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# Certain Properties and Applications of Convolved $\Delta_h$ Multi-Variate Hermite and Appell Sequences

Shahid Ahmad Wani <sup>1,\*</sup>, Ibtihal Alazman <sup>2,\*</sup> and Badr Saad T. Alkahtani <sup>3,†</sup>

<sup>1</sup> Department of Applied Sciences, Symbiosis Institute of Technology, Symbiosis International (Deemed University) (SIU), Lavale, Pune 412115, Maharashtra, India

<sup>2</sup> Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11566, Saudi Arabia

<sup>3</sup> Department of Mathematics, College of Science, King Saud University, Riyadh 11989, Saudi Arabia

\* Correspondence: shahidwani177@gmail.com or shahid.wani@sitpune.edu.in (S.A.W.); iaazman@imamu.edu.sa (I.A.)

† These authors contributed equally to this work.

**Abstract:** This study follows the line of research that by employing the monomiality principle, new outcomes are produced. Thus, in line with prior facts, our aim is to introduce the  $\Delta_h$  multi-variate Hermite Appell polynomials  $\Delta_{h,S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)$ . Further, we obtain their recurrence sort of relations by using difference operators. Furthermore, symmetric identities satisfied by these polynomials are established. The operational rules are helpful in demonstrating the novel characteristics of the polynomial families and thus operational principle satisfied by these polynomials is derived and will prove beneficial for future observations. Further, a few members of the  $\Delta_h$  Appell polynomial family are considered and their corresponding results are derived accordingly.

**Keywords:**  $\Delta_h$  Appell sequences;  $\Delta_h$  multivariate Hermite polynomials; symmetric identities; monomiality principle; operational formalism

**MSC:** 26A33; 33E20; 33B10; 33E30; 33C45



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## 1. Introduction

The Hermite polynomial sequences [1] is one of the important classes of polynomial sequences, and they arise in many issues in applied mathematics, theoretical physics, approximation theory, and other branches of mathematics. Particularly in recent years, a number of extensions of special functions in mathematical physics have seen significant evolution. The bulk of precisely solved problems in mathematical physics and engineering, which have several broad applications, now have an analytical foundation thanks to this new discovery. A notable development in the theory of generalized special functions is the introduction of multitudinous-index and variable special functions. Both in pure mathematics and practical situations, the importance of these functions has been recognized. It is recognized that these polynomials with multitudinous variables and multitudinous indices are necessary to address the problems that are being raised in a variety of mathematical fields, from the theory of partial differential equations to abstract group theory. Hermite polynomials with multiple indices and variables were initially proposed by Hermite himself. The Hermite polynomials may be found in physics, numerical analysis, the quantum harmonic oscillator, and Schrödinger's equation, as well as in Gaussian quadrature.

A large number of authors are taking interest in introducing and finding several characteristics of  $\Delta_h$  special polynomials, see for example [2–6]. Recently, Shahid Wani et al., established various doped polynomials of a special type and derived their numerous characteristics and properties, which are important from an engineering point of view, see e.g., [7–11]. These properties include summation formulae, determinant forms, approximation properties, explicit and implicit formulae, generating expressions, etc.

Let  $g : I \subset \mathbb{R} \rightarrow \mathbb{R}$  and  $h \in \mathbb{R}_+$ , then the forward difference operator represented by  $\Delta_h$  ([12] p. 2) is given by

$$\Delta_h[g](u) = g(u + h) - g(u). \tag{1}$$

Thus, for finite difference of order  $i \in \mathbb{N}$ , it follows

$$\Delta_h^i[g](u) = \Delta_h(\Delta_h^{i-1}[g](u)) = \sum_{l=0}^i (-1)^{i-l} \binom{i}{l} g(u + lh), \tag{2}$$

where  $\Delta_h^0 = I$  and  $\Delta_h^1 = \Delta_h$ , with  $I$  as the identity operator.

Costabile et al. [3,13,14] recently made the first attempt to introduce  $\Delta_h$  polynomial sequences, namely  $\Delta_h$  Appell polynomials, and they explored their many features, including generating function, differential equation, determinant form, etc.

Further, in [3],  $\Delta_h$  Appell sequences  $\mathcal{Q}_m(q)$ ,  $m \in \mathbb{N}$  were defined by the power series of the product of two functions  $\gamma(t)(1 + h\xi)^{\frac{q_1}{h}}$  by

$$\eta(\xi)(1 + h\xi)^{\frac{q_1}{h}} = \mathcal{Q}_0(q; h) + \mathcal{Q}_1(q; h) \frac{\xi}{1!} + \mathcal{Q}_2(q; h) \frac{\xi^2}{2!} + \dots + \mathcal{Q}_m(q; h) \frac{\xi^m}{m!} + \dots, \tag{3}$$

where

$$\eta(\xi) = \eta_{0,h} + \eta_{1,h} \frac{\xi}{1!} + \eta_{2,h} \frac{\xi^2}{2!} + \dots + \eta_{m,h} \frac{\xi^m}{m!} \dots \tag{4}$$

$\Delta_h$  Appell sequences transform into popular sequences and polynomials such as extended falling factorials  $(q)^h m \equiv (q)m$  [12], Bernoulli sequence of the second kind  $b_m(q)$  [12], Boole sequence  $B_{lm}(q; \lambda)$  [12], and Poisson–Charlier sequence  $C_m(q; \gamma)$  ([12] p. 2).

The origins of monomiality can be traced to 1941 when Steffenson developed the poweroid notion [15], which was later refined by Dattoli [16]. The  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{D}}$  operators exist and function as multiplicative and derivative operators for a polynomial set  $\{b_m(u)\}_{m \in \mathbb{N}}$ , which means that they satisfy the following expressions:

$$b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\} \tag{5}$$

and

$$m b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}. \tag{6}$$

Then, the set  $\{b_m(q)\}_{m \in \mathbb{N}}$  manipulated by multiplicative and derivative operators is referred to as a quasi-monomial and is required to obey the formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{7}$$

thus displays a Weyl group structure as a result.

The properties of the operators  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{D}}$  can be used to determine the properties of the underlying set  $\{b_m(q)\}_{m \in \mathbb{N}}$  when it is quasi-monomial. Thus, the following traits are accurate:

- (i)  $b_m(q)$  demonstrate the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(q)\} = m b_m(q), \tag{8}$$

if  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{D}}$  possesses differential realizations.

- (ii) The explicit form of  $b_m(u)$ , can be cast in the form as listed:

$$b_m(q) = \hat{\mathcal{M}}^m \{1\}, \tag{9}$$

while taking,  $b_0(q) = 1$ .

(iii) Furthermore, generating relation in exponential form for  $b_m(q)$  can be cast in the form

$$e^{t\hat{\mathcal{M}}\{1\}} = \sum_{m=0}^{\infty} b_m(q) \frac{t^m}{m!}, \quad |t| < \infty, \tag{10}$$

by usage of identity (9).

These operational approaches are still used today in many areas of mathematical physics, quantum mechanics, and classical optics. Therefore, these techniques provide effective and potent tools of research, see for example [17–19].

Motivated by the work of Costabile and Longo [3], who introduced  $\Delta_h$  Appell sequences and established their several characterizations including generating relation, series expansion, and determinant form. Here, we are introducing the  $\Delta_h$  multi-variate Hermite Appell polynomial sequence, which contains the whole Appell family and multivariate Hermite polynomials and thus contains a larger family of polynomial sequences and establishes their several characterizations, which will prove beneficial and significant in the long run of  $\Delta_h$  polynomial sequences. The introduction of  $\Delta_h$  multi-variate Hermite Appell polynomials will pay a way to introduce different hybrid and doped  $\Delta_h$  special polynomials, being important in the field of engineering and mathematical sciences. Therefore, considering the importance of the above-mentioned class of polynomials, we give the generating expression for these  $\Delta_h$  multi-variate Hermite Appell polynomials as listed:

$$\eta(\zeta) (1 + h\zeta)^{\frac{q_1}{h}} (1 + h\zeta^2)^{\frac{q_2}{h}} (1 + h\zeta^3)^{\frac{q_3}{h}} \cdots (1 + h\zeta^r)^{\frac{q_r}{h}} = \sum_{m=0}^{\infty} \Delta_h \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \tag{11}$$

and derive their several properties. The rest of the manuscript is written as follows: multi-variate  $\Delta_h$  Hermite Appell polynomials are introduced in Section 2 along with some of their specific features. In Section 3, symmetric identities for these polynomials are established. Quasi-monomial characteristics for these polynomials are established in Section 4. Corresponding results for certain members are given by taking an appropriate choice of  $\Delta_h$  Appell polynomial family. A conclusion part is added in the last section.

### 2. $\Delta_h$ Multi-Variate Hermite Appell Polynomials

In this section, we offer an alternative generic technique for identifying multivariate  $\Delta_h$  Hermite Appell sequences. In actuality, we have

**Theorem 1.** *Since, we observe  $\Delta_h$  multi-variate Hermite Appell sequences are given by (11); therefore, we have*

$$\begin{aligned} q_1 \Delta_h [\Delta_h \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= mh \Delta_h \mathfrak{A}_{m-1}^{[r]}(q_1, q_2, \dots, q_r; h) \\ q_2 \Delta_h [\Delta_h \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)h \Delta_h \mathfrak{A}_{m-2}^{[r]}(q_1, q_2, \dots, q_r; h) \\ q_3 \Delta_h [\Delta_h \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)(m-2)h \Delta_h \mathfrak{A}_{m-3}^{[r]}(q_1, q_2, \dots, q_r; h) \\ &\vdots \\ &\vdots \\ q_r \Delta_h [\Delta_h \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)(m-2) \cdots (m-r+1)h \\ &\times \Delta_h \mathfrak{A}_{m-r}^{[r]}(q_1, q_2, \dots, q_r; h). \end{aligned} \tag{12}$$

**Proof.** Differentiating Equation (11) with regard to  $q_1$  via difference operators, we have

$$\begin{aligned} q_1 \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} &= q_1 \Delta_h \left\{ \eta(\zeta) (1 + h\zeta)^{\frac{q_1}{h}} (1 + h\zeta^2)^{\frac{q_2}{h}} (1 + h\zeta^3)^{\frac{q_3}{h}} \cdots \right. \\ &\quad \left. \times (1 + h\zeta^r)^{\frac{q_r}{h}} \right\} \end{aligned}$$

thus, in view of difference operators given by Equations (1) and (2), it follows that

$$\begin{aligned}
 q_1 \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} &= \eta(\zeta)(1+h\zeta)^{\frac{q_1+h}{h}} \\
 &\times (1+h\zeta^2)^{\frac{q_2}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} - \eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} \\
 &= (1+h\zeta-1)\eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} (1+h\zeta^3)^{\frac{q_3}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} \\
 &= (h\zeta)\eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} (1+h\zeta^3)^{\frac{q_3}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} \\
 &= h \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{t^{m+1}}{m!}.
 \end{aligned}$$

Replace  $m$  with  $m - 1$ , and then equalize the coefficients of the same powers of  $\zeta$  in the resultant expression, we have

$$q_1 \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} = \sum_{m=0}^{\infty} m \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_{m-1}^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\},$$

the proof of the first equation of the system of Equation (12) is obtained by comparing the coefficients of the same powers of  $\zeta$  on b/s of the previous equation.

Next, differentiating Equation (11) with regard to  $q_2$  via difference operators, we have

$$\begin{aligned}
 q_2 \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} &= q_2 \Delta_h \left\{ \eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} (1+h\zeta^3)^{\frac{q_3}{h}} \dots \right. \\
 &\quad \left. \times (1+h\zeta^r)^{\frac{q_r}{h}} \right\}
 \end{aligned}$$

thus, in view of difference operators given by Equations (1) and (2), it follows that

$$\begin{aligned}
 q_2 \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} &= \eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} \\
 &\times (1+h\zeta^2)^{\frac{q_2+h}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} - \eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} \\
 &= (1+h\zeta^2-1)\eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} (1+h\zeta^3)^{\frac{q_3}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} \\
 &= (h\zeta^2)\eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} (1+h\zeta^3)^{\frac{q_3}{h}} \dots (1+h\zeta^r)^{\frac{q_r}{h}} \\
 &= h \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{t^{m+2}}{m!}.
 \end{aligned}$$

Replace  $m$  with  $m - 2$ , and then equalize the coefficients of the same powers of  $\zeta$  in the resultant expression, we have

$$q_1 \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} = \sum_{m=0}^{\infty} m(m-1) \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_{m-2}^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\},$$

the proof of the second equation of the system of Equation (12) is obtained by comparing the coefficients of the same powers of  $\zeta$  on b/s of the previous equation.

By using a similar fashion, differentiating Equation (11) with regard to  $q_r$  via difference operators, we have

$$\begin{aligned}
 q_r \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!} \right\} &= q_r \Delta_h \left\{ \eta(\zeta)(1+h\zeta)^{\frac{q_1}{h}} (1+h\zeta^2)^{\frac{q_2}{h}} (1+h\zeta^3)^{\frac{q_3}{h}} \dots \right. \\
 &\quad \left. \times (1+h\zeta^r)^{\frac{q_r}{h}} \right\}
 \end{aligned}$$

thus, in view of difference operators given by Equations (1) and (2), it follows that

$$\begin{aligned} q_r \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\xi^m}{m!} \right\} &= \eta(\xi)(1+h\xi)^{\frac{q_1}{h}} \\ &\times (1+h\xi^2)^{\frac{q_2}{h}} \dots (1+h\xi^r)^{\frac{q_r}{h}} - \eta(\xi)(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} \dots (1+h\xi^r)^{\frac{q_r+h}{h}} \\ &= (1+h\xi^r - 1) \eta(\xi)(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} (1+h\xi^3)^{\frac{q_3}{h}} \dots (1+h\xi^r)^{\frac{q_r}{h}} \\ &= (h\xi^r) \eta(\xi)(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} (1+h\xi^3)^{\frac{q_3}{h}} \dots (1+h\xi^r)^{\frac{q_r}{h}} \\ &= h \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{h^{m+r}}{m!}. \end{aligned}$$

Replace  $m$  with  $m - r$ , and then equalize the coefficients of the same powers of  $\xi$  in the resultant expression, we have

$$\begin{aligned} q_r \Delta_h \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\xi^m}{m!} \right\} &= \sum_{m=0}^{\infty} m(m-1)(m-2) \dots (m-r+1) \\ &\times \left\{ \sum_{m=0}^{\infty} \Delta_h \mathfrak{S} \mathfrak{A}_{m-r}^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\xi^m}{m!} \right\}, \end{aligned}$$

the proof of the  $r$ th equation of the system of Equation (12) is obtained by comparing the coefficients of the same powers of  $\xi$  on b/s of the previous equation.  $\square$

**Theorem 2.** Further, the  $\Delta_h$  multi-variate Hermite Appell polynomials  $\Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)$ ,  $m \in \mathbb{N}$  is determined by the power series expansion of the product  $\eta(\xi)(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} (1+h\xi^3)^{\frac{q_3}{h}} \dots (1+h\xi^r)^{\frac{q_r}{h}}$ , that is

$$\begin{aligned} \eta(\xi)(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} (1+h\xi^3)^{\frac{q_3}{h}} &= \Delta_h \mathfrak{S} \mathfrak{A}_0^{[r]}(q_1, q_2, \dots, q_r; h) + \Delta_h \mathfrak{S} \mathfrak{A}_1^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\xi}{1!} \\ + \Delta_h \mathfrak{S} \mathfrak{A}_2^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\xi^2}{2!} &+ \dots + \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\xi^m}{m!} + \dots \end{aligned} \tag{13}$$

**Proof.** Expanding  $\eta(\xi)(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} (1+h\xi^3)^{\frac{q_3}{h}} \dots (1+h\xi^r)^{\frac{q_r}{h}}$  by Newton series for finite differences at  $q_1 = q_2 = \dots = q_r = 0$  and order the product of the developments of functions  $\eta(\xi)$  and  $(1+h\xi)^{\frac{q_1}{h}} (1+h\xi^2)^{\frac{q_2}{h}} (1+h\xi^3)^{\frac{q_3}{h}} \dots (1+h\xi^r)^{\frac{q_r}{h}}$  with regard to the powers of  $\xi$ , then in view of expression (3), we observe the polynomials  $\Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)$  are expressed in Equation (13) as coefficients of  $\frac{\xi^m}{m!}$  as the generating function of  $\Delta_h$  multi-variate Hermite Appell polynomials.  $\square$

Next, we establish the quasi-monomial properties satisfied by  $\Delta_h$  multi-variate Hermite Appell polynomials, by proving the following results:

**Theorem 3.** The  $\Delta_h$  multi-variate Hermite Appell polynomials satisfy the following multiplicative and derivative operators:

$$\begin{aligned} \Delta_h \mathfrak{S} \mathfrak{A}_{m+1}^{[r]}(q_1, q_2, \dots, q_r; h) &= \hat{\mathcal{M}}_{\Delta_h} \{ \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \} = \left( \eta\left(\frac{q_1 \Delta_h}{h}\right) + q_1 \frac{1}{1+q_1 \Delta_h} \right. \\ &\left. + 2q_2 \frac{q_1 \Delta_h}{h+q_1 \Delta_h^2} + 3q_3 \frac{q_1 \Delta_h}{h^2+q_1 \Delta_h^3} + \dots + r q_r \frac{q_1 \Delta_h^{r-1}}{h^{r-1}+q_1 \Delta_h^r} \right) \{ \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \Delta_h \mathfrak{S} \mathfrak{A}_{m-1}^{[r]}(q_1, q_2, \dots, q_r; h) &= \hat{\mathcal{D}}_{\Delta_h} \{ \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \} = \\ &= \frac{\log(1+q_1 \Delta_h)}{mh} \{ \Delta_h \mathfrak{S} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) \}, \end{aligned} \tag{15}$$

respectively.

**Proof.** In view of finite difference operator  $\Delta_h$ , we have

$$q_1 \Delta_h \left[ \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) \right] = h \xi \left[ \Delta_h \mathfrak{H}_{m-1}^{[r]}(q_1, q_2, \dots, q_r; h) \right], \tag{16}$$

or

$$\frac{q_1 \Delta_h}{h} \left[ \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) \right] = \xi \left[ \Delta_h \mathfrak{H}_{m-1}^{[r]}(q_1, q_2, \dots, q_r; h) \right]. \tag{17}$$

Differentiating (11) with regard to  $\xi$  and  $q_1$ , separately, we find

$$\Delta_h \mathfrak{H}_{m+1}^{[r]}(q_1, q_2, \dots, q_r; h) = \mathcal{M}_{\Delta_h} \left\{ \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) \right\} = \left( \frac{\eta'(\xi)}{\eta(\xi)} + \frac{q_1}{1+h\xi} + \frac{2q_2\xi}{1+h\xi^2} + \frac{3q_3\xi^2}{1+h\xi^3} + \dots + \frac{r q_r \xi^{r-1}}{1+h\xi^r} \right) \left\{ \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) \right\} \tag{18}$$

and

$$\Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) = \mathcal{D}_{\Delta_h} \left\{ \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) \right\} = \frac{\log(1+h\xi)}{mh} \left\{ \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) \right\}. \tag{19}$$

Using identity (17) in view of (5) and (6) in above Equations (18) and (19), we are lead to assertions (14) and (15).  $\square$

**Theorem 4.** The  $\Delta_h$  multi-variate Hermite polynomials satisfy the differential equation listed as:

$$\left( \frac{\eta'(\frac{q_1 \Delta_h}{h})}{\eta(\frac{q_1 \Delta_h}{h})} + q_1 \frac{1}{1+q_1 \Delta_h} + 2q_2 \frac{q_1 \Delta_h}{h+q_1 \Delta_h^2} + 3q_3 \frac{q_1 \Delta_h}{h^2+q_1 \Delta_h^3} + \dots + r q_r \frac{q_1 \Delta_h^{r-1}}{h^{r-1}+q_1 \Delta_h^r} - \frac{m^2 h}{\log(1+q_1 \Delta_h)} \right) \times \Delta_h \mathfrak{H}_m^{[r]}(q_1, q_2, \dots, q_r; h) = 0. \tag{20}$$

**Proof.** Making use of expressions (14) and (15) in (8), we are lead to assertion (20).  $\square$

### 3. Symmetric Identities

We provide symmetric identities for the multivariate Hermite Kampé de Fériet Appell polynomials in this section. Furthermore, we learn some of the multi-variate Hermite Kampé de Fériet Appell polynomials' formulae and characteristics.

**Theorem 5.** For,  $\beta \neq \alpha$  and  $\beta, \alpha > 0$ , we have

$$\alpha^m \Delta_h \mathfrak{H}_m^{[r]}(\beta q_1, \beta q_2, \dots, \beta q_r; h) = \beta^m \Delta_h \mathfrak{H}_m^{[r]}(\alpha q_1, \alpha q_2, \dots, \alpha q_r; h). \tag{21}$$

**Proof.** Since,  $\beta \neq \alpha$  and  $\beta, \alpha > 0$ , we start by writing:

$$\mathfrak{R}(\xi; q_1, q_2, q_3, \dots, q_r; h) = \eta(\xi) (1+h\xi)^{\frac{\alpha \beta q_1}{h}} (1+h\xi^2)^{\frac{\alpha^2 \beta^2 q_2}{h}} \dots (1+h\xi^r)^{\frac{\alpha^r \beta^r q_r}{h}}. \tag{22}$$

Therefore, the above expression  $\mathfrak{R}(\xi; q_1, q_2, q_3, \dots, q_r; h)$  is symmetric in  $\alpha$  and  $\beta$ . Further, we can write

$$\mathfrak{R}(\xi; q_1, q_2, q_3, \dots, q_r; h) = \Delta_h \mathfrak{H}_m^{[r]}(\alpha q_1, \alpha^2 q_2, \dots, \alpha^r q_r; h) \frac{(\beta \xi)^m}{m!} = \beta^m \Delta_h \mathfrak{H}_m^{[r]}(\alpha q_1, \alpha^2 q_2, \dots, \alpha^r q_r; h) \frac{\xi^m}{m!}. \tag{23}$$

Thus, it follows that

$$\mathfrak{R}(\xi; q_1, q_2, q_3, \dots, q_r; h) = \Delta_h \mathfrak{H}_m^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \frac{(\alpha \xi)^m}{m!} = \alpha^m \Delta_h \mathfrak{H}_m^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \frac{\xi^m}{m!}. \tag{24}$$

The assertion (21) is obtained by equating the coefficients of the similar term of  $xi$  in the final two Equations (23) and (24).  $\square$

**Theorem 6.** For,  $\beta \neq \alpha, \alpha > 0$  and, it follows that

$$\sum_{i=0}^m \sum_{n=0}^i \binom{m}{i} \binom{i}{n} \alpha^i \beta^{m+1-i} {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \mathcal{P}_{m-i}(\alpha - 1; h) = \sum_{i=0}^m \sum_{n=0}^i \binom{m}{i} \binom{i}{n} \beta^i \alpha^{m+1-i} {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(\alpha q_1, \alpha^2 q_2, \dots, \alpha^r q_r; h) \mathcal{P}_{m-i}(\beta - 1; h). \tag{25}$$

**Proof.** Since,  $\beta \neq \alpha, \alpha > 0$ , we start by writing:

$$\mathfrak{G}(\xi; q_1, q_2, q_3, \dots, q_r; h) = \frac{\alpha \beta \xi \eta(\xi) (1 + h\xi)^{\frac{\alpha \beta q_1}{h}} (1 + h\xi^2)^{\frac{\alpha^2 \beta^2 q_2}{h}} \dots (1 + h\xi^r)^{\frac{\alpha^r \beta^r q_r}{h}} \left( (1 + h\xi)^{\frac{\alpha \beta q_1}{h} - 1} \right)}{\left( (1 + h\xi)^{\frac{\alpha}{h} - 1} \right) \left( (1 + h\xi)^{\frac{\beta}{h} - 1} \right)}. \tag{26}$$

In a similar way to the previous theorem, we obtain statement (25).  $\square$

**Theorem 7.** For,  $\beta \neq \alpha, \alpha > 0$ , we have

$$\sum_{i=0}^m \sum_{n=0}^i \binom{m}{i} \binom{i}{n} \alpha^i \beta^{m+1-i} \mathcal{B}_m(h) {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{i-m}^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \mathcal{P}_{m-i}(\alpha - 1; h) \sigma_{m-i}(\alpha - 1; h) = \sum_{i=0}^m \sum_{n=0}^i \binom{m}{i} \binom{i}{n} \beta^i \alpha^{m+1-i} \mathcal{B}_m(h) {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{i-m}^{[r]}(\alpha q_1, \alpha^2 q_2, \dots, \alpha^r q_r; h) \sigma_{m-i}(\beta - 1; h). \tag{27}$$

**Proof.** Since,  $\beta \neq \alpha, \alpha > 0$ , we start by writing:

$$\mathfrak{T}(\xi; q_1, q_2, q_3, \dots, q_r; h) = \frac{\alpha \beta \xi \eta(\xi) (1 + h\xi)^{\frac{\alpha \beta q_1}{h}} (1 + h\xi^2)^{\frac{\alpha^2 \beta^2 q_2}{h}} \dots (1 + h\xi^r)^{\frac{\alpha^r \beta^r q_r}{h}} \left( (1 + h\xi)^{\frac{\alpha \beta q_1}{h} - 1} \right)}{\left( (1 + h\xi)^{\frac{\alpha}{h} - 1} \right) \left( (1 + h\xi)^{\frac{\beta}{h} - 1} \right)}. \tag{28}$$

The preceding equation may be expressed as

$$\mathfrak{T}(\xi; q_1, q_2, q_3, \dots, q_r; h) = \frac{\alpha \beta \xi}{\left( (1 + h\xi)^{\frac{\alpha}{h} - 1} \right)} \eta(\xi) (1 + h\xi)^{\frac{\alpha \beta q_1}{h}} (1 + h\xi^2)^{\frac{\alpha^2 \beta^2 q_2}{h}} \dots (1 + h\xi^r)^{\frac{\alpha^r \beta^r q_r}{h}} \times \frac{\left( (1 + h\xi)^{\frac{\alpha \beta q_1}{h} - 1} \right)}{\left( (1 + h\xi)^{\frac{\beta}{h} - 1} \right)}. \tag{29}$$

Using  $\frac{\alpha \beta \xi}{\left( (1 + h\xi)^{\frac{\alpha}{h} - 1} \right)} = \alpha \sum_{m=0}^{\infty} \mathcal{B}_m(h) \frac{(\beta \xi)^m}{m!}$ ,  $\frac{\left( (1 + h\xi)^{\frac{\alpha \beta q_1}{h} - 1} \right)}{\left( (1 + h\xi)^{\frac{\beta}{h} - 1} \right)} = \sigma_{m-i}(\alpha - 1; h) \frac{(\beta \xi)^m}{m!}$  and (11), we have

$$\begin{aligned} \mathfrak{T}(\xi; q_1, q_2, q_3, \dots, q_r; h) &= \alpha \sum_{m=0}^{\infty} \mathcal{B}_m(h) \frac{(\beta \xi)^m}{m!} \sum_{m=0}^{\infty} {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \frac{(\alpha \xi)^m}{m!} \\ &\quad \times \sum_{m=0}^{\infty} \sigma_{m-i}(\alpha - 1; h) \frac{(\beta \xi)^m}{m!} \\ &= \sum_{m=0}^{\infty} \left( \sum_{i=0}^m \sum_{n=0}^i \binom{m}{i} \binom{i}{n} \alpha^i \beta^{m+1-i} \mathcal{B}_m(h) {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{i-m}^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \sigma_{m-i}(\alpha - 1; h) \right) \frac{\xi^m}{m!}. \end{aligned} \tag{30}$$

If we proceed in the same way, we have

$$\begin{aligned} \mathfrak{A}(\xi; q_1, q_2, q_3, \dots, q_r; h) &= \sum_{m=0}^{\infty} \left( \sum_{i=0}^m \sum_{n=0}^i \binom{m}{i} \binom{i}{n} \beta^i \alpha^{m+1-i} \mathcal{B}_m(h) \right. \\ &\quad \left. \times {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{i-m}^{[r]}(\beta q_1, \beta^2 q_2, \dots, \beta^r q_r; h) \sigma_{m-i}(\beta - 1; h) \right) \frac{\xi^m}{m!}. \end{aligned} \tag{31}$$

If we compare the coefficients of expressions (30) and (31), we obtained the result (27). □

#### 4. Operational Formalism and Examples

Operational techniques can be used to construct new families of functions and to make the derivation of the attributes connected to regular and generalized special functions easier. Dattoli and his coworkers see for example [15,16,20–23] have acknowledged the significance of the employment of operational approaches in the study of special functions that attempt to provide explicit solutions for families of partial differential equations, including those of the Heat and D’Alembert type, and their applications.

Differentiating successively (11) with regard to  $q_1$  via forward difference operator concept by taking into consideration expression (2), we find

$$\begin{aligned} q_1 \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= mh {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-1}^{[r]}(q_1, q_2, \dots, q_r; h) \\ q_1 \Delta_h^2 [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)h {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-2}^{[r]}(q_1, q_2, \dots, q_r; h) \\ &\vdots \\ q_1 \Delta_h^r [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)(m-2) \dots (m-r+1)h \\ &\quad \times {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-r}^{[r]}(q_1, q_2, \dots, q_r; h). \end{aligned} \tag{32}$$

Next, differentiating (11) with regard to  $q_2, q_3, \dots, q_r$  via forward difference operator concept by taking into consideration expression (2), we find

$$\begin{aligned} q_2 \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)h {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-2}^{[r]}(q_1, q_2, \dots, q_r; h) \\ q_3 \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)(m-2)h {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-3}^{[r]}(q_1, q_2, \dots, q_r; h) \\ &\vdots \\ &\vdots \\ q_r \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= m(m-1)(m-2) \dots (m-r+1)h \\ &\quad \times {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-r}^{[r]}(q_1, q_2, \dots, q_r; h). \end{aligned} \tag{33}$$

In view of system of expressions (32) and (33), we find  ${}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-r}^{[r]}(q_1, q_2, \dots, q_r; h)$  are solutions of the expressions:

$$\begin{aligned} q_2 \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= q_1 \Delta_h^2 {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-2}^{[r]}(q_1, q_2, \dots, q_r; h) \\ q_3 \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= q_1 \Delta_h^3 {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-3}^{[r]}(q_1, q_2, \dots, q_r; h) \\ &\vdots \\ &\vdots \\ q_r \Delta_h [{}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)] &= q_1 \Delta_h^r {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_{m-r}^{[r]}(q_1, q_2, \dots, q_r; h), \end{aligned} \tag{34}$$

under the listed initial condition

$${}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(0, 0, 0, \dots, 0; h) = {}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(0; h) = {}_{\Delta_h} \mathfrak{A}_m(\xi; h). \tag{35}$$

Therefore, from system of expressions (34) and expression (35), it states that

$${}_{\Delta_h \mathfrak{S}} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h) = \exp \left( q_1 + q_2 q_1 \Delta_h^2 + q_3 q_1 \Delta_h^3 + \dots + q_r q_1 \Delta_h^r \right) {}_{\Delta_h} \mathfrak{A}_m(\xi; h). \tag{36}$$



In light of previous mentioned expression, the  $\Delta_h$  multi-variate Hermite Appell polynomials  $\Delta_{h, \zeta} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)$  can be constructed from the the  $\Delta_h$  polynomial  $\Delta_h \mathfrak{A}_m(\zeta; h)$  by applying the operational rule (36).

Numerous applications in number theory, combinatorics, numerical analysis, and other areas of practical mathematics make use of the Bernoulli, Euler, and Genocchi numbers. The Genocchi numbers have significance in graph theory, automata theory, and counting the number of up-down ascent sequences. The trigonometric and hyperbolic secant functions' origins are close to where the Euler numbers enter the Taylor expansion. Numerous mathematical formulae contain the Bernoulli numbers, including the Taylor expansion, sums of powers of natural numbers, trigonometric and hyperbolic tangent and cotangent functions, and many others.

By taking Bernoulli, Euler, and Genocchi polynomials as members of the Appell family, we obtain different members of  $\Delta_h$  MHAP family as multivariate  $\Delta_h$  Hermite—Bernoulli polynomials  $\Delta_{h, \zeta} \mathfrak{B}_m^{[r]}(q_1, q_2, \dots, q_r; h)$ , multivariate  $\Delta_h$  Hermite–Euler polynomials  $\Delta_{h, \zeta} \mathfrak{E}_m^{[r]}(q_1, q_2, \dots, q_r; h)$  and multivariate  $\Delta_h$  Hermite-Genocchi polynomials  $\Delta_{h, \zeta} \mathfrak{G}_m^{[r]}(q_1, q_2, \dots, q_r; h)$ . These polynomials are given by generating expression as listed:

$$\frac{\log(1 + h\zeta)}{h(1 + h\zeta)^{\frac{1}{h}} - 1} (1 + h\zeta)^{\frac{q_1}{h}} (1 + h\zeta^2)^{\frac{q_2}{h}} (1 + h\zeta^3)^{\frac{q_3}{h}} = \sum_{m=0}^{\infty} \Delta_{h, \zeta} \mathfrak{B}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!}. \tag{37}$$

$$\frac{2}{(1 + h\zeta)^{\frac{1}{h}} + 1} (1 + h\zeta)^{\frac{q_1}{h}} (1 + h\zeta^2)^{\frac{q_2}{h}} (1 + h\zeta^3)^{\frac{q_3}{h}} = \sum_{m=0}^{\infty} \Delta_{h, \zeta} \mathfrak{E}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!}. \tag{38}$$

and

$$\frac{2\log(1 + h\zeta)}{h(1 + h\zeta)^{\frac{1}{h}} + 1} (1 + h\zeta)^{\frac{q_1}{h}} (1 + h\zeta^2)^{\frac{q_2}{h}} (1 + h\zeta^3)^{\frac{q_3}{h}} = \sum_{m=0}^{\infty} \Delta_{h, \zeta} \mathfrak{G}_m^{[r]}(q_1, q_2, \dots, q_r; h) \frac{\zeta^m}{m!}. \tag{39}$$

respectively.

Using the results derived for  $\Delta_h$  multivariate Hermite Appell polynomials  $\Delta_{h, \zeta} \mathfrak{A}_m^{[r]}(q_1, q_2, \dots, q_r; h)$ , we can derive the corresponding results for the  $\Delta_h$  multivariate Hermite Bernoulli, Euler and Genocchi polynomials.

### 5. Conclusions

Here, we introduced the  $\Delta_h$  multi-variate Hermite Appell polynomials, and some of their specific features are presented: Quasi-monomial characteristics for these polynomials are established in Section 2, and forward difference relations are established in Theorem 1. Furthermore, symmetric identities are given in Section 3 and the operational rule is established in Section 4. In addition to Gaussian quadrature, these polynomials may be found in physics, numerical analysis, the quantum harmonic oscillator, and Schrödinger’s equation. They also come up often in problems involving theoretical physics, approximation theory, applied mathematics, and other disciplines of mathematics including Biological and medical sciences.

Further, future investigations and observations can be used to establish extended, generalized forms, integral representations, determinant and series representations, and other properties of the above-mentioned polynomials. Furthermore, the interpolation form, recurrence relations, shift operators, and summation formulae can also be a problem for new observations. Moreover, the hybrid form of these polynomials can be taken as a future investigation.

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