On the Ideal Convergent Sequences in Fuzzy Normed Space

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Abstract: This article discusses a variety of important notions, including ideal convergence and ideal Cauchyness of topological sequences produced by fuzzy normed spaces. Furthermore, the connections between the concepts of the ideal limit and ideal cluster points of a sequence in a fuzzy normed linear space are investigated. In a fuzzy normed space, we investigated additional effects, such as describing compactness in terms of ideal cluster points and other relevant but previously unresearched ideal convergence and adjoint ideal convergence aspects of sequences and nets. The countable compactness of a fuzzy normed space and its link to it were also defined. The terms ideal and its adjoint divergent sequences are then introduced, and specific aspects of them are explored in a fuzzy normed space. Our study supports the importance of condition (AP) in examining summability via ideals. It is suggested to use a fuzzy point symmetry-based genetic clustering method to automatically count the number of clusters in a data set and determine how well the data are fuzzy partitioned. As long as the clusters have the attribute of symmetry, they can be any size, form, or convexity. One of the crucial ways that symmetry is used in fuzzy systems is in the solution of the linear Fuzzy Fredholm Integral Equation (FFIE), which has symmetric triangular (Fuzzy Interval) output and any fuzzy function input.

Keywords: fuzzy norm space; ideal; ideal convergence; ideal Cauchy; ideal limit; ideal cluster

1. Introduction

One of the most fundamental and significant ideas in mathematics is convergence (of sequences). It was generalized in a number of ways. Even so, the first concept of statistical convergence, also known as nearly convergence, initially appeared in the famed Zygmund’s monograph’s first edition in 1935, H. Fast [1] and H. Steinhaus [2] independently discovered the concept of statistical convergence of real number sequences. The idea of the asymptotic density of a subset of the natural numbers serves as its basis. Let $E(m):= \{ k \in E : k \leq m \}$ denote the cardinality of $E(m)$ and $|E(m)|$ the cardinality of $E(m)$ for $E \subset \mathbb{N}$ and $m \in \mathbb{N}$. The natural density of $E$ is defined by

$$\delta(E) = \lim_{m \to \infty} \frac{\text{card}(E(m))}{m}.$$  

Many disciplines of mathematics contain multiple applications of statistical convergence (see, for instance, [3,4] and references therein). Let us remark that [5,6] explored statistical convergence in function spaces. The concept of sequence convergence with respect to a filter $\mathcal{F}$ on $\mathbb{N}$ was first developed by Bernstein [7] in 1970. Using the idea of an ideal, Kostyrko et al. [8] developed the concept of ideal convergence, this is a typical generalization of statistical and ordinary convergence (see also [9]). The ideal convergence offers a wide framework for analyzing the traits of various types of convergence. It should be noted that in [8], several findings for the set of ideal cluster points and ideal limit points...
were discovered. This article also explains the concepts of ideal cluster points and ideal limit points for sequences in metric spaces. Be aware that the interesting generalization of statistical convergence known as ideal convergence (see [10,11]) exists.

“The concept of a fuzzy norm on a linear space was initially put forth in 1984 by Katsaras [12] while researching fuzzy topological vector spaces (see [10,11]). By giving each element a fuzzy real number, Felbin [13] established the concept of a fuzzy norm on a linear space in 1992. This resulted in the related fuzzy metric for this fuzzy norm, which is of the Kaleva and Seikkala type [14]. In order for the associated fuzzy metric to be of the Kramosil and Michalek type, Cheng and Mordeson in 1994 [15] suggested an alternative notion of a fuzzy norm on a linear space”.

T. Bag and S. K. Samanta introduced the idea of a fuzzy norm in [16], whose associated fuzzy metric is similar to the Kramosil and Michalek type [17]. This theory was developed after Cheng and Mordeson. The distinctiveness of this formulation comes from the fact that this kind of fuzzy norm can be successfully decomposed into a family of crisp norms according to the theory set forth in [16]. This idea has been applied in numerous papers by numerous writers to create fuzzy functional analysis and its applications (for references, see [16,18,19]).

One of Nature’s fundamental characteristics is symmetry. An object or process is said to be symmetric if it is invariant to a particular set of transformations, known as “symmetry operations” (e.g., translation, rotation, reflection, inversion, etc.) that together make up a mathematical group. To estimate the fuzzy solution to the linear equation, symmetric fuzzy numbers and expected intervals are employed. It is also known as a model for solving a fuzzy Fredholm integral equation (FFIE), which has an arbitrary fuzzy function as its input and produces symmetric or interval fuzzy function as its output. The use of an analytical method, namely the homotopy analysis method, falls within the category of analytical methods. A solution of a linear Fuzzy Fredholm Integral Equation (FFIE) with any Fuzzy Function input and symmetric triangular (Fuzzy Interval) output is one of the significant applications of symmetry in fuzzy systems.

**Definition 1.** A non-void class \( \mathcal{I} \subset 2^\mathbb{N} \) is said to be an ideal if the following conditions hold:

(i) \( S, T \in \mathcal{I} \) implies \( T \cup S \in \mathcal{I} \) (additive property);

(ii) \( T \in \mathcal{I} \) and \( S \subseteq T \) implies \( S \in \mathcal{I} \) (hereditary property) [20].

An ideal \( \mathcal{I} \) is non-trivial provided that \( \mathcal{I} \neq 2^\mathbb{N} \) and called admissible if its non-trivial and \( \{ l \} \in \mathcal{I} \) for all \( l \in \mathbb{N} \). For any ideal, there is a filter \( \mathfrak{F}(\mathcal{I}) \) corresponding with \( \mathcal{I} \), given by

\[
\mathfrak{F}(\mathcal{I}) = \{ K \subseteq \mathbb{N} : \mathbb{N} \setminus K \in \mathcal{I} \}.
\]

“An admissible ideal \( \mathcal{I} \subset 2^\mathbb{N} \) possesses property (AP) if for any sequence \( \{ \mathcal{W}_1, \mathcal{W}_2, \ldots \} \) mutually exclusive sets of \( \mathcal{I} \), there is a sequence \( \{ \mathcal{W}_1, \mathcal{W}_2, \ldots \} \) of subsets of \( \mathbb{N} \) such that each symmetric difference \( \mathcal{W}_i \Delta \mathcal{W}_j \) \( (i = 1, 2, \ldots) \) is finite and \( \bigcup_{i=1}^{\infty} \mathcal{W}_i \in \mathcal{I} \) [8,21].”

In contrast to fuzzy settings, ideal convergence in normed spaces is the focus of this study. The paper is structured as follows: In Section 2, we provide some preliminary definitions and conclusions regarding fuzzy normed spaces. In Section 3 of this essay, we primarily demonstrate that, under a general assumption, the condition (AP) is both required and sufficient for the ideal Cauchy condition and its adjoint Cauchy condition to be equivalent. We also provide an example to demonstrate that the ideal Cauchy condition is typically true. Not every sequence must have adjoint ideal Cauchy relations. Section 4 looks into some important, little-studied aspects of ideal convergence and adjoint ideal convergence of sequences and nets in a fuzzy normed space, as well as some additional implications in a fuzzy normed space, such as the characterization of compactness in terms of ideal cluster points. Furthermore, we established the definition of ideal sequential compactness and its relation to countable compactness in a fuzzy normed space. The
concepts of ideal divergent and adjoint ideal divergent sequences in a fuzzy normed space are introduced and established in Section 5 of this study. We mainly demonstrate that condition (AP) is the necessary and sufficient condition for the equivalence of the ideal and its adjoint divergence under specific conditions, similar to convergence and the Cauchy condition. This strengthens and validates the role of condition (AP) in the evaluation of summability via ideals.

2. Definitions and Preliminaries

We first go through some of the fundamental ideas that will be applied in the follow-up.

**Lemma 1 ([9]).** Let $J$ be a proper ideal in $\mathcal{W}$, i.e., $\mathcal{W} \not\in J$, $\mathcal{W} \not= \emptyset$. Then the family of sets

$$\mathcal{F}(J) = \{ H \subset \mathcal{W} : \exists G \in J : H = \mathcal{W} \setminus G \}$$

is a filter in $\mathcal{W}$. It is referred to as the filter associated with the ideal $J$.

**Definition 2 ([8]).** Let $I \subset N$ be a proper ideal in $\mathbb{N}$ and $(\mathcal{X}, \rho)$ be a metric space. The sequence $\{ \kappa_n \}$ of elements of $\mathcal{X}$ is said to be $I$-convergent to $\xi \in \mathcal{X}$ if for each $\epsilon > 0$ the set $\Gamma(\epsilon) = \{ n \in \mathbb{N} : \rho(\kappa_n, \xi) \geq \epsilon \}$ belongs to $I$.

If $\{ \kappa_n \}$ is $I$-convergent to $\xi$, then we write $\lim_{n \to \infty} \kappa_n = \xi$. In this case, the element $\xi \in \mathcal{X}$ is called $I$-limit of the sequence $\{ \kappa_n \} \subset \mathcal{X}$.

A concept of adjoint ideal, which is closely related to ideal convergence, was introduced by Kostyrko et al. [8].

**Definition 3.** A sequence $\{ \kappa_n \}$ of elements of $\mathcal{X}$ is said to be $I^*\convergent$ to $\xi \in \mathcal{X}$ if and only if there exists a set $Q \in \mathcal{F}(I)$, $M = \{ q_1 < q_2 < \cdots < q_k < \cdots \} \subset N$ such that $\lim_{k \to \infty} d(\kappa_{q_k}, \xi) = 0$.

Under the condition (AP), the notions of ideal convergence and its adjoint are equivalent, as shown by [8], if the ideal is admissible.

**Lemma 2.** Let $(\mathcal{X}, \rho)$ be an arbitrary metric space. If $I$ is an admissible ideal that possesses property (AP), then $\lim_{n \to \infty} \kappa_n = \xi$ if and only if there exists a set $P \in \mathcal{F}(I)$, $P = \{ p_1 < p_2 < \cdots < p_k < \cdots \}$ such that $\lim_{k \to \infty} \rho(p_k, \xi) = 0$.

**Definition 4 ([22]).** A binary operation $T : [0,1]^2 \to [0,1]$ is said to be a continuous $\tau$-norm if $([0,1], T)$ a topological monoid with unit 1 that has the property $T(x_1, x_2) \leq T(y_1, y_2)$ whenever $x_1 \leq y_1, x_2 \leq y_2$ for all $x_1, x_2, x_3, x_4 \in [0,1]$.

**Definition 5 ([22]).** The triple $(\mathcal{X}, \psi, T)$ is called a fuzzy metric space if $\mathcal{X}$ is an arbitrary set, $T$ is a continuous $\tau$-norm and $\psi$ is a fuzzy set on $\mathcal{X}^2 \times (0, \infty)$ satisfying the following conditions for all $u, v, w \in \mathcal{X}$ and $\tau, \eta > 0$,

(FM1) $\psi(u, v, \tau) > 0$,
(FM2) $\psi(u, v, \tau) = 1$ for all $\tau > 0$ if and only if $u = v$,
(FM3) $\psi(u, v, \tau) = \psi(v, u, \tau)$,
(FM4) $\psi(u, v, \tau + \eta) \geq T(\psi(u, v, \tau), \psi(v, w, \eta))$ for all $\tau, \eta > 0$,
(FM5) $\psi(u, v, \cdot) : (0, \infty) \to [0,1]$ is continuous.

(FM6) $\lim_{\tau \to \infty} \psi(u, v, \tau) = 1$.

**Definition 6 ([22]).** Let $(\mathcal{X}, \psi, T)$ be a fuzzy metric space. We a sequence $\{ \kappa_n \}$ is as follows:
(i) a convergent to a point \( \xi \in X \) if and only if for each \( \omega \in (0, 1) \) and \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) we have,
\[
\psi(n, \xi, \varepsilon) > 1 - \omega.
\]
(ii) a Cauchy sequence if and only if for each \( \omega \in (0, 1) \) and \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n, m \geq n_0 \) we have,
\[
\psi(n, \xi, \varepsilon) > 1 - \omega.
\]
(iii) If every a Cauchy sequence in \( X \) is convergent, then we say that a fuzzy metric space \((X, \psi, T)\) is complete.

**Definition 7** ([19]). If \( X \) is a vector space, \( T \) is a continuous \( \tau \)-norm, and \( \mathfrak{N} \) is a fuzzy set on \( X \times (0, \infty) \), then the triple \((X, \mathfrak{N}, T)\) is said to be a fuzzy normed space and meets the following criteria for each \( u, v \) and \( \tau, \eta \in X \):

(FN1) \( \mathfrak{N}(u, \tau) > 0 \),
(FN2) \( \mathfrak{N}(u, \tau) = 1 \) if and only if \( u = 0 \),
(FN3) \( \mathfrak{N}(cu, \tau) = \mathfrak{N}(u, \frac{\tau}{|c|}) \) for all \( c \neq 0 \),
(FN4) \( \mathfrak{N}(u + v, \tau + \eta) \geq T(\mathfrak{N}(u, \tau), \mathfrak{N}(v, \eta)) \),
(FN5) \( \mathfrak{N} : (0, \infty) \to [0, 1] \) is continuous,
(FN6) \( \lim_{\tau \to 0} \mathfrak{N}(u, \tau) = 1 \),
(FN7) \( \mathfrak{N}(u, \cdot) : (0, \infty) \to [0, 1] \) is continuous.

**Definition 8** ([18]). Let \((X, \mathfrak{N}, T)\) be a fuzzy normed space. We define the open ball \( B(\xi, r, \tau) \) and the closed ball \( B[\xi, r, \tau] \) with center \( \xi \in X \) and radius \( 0 < r < 1, \tau > 0 \) as follows:
\[
B(\xi, r, \tau) = \{ \xi \in X : \mathfrak{N}(\xi - \xi, \tau) > 1 - r \}
\]
\[
B[\xi, r, \tau] = \{ \xi \in X : \mathfrak{N}(\xi - \xi, \tau) \geq 1 - r \}
\]

**Proposition 1** ([18]). Let \((X, \mathfrak{N})\) be a fuzzy normed linear space. If we define
\[
\mathcal{I} = \{ G \subseteq U : \xi \in G \iff \exists \tau > 0 \text{ and } 0 < r < 1 \text{ such that } B(\xi - \xi, \tau) \subseteq G \},
\]
then \( \mathcal{I} \) is a topology on \((X, \mathfrak{N})\).

Considering a fuzzy normed space’s topological structure \((X, \mathfrak{N})\). For \( \xi \in X, \varepsilon > 0 \) and \( \omega \in (0, 1) \), the \((\varepsilon, \omega)\)-neighborhood of the set
\[
U_\xi(\varepsilon, \omega) = \{ \xi \in X : \mathfrak{N}(\xi - \xi, \tau) > 1 - \omega \}.
\]
The \((\varepsilon, \omega)\)-neighborhood system at \( x \) is the collection
\[
U_x = \{ U_\xi(\varepsilon, \omega) : \varepsilon > 0, \omega \in (0, 1) \},
\]
and the \((\varepsilon, \omega)\)-neighborhood system for \( X \) is the union \( U = \bigcup_{\xi \in X} U_\xi \). It is clear that \( U \) is the first countable and determines a Hausdorff topology for \( X \).

**Definition 9** ([19]). A fuzzy normed space is defined as \((X, \mathfrak{N})\). If there is an integer \( n_0 \) such that \( \mathfrak{N}(k_n - \xi, \varepsilon) > 1 - \omega \) for every \( n \geq n_0 \), then a sequence \( \kappa_n \) in \( X \) is said to converge to \( \xi \) in \( X \).

**Definition 10** ([1]). If the set \( Q(\varepsilon) = \{ k \in \mathbb{N} : |k_n - \xi| \geq \varepsilon \} \) belongs to \( I \) for each \( \varepsilon > 0 \), the real sequence \( \kappa_n \) is said to be \( I \)-convergent to \( \xi \) in \( \mathbb{R} \). In this instance, we use the notation \( I-\lim \kappa_n = \xi \). The sequence \( \kappa_n \)'s \( I \)-limit is designated as the integer \( \xi \).
Definition 11 ([8]). The natural density of a set $E$ of positive integers is defined by

$$\delta(E) = \frac{1}{n} \text{card}\{k \in E : k \leq n\},$$

where card $\{k \in E : k \leq n\}$ indicates that the set $E$’s maximum number of items is $n$. It is evident that we have $\delta(E) = 0$ for a finite collection $E$.

Definition 12 ([1]). If the collection $A(\varepsilon) = \{n \in \mathbb{N} : |\xi_n - \zeta| \geq \varepsilon\}$ has a natural density of $0$ for any $\varepsilon > 0$, then the real sequence $\xi_n$ is said to be statistically convergent to $\zeta \in \mathbb{R}$. In this instance, we use the notation $\text{st-lim} \xi_n = \zeta$.

Throughout this study, we assume that $\mathcal{I}$ is an admissible ideal and $\mathcal{X}$ is a fuzzy normed space.

3. $\mathcal{I}$-Convergence in Fuzzy Normed Linear Space

In this part, ideal convergence and ideal Cauchy are defined in terms of the fuzzy normed space $(\mathcal{X}, \mathcal{N})$, along with several important findings. The ideal limit point and ideal cluster point of a real sequence in fuzzy normed linear space are other concepts that we introduce.

Definition 13. According to the fuzzy norm on $\mathcal{N}$, a sequence $\{\xi_n\}$ in $\mathcal{X}$ is said to be $\mathcal{I}$-convergent to $\zeta \in \mathcal{X}$ if the set $\Gamma(\varepsilon) = \{n \in \mathbb{N} : \mathcal{N}(\xi_n - \zeta, \varepsilon) < 1 - \omega\}$ belongs to $\mathcal{I}$ for each $\varepsilon > 0$ and $\omega \in (0, 1)$. In this instance, we write $\xi_n \xrightarrow{\mathcal{I}} \zeta$. The $\mathcal{I}$-limit of $\{\xi_n\}$ in $\mathcal{N}$ is the name given to the element $\zeta$.

Remark 1. In terms of neighborhoods, we have $\xi_n \xrightarrow{\mathcal{I}} \zeta$, provided that for each $\varepsilon > 0$ and $\omega \in (0, 1)$,

$$\{n \in \mathbb{N} : \xi_n \notin \mathcal{U}_\omega(\varepsilon, \omega)\} \in \mathcal{I}.$$

Example 1. (1) If we take $\mathcal{I} = \mathcal{I}_{\text{fin}} = \{K \subset \mathbb{N} : K$ is finite subset$\}$, then $\mathcal{I}_{\text{fin}}$ is a non-trivial admissible ideal of $\mathbb{N}$, and the accompanying convergence is ordinarily converging with respect to the fuzzy norm on $\mathcal{X}$.

(2) If we take $\mathcal{I} = \mathcal{I}_\delta = \{Q \subset \mathbb{N} : \delta(Q) = 0\}$. The accompanying convergence takes place concurrently with statistical convergence with respect to the fuzzy norm on $\mathcal{X}$, making $\mathcal{I}_\delta$ a non-trivial admissible ideal of $\mathbb{N}$.

Lemma 3. Let $(\mathcal{X}, \|\cdot\|)$ be a real norm and $(\mathcal{X}, \mathcal{N}, T)$ be a setting where the fuzzy norm creates its own fuzzy norms $\mathcal{N}(\xi, \tau) = \frac{\tau}{\tau + \|\xi\|}$, where $\xi \in \mathcal{X}$ and $\tau > 0$. Then for every sequence $\{\xi_n\}$

$$\lim_{n \to \infty} \|\xi_n - \xi\| = 0 \Rightarrow \mathcal{I} - \lim(\xi_n - \xi) = 0.$$

Proof. Suppose that $\lim_{n \to \infty} \|\xi_n - \xi\| = 0$. Then, for every $\tau > 0$ and for every $\tau > 0$, there exists a positive integer $N = \mathcal{N}(\tau)$ such that

$$\|\xi_n - \xi\| < \tau \text{ for each } n \geq N.$$

We observe that for any specific $\varepsilon > 0$,

$$\frac{\varepsilon + \|\xi_n - \xi\|}{\varepsilon} < \frac{\varepsilon + \tau}{\varepsilon}$$

it is comparable to

$$\frac{\varepsilon}{\varepsilon + \|\xi_n - \xi\|} > \frac{\varepsilon}{\varepsilon + \tau} = 1 - \frac{\tau}{\varepsilon + \tau}.$$
Whence, by letting \( \omega = \frac{\tau}{\tau + \epsilon} \), we have
\[
\mathfrak{N}(\kappa_n - \zeta, \epsilon) > 1 - \omega \quad \text{for each } n \geq N.
\]
This suggests that \( \kappa_n \in U_\mathfrak{I}(\epsilon, \omega) \) for each \( n \geq N \).

**Definition 14.** A sequence \( \{\kappa_n\} \) in \( \mathfrak{X} \) is said to be ideal Cauchy (abbreviation \( \mathcal{I}\)-Cauchy) with respect to the fuzzy norm on \( \mathfrak{X} \) if for each \( \epsilon > 0 \) and \( \omega \in (0, 1) \) there exists an integer \( k = k(\epsilon, \omega) \) in \( \mathbb{N} \) such that the set \( \{n \in \mathbb{N} : \mathfrak{N}(\kappa_n - \kappa_k, \epsilon) < 1 - \omega\} \) belongs to \( \mathcal{I} \).

For information on \( \mathcal{I}\)-Cauchy sequences, (see [23, 24]).

**Lemma 4** ([8]). Let \( \mathcal{I} \subset 2^{\mathbb{N}} \) be an admissible ideal with the property (AP) and let \((\mathfrak{X}, \varrho)\) be an ordinary metric space. Then, \( \mathcal{I}\)-\( \lim_{n \to \infty} \kappa_n = \zeta \) if and only if there exists a set \( P \in \mathcal{C}(\mathcal{I}) \), \( P = \{p_1 < p_2 < \cdots < p_n < \cdots\} \) such that \( \lim_{n \to \infty} \kappa_{p_n} = \zeta \).

**Lemma 5** ([8]). For each \( j, \Gamma_j \in \mathcal{C}(\mathcal{I}) \), where \( \mathcal{I} \) is an admissible ideal with the property (AP), let \( \{\Gamma_j\}_{j=1}^{\infty} \) be a countable collection of subsets of \( \mathbb{N} \). The set \( \Gamma \subseteq \mathbb{N} \) is then such that \( \Gamma \in \mathcal{C}(\mathcal{I}) \) and the set \( \Gamma \setminus \Gamma_j \) are both finite for all \( j \).

**Lemma 6.** If a sequence \( \{\kappa_n\} \) is \( \mathcal{I}\)-convergent in \( \mathfrak{X} \) with respect to the fuzzy norm, then \( \mathcal{I}\)-limit is unique.

**Proof.** For now, let us assume \( \kappa_n \xrightarrow{\mathcal{I}} \eta_1 \) and \( \kappa_n \xrightarrow{\mathcal{I}} \eta_2 \), where \( \eta_1 \neq \eta_2 \), select \( \epsilon > 0, \omega \in (0, 1) \) such that \( U_{\eta_1}(\epsilon, \omega) \) and \( U_{\eta_2}(\epsilon, \omega) \) are disjoint neighborhoods of \( \eta_1 \) and \( \eta_2 \). Since \( \eta_1 \neq \eta_2 \) are both \( \mathcal{I}\)-limit of the sequence \( \{\kappa_n\} \), we have
\[
\Gamma = \{n \in \mathbb{N} : \kappa_n \notin U_{\eta_1}(\epsilon, \omega)\}
\]
and
\[
\Lambda = \{n \in \mathbb{N} : \kappa_n \notin U_{\eta_2}(\epsilon, \omega)\}
\]
both belong to \( \mathcal{I} \). This allows us to infer that the sets
\[
\Gamma^c = \{n \in \mathbb{N} : \kappa_n \in U_{\eta_1}(\epsilon, \omega)\}
\]
and
\[
\Lambda^c = \{n \in \mathbb{N} : \kappa_n \in U_{\eta_2}(\epsilon, \omega)\}
\]
belong to \( \mathcal{C}(\mathcal{I}) \). This contradicts the notion that the neighborhoods \( U_{\eta_1}(\epsilon, \omega) \) and \( U_{\eta_2}(\epsilon, \omega) \) of \( \eta_1 \) and \( \eta_2 \) are disjoint. Hence, we have \( \eta_1 = \eta_2 \). \( \square \)

**Theorem 1.** Let \( \{\kappa_n\} \) and \( \{\gamma_n\} \) be sequences in \( \mathfrak{X} \) such that \( \kappa_n \xrightarrow{\mathcal{I}} \zeta \) and \( \gamma_n \xrightarrow{\mathcal{I}} \gamma \), where \( \zeta, \gamma \in \mathfrak{X} \). Then
(a) \( \kappa_n + \gamma_n \xrightarrow{\mathcal{I}} \zeta + \gamma \);
(b) \( c\kappa_n \xrightarrow{\mathcal{I}} c\zeta \) for \( c \in \mathbb{R} \).

**Proof.** (a) Suppose that \( \kappa_n \xrightarrow{\mathcal{I}} \zeta \) and \( \gamma_n \xrightarrow{\mathcal{I}} \gamma \). Given \( \epsilon > 0, \omega \in (0, 1) \). Choose \( \eta \in (0, 1) \) such that \( T(1 - \eta, 1 - \eta) > 1 - \omega \). Let
\[
\Gamma(\epsilon/2) = \{n \in \mathbb{N} : \kappa_n \notin U_\mathfrak{I}(\epsilon/2, \eta)\}
\]
\[
\Lambda(\epsilon/2) = \{n \in \mathbb{N} : \kappa_n \notin U_\mathfrak{I}(\epsilon/2, \eta)\}
\]
and
\[
\Omega(\epsilon) = \{n \in \mathbb{N} : \kappa_n + \gamma_n \notin U_{\mathfrak{I}+\gamma}(\epsilon, \omega)\}.
\]
By assumption, $\Gamma(\epsilon/2)$ and $\Lambda(\epsilon/2)$ belong to $\mathcal{I}$ and so, $\Gamma(\epsilon/2) \cup \Lambda(\epsilon/2) \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, it is sufficient to show that $\Omega(\epsilon) \subset \Gamma(\epsilon/2) \cup \Lambda(\epsilon/2)$. Let $n \notin (\Gamma(\epsilon/2) \cup \Lambda(\epsilon/2)$. Then $n \notin \Gamma(\epsilon/2)$ and $n \notin \Lambda(\epsilon/2)$ and so, $\kappa_n \in U_{\xi}(\epsilon/2, \eta)$ and this implies that $\Omega(\kappa_n - \xi, \epsilon/2) > 1 - \eta$. Similarly, $\gamma_n \in U_{\xi}(\epsilon/2, \eta)$ implies that $\Omega(\gamma_n - \xi, \epsilon/2) > 1 - \eta$. Consequently,

$$\Omega(\kappa_n + \gamma_n - (\xi + \gamma), \epsilon) \geq T(\Omega(\kappa_n - \xi, \epsilon/2), \Omega(\gamma_n - \xi, \epsilon/2) > 1 - \eta)$$

Hence, $\kappa_n + \gamma_n \in U_{\xi + \gamma}(\epsilon, \alpha)$ and so, $n \notin \Omega(\epsilon)$, i.e., $\Omega(\epsilon) \subseteq \Gamma(\epsilon/2) \cup \Lambda(\epsilon/2)$. Therefore, $\Omega(\epsilon) \in \mathcal{I}$ and hence, $\kappa_n + \gamma_n \xrightarrow{I} \xi + \gamma$.

(b) It is trivial for $c = 0$. Now let $c \neq 0, \epsilon > 0, \alpha \in (0, 1)$. Since $\kappa_n \xrightarrow{I} \xi$, we have

$$\Gamma = \{ n \in \mathbb{N} : \kappa_n \notin U_{\xi}(\epsilon, \alpha) \} \in \mathcal{I}.$$ 

This implies that

$$\Gamma^c = \{ n \in \mathbb{N} : \kappa_n \in U_{\xi}(\epsilon, \alpha) \} \in \mathcal{F}(\mathcal{I}).$$

Let $n \in \Gamma^c$. Then we have

$$\Omega(c\kappa_n - c\xi, \epsilon) = \Omega(\kappa_n - \xi, \frac{\epsilon}{|c|}) \geq T(\Omega(\kappa_n - \xi, \epsilon), \Omega(0, \frac{\epsilon}{|c|}))$$

$$> T(1 - \alpha, 1) = 1 - \alpha.$$ 

So $\{ n \in \mathbb{N} : c\kappa_n \notin U_{\xi}(\epsilon, \alpha) \} \in \mathcal{I}$. Hence $c\kappa_n \xrightarrow{I} c\xi$ for $c \in \mathbb{R}$. 

\textbf{Theorem 2.} If $\mathcal{I}$ possesses property (AP), then a sequence $\{ \kappa_k \}$ in $\mathcal{X}$ is an ideal convergent sequence in $\mathcal{X}$ if and only if there is a sequence $\{ \gamma_k \}$ such that $\{ k \in \mathbb{N} : \kappa_k \neq \gamma_k \} \in \mathcal{I}$.

\textbf{Proof.} Let us say $\kappa_k \xrightarrow{I} x$. For each $n \in \mathbb{N}$ and $\alpha \in (0, 1)$, let

$$\Gamma_n = \{ k \in \mathbb{N} : \Omega(\kappa_k - \xi, \frac{1}{n}) > 1 - \alpha \},$$

then $\Gamma_n \in \mathcal{F}(\mathcal{I})$ for each $n \in \mathbb{N}$.

Considering that the admissible ideal $\mathcal{I}$ has the property (AP), Lemma 5 provides us with there exists an integer $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $k \geq n_0$ and $k \in \Gamma$ implies $k \in \Gamma$ implies $\Omega(\kappa_k - \xi, \epsilon) > 1 - \alpha$.

Define a sequence $\{ \gamma_k \}$ in $\mathcal{X}$ as

$$\gamma_k = \begin{cases} \kappa_k, & \text{for each } k \in \Gamma; \\ \xi, & \text{for each } k \in \mathbb{N} \setminus \Gamma. \end{cases}$$

This shows that the sequence $\{ \gamma_k \}$ is convergent to $\xi$ with respect to the fuzzy norm on $\mathcal{X}$. Thus, we have $\{ k \in \mathbb{N} : \kappa_k \neq \gamma_k \} \in \mathcal{I}$. 

Next, suppose that \( \{k \in \mathbb{N} : \kappa_k \neq \gamma_k \} \in \mathcal{I} \) and \( \gamma_k \to \xi \). Let \( \varepsilon > 0, \omega \in (0, 1) \) be given. Then, for each \( n \), we can write
\[
\{k \leq n : \mathcal{N}(\kappa_k - \xi, \varepsilon) \leq 1 - \omega \} \subseteq \{k \leq n : \kappa_k \neq \gamma_k \} \cup \{k \leq n : \mathcal{N}(\gamma_k - \xi, \varepsilon) < 1 - \omega \}.
\] (1)

As the second set contains a constant number of integers, it too belongs to \( \mathcal{I} \), just like the first set on the right side of Equation (1). This implies that \( \{k \in \mathbb{N} : \mathcal{N}(\kappa_n - \xi, \varepsilon) \leq 1 - \omega \} \) belongs to \( \mathcal{I} \). Hence, the proof is obtained. \( \square \)

**Theorem 3.** Let \( \{\kappa_k\} \) be a sequence in \( \mathcal{X} \) and let \( \mathcal{I} \) be an admissible ideal with the property (AP). Hence, the following claims are equivalent:

(i) \( \kappa_k \xrightarrow{\mathcal{I}} \xi \);

(ii) There exist \( \{\kappa_k\} \) and \( \{\omega_k\} \) in \( X \) such that \( \kappa_k = \gamma_k + \omega_k \); \( \gamma_k \to \xi \) and \( \text{supp}(\omega_k) = \{k \in \mathbb{N} : \omega_k \neq 0\} \in \mathcal{I} \), where \( \theta \) is the zero element of the linear space \( \mathcal{X} \).

**Proof.** (i) \( \implies \) (ii). Suppose \( \kappa_k \xrightarrow{\mathcal{I}} \xi \). Thus, we can deduce that there is a set by using Lemma 4, \( H \in \mathcal{S}(\mathcal{I}) \), \( H = \{h_1 < h_2 < \cdots < h_k < \cdots \} \subseteq \mathbb{N} \) such that \( \kappa_{h_k} \to \xi \).

Define the sequence \( \{\gamma_k\} \) in \( \mathcal{X} \) as
\[
\gamma_k = \begin{cases}
\kappa_k, & \text{for each } k \in H; \\
\xi, & \text{for each } k \in \mathbb{N} \setminus H.
\end{cases}
\] (2)

This shows that \( \gamma_k \to \xi \). Further, we get \( \omega_k = \kappa_k - \gamma_k \) for each \( k \in \mathbb{N} \). Since \( \{k \in \mathbb{N} : \kappa_k \neq \gamma_k\} \in \mathcal{N} \setminus \mathcal{H} \in \mathcal{I} \). So, we have \( \{k \in \mathbb{N} : \omega_k \neq 0\} \in \mathcal{I} \). It follows that \( \text{supp}(\omega_k) \in \mathcal{I} \) and by Equation (2), we get \( \kappa_k = \gamma_k + \omega_k \).

(ii) \( \implies \) (i). Suppose that there exist two sequences \( \{\kappa_k\} \) and \( \{\omega_k\} \) in \( \mathcal{X} \) such that \( \kappa_k = \gamma_k + \omega_k \); \( \gamma_k \to \xi \) and \( \text{supp}(\omega_k) \in \mathcal{I} \). We proved that \( \kappa_k \xrightarrow{\mathcal{I}} \xi \). Let \( H = \{h \in H : \omega_h = 0\} \). Since \( \text{supp}(\omega_k) = \{h \in \mathbb{N} : \omega_h \neq 0\} \in \mathcal{I} \), we have \( H \in \mathcal{S}(\mathcal{I}) \); therefore, \( \kappa_k = \gamma_k \) if \( k \in H \).

Thus, we conclude that there exists a set \( H = \{h_1 < h_2 < \cdots < h_k < \cdots \} \subseteq \mathbb{N} \), \( H \in \mathcal{S}(\mathcal{I}) \), such that \( \kappa_{h_k} \to \xi \). By Lemma 4, it follows that \( \kappa_k \xrightarrow{\mathcal{I}} \xi \). This accomplishes the proof. \( \square \)

**Corollary 1.** Suppose that \( \mathcal{I} \) is an admissible ideal and assume that \( \{\kappa_k\} \) is a sequence in \( \mathcal{X} \). Then, \( \kappa_k \xrightarrow{\mathcal{I}} \xi \) if and only if only if \( \{\gamma_k\}, \{\omega_k\} \in \mathcal{X} \) such that \( \kappa_k = \gamma_k + \omega_k \); \( \gamma_k \to \xi \) and \( \omega_k \xrightarrow{\mathcal{I}} \theta \).

**Proof.** Let \( \omega_k = \kappa_k - \gamma_k \), where \( \{\gamma_k\} \) is the sequence defined by (2). Then, \( \gamma_k \to \xi \) and for the inclusion \( \mathcal{I}_{fin} \subseteq \mathcal{I} \), we conclude that \( \omega_k \xrightarrow{\mathcal{I}} \theta \).

Conversely, let \( \kappa_k = \gamma_k + \omega_k \), where \( \gamma_k \to \xi \). By the inclusion \( \mathcal{I}_{fin} \subseteq \mathcal{I} \), we get \( \kappa_k \to \xi \). This implies \( \kappa_k \xrightarrow{\mathcal{I}} \xi \). This accomplishes the proof. \( \square \)

**Remark 2.** “It is evident from the Theorem 3 argument that if (ii) is satisfied, the ideal \( \mathcal{I} \) need not possess the property (AP). In fact, let \( \kappa_k = \gamma_k + \omega_k, \gamma_k \to \xi \) and \( \text{supp}(\omega_k) \in \mathcal{I} \), where \( \mathcal{I} \) is an admissible ideal which does not have the property (AP). Since \( \mathcal{I}(\varepsilon) = \{k \in \mathbb{N} : \mathcal{N}(\omega_k - \theta, \varepsilon) \leq 1 - \omega \} \subseteq \{k \in \mathbb{N} : \omega_k \neq 0\} \in \mathcal{I} \). For each \( \varepsilon > 0, \omega \in (0, 1) \), we have \( \omega_k \xrightarrow{\mathcal{I}} \theta \). Then we have \( \kappa_k \xrightarrow{\mathcal{I}} \xi \).”

By Theorem 3 and Remark 2, we discover the subsequent theorem.

**Theorem 4.** Let \( c^\mathcal{I}_f(\mathcal{X}) \) be the set of all sequences which are \( \mathcal{I} \)-convergent to the zero element of the fuzzy norm \( \mathcal{X} \) and let \( \text{supp}\mathcal{I}(\mathcal{X}) \) be the set of all sequences \( \{\omega_k\} \in c^\mathcal{I}_f(\mathcal{X}) \) with \( \text{supp}(\omega_k) \in \mathcal{I} \). Then, \( c^\mathcal{I}_f(\mathcal{X}) \supset \text{supp}\mathcal{I}(\mathcal{X}) \) for each admissible ideal \( \mathcal{I} \).
Theorem 7. If a sequence \( \{ \kappa_n \} \) of elements in \( \mathcal{X} \) is said to be \( \mathcal{I} \)-Cauchy sequence in \( \mathcal{X} \) if for every \( \epsilon > 0 \) and \( \omega \in (0, 1) \), there exists \( m = m(\epsilon) \) such that
\[
\{ k \in \mathbb{N} : \kappa_k - \kappa_m \notin U_\theta(\epsilon, \omega) \} \in \mathcal{I}.
\]

Definition 15. A sequence \( \{ \kappa_n \} \) of elements in \( \mathcal{X} \) is called \( \mathcal{I}^* \)-Cauchy sequence in \( \mathcal{X} \) if for every \( \epsilon > 0 \) and \( \omega \in (0, 1) \), there exists \( \mathcal{I}^* \)-Cauchy sequence in \( \mathcal{X} \) such that
\[
\{ n_m : n_1 < n_2 < \cdots \} \subset \mathbb{N}
\]
where \( n \in \mathbb{N} \) is such that \( \epsilon_{n} \) for every \( \eta \).\( \quad \)

Definition 16. We say that a sequence \( \{ \kappa_n \} \) of elements in \( \mathcal{X} \) is said to be \( \mathcal{I}^* \)-Cauchy sequence in \( \mathcal{X} \) if for every \( \epsilon > 0 \) and \( \omega \in (0, 1) \), there exists a set
\[
\mathcal{W} = \{ n_m : n_1 < n_2 < \cdots \} \subset \mathbb{N}
\]
such that \( \mathcal{W} \in \mathcal{I}(\mathcal{I}) \) and \( \{ \kappa_{n_m} \} \) is an ordinary \( \mathcal{N} \)-Cauchy in \( \mathcal{X} \).

According to the following theorem, the notions of \( \mathcal{I}^* \)-Cauchy sequence and \( \mathcal{I} \)-Cauchy sequence coincide.

Theorem 5. If \( \{ \kappa_k \} \) is an \( \mathcal{I}^* \)-Cauchy sequence in \( \mathcal{X} \), then \( \{ \kappa_k \} \) is an \( \mathcal{I} \)-Cauchy sequence in \( \mathcal{X} \).

Proof. Let \( \{ \kappa_k \} \) be an \( \mathcal{I}^* \)-Cauchy sequence. Then, for every \( \epsilon > 0 \) and \( \omega \in (0, 1) \), there exists
\[
K = \{ k_m \in \mathbb{N} : k_1 < k_2 < \cdots \} \in \mathcal{I}(\mathcal{I})
\]
and a number \( N_0 \in \mathbb{N} \) such that
\[
\kappa_k - \kappa_p \notin U_\theta(\epsilon, \omega)
\]
for every \( m, p \geq N_0 \).

Let \( H = \mathbb{N} \setminus K \). It is obvious that \( H \in \mathcal{I} \) and
\[
\Gamma_\epsilon(\epsilon, \omega) = \{ k \in \mathbb{N} : k_k - k_p \notin U_\theta(\epsilon, \omega) \} \subset H \cup \{ k_1 < k_2 < \cdots < k_N \} \in \mathcal{I}.
\]
Therefore, for every \( \epsilon > 0 \) and \( \omega \in (0, 1) \), we can find \( p \in \mathbb{N} \) such that \( \Gamma_\epsilon(\epsilon, \omega) \in \mathcal{I} \), i.e., \( \{ \kappa_k \} \) is a \( \mathcal{I} \)-Cauchy sequence. \( \square \)

Theorem 6. If a sequence \( \kappa_n \) is an ideal convergent in \( \mathcal{X} \), then its ideal Cauchy is \( \mathcal{X} \).

Proof. Suppose that \( \kappa_k \overset{\mathcal{I}}{\to} \xi \). Let \( \epsilon > 0 \), \( \omega \in (0, 1) \) be given. Then we have
\[
\Gamma = \{ k \in \mathbb{N} : \kappa_k \notin U_\theta(\epsilon/2, \omega) \} \in \mathcal{I}.
\]
This implies that
\[
\Gamma^c = \{ k \in \mathbb{N} : k_k \in U_\theta(\epsilon/2, \omega) \} \in \mathcal{I}(\mathcal{I}).
\]
Choose \( \eta \in (0, 1) \) such that \( T(1 - \eta, 1 - \eta) > 1 - \omega \). Then, for every \( k, m \in \Gamma^c \),
\[
\Theta(k_k - k_m, \epsilon) \geq \Theta(\Theta(k_k - \xi, \epsilon/2), \Theta(k_m - \xi, \epsilon/2)) > T(1 - \eta, 1 - \eta) > 1 - \omega.
\]
Hence, \( \{ k \in \mathbb{N} : \kappa_k - k_m \notin U_\theta(\epsilon, \omega) \} \in \mathcal{I}(\mathcal{I}) \). This implies that
\[
\{ k \in \mathbb{N} : \kappa_k - k_m \notin U_\theta(\epsilon, \omega) \} \in \mathcal{I},
\]
that is, \( \{ \kappa_k \} \) is an ideal Cauchy sequence. \( \square \)

Theorem 7. Let \( \{ \kappa_k \} \) be a sequence in \( \mathcal{X} \) and denote \( \Gamma_n(\epsilon) = \{ k \in \mathbb{N} : \Theta(k_k - k_n, \epsilon) \leq 1 - \omega \} \), where \( n \in \mathbb{N} \). If \( \{ \kappa_k \} \) is an \( \mathcal{I} \)-Cauchy sequence, then for every \( \epsilon > 0 \), \( \omega \in (0, 1) \), there exists \( \Lambda \subset \mathbb{N} \) with \( \Lambda \in \mathcal{I} \) such that \( \Theta(k_k, \epsilon) > 1 - \omega \) for all \( k \neq \Lambda \).
Proof. Let $\varepsilon > 0, \omega \in (0, 1)$ be given and set $\Lambda = \Gamma_n(\varepsilon/2)$ and choose $\eta \in (0, 1)$ such that $T(1 - \eta, 1 - \eta) > 1 - \omega$. Since $\{\kappa_k\}$ is an $I$-Cauchy sequence, we have $\Lambda \in I$ and for all $l, k \notin \Lambda$, we get

$$\mathfrak{N}(\kappa_k - \kappa_n, \varepsilon/2) > 1 - \eta \quad \text{and} \quad \mathfrak{N}(\kappa_l - \kappa_n, \varepsilon/2) > 1 - \eta.$$  

Thus, we have $\mathfrak{N}(\kappa_k - \kappa_l, \varepsilon) > 1 - \omega$, for all $l, k \notin \Lambda$. This accomplishes the proof. □

Definition 17. A sequence $\{\kappa_k\}$ in $X$ is said to be $I^*$-convergent to $\xi \in X$ with respect to the fuzzy norm if there exists a set $H \in \mathfrak{F}(I)$, $H = \{h_1 < h_2 < \cdots < h_k < \cdots\} \subset \mathbb{N}$ such that $I - \lim_{k \to \infty} \kappa_h = \xi$.

Theorem 8. If a sequence $\{\kappa_k\}$ is $I^*$-convergent in $X$, then it is an $I$-Cauchy sequence in $X$.

Proof. Considering that there exists a set

$$H = \{h_m : j_1 < j_2 < \cdots < j_k < \cdots\} \subset \mathbb{N}$$

such that $H \in \mathfrak{F}(I)$ and $I - \lim_{m \to \infty} \kappa_m = \xi$, i.e., there exists $N \in \mathbb{N}$ such that $\kappa_m \in U_{\xi}(\varepsilon, \omega)$ for every $\varepsilon > 0$ and $\omega \in (0, 1)$ and $m > N$. Choose $\eta \in (0, 1)$ such that $T(1 - \eta, 1 - \eta) > 1 - \omega$.

$$\mathfrak{N}(\kappa_m \in U_{\xi}(\varepsilon, \omega)) \geq T(\mathfrak{N}(\kappa_m - \xi, \varepsilon/2), \mathfrak{N}(\kappa_m - \xi, \varepsilon/2)) \geq T(1 - \eta, 1 - \eta) > 1 - \omega$$

for every $\varepsilon > 0, \omega \in (0, 1)$, and $m, p > N$, we have $\kappa_n \in \kappa_p \notin U_{\xi}(\varepsilon, \omega)$ for every $m, p > N$, i.e., $\{\kappa_k\}$ in $X$ is an $I^*$-Cauchy sequence in $X$. Then, by Theorem 5, $\{\kappa_k\}$ is an $I$-Cauchy sequence in $X$. □

Theorem 9. If the sequence $\{\kappa_k\}$ is $I^*$-convergent to $\xi$, then $\{\kappa_k\}$ is $I$-convergent to $\xi$, i.e., $\kappa_k \overset{I^*}{\to} \xi \Rightarrow \kappa_k \overset{I}{\to} \xi$.

Proof. Letting $\kappa_k \overset{I^*}{\to} \xi$. Hence, by definition, there exists

$$H = \{h_1 < h_2 < \cdots < h_k < \cdots\} \subset \mathfrak{F}(I)$$

such that $I - \lim_{k \to \infty} \kappa_h = \xi$. Let $\varepsilon > 0$ and $\omega \in (0, 1)$ be given. Since $I - \lim_{k \to \infty} \kappa_h = \xi$, there exists $N \in \mathbb{N}$ such that $\kappa_h \in U_{\xi}(\varepsilon, \omega)$ for every $k \geq N$. Since

$$\Gamma = \{h_k \in H : \kappa_h \notin U_{\xi}(\varepsilon, \omega)\}$$

is contained in

$$\Lambda = \{h_1, h_2, \cdots, h_{N-1}\}$$

and the ideal $I$ is admissible, we have $\Gamma \in I$. Hence,

$$\{k \in \mathbb{N} : \kappa_k \notin U_{\xi}(\varepsilon, \omega)\} \subset H \cup \Lambda \in I$$

for every $\varepsilon > 0$ and $\omega \in (0, 1)$. Whence, we draw the conclusion that $\kappa_k \overset{I}{\to} x$. □

The converse of Theorem 9 need not be true, as the following example demonstrates.

Example 2. Consider $X = \mathbb{R}$ with $||\xi|| = |\xi|$ and let $T(a, b) = ab$ for all $a, b \in [0, 1]$. For $x \in \mathbb{R}$ and $\tau > 0$, Consider

$$\mathfrak{N}(\xi, \tau) = \frac{\tau}{\tau + ||\xi||}.$$
Thus, \( \Delta \) is a fuzzy normed space. Have a look at the decomposition of \( \mathbb{N} \) as \( \mathbb{N} = \bigcup \Delta_i \)'s where \( i \geq m \) and \( \Delta_i \cap \Delta_m = \emptyset \) for \( i \neq m \). Let \( \mathcal{I} \) be the class of all subsets of \( \mathbb{N} \) which intersect at the most finite number of \( \Delta_i \)'s. Then, \( \mathcal{I} \) is an admissible ideal. We define a sequence \( \{\kappa_m\} \) as follows: \( \kappa_m = \frac{1}{m} \in \mathbb{R} \) if \( m \in \Delta_i \). Then, we have

\[
\mathcal{N}(\kappa_m, t) = \frac{t}{t + ||\kappa_m||} \to 1
\]

as \( m \to \infty \). Hence, \( \mathcal{I} \)-lim \( \kappa_m = 0 \).

Now, we show that \( \mathcal{I}^* \)-lim \( \kappa_m \neq 0 \). Suppose that \( \mathcal{I}^* \)-lim \( \kappa_m = 0 \). Then, by definition, there exists a subset

\[
K = \{m_j : m_1 < m_2 < \cdots \} \subset \mathbb{N}
\]

such that \( H = \mathcal{S}(\mathcal{I}) \) and \( \mathcal{I} \)-lim \( \kappa_m = 0 \). Since \( K \in \mathcal{S}(\mathcal{I}) \), there exists \( H \in \mathcal{I} \) such that \( K = \mathbb{N} \setminus H \). Then, there exists positive integer \( g \) such that

\[
H \subset \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \Delta_{n_m}.
\]

Thus, \( \Delta_{n_{g+1}} \subset K \) and so \( \kappa_{n_j} = \frac{1}{g+1} > 0 \) for infinitely many values \( n_j \)'s in \( K \). Thus, it contradicts the assumption that \( \mathcal{I} \)-lim \( \kappa_{m_j} = 0 \). Hence, \( \mathcal{I}^* \)-lim \( \kappa_m \neq 0 \). Therefore, the converse of Theorem 9 need not be true.

If the ideal \( \mathcal{I} \) meets the requirement (AP), the following theorem demonstrates that the converse is true.

**Theorem 10.** If the admissible ideal \( \mathcal{I} \) satisfies the condition (AP), then the sequence \( \{\kappa_k\} \) in \( X \) such that \( \kappa_k \xrightarrow{\mathcal{I}} \xi \) implies \( \kappa_k \xrightarrow{\mathcal{I}^*} \xi \).

**Proof.** Since \( \kappa_k \xrightarrow{\mathcal{I}} \xi \), so for every \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), the set \( \{k \in \mathbb{N} : \kappa_k \notin \mathcal{U}_\varepsilon(\varepsilon, \omega)\} \) is \( \mathcal{I} \).

We define the set \( \Gamma_p \) for \( p \in \mathbb{N} \) as

\[
\Gamma_p = \{k \in \mathbb{N} : 1 - \frac{1}{p} < \mathcal{N}(\kappa_k - \xi) < 1 - \frac{1}{p + 1}\}.
\]

Then, it is clear that \( \{\Gamma_1, \Gamma_2, \cdots\} \) is a countable family of sets that belong to \( \mathcal{I} \) that are mutually disjoint, and by the property (AP), there is a countable family of sets \( \{\Lambda_1, \Lambda_2, \cdots\} \in \mathcal{I} \) such that the symmetric difference \( \Gamma_i \Delta \Lambda_j \) is a finite set for each \( i \in \mathbb{N} \) and \( \Lambda = \bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{I} \).

Since \( \Lambda \in \mathcal{I} \), there is a set \( K \in \mathcal{S}(\mathcal{I}) \) such that \( K = \mathbb{N} \setminus \Lambda \). Now we prove that the subsequence \( \{\kappa_k\}_{k \in K} \) is convergent to \( \xi \) with respect to the fuzzy norm \( \mathcal{N} \). Let \( \eta \in (0, 1) \) and \( \varepsilon > 0 \). Choose a positive \( \nu \) such that \( \nu^{-1} < \eta \). Then

\[
\{k \in \mathbb{N} : \kappa_k \notin \mathcal{U}_\varepsilon(\varepsilon, \eta)\} \subset \{k \in \mathbb{N} : \kappa_k \notin \mathcal{U}_\varepsilon(\varepsilon, 1/\nu)\} \subset \bigcup_{i=1}^{\nu-1} \Gamma_i.
\]
Since $\Gamma_i \Delta \Lambda_i$ is a finite set for each $i = 1, 2, \cdots, \nu - 1$, there exists $k_0 \in \mathbb{N}$ such that
\[
\left( \bigcup_{i=1}^{\nu-1} \Lambda_i \right) \cap \{ k \in \mathbb{N} : k \geq k_0 \} = \left( \bigcup_{i=1}^{\nu-1} \Gamma_i \right) \cap \{ k \in \mathbb{N} : k \geq k_0 \}.
\]

If $k \geq k_0$ and $k \in K$, then $k \notin \bigcup_{i=1}^{\nu-1} \Lambda_i$ and $k \notin \bigcup_{i=1}^{\nu-1} \Gamma_i$. Hence, for every $k \geq k_0$ and $k \in K$ we have $\kappa_k \notin U_2(\epsilon, \eta)$. Since this holds for every $\epsilon > 0$ and $\eta \in (0, 1)$, so we have $\mathcal{I}^*\lim_{k} \kappa_k = x$. This accomplishes the proof.  \( \square \)

**Theorem 11.** Assume that $X$ has at least one accumulation point. If for every sequence $\{k_n\}$, $\mathcal{I}$-Cauchy condition implies $\mathcal{I}^*$-Cauchy condition then $\mathcal{I}$ possesses property (AP).

**Proof.** Assume that $X$ is accumulated up to $k_0$. Then, a sequence of distinct points in $X$ called $\{k_n\}_{n \in \mathbb{N}}$ occurs such that $\{k_n\}_{n \in \mathbb{N}}$ converges to $k_0$ and $k_n \neq k_0$ for all $n \in \mathbb{N}$. Assume that the sequence of mutually exclusive non-empty sets from $\mathcal{I}$ is $\{\Gamma_j : j \in \mathbb{N}\}$. Define a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ by
\[
\gamma_n = \begin{cases} 
\kappa_j, & \text{if } n \in \Gamma_j; \\
\kappa_0, & \text{if } n \notin \Gamma_j,
\end{cases}
\]

For any $n \in \mathbb{N}$. Let $\epsilon > 0, \omega \in (0, 1)$. Choose $\eta \in (0, 1)$ such that $T(1 - \eta, 1 - \eta) > 1 - \omega$. Then, there exists $l \in \mathbb{N}$ such that $\mathcal{I}(\kappa_n - \kappa_0, \frac{\epsilon}{2}) > 1 - \eta$ for all $n \geq l$. Then, $\Gamma_j(\frac{\epsilon}{2}) = \{n \in \mathbb{N} : \mathcal{I}(\gamma_n - \kappa_0, \frac{\epsilon}{2}) \leq 1 - \eta \} \subset \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_j$ and $\Gamma_j \cup \Gamma_2 \cup \cdots \cup \Gamma_l \in \mathcal{I}$. Now, clearly, $i, j \notin \Gamma_j(\frac{\epsilon}{2})$ implies that $\mathcal{I}(\gamma_i - \kappa_0, \frac{\epsilon}{2}) > 1 - \eta$ and $\mathcal{I}(\gamma_j - \kappa_0, \frac{\epsilon}{2}) > 1 - \eta$. So,
\[
\mathcal{I}(\gamma_i - \gamma_j, \epsilon) \geq T(\mathcal{I}(\gamma_i - \kappa_0, \frac{\epsilon}{2}), \mathcal{I}(\gamma_j - \kappa_0, \frac{\epsilon}{2})) > T(1 - \eta, 1 - \eta) > 1 - \omega.
\]

By doing so, it can be seen that the sequence $\gamma_n$ is a $\mathcal{I}$-Cauchy sequence. Our assumption is that $\gamma_n$ is $\mathcal{I}^*$-Cauchy. Since $\Lambda = \mathbb{N} \setminus H \in \mathcal{F}(\mathcal{I})$ and $\{\gamma_n\}_{n \in \Lambda}$ are Cauchy, $H \subset \mathcal{I}$ exists. For $j \in \mathbb{N}$, let $\Lambda_j = \Gamma_j \cap H$. After that, each $\Lambda_j \in \mathcal{I}$. Additionally, $\bigcup \Lambda_j = H \cap \bigcup \Gamma_j \subset H$. Therefore, $\bigcup \Lambda_j \subset \mathcal{I}$. By the sets $\Gamma_j \cap \Lambda, i \in \mathbb{N}$, in the following three situations could occur:

Case (1) Each $\Gamma_j \cap \Lambda$ is included in a finite subset of $\mathbb{N}$.
Case (2) In a finite subset of $\mathbb{N}$, only one of the $\Gamma_j \cap \Lambda$'s, let us say $\Gamma_k \cap \Lambda$, is excluded.
Case (3) The finite subset of $\mathbb{N}$ does not include more than one $\Gamma_i \cap \Lambda$'s.

If scenario (1) occurs, then a finite subset of $\mathbb{N}$ includes the
\[
\Gamma_j \Delta \Lambda_j = \Gamma_j \setminus \Lambda_j = \Gamma_j \setminus H = \Gamma_j \cap \Lambda
\]
which indicates that $\mathcal{I}$ has the (AP) condition.

If case (2) occurs, we redefine $\Lambda_k = \Gamma_k$ and $\Lambda_j = \Gamma_j \cap H$ for $j \neq k$. Then,
\[
\bigcup_{j \in \mathbb{N}} \Lambda_j = \left[ H \cap \left( \bigcup_{j \neq k} \Gamma_j \right) \right] \subset H \cup \Gamma_k.
\]

Additionally, if $i > k$ and $\Gamma_k \cap \Lambda_k = \emptyset$, $\Gamma_k \Delta \Lambda_i = \Gamma_i \cap \Lambda$ holds. The prerequisites for the (AP) condition are thus satisfied, just as in Case (1).

If Case 3 occurs, then $k, l \in \mathbb{N}$ with $k \neq l$ exists such that $\Gamma_k \cap \Lambda$ and $\Gamma_l \cap \Lambda$ are not included in any finite subset of $\mathbb{N}$. Allow $\mathcal{I}(\kappa_l - \kappa_j, 2\epsilon_0) = 1 - \omega$. Given that $\{\gamma_n\}_{n \in \Lambda}$ is a Cauchy sequence, $k_0 \in \mathbb{N}$ exists such that $\mathcal{I}(\kappa_l - \kappa_j, \epsilon_0) > 1 - \omega$ for all $i, j \geq k_0$ and $i, j \in \Lambda$. We can choose $i \in \Gamma_k \cap \Lambda$ and $j \in \Gamma_l \cap \Lambda$ with $i, j \geq k_0$ since $\Gamma_k$ and $\Gamma_l \cap \Lambda$ are not included in any finite subset of $\mathbb{N}$. However, since $\gamma_i = \kappa_k$ and $\gamma_j = \kappa_l$, 

\( \mathfrak{M}(\gamma_i - \gamma_j, \varepsilon_0) = \mathfrak{M}(\kappa_i - \kappa_j, \varepsilon_0) < 1 - \omega \) (there are an unlimited number of \( \Lambda \) indices with that property). The Cauchy character of \( \{ \gamma_n \}_{n \in \Lambda} \) is violated by this. Therefore, Case (3) is ruled out. Additionally, \( \mathcal{I} \) satisfies the possession condition (AP) in light of Cases (1) and (2). \( \square \)

The following theorem is implied by the following Theorem 3.

**Theorem 12.** Let \( \{ \kappa_i \} \) be a sequence in \( \mathcal{X} \). If there exist two sequences \( \{ \gamma_k \} \) and \( \{ \omega_k \} \) in \( \mathcal{X} \) such that \( \kappa_k = \gamma_k + \omega_k; \gamma_k \to \xi \) and \( \text{supp}(\omega_k) = \{ k \in \mathbb{N} : \omega_k \neq \emptyset \} \in \mathcal{I} \), then \( \kappa_k \xrightarrow{\mathcal{I}} \xi \).

**Definition 18.** Let \( \{ \kappa_k \} \) be a sequence in \( \mathcal{X} \). An element \( \rho \in \mathcal{X} \) is said to be an \( \mathcal{I} \)-limit point of \( \{ \kappa_k \} \) provided that for each \( \varepsilon > 0 \) there is a set \( H = \{ h_1 < h_2 < \cdots < h_n < \cdots \} \subset \mathbb{N} \) such that \( H \notin I \) and \( \lim_{n \to \infty} \mathfrak{M}(\kappa_{h_n} - \xi, \varepsilon) = 1 \).

**Definition 19.** Let \( \{ \kappa_k \} \) be a sequence in \( \mathcal{X} \). An element \( \nu \in \mathcal{X} \) is said to be an \( \mathcal{I} \)-cluster point of \( \{ \kappa_k \} \) provided that for each \( \varepsilon > 0 \) and \( \omega \in (0, 1), \) the set \( \{ k \in \mathbb{N} : \mathfrak{N}(\kappa_k - \nu, \varepsilon) > 1 - \omega \} \notin I. \) We denote \( \mathcal{L}_{FS}^I(k) \) and \( \mathcal{C}_{FS}^I(k) \) the set of all \( \mathcal{I} \)-limit points and \( \mathcal{I} \)-cluster points of a sequence \( \{ \kappa_k \} \) in \( (\mathcal{X}, \mathfrak{M}) \).
For every net \( \{ x_n \} \in X \), these concepts can be combined to better define compactness. We introduce the notion of a filter based on \( X \) and in that case, \( \{ x_n \} \) converges in \( X \) to \( x \). Moreover, if \( \{ x_n \} \) is a net in \( D \), then \( \{ x_n \} \) forms a filter in \( X \). Let \( I \) be a non-trivial ideal of \( \mathcal{F}_N(k) \). Then, \( \{ x_n \} \) is closed in \( X \) if and only if \( \exists \eta > 0 \) such that \( \mathcal{U}_\eta(x, \omega) \subseteq \mathcal{U}_\omega(x, \omega) \). Obviously, we have

\[
\{ k \in \mathbb{N} : \mathcal{U}_\xi(x, \omega) \subseteq \mathcal{U}_\omega(x, \omega) \} \subseteq \{ k \in \mathbb{N} : \mathcal{U}_\eta(x, \omega) \supseteq \mathcal{U}_\omega(x, \omega) \}.
\]

This indicates that \( \{ k \in \mathbb{N} : \mathcal{U}_\eta(x, \omega) \supseteq \mathcal{U}_\omega(x, \omega) \} \neq \emptyset \). Therefore, \( w \in \mathcal{F}_N(k) \). Hence, \( \mathcal{F}_N(k) \) is closed in \( X \). □

4. \( \mathcal{I} \)-Convergence and \( \mathcal{I}^* \)-Convergence of Nets in Normed Fuzzy Space

In this section, some important topological properties are investigated using the notion of ideal and its adjoint convergence of sequences and nets in a normed fuzzy space. These concepts can be combined to better define compactness. We introduce the notion of ideal sequentially compactness and derive some basic topological space characteristics.

The first two of the following definitions are well-known.

**Definition 20** ([25]). Let \( \omega \geq \) be a binary relation on a non-void set \( D \) such that the binary relation is reflexive, antisymmetric, and transitive, and \( \exists p \in D \) such that \( p \geq m \) and \( p \geq n \) for any two elements \( m, n \in D \). Then, we call \( (D, \geq) \) a directed set.

**Definition 21** ([25]). Let \( X \) be a nonempty set and \( (D, \geq) \) be a directed set. A mapping \( \vartheta \) from \( D \) into \( X \) is said to be a net in \( X \), indicated by \( \{ \vartheta_n : n \in D \} \) or just \( \{ \vartheta_n \} \) when the set \( D \) is obvious from the context.

“For \( n \in D \) let \( D_n = \{ k \in D : k \geq n \} \). Then, the collection

\[
\mathcal{F}_0 = \{ \Gamma \subseteq D : \Gamma \supseteq D_n, \text{ for some } n \in D \}
\]

forms a filter in \( D \). Let \( \mathcal{I}_0 = \{ \Lambda \subseteq D : D \setminus \Lambda \not\in \mathcal{F}_0 \} \). Then, \( \mathcal{I}_0 \) is also a non-trivial ideal in \( D \).”

**Definition 22** ([26]). A \( D \)-admissible is a non-trivial ideal \( \mathcal{I} \) of \( D \) for which \( D_n \in \mathcal{F}_0(\mathcal{I}) \) for all \( n \in D \).

**Definition 23.** A net \( \{ \vartheta_n : n \in D \} \) in \( X \) is said to be ideal convergent to \( \xi \in X \) provided for each \( \epsilon > 0 \) and \( \omega \in (0, 1) \),

\[
\{ n \in D : \vartheta_n \notin \mathcal{U}_\epsilon(x, \omega) \} \in \mathcal{I}.
\]

Symbolically we write \( \mathcal{I} \)-lim \( \vartheta_n = \xi \) and we say that \( \xi \) is an \( \mathcal{I} \)-limit of the net \( \{ \vartheta_n \} \).

**Remark 3.** “If \( \mathcal{I} \) is \( D \)-admissible, a net will converge in \( X \), whereas the opposite is true if \( \mathcal{I} = \mathcal{I}_0 \). Moreover, if \( D = \mathbb{N} \) with the natural ordering, the words \( D \)-admissibility and admissibility overlap, and in that case, \( \mathcal{I}_0 \) is the ideal of all finite subsets of \( \mathbb{N} \).”

The following gives the definition of an \( \mathcal{I} \)-cluster point of a net with \( \{ \vartheta_n : n \in D \} \) in \( X \).

**Definition 24.** An element \( \xi \in X \) is called an \( \mathcal{I} \)-cluster point of a net \( \{ \vartheta_n : n \in D \} \) if for each \( \epsilon > 0 \) and \( \omega \in (0, 1) \), \( \{ n \in D : \vartheta_n \notin \mathcal{U}_\epsilon(x, \omega) \} \notin \mathcal{I} \).

**Theorem 13.** For every net \( \{ \vartheta_n : n \in D \} \) in \( X \), there is a filter \( \mathcal{F}_0 \) on \( X \) such that \( \xi \) is an \( \mathcal{I} \)-limit of the net \( \{ \vartheta_n : n \in D \} \) if and only if \( \xi \) is the limit of the filter \( \mathcal{F}_0 \) and \( \xi \) is an \( \mathcal{I} \)-cluster point of the net \( \{ \vartheta_n : n \in D \} \) if and only if \( \xi \) is the cluster point of the filter \( \mathcal{F}_0 \).

**Proof.** Let \( X \) be the space, and \( \{ \vartheta_n : n \in D \} \). Let \( \mathcal{F}_0(\mathcal{I}) \) be the associated filter on \( D \) and \( \mathcal{I} \) be a non-trivial ideal of \( D \). Let us create the set \( \Gamma_M = \{ \vartheta_n : n \in M \} \) for each \( M \in \mathcal{F}_0(\mathcal{I}) \). Then, a filter based on \( X \) is formed by the family \( \Lambda = \{ \Gamma_M : M \in \mathcal{F}_0(\mathcal{I}) \} \). In fact, each \( \Gamma_M \) is not
Theorem 14. In a compact space \( I \), if every sequence in a space \( V \) converges to \( \Gamma \) in \( I \), then \( \Gamma \) is a filter on \( I \) that is compact. Hence, our conclusion is correct. Let \( \mathcal{F} \) represent the filter that this filter base \( \Lambda \) produces. We now demonstrate that \( \mathcal{F} \) has the necessary property.

Let the net \( \{ \theta_n : n \in \mathcal{D} \} \) be \( I \)-convergent to \( \zeta \). Then, for any \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \) is non-empty. Hence, our conclusion is correct. Let \( \mathcal{F} \) represent the filter that this filter base \( \Lambda \) produces. We now demonstrate that \( \mathcal{F} \) has the necessary property.

Again, let the filter \( \mathcal{F} \) be convergent to \( \zeta \). Then, the neighbourhood filter \( \rho_\varepsilon \) of the point \( \zeta \) is a subfamily of \( \mathcal{F} \), i.e., \( \rho_\varepsilon \subset \mathcal{F} \). Let \( \mathcal{V}_\varepsilon(\omega, \omega) \in \rho_\varepsilon \) for \( \varepsilon \) and \( \omega \) be arbitrary. Then, \( \mathcal{F} \subset \mathcal{V}_\varepsilon(\omega, \omega) \) for some \( \mathcal{F} \in \mathcal{F}(I) \). This implies that \( \mathcal{F} \subset \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \), which further implies that \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \), i.e., \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \). This shows that the set \( \{ \theta_n : n \in \mathcal{D} \} \) is also \( I \)-convergent to \( \zeta \).

Now suppose that \( \zeta \) is an \( I \)-cluster point of the net \( \{ \theta_n : n \in \mathcal{D} \} \). Then, for any \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), we have \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \), i.e., \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \). Hence, we conclude that the set \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \). This implies that \( \mathcal{F} \subset \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \), i.e., \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \subset \mathcal{F}(I) \). This shows that the set \( \{ \theta_n : n \in \mathcal{D} \} \) is also \( I \)-convergent to \( \zeta \).

Next let \( \zeta \) be a cluster point of the filter \( \mathcal{F} \). Then, for any \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), we have \( \mathcal{V}_\varepsilon(\omega, \omega) \cap A_M \neq \emptyset \) for all \( M \in \mathcal{F}(I) \), i.e., \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \cap A_M \neq \emptyset \) for all \( M \in \mathcal{F}(I) \). We conclude that \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \neq \emptyset \). For, if \( \{ n \in \mathcal{D} : \theta_n \in \mathcal{V}_\varepsilon(\omega, \omega) \} \neq \emptyset \), then this would imply that \( \mathcal{V}_\varepsilon(\omega, \omega) \cap A_M \neq \emptyset \). Thus, \( \mathcal{F} \subset \mathcal{F}(I) \) so that \( \mathcal{F} \) becomes an \( I \)-cluster point of the filter \( \mathcal{F} \).

Theorem 14. In a compact space \( X \), each net \( \{ \theta_n : n \in \mathcal{D} \} \) has a \( I \)-cluster point that corresponds to every non-trivial ideal \( I \) of \( X \).

Proof. A net in \( X \) is defined as \( \{ \theta_n : n \in \mathcal{D} \} \), where \( X \) is a compact space. Assume that \( I \) is a non-trivial ideal of \( X \) and that \( \mathcal{F}(I) \) is the filter on \( I \) that is compact. Take into account the set \( \Gamma_S = \{ \theta_n : n \in S \} \) for each \( S \in \mathcal{F}(I) \). In light of the fact that \( F \) is a filter, the family that includes all such \( \Gamma_S \) has FIP. As a result, the family \( \Lambda = \{ \Gamma_S : S \in \mathcal{F}(I) \} \) is a family of closed sets that possesses FIP. Since \( X \) is a compact space, \( \bigcap \{ \Gamma_S : S \in \mathcal{F}(I) \} \neq \emptyset \). So there is some \( \kappa_0 \in X \) such that \( \kappa_0 \in \bigcap \{ \Gamma_S : S \in \mathcal{F}(I) \} \). Then, for every \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), we have \( \mathcal{V}_{\varepsilon}(\omega, \omega) \neq \emptyset \). Now, we consider the set \( K = \{ n \in \mathcal{D} : \theta_n \not\in \mathcal{V}_{\varepsilon}(\omega, \omega) \} \). If \( K \in \mathcal{F}(I) \), then the corresponding set \( \Gamma_K = \{ \theta_n : n \in K \} \) does not intersect \( \mathcal{V}_{\varepsilon}(\omega, \omega) \), i.e., \( \Gamma_K \cap \mathcal{V}_{\varepsilon}(\omega, \omega) = \emptyset \), which runs counter to the previously inferred fact. Hence, \( K \notin \mathcal{F}(I) \), which implies that \( \{ n \in \mathcal{D} : \theta_n \not\in \mathcal{V}_{\varepsilon}(\omega, \omega) \} \neq \emptyset \). Thus, \( \kappa_0 \) becomes an \( I \)-cluster point of the net \( \{ \theta_n : n \in \mathcal{D} \} \).

Definition 25. If every sequence in a space \( X \) has an \( I \)-cluster point, where \( I \) is a non-trivial ideal of the set \( \mathbb{N} \), then the space is said to be ideal sequentially compact.

“The notations of sequential compactness and ideal sequential compactness of a fuzzy normed space are distinct, as shown by the next two instances.”
Example 3. This example shows how a sequence in a space with fuzzy norms can have a cluster point without simultaneously having an \( I \)-cluster point, which corresponds to a non-trivial ideal \( I \) of \( \mathbb{N} \), the set of natural numbers. The set of all even positive integer subsets and the set of all odd positive integer finite subsets will provide \( I \), the non-trivial ideal of \( \mathbb{N} \).

Consider \( \mathcal{X} = \mathbb{R} \) with \( \| \xi \| = | \xi | \) and let \( T(a, b) = ab \) for all \( a, b \in [0, 1] \). For \( \xi \in \mathbb{R} \) and \( \tau > 0 \), consider

\[
\mathcal{N}(\xi, \tau) = \frac{\tau}{\tau + | \xi |}.
\]

Then, \((\mathbb{R}, \mathfrak{N}, T)\) is a fuzzy normed space. Let \( \{\kappa_n\} \) be a sequence in \( \mathcal{X} \) defined by

\[
\kappa_n = \begin{cases} 
0, & \text{if } n \text{ is even;} \\
 n + 1, & \text{if } n \text{ is odd.}
\end{cases}
\]

So, it is obvious that a convergent subsequence exists for \( \kappa_n \). Nevertheless, there is no \( I \)-cluster point in \( \{\kappa_n\} \).

Example 4. With the help of this example, we can see that there is a sequence in a fuzzy normed space that lacks a cluster point but has an \( I \)-cluster point corresponding to a non-trivial ideal of the set \( \mathbb{N} \). Make \( I \) a non-trivial ideal of \( \mathbb{N} \) that includes all the sets of even positive integers as subsets.

Consider \( \mathcal{X} = \mathbb{R} \) with \( \| \xi \| = | \xi | \) and let \( T(a, b) = ab \) for all \( a, b \in [0, 1] \). For \( \xi \in \mathbb{R} \) and \( \tau > 0 \), consider

\[
\mathcal{N}(\xi, \tau) = \frac{\tau}{\tau + | \xi |}.
\]

Then, \((\mathbb{R}, \mathfrak{N}, T)\) is a fuzzy normed space. Let \( \{\kappa_n\} \) be a sequence in \( \mathcal{X} \) defined by \( \kappa_n = n \) for all \( n \in \mathbb{N} \). Now, it is obvious that the sequence \( \{\kappa_n\} \) does not have a cluster point in \( \mathbb{R} \), but every odd positive integer turns into a \( I \)-cluster point for the sequence \( \{\kappa_n\} \).

“In the following, we show how, in some situations, the countable compactness and ideal sequential compactness of a fuzzy normed space relate to one another. Currently, we can recall the outcome”.

Lemma 7. For a fuzzy normed space \( \mathcal{X} \). The following are interchangeable.

(i) \( \mathcal{X} \) is countably compact.

(ii) There is a non-empty intersection for any countable set of closed subsets of \( \mathcal{X} \) satisfying the FIP.

(iii) If \( \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \mathfrak{F}_3 \supset \cdots \supset \mathfrak{F}_n \supset \cdots \) is a family of non-empty closed subsets that descends from \( \mathcal{X} \), then \( \bigcap_{n=1}^{\infty} \mathfrak{F}_n \neq \emptyset \).

Theorem 15. Let \((\mathcal{X}, \mathfrak{N}, T)\) be fuzzy normed space and let \( I \) be an admissible ideal. Then, \((\mathcal{X}, \mathfrak{N}, T)\) is a countably compact space if and only if \((\mathcal{X}, \mathfrak{N}, T)\) is sequentially compact in the ideal sense.

Proof. Assume that \((\mathcal{X}, \mathfrak{N}, T)\) is a space that is progressively compact from \( I \). Let \( \{\mathfrak{V}_n(\epsilon, \omega)\}_{n \in \mathbb{N}} \) be a countable open cover of \( \mathcal{X} \) that has no finite subcover for \( \epsilon > 0 \) and \( \omega \in (0, 1) \). Then, we can choose \( \kappa_n \in \mathcal{X} \setminus \bigcup_{j=1}^{n} \mathfrak{V}_j(\epsilon, \omega) \). The sequence \( \kappa_n \) must now contain an \( I \)-cluster point; let us say \( \kappa_0 \in \mathcal{X} \). For some \( r \in \mathbb{N} \), let \( \kappa_0 \in \mathfrak{V}_r(\epsilon, \omega) \). Accordingly, \( \{n \in \mathbb{N} : \kappa_n \in \mathfrak{V}_r(\epsilon, \omega)\} \notin I \). The set \( \Gamma = \{n \in \mathbb{N} : \kappa_n \in \mathfrak{V}_r(\epsilon, \omega)\} \) must be an infinite subset of \( \mathbb{N} \) since \( I \) is an admissible ideal of \( \mathbb{N} \). As a result, there exists some \( m > r \) such that \( \kappa_m \in \mathfrak{V}_r(\epsilon, \omega) \). However, according to our method of construction, \( \kappa_m \notin \mathfrak{V}_r(\epsilon, \omega) \), we arrive at a contradiction. As a result, \((\mathcal{X}, \mathfrak{N}, T)\) needs to be countably compact.

Assume that \((\mathcal{X}, \mathfrak{N}, T)\) is a countably compact space to demonstrate the opposite. We will first demonstrate that \((\mathcal{X}, \mathfrak{N}, T)\) is a first countable. For this, let \( \epsilon > 0, \xi \in \mathcal{X} \). We will show that \( U_{\xi} = \left\{ N_\epsilon \left( \frac{1}{n}, \frac{\epsilon}{n} \right) : n \in \mathbb{N} \right\} \) is a local basis for \( \xi \in \mathcal{X} \). Let \( \mathcal{V} \) be
open set and $\xi \in \mathcal{V}$. Since $\mathcal{V}$ is open, then there exists $\omega \in (0, 1)$ and $\varepsilon > 0$ such that $N_{\varepsilon}(\xi, \omega) \subset \mathcal{V}$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \omega$ and $\frac{\varepsilon}{n} < \varepsilon$. Now, we just need to show $N_{\varepsilon}(\frac{\varepsilon}{n} - \frac{1}{n}, \xi) \subset N_{\varepsilon}(\xi, \omega)$. Let $\nu \in N_{\varepsilon}(\xi, \omega)$. Then, $N(\frac{\varepsilon}{n}, \xi) > 1 - \frac{1}{n} > 1 - \omega$. Since $\frac{\varepsilon}{n} \leq \varepsilon$, we have $1 - \omega < N(\xi - \nu, \frac{\varepsilon}{n}) \leq \mathcal{V}(\xi - \nu, \varepsilon)$. Hence, $\nu \in N_{\varepsilon}(\xi, \omega)$ which implies that $N_{\varepsilon}(\frac{\varepsilon}{n} - \frac{1}{n}, \xi) \subset N_{\varepsilon}(\xi, \omega) \subset \mathcal{V}$. Consequently, $U_{\varepsilon}$ is a countable local basis for $\mathcal{X}$.

Therefore, $(\mathcal{X}, \mathcal{V}, T)$ is first countable.

Assume that $\mathcal{X}$ consists of a sequence of distinct points, $\{\xi_n\}_{n \in \mathbb{N}}$. Let us assume that for each positive integer $n$, $W_n = \{\xi_m : m \geq n\}$. Hence, by Lemma 7 is a descending sequence of non-empty closed sets, and we get $\bigcap_{n=1}^{\infty} W_n \neq \emptyset$. Let $\nu \in \bigcap_{n=1}^{\infty} W_n$. Since $(\mathcal{X}, \mathcal{V}, T)$ is a first countable space, suppose that for $\varepsilon > 0$ and $\omega \in (0, 1)$, $\{B_\varepsilon(\nu)\}_{n \in \mathbb{N}}$ is a countable local base at the point $\nu \in \mathcal{X}$, where $\Lambda_0(\nu) = N_\omega(\frac{1}{n} \xi_n) \subset N_\omega(\xi_n, \omega)$. Observe that $\Lambda_m \supset \Lambda_{m+1}$ for all $n \in \mathbb{N}$. Now, $\Lambda_m(\nu) \cap W_n \neq \emptyset$. So, there exists some $k_m \geq m$ such that $\xi_{k_m} \in \Lambda_m(\nu)$. Since $\Lambda_0(v) \cap W_1 \neq \emptyset$, we select an integer that is positive, $k_1$, such that $\xi_{k_1} \in \Lambda_0(v)$. Again, since $\Lambda_2(v) \cap W_{k_1} \neq \emptyset$, choose a positive integer $k_2 > k_1$ such that $\xi_{k_2} \in \Lambda_2(v)$. Suppose $k_1 < k_2 < \cdots < k_n$ have been chosen such that $\xi_{k_i} \in \Lambda_i(v)$ for $i = 1, 2, \ldots, n$. Again, since $\Lambda_{n+1}(v) \cap W_{k_n+1} \neq \emptyset$, there is some $k_{n+1} > k_n$ such that $\xi_{k_{n+1}} \in \Lambda_{n+1}(v)$. Consequently, we obtain a subsequence $\{\xi_{k_n}\}_{n=1}^{\infty}$ of the sequence $\{\xi_n\}$, which is such that $\xi_{k_n} \in \Lambda_n(v)$, for all $r \in \mathbb{N}$ for every $r \in \mathbb{N}$. We demonstrate the convergence of this subsequence to $\nu$. Given that $\mathcal{V}$ is an open subset of $\mathcal{X}$, let $\nu \in \mathcal{V}$. Hence, if $\Lambda_m(\nu) \subset \mathcal{V}$, there exists some positive integer $m$. So, we get $\xi_{k_n} \in \Lambda_m(\nu) \subset \Lambda_m(\nu) \subset \mathcal{V}$ for all $n > m$. Since $\mathcal{I}$ is an admissible, the sequence $\xi_{k_n}$ converges to $\nu$ under $\mathcal{I}$. This implies that we have $\{n \in \mathbb{N} : k_n \not\in U\} \in \mathcal{I}$ for any open set $U$ containing $\nu$. Since $\mathcal{I}$ is a non-trivial ideal, $\nu$ becomes an $\mathcal{I}$-cluster point of the sequence $\{\xi_{k_n}\}_{n=1}^{\infty}$ when $\{n \in \mathbb{N} : k_n \not\in U\} \not\in \mathcal{I}$. Now, that $\{n \in \mathbb{N} : k_n \not\in U\} \supset \{n \in \mathbb{N} : k_n \not\in U\} \not\in \mathcal{I}$. Hence, we have $\{n \in \mathbb{N} : k_n \not\in U\} \not\in \mathcal{I}$, which means that $\nu$ becomes an $\mathcal{I}$-cluster point of the sequence $\{\xi_n\}$. As a result, $(\mathcal{X}, \mathcal{V}, T)$ is an $\mathcal{I}$-sequentially compact space.

5. $\mathcal{I}$-Divergences and $\mathcal{I}^*$-Divergence

In their article [27], Macaj and Salat expanded the definition of divergent sequences of real numbers to include statistically divergent sequences of real numbers. Notwithstanding the eventual addition of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence to the definition of statistical convergence (as already indicated), no comparable approach has been taken in the case of divergence. Later, in their publication cited as [28], Das and Ghosal conceived and developed the notion of a sequence diverging in a metric space. That is what this paragraph aspires to achieve. As an alternative to limiting the idea of divergence to only real sequences (note that our definition includes the basic definition of real divergent sequences as a particular case), with the aid of ideals, we establish it in a fuzzy normed space and broaden it. According to our research, the condition (AP), exactly like it was in the cases of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy condition, is once more significant.

**Definition 26.** A sequence $\{\xi_n\}_{n \in \mathbb{N}}$ in a fuzzy normed space $(\mathcal{X}, \mathcal{V}, T)$ is said to be divergent (or properly divergent) if there exists an element $\xi \in \mathcal{X}$ such that $\mathcal{V}(\xi_n - \xi, \omega) < 1 - \omega$ for $\omega \in (0, 1)$ and $\varepsilon > 0$.

“It should be noted that in a fuzzy normed space, a divergent sequence cannot have any convergent subsequence”.

**Definition 27.** A sequence $\{\xi_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$ is said to be $\mathcal{I}$-divergent if there exists an element $\xi \in \mathcal{X}$ such that $\Gamma(\epsilon) = \{n \in \mathbb{N} : \mathcal{V}(\xi_n - \xi, \epsilon) \geq 1 - \omega\} \in \mathcal{I}$ for each $\epsilon > 0$ and $\omega \in (0, 1)$.
Definition 28. A sequence \( \{ \kappa_n \}_{n \in \mathbb{N}} \) in \( \mathbb{X} \) is said to be \( \mathcal{I}^* \)-divergent if there is at least one \( \xi \in \mathbb{X} \) such that \( \kappa_n \not\rightarrow \xi \) exists. This is defined as there being at least one \( H \in \mathcal{G} \) or \( \mathbb{N} \setminus H \in \mathcal{I} \).

Theorem 16. Let \( \{ \kappa_k \} \) be a sequence in \( \mathbb{X} \). If \( \{ \kappa_n \}_{n \in \mathbb{N}} \) is \( \mathcal{I}^* \)-divergent, then \( \{ \kappa_n \}_{n \in \mathbb{N}} \) is \( \mathcal{I} \)-divergent.

Proof. Since \( \{ \kappa_n \}_{n \in \mathbb{N}} \) is \( \mathcal{I}^* \)-divergent, so there exists \( H \in \mathcal{G}(M) \), i.e., \( \mathbb{N} \setminus H \in \mathcal{I} \) such that \( \{ \kappa_n \}_{n \in H} \) is divergent, i.e., there exists at least one \( \xi \in \mathbb{X} \) such that \( \kappa_n \not\rightarrow \xi \). Then, for any \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), there exists \( m \in \mathbb{N} \) such that \( \mathcal{G}(\kappa_n - \xi, \varepsilon) < 1 - \omega \) for all \( k \geq m \) and \( k \in H \). Hence, we have \( \{ n \in \mathbb{N} : \mathcal{G}(\kappa_n - \xi, \varepsilon) \geq 1 - \omega \} \subset (\mathbb{N} \setminus H) \cup \{ 1, 2, \ldots, m \} \in \mathcal{I} \). This implies that \( \{ \kappa_n \}_{n \in H} \) is \( \mathcal{I} \)-divergent. \( \square \)

"The opposite of the aforementioned theorem is not always true, as demonstrated by the example below".

Example 5. Consider \( \mathbb{X} = \mathbb{R} \) with \( ||\xi|| = |\xi| \) and let \( T(a, b) = ab \) for all \( a, b \in [0, 1] \). For \( \xi \in \mathbb{R} \) and \( \tau > 0 \), consider

\[
\mathcal{G}(\xi, \tau) = \frac{\tau}{\tau + ||\xi||}.
\]

Then, \( (\mathbb{R}, \mathcal{G}, T) \) is a fuzzy normed space. Let \( \mathbb{N} = \bigcup_{j \in \mathbb{N}} \Delta_j \) be a decomposition of \( \mathbb{N} \) such that each \( \Delta_j \) is infinite and \( \Delta_i \cap \Delta_j = \emptyset \) for \( i \neq j \). Let \( \mathcal{I} \) be the class of all subsets \( \Gamma \) of \( \mathbb{N} \) that can only intersect \( \Delta_j \)'s. If so, \( \mathcal{I} \) is a non-trivial ideal of \( \mathbb{N} \) that is admissible ideal of \( \mathbb{N} \). Let \( \gamma_i = n \) if \( i \in \Delta_n \). Now, for any \( \varepsilon > 0 \) and \( \omega \in (0, 1) \), there exists a natural number \( m \) such that the set \( \{ n \in \mathbb{N} : \mathcal{G}(\gamma_i - 0, \varepsilon) \leq 1 - \omega \} \subset \sum_{k=1}^{m} \Delta_k \subset \mathcal{I} \) and so \( \{ \gamma_i \}_{i \in H} \) is \( \mathcal{I} \)-divergent. Next, we will demonstrate that \( \{ \gamma_i \}_{i \in \mathcal{I}} \) is not \( \mathcal{I}^* \)-divergent. Assume that it is \( \mathcal{I}^* \)-divergent if at all possible. Consequently, an \( H \in \mathcal{G}(\mathcal{I}) \) such that \( \{ \gamma_i \}_{i \in H} \) is divergent. There is an \( l \in \mathbb{N} \) such that \( \mathbb{N} \setminus H \subset \sum_{k=1}^{l} \Delta_k \) since \( \mathbb{N} \setminus H \in \mathcal{I} \) exists. But after that, for any \( i > l \), \( \Delta_i \subset H \). More specifically, \( \Delta_{l+1} \subset H \). However, this means that the constant sequence \( \{ \gamma_i \}_{i \in H} \) is convergent to \( l + 1 \) and is a constant subsequence of \( \{ \gamma_i \}_{i \in H} \). The divergence of \( \{ \gamma_i \}_{i \in H} \) is contradicted by this.

Theorem 17. Let \( \{ \kappa_n \}_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{X} \). If \( \mathcal{I} \) is an admissible ideal with property (AP), then for any sequence \( \{ \kappa_n \}_{n \in \mathbb{N}} \) in \( \mathbb{X} \), \( \mathcal{I} \)-divergence implies \( \mathcal{I}^* \)-divergence.

Proof. Let us assume for a moment that \( \mathcal{I} \) has property (AP). Since \( \{ \kappa_n \}_{n \in \mathbb{N}} \) is \( \mathcal{I} \)-divergent, there must be at least one \( \xi \in \mathbb{X} \) such that \( \mathcal{I}(\varepsilon) = \{ n \in \mathbb{N} : \mathcal{G}(\kappa_n - \xi, \varepsilon) \geq 1 - \omega \} \subset \mathcal{I} \) for \( \varepsilon > 0 \) and \( \omega \in (0, 1) \). For \( q \in \mathbb{N} \), we define the set \( \Gamma_q \) as follows:

\[
\Gamma_q = \{ n \in \mathbb{N} : 1 - \frac{1}{q} < \mathcal{G}(\kappa_n - \xi, \varepsilon) < 1 - \frac{1}{q+1} \}.
\]

As a result, we have a collection of mutually disjoint sets \( \{ \Gamma_i \}_{i \in \mathbb{N}} \) with \( \Gamma_i \in \mathcal{I} \) for all \( i \in \mathbb{N} \). According to the property (AP), a family of sets \( \{ \Lambda_i \}_{i \in \mathbb{N}} \) exists where \( \Lambda = \bigcup_{i \in \mathbb{N}} \Lambda_i \in \mathcal{I} \) and \( \Gamma_i \Delta \Lambda_i \) is finite for all \( i \)'s. Let \( H = \mathbb{N} \setminus \Lambda \). Then, \( H \in \mathcal{G}(\mathcal{I}) \). Let \( \eta \in (0, 1) \) and \( \varepsilon > 0 \). Select a positive \( k \) such that \( \frac{1}{k} > \eta \). Then, \( \{ n \in \mathbb{N} : \mathcal{G}(\kappa_n - \xi, \varepsilon) \geq 1 - \eta \} \subset \bigcup_{i=1}^{k} \Gamma_i \). Since \( \Gamma_i \Delta \Lambda_i \) is finite, so there exists \( n_0 \in \mathbb{N} \) such that

\[
\left( \bigcup_{i=1}^{k} \Lambda_i \right) \cap \{ n \in \mathbb{N} : n \geq n_0 \} = \left( \bigcup_{i=1}^{k} \Gamma_i \right) \cap \{ n \in \mathbb{N} : n \geq n_0 \}.
\]
Obviously, if \( n \geq n_0 \) and \( n \in H \), then \( n \not\in \bigcup_{i=1}^{k} \Lambda_i \) implies that \( n \not\in \bigcup_{i=1}^{k} \Gamma_i \). Therefore, 
\[ \forall (\kappa_n - x, \varepsilon) < 1 - \frac{1}{k} < 1 - \eta. \] Thus, \( \{\kappa_n\}_{n \in H} \) is divergent. \( \square \)

**Theorem 18.** Let \( \mathcal{I} \) be an admissible ideal and \((X, \mathcal{F}, \mathcal{T})\) be a fuzzy normed space with at least one divergent sequence. If \( \mathcal{I} \)-divergence implies \( \mathcal{I}^* \)-divergence for any sequence \( \{\kappa_n\}_{n \in \mathbb{N}} \), then \( \mathcal{I} \) possesses property (AP).

**Proof.** Suppose that the divergent sequence \( \{\kappa_n\}_{n \in \mathbb{N}} \) in \( X \). The element \( \xi \in X \) is thus such that \( \kappa_n \rightharpoonup \xi \). Assume that \( \{\Gamma_i : i \in \mathbb{N}\} \) is a list of non-empty sets from \( \mathcal{I} \) that are mutually disjoint. Define a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) by

\[
\lambda_n = \begin{cases} 
\lambda_n = \kappa_j, & \text{if } n \in \Gamma_j; \\
\lambda_n = \kappa_n, & \text{if } n \not\in \Gamma_j.
\end{cases}
\]

for any \( j \in \mathbb{N} \). Let \( \eta \in (0,1) \) and \( \varepsilon > 0 \). Choose a positive \( k \in \mathbb{N} \) such that \( \forall (\kappa_n - \xi, \varepsilon) > 1 - \eta \) for all \( n \geq k \). Now, \( \Gamma(\varepsilon) = \{n \in \mathbb{N} : \forall (\kappa_n - \xi, \varepsilon) < 1 - \eta\} \subset \bigcup_{i=1}^{k} \Gamma_i \cup \{1, 2, \ldots, k\} \in \mathcal{I} \).

So, \( \{\lambda_n\}_{n \in \mathbb{N}} \) is \( \mathcal{I} \)-divergent. According to our presumption, \( \{\lambda_n\}_{n \in \mathbb{N}} \) is \( \mathcal{I}^* \)-divergent. As a result, there is a \( H \subset \mathbb{N} \) where \( H \not\in \mathcal{F}(\mathcal{I}) \) and \( \{\lambda_n\}_{n \in H} \) is divergent. Let's say \( \Lambda = \mathbb{N} \setminus H \).

Next, \( \Lambda \in \mathcal{I} \). Insert \( j \in \mathbb{N} \) for all \( \Lambda_j = \Gamma_j \cap \Lambda \). Indeed, \( \bigcup_{i \in \mathbb{N}} \Lambda_i \subset \Lambda \), \( \bigcup_{i \in \mathbb{N}} \Lambda_i \in \mathcal{I} \). We assert that a finite set, \( \Gamma_i \cap H \), exists. If not, \( H \) must contain a convergent subsequence of \( \{\lambda_n\}_{n \in H} \), which is an infinite series of objects \( \lambda_{n_k} = \kappa_j \) for every \( k \in \mathbb{N} \). This, however, contradicts the divergence of \( \{\lambda_n\}_{n \in H} \). \( \Gamma_i \triangle \Lambda_i = \Gamma_i \setminus \Lambda_i = \Gamma_i \setminus \Lambda \) is a finite subset of \( N \). This proves that \( \mathcal{I} \) satisfies the requirement (AP). \( \square \)

6. Conclusions

A substantial amount of recent research on the topic of fuzzy numbers and fuzzy normed spaces in relation to convergence has made use of a variety of methodologies. By focusing on the convergence and Cauchyness of sequences and nets in fuzzy normed spaces with respect to an ideal on \( \mathbb{N} \), this work verified a number of previous findings. Instead of restricting the concept of divergence to only real sequences, we widened it with the help of ideals and established it in a fuzzy normed space. An analogous investigation into the convergence of double sequences in fuzzy anti-2-normed (non-Archimedean) spaces, fuzzy 2-normed spaces, and related structures, in our opinion, would be intriguing. In [29] fuzzy relations for MISO and MIMO fuzzy systems are examined as algebraic formulations. To schedule tasks, manage manufacturing resources, monitor the execution of plans, and provide online decision support, production planning and scheduling systems are used. The same justification that underpins the use of diagnostic expert systems also justifies the use of fuzzy expert systems in the future.


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