 0(mod\ 3)$	when $n \equiv 1(mod\ 3)$	and when $n \equiv 2(mod\ 3)$, n odd
$h(u) = i$ $h(v_r) = \begin{cases} i & ;0\ mod\ 3 \\ f & ;1\ mod\ 3 \\ g & ;2\ mod\ 3 \end{cases}$	$h(u) = g$ $h(v_r) = \begin{cases} i & ;0\ mod\ 3 \\ f & ;1\ mod\ 3,\ r \neq n \\ g & ;2\ mod\ 3 \\ i & ;\quad r = n \end{cases}$	$h(u) = i$ $h(v_r) = \begin{cases} g & ;r = 0\ mod\ 3, r \neq n-1 \\ i & ;r = 1\ mod\ 3, r \neq n \\ f & ;r = 2\ mod\ 3 \\ f & ;r = n-1 \\ g & ;r = n \end{cases}$

Now, we defined the labeling of the edge set of W_n using the function as follows in Table 6.

Table 6. Labeling of Edge set.

	.	$v(i)$	$v(f)$	$v(g)$
$h^*(e)$	$v(i)$	i	f	g
	$v(f)$	f	g	i
	$v(g)$	g	i	f

It is follows that W_n, $n \geq 3$ admits permuted cordial labeling as required in Table 7.

Table 7. Vertex and edge conditions of a Wheel graph.

n	$v(i)$	$v(f)$	$v(g)$	$\lvert v(x) - v(y) \rvert x \neq y$ and $x, y \in \{i, f, g\}$
$n \equiv 0 \mod 3$ i.e., $n = 3r$,	$r+1$	r	r	$1, 1, 0$
$n \equiv 1 \mod 3$ i.e., $n = 3r+1$,	$r+1$	r	$r+1$	$1, 0, 1$
$n \equiv 2 \mod 3$ i.e., $n = 3r+2$, n odd	$r+1$	$r+1$	$r+1$	$0, 0, 0$
n	$e(i)$	$e(f)$	$e(g)$	$\lvert e(x) - e(y) \rvert x \neq y$ and $x, y \in \{i, f, g\}$
$n \equiv 0 \mod 3$ i.e., $n = 3r$,	$2r$	$2r$	$2r$	$0, 0, 0$
$n \equiv 1 \mod 3$ i.e., $n = 3r+1$,	$2r+1$	$2r$	$2r+1$	$1, 0, 1$
$n \equiv 2 \mod 3$ i.e., $n = 3r+2$, n odd	$2r+2$	$2r+1$	$2r+1$	$1, 1, 0$

Hence the Wheel graph $W_n, n \geq 3$ admits permuted cordial labeling except $n \equiv 2 \mod 3$ and n even. □

4. The Permuted Cordial Labeling for Union of Paths and Cycles

In this section, we will study the permuted cordial labeling of a union of two paths, and a similar study will be performed of two union cycles. We end this section by studying the permuted cordial labeling of the union paths with cycles.

Theorem 5. *The union of $P_n \cup P_m$, $n, m \geq 2$ admits a permuted cordial labeling for all n, m.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_m\}$ be the vertex set of $P_n \cup P_m$ and $E = E_1 \cup E_2$, where $E_1 = \{v_r v_{r+1}; 1 \leq r \leq n-1\}$, $E_2 = \{v'_s v'_{s+1}; 1 \leq r \leq m-1\}$, be the edge set of $P_n \cup P_m$. Define vertex labeling $h : V \to \{i, g, f\}$ of the $P_n \cup P_m$ as given in Table 8.

Table 8. Labeling of P_n and P_m.

$n = 3r + j$ $r \geq 0$, $j = 0, 1, 2$	Labeling of P_n	$v(i)$	$v(f)$	$v(g)$	$e(i)$	$e(f)$	$e(g)$
$j = 0$	$A_0 = A_{3r}$ $A'_0 = D_{3r}$	r r	r r	r r	r r	r $r-1$	$r-1$ r
$j = 1$	$A_1 = D_{3r}(f)$	r	$r+1$	r	r	r	r
$j = 2$	$A_2 = A_{3r}(gi)$	$r+1$	r	$r+1$	r	$r+1$	r
$m = 3s + j$ $s \geq 0$, $j = 0, 1, 2$	Labeling of P_m	$v'(i)$	$v'(f)$	$v'(g)$	$e'(i)$	$e'(f)$	$e'(g)$
$j = 0$	$B_0 = A_{3s}$ $B'_0 = D_{3s}$	s s	s s	s s	s s	s $s-1$	$s-1$ s
$j = 1$	$B_1 = A_{3s}(i)$ $B'_1 = A_{3s}(g)$	$s+1$ s	s s	s $s+1$	s s	s s	s s
$j = 2$	$B_2 = A_{3s}(gi)$ $B'_2 = D_{3s}(fi)$	$s+1$ $s+1$	s $s+1$	$s+1$ s	s s	$s+1$ s	s $s+1$

Now, we defined the labeling of the edge set of P_n using the function h^* can be defined as in Table 2.

Hence the deduced labeling of the union is shown in Table 9.

Table 9. Vertex and edge conditions of $\boldsymbol{P_n \cup P_m}$.

$n = 3r + j$ $r \geq 0$, , $j = 0, 1, 2$	$m = 3s + j$, $s \geq 0$, $j = 0, 1, 2$	P_n	P_m	$\lvert v(x) - v(y) \rvert$, $x \neq y$ and $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y) \rvert$, $x \neq y$ and $x, y \in \{i, f, g\}$
0	0	A_0	B'_0	$0, 0, 0$	$1, 1, 0$
0	1	A'_0	B_1	$1, 1, 0$	$1, 0, 1$
0	2	A'_0	B_2	$1, 0, 1$	$0, 0, 0$
1	1	A_1	B'_1	$1, 1, 0$	$0, 0, 0$
1	2	A_1	B_2	$0, 0, 0$	$1, 0, 1$
2	2	A_2	B'_2	$1, 1, 0$	$1, 1, 0$

It is obvious that the difference $|v(x) - v(y)|$ and $|e(x) - e(y)|$,where $x \neq y$ and $x, y \in \{i, f, g\}$ are always do not exceed one. Therefore $P_n \cup P_m$, $n, m \geq 2$ admits a permuted cordial labeling.

Now, we denote our attention to study the union of two cycles. □

Theorem 6. *The union of $C_n \cup C_m$, $n, m \geq 3$ admits a permuted cordial labeling for all n, m.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_m\}$ be the vertex set of $C_n \cup C_m$ and $E = E_1 \cup E_2$, where $E_1 = \{v_r v_{r+1}; 1 \leq r \leq n-1\} \cup v_1 v_n$, $E_2 = \{v'_s v'_{s+1}; 1 \leq r \leq m-1\} \cup v'_1 v'_m$ be the edge set of $C_n \cup C_m$. Define $h : V \to \{i, g, f\}$ to be the chosen labeling for the vertex set of each C_n and C_m as seen in Table 10.

Table 10. Labeling of $\boldsymbol{C_n}$ and $\boldsymbol{C_m}$.

$n = 3r + j$, $j = 0, 1, 2$	Labeling of C_n	$v(i)$	$v(f)$	$v(g)$	$e(i)$	$e(f)$	$e(g)$
$j = 0$	$A_0 = A_{3r}$ $A'_0 = D_{3r}$	r r	r r	r r	r r	r r	r r
$j = 1$	$A_1 = D_{3r}(f)$ $A'_1 = A_{3r}(i)$	r $r+1$	$r+1$ r	r r	r $r+1$	$r+1$ r	r r
$j = 2$	$A_2 = A_{3r}(gi)$ $A'_2 = D_{3s}(fi)$	$r+1$ $r+1$	r $r+1$	$r+1$ r	$r+1$ $r+1$	$r+1$ r	r $r+1$
$m = 3s + j$, $j = 0, 1, 2$	Labeling of C_n	$v'(i)$	$v'(f)$	$v'(g)$	$e'(i)$	$e'(f)$	$e'(g)$
$j = 0$	$B_0 = A_{3s}$ $B'_0 = D_{3s}$	s s	s s	s s	s s	s s	s s
$j = 1$	$B_1 = A_{3s}(i)$ $B'_1 = A_{3s}(g)$	$s+1$ s	s s	s $s+1$	$s+1$ s	s $s+1$	s s
$j = 2$	$B_2 = A_{3s}(gi)$ $B'_2 = D_{3s}(fi)$ $B''_2 = B_{3s}(fg)$	$s+1$ $s+1$ s	s $s+1$ $s+1$	$s+1$ s $s+1$	$s+1$ $s+1$ s	$s+1$ s $s+1$	s $s+1$ $s+1$

So, one can define the edge labeling h^* for $C_n \cup C_m$ as follows in Table 2.

In view of the above labeling pattern, we have the vertex labeling of $C_n \cup C_m$ and also the edges labeling of it as indicated in the following Table 11.

Table 11. Vertex and edge conditions of $C_n \cup C_m$.

$n = 3r + j$ $r \geq 1,$ $j = 0, 1, 2$,	$m = 3s + j$ $s \geq 1,$ $j = 0, 1, 2$	C_n	C_m	$\lvert v(x) - v(y) \rvert$ $x \neq y$ and $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y) \rvert$ $x \neq y$ and $x, y \in \{i, f, g\}$
0	0	A_0	B_0	0, 0, 0	0, 0, 0
0	1	A_0	B_1	1, 1, 0	1, 1, 0
0	2	A_0	B_2	1, 0, 1	0, 1, 1
1	1	A_1	B_1	0, 1, 1	0, 1, 1
1	2	A'_1	B''_2	0, 0, 0	0, 0, 0
2	2	A_2	B'_2	1, 1, 0	1, 1, 0

It is obvious that the difference $|v(x) - v(y)|$ and $|e(x) - e(y)|$,where $x \neq y$ and $x, y \in \{i, f, g\}$ are always donated to exceed one. Therefore $C_n \cup C_m$, $n, m \geq 3$ admits a permuted cordial labeling.

Finally, we study the permuted cordial labeling for $P_n \cup C_m$. □

Theorem 7. *The union of $P_n \cup C_m$, $n \geq 2, m \geq 3$ admits a permuted cordial by for all n, m.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_m\}$ be the vertex set of union of $P_n \cup C_m$ and $E = E_1 \cup E_2$, where $E_1 = \{v_r v_{r+1}; 1 \leq r \leq n-1\}$, $E_2 = \{v'_s v'_{s+1}; 1 \leq r \leq m-1\} \cup v'_1 v'_m$ be the edge set of the union of $P_n \cup C_m$. Define vertex labeling $h : V \to \{i, g, f\}$ to be the chosen labeling for the vertex set of each P_n and C_n as seen in Table 12.

Table 12. Labeling of P_n and C_m.

$n = 3r + j$ $j = 0, 1, 2$,	Labeling of P_n	$v(i)$	$v(f)$	$v(g)$	$e(i)$	$e(f)$	$e(g)$
$j = 0$	$A_0 = A_{3r}$ $A'_0 = B_{3r}$	r r	r r	r r	r $r-1$	r r	$r-1$ r
$j = 1$	$A_1 = A_{3r}(i)$	$r+1$	r	r	r	r	r
$j = 2$	$A_2 = A_{3r}(if)$ $A'_2 = A_{3r}(gf)$	$r+1$ r	$r+1$ $r+1$	r $r+1$	r $r+1$	r $r+1$	$r+1$ $r-1$
$m = 3s + j$ $j = 0, 1, 2$,	Labeling of C_m	$v'(i)$	$v'(f)$	$v'(g)$	$e'(i)$	$e'(f)$	$e'(g)$
$j = 0$	$B_0 = A_{3s}$	s	s	s	s	s	s
$j = 1$	$B_1 = A_{3s}(i)$ $B'_1 = A_{3s}(g)$ $B^*_1 = D_{3s}(g)$	$s+1$ s s	s s s	s $s+1$ $s+1$	$s+1$ s $s+1$	s $s+1$ $s-1$	s s $s+1$
$j = 2$	$B_2 = B_{3s}(gi)$ $B'_2 = B_{3s}(fg)$ $B^*_2 = A_{3s}(gi)$	$s+1$ s $s+1$	s $s+1$ s	$s+1$ $s+1$ $s+1$	s s $s+1$	s $s+1$ $s+1$	$s+2$ $s+1$ s

The edge labeling follows in Table 2. Hence the deduced labeling of the union is shown in Table 13.

Table 13. Vertex and edge conditions of $P_n \cup C_m$.

$n = 3r + j$ $r \geq 0,$ $j = 0, 1, 2$,	$m = 3s + j,$ $s \geq 1,$ $j = 0, 1, 2$	P_n	C_m	$\lvert v(x) - v(y) \rvert$ $, x \neq y$ and $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y) \rvert$ $, x \neq y$ and $x, y \in \{i, f, g\}$
0	0	A_0	B_0	0, 0, 0	0, 1, 1
0	1	A'_0	B_1	1, 1, 0	0, 0, 0

Table 13. *Cont.*

0	2	A_0	B_2	$1,0,1$	$0,1,1$
1	0	A_1	B_0	$1,1,0$	$0,0,0$
1	1	A_1	B'_1	$1,0,1$	$1,0,1$
1	2	A_1	B'_2	$0,0,0$	$1,1,0$
2	0	A_2	B_0	$0,1,1$	$0,1,1$
2	1	A_2	B'_1	$0,0,0$	$1,1,0$
2	2	A_2	B^*_2	$1,1,0$	$0,0,0$

It is obvious that the difference $|v(x) - v(y)|$ and $|e(x) - e(y)|$, where $x \neq y$ and $x, y \in \{i, f, g\}$ are always donated to exceed one. Therefore, $P_n \cup C_m$, $n \geq 2, m \geq 3$ admits a permuted cordial labeling. □

5. Conclusions

We proved that each path P_n, $n \geq 2$ admits permuted cordial labeling for all n. Each cycle C_n, $n \geq 3$ admits a permuted cordial labeling. Each Fan F_n, $n \geq 2$ admits permuted cordial labeling for all n. The Wheel graph $W_n, n \geq 3$ admits permuted cordial labeling except $n \equiv 2 \ mod\ 3$ and n even.

Moreover, we proved that the union of $P_n \cup P_m$, $n, m \geq 2$ admits a permuted cordial labeling for all n, m. The union of $C_n \cup C_m$, $n, m \geq 3$ admits a permuted cordial labeling for all n, m. The union of $P_n \cup C_m$, $n \geq 2, m \geq 3$ admits a permuted cordial labeling for all n, m. In the future, we will apply permuted cordial labeling to other types of graphs.

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