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On the Nature of Bondi–Metzner–Sachs Transformations

Zahra Mirzaiyan [1] and Giampiero Esposito [1,2,*]

[1] INFN Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, 80126 Napoli, Italy; zahra.mirzaiyan@na.infn.it
[2] Dipartimento di Fisica "Ettore Pancini", Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, 80126 Napoli, Italy
* Correspondence: gesposito@na.infn.it

Abstract: This paper investigates, as a first step, the four branches of BMS transformations, motivated by the classification into elliptic, parabolic, hyperbolic and loxodromic proposed a few years ago in the literature. We first prove that to each normal elliptic transformation of the complex variable ζ used in the metric for cuts of null infinity, there is a corresponding BMS supertranslation. We then study the conformal factor in the BMS transformation of the u variable as a function of the squared modulus of ζ. In the loxodromic and hyperbolic cases, this conformal factor is either monotonically increasing or monotonically decreasing as a function of the real variable given by the modulus of ζ. The Killing vector field of the Bondi metric is also studied in correspondence with the four admissible families of BMS transformations. Eventually, all BMS transformations are re-expressed in the homogeneous coordinates suggested by projective geometry. It is then found that BMS transformations are the restriction to a pair of unit circles of a more general set of transformations. Within this broader framework, the geometry of such transformations is studied by means of its Segre manifold.

Keywords: Bondi–Metzner–Sachs group; fractional linear maps; supertranslations

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1. Introduction

The recent developments on the applications of the Bondi–Metzner–Sachs (hereafter BMS) group, i.e., the asymptotic symmetry group of an asymptotically flat space-time (see Equations (A1)–(A3) of Appendix A), have been motivated by black hole physics, quantum gravity and gauge theories, as is well described in many outstanding works (e.g., Refs. [1–15]. However, a purely classical investigation may still lead to a neater understanding of the mathematical operations frequently performed. Within this framework, at least four properties can be mentioned in our opinion:

(i) The proof by F. Alessio and one of us [16] that the BMS group is the right semidirect product of the proper orthocronous Lorentz group $SO^{+}(3,1)$ with supertranslations (cf. Appendix A).
(ii) The division of BMS transformations into parabolic, elliptic, hyperbolic and loxodromic, since the first half of them consists of fractional linear maps which can be classified by studying their fixed points [17].
(iii) The investigation of fractional linear maps in general relativity and quantum mechanics performed in Ref. [18].
(iv) The proof that the BMS group is not real analytic, and the related suggestion that it is not locally exponential [19].
(v) The recent discovery that groups of BMS type arise not only as macroscopic asymptotic symmetry groups in cosmology but describe also a fundamental microscopic symmetry of pseudo-Riemannian geometry [20].

In the following sections, we aim at presenting a detailed investigation of the four branches of the BMS group and of yet other properties. For this purpose, Section 2 defines our basic framework, Section 3 obtains a new theorem on supertranslations, Section 4 studies the conformal factor in the second half of BMS transformations, while Section 5 studies Killing vector fields of the Bondi metric and their behavior under BMS transformations, obtaining a novel classification. Eventually, BMS transformations in homogeneous coordinates are studied in Section 6, concluding remarks are presented in Section 7, while relevant details are provided in the Appendices A and B. The reader is referred to Refs. [21–23] for the basic concepts of causal and asymptotic structure of a space-time manifold.

2. Basic Framework

The cuts of null infinity are spacelike 2-surfaces orthogonal to the generators of null infinity. Lengths within a cut scale by a variable factor K under holomorphic bijections of the 2-sphere S^2 to itself (hereafter, we use the complex variable $\zeta = e^{i\phi}\cot\frac{\theta}{2}$, ϕ and θ being the standard coordinates for S^2):

$$\zeta' = f(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = f_\Lambda(\zeta), \tag{1}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is a fractional linear (or Möbius) map, and the matrix Λ can be always taken to belong to $\mathrm{SL}(2,\mathbb{C})$, because the ratio is unaffected by rescalings of a, b, c, d by the same factor, so that the passage from $\mathrm{GL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$ is eventually achieved. In particular, $f_\Lambda(\zeta) = f_{-\Lambda}(\zeta)$, because

$$\frac{(a\zeta + b)}{(c\zeta + d)} = \frac{(-a\zeta - b)}{(-c\zeta - d)}.$$

Thus, two matrices of $\mathrm{SL}(2,\mathbb{C})$ yield the same fractional linear map if and only if the one is the opposite of the other. At this stage, we are actually dealing with the projective version of the $\mathrm{SL}(2,\mathbb{C})$ group, i.e.,

$$\begin{aligned} \mathrm{PSL}(2,\mathbb{C}) &= \left\{ (f,\Lambda) | \, f : \zeta \in \mathbb{C} \to f(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)}, \; ad - bc = 1 \right\} \\ &= \mathrm{SL}(2,\mathbb{C})/\delta, \end{aligned} \tag{2}$$

where δ is the homeomorphism such that

$$\delta(a,b,c,d) = (-a,-b,-c,-d). \tag{3}$$

Our Equation (2) for the definition of $\mathrm{PSL}(2,\mathbb{C})$ as a space of maps is formally analogous to the definition of $\mathrm{PSL}(2,\mathbb{R})$ by S. Katok [24], and it puts the emphasis on the fractional linear map associated to any matrix of $\mathrm{SL}(2,\mathbb{C})$. Such maps can be extended to the whole complex plane by defining [19]

$$f(\infty) = \frac{a}{c}, \; f\left(-\frac{d}{c}\right) = \infty. \tag{4}$$

By virtue of the above considerations, we can consider the equivalence relation

$$(f_\Lambda, \Lambda) \sim (f_{\Lambda'}, \Lambda') \Longleftrightarrow \Lambda' = \pm\Lambda \Longrightarrow f_{\Lambda'}(\zeta) = f_\Lambda(\zeta).$$

A cut remains the unit 2-sphere under $f_\Lambda(\zeta)$ provided that its metric is subject to the conformal rescaling

$$K^2(\Lambda,\zeta)(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi), \tag{5}$$

having defined

$$K(\Lambda,\zeta) = K_\Lambda(\zeta) = \frac{1+|\zeta|^2}{|a\zeta+b|^2+|c\zeta+d|^2}, \tag{6}$$

where of course $|\gamma|^2 = \gamma\overline{\gamma}$ for all $\gamma \in \mathbb{C}$. The asymptotic theory remains invariant under this rescaling provided that lengths along the generators of null infinity scale by the same amount, i.e.,

$$du' = K_\Lambda(\zeta)du, \tag{7}$$

which can be integrated to find

$$u' = K_\Lambda(\zeta)\left[u + \alpha(\zeta,\overline{\zeta})\right], \tag{8}$$

where α is a suitably smooth function of ζ and of its complex conjugate $\overline{\zeta}$. The transformations (1) and (8) are related in such a way that they define the group of BMS transformations

$$T(\zeta) = f_\Lambda(\zeta) = \frac{(a\zeta+b)}{(c\zeta+d)}, \tag{9}$$

$$T(u) = K_\Lambda(\zeta)\left[u + \alpha(\zeta,\overline{\zeta})\right]. \tag{10}$$

In a concise form, one can write [16]

$$T(\zeta,u) = (T(\zeta),T(u)) = \left(f_\Lambda(\zeta), K_\Lambda(\zeta)\left[u+\alpha(\zeta,\overline{\zeta}\right]\right). \tag{11}$$

As pointed out in Ref. [17], the transformations (9) can be classified according to their fixed points, for which $f(\zeta) = \zeta$. Hence, only four families of fractional linear maps are found to exist

(i) **Parabolic.** Only one fixed point exists, for which $(a+d)^2 = 4$, while

$$\Lambda = A_P = \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}, \tag{12}$$

$$f_\Lambda(\zeta) = f_P(\zeta) = \zeta \pm \beta. \tag{13}$$

(ii) **Elliptic.** Two fixed points exist, for which $(a+d)^2 < 4$, while

$$\Lambda = A_E = \begin{pmatrix} e^{i\frac{\chi}{2}} & 0 \\ 0 & e^{-i\frac{\chi}{2}} \end{pmatrix}, \tag{14}$$

$$f_\Lambda(\zeta) = f_E(\zeta) = e^{i\chi}\zeta. \tag{15}$$

(iii) **Hyperbolic.** Two fixed points exist, for which $(a+d)^2 > 4$, while

$$\Lambda = A_H = \begin{pmatrix} \sqrt{|\kappa|} & 0 \\ 0 & \frac{1}{\sqrt{|\kappa|}} \end{pmatrix}, \tag{16}$$

$$f_\Lambda(\zeta) = f_H(\zeta) = |\kappa|\zeta. \tag{17}$$

(iv) **Loxodromic.** Two fixed points exist, for which $(a+d)^2 \in \mathbb{C} - \mathbb{R}$, and

$$(a+d) = \sqrt{k} + \frac{1}{\sqrt{k}}, \; k = \rho e^{i\sigma}, \; \rho \neq 1,$$

while

$$\Lambda = A_L = \begin{pmatrix} \sqrt{\rho} e^{i\frac{\sigma}{2}} & 0 \\ 0 & \frac{1}{\sqrt{\rho}} e^{-i\frac{\sigma}{2}} \end{pmatrix}, \tag{18}$$

$$f_\Lambda(\zeta) = f_L(\zeta) = \rho e^{i\sigma} \zeta. \tag{19}$$

Note that our matrices (12), (14), (16) and (18) belong to $SL(2, \mathbb{C})$, whereas in Section 2 of Ref. [17], only $A_P = M_P$ was in $SL(2, \mathbb{C})$, whereas the matrices M_E, M_H and M_L therein were elements of $GL(2, \mathbb{C})$.

Since also the transformation $T(u)$ depends on the matrix Λ through the conformal factor $K_\Lambda(\zeta)$, the work in Ref. [17] proposed the same nomenclature, from parabolic to loxodromic, for the whole group of BMS transformations in Equation (11). By virtue of Equations (6), (12), (14), (16) and (18), one finds therefore

$$K_P(\zeta) = \frac{1 + |\zeta|^2}{\left(1 + |\pm\beta + \zeta|^2\right)}, \tag{20}$$

$$K_E(\zeta) = 1, \tag{21}$$

$$K_H(\zeta) = \frac{|\kappa|(1 + |\zeta|^2)}{\left(1 + |\kappa|^2 |\zeta|^2\right)}, \tag{22}$$

$$K_L(\zeta) = \frac{\rho(1 + |\zeta|^2)}{\left(1 + \rho^2 |\zeta|^2\right)}, \tag{23}$$

in the parabolic, elliptic, hyperbolic and loxodromic cases, respectively. Once more, our Equations (22) and (23) differ by a multiplicative factor from the equations in Section 4 of Ref. [17] because all our matrices are in $SL(2, \mathbb{C})$.

3. A New Theorem on Supertranslations

At this stage, we can immediately prove the following theorem:

Theorem 1. *A normal elliptic transformation, where the phase factor χ in Equation (15) is an integer multiple of 2π, engenders a BMS supertranslation.*

Proof. If $\chi = 2\pi l$, l being a relative integer, one finds the BMS transformation

$$T(\zeta) = f_E(\zeta) = \zeta,$$

which implies that

$$T(\theta) = \theta, \; T(\phi) = \phi, \tag{24}$$

as well as (see Equations (10) and (21))

$$K_E(\zeta) = 1 \Longrightarrow T(u) = u + \alpha(\zeta, \overline{\zeta}). \tag{25}$$

Equations (24) and (25) are precisely the defining equations of the Abelian subgroup of supertranslations [16]. In other words, restriction to normal elliptic transformations, jointly with a choice of the function α, engenders all supertranslations, proving the statement made here.

As an explicit example, let us consider the most general metric in four dimensions in Bondi coordinates $(u, r, \zeta, \overline{\zeta})$:

$$g = -U du \otimes du - e^\beta (du \otimes dr + dr \otimes du) + g_{AB}(dx^A + \frac{1}{2} U^A du) \otimes (dx^B + \frac{1}{2} U^B du), \tag{26}$$

where $x^A = (\zeta, \bar{\zeta})$. The local diffeomorphism invariance is fixed by the following conditions:

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0, \quad g_{rr} = g_{rA} = 0. \tag{27}$$

In order to eliminate six Lorentz generators and thereby eliminating boosts and rotations that grow with r at infinity, we restrict ourselves to the diffeomorphisms generated by the vector field ε whose components have the large-r falloffs:

$$^{(u)}\varepsilon,\ ^{(r)}\varepsilon \sim \mathcal{O}(r^0), \quad ^{(\zeta)}\varepsilon,\ ^{(\bar{\zeta})}\varepsilon \sim \mathcal{O}(\frac{1}{r}). \tag{28}$$

By definition, the asymptotic symmetries must preserve the falloff conditions:

$$\begin{aligned} g_{uu} &= -1 + \frac{2m_B}{r} + \mathcal{O}(r^{-2}), \\ g_{ur} &= -1 + \mathcal{O}(r^{-2}), \\ g_{uA} &= \frac{1}{2} D^B C_{BA} + \mathcal{O}(r^{-1}), \\ g_{AB} &= r^2 \gamma_{AB} + r C_{AB} + \mathcal{O}(r^0), \end{aligned} \tag{29}$$

where m_B is known as the Bondi mass aspect and $C_{\zeta\zeta}$ describes gravitational waves at large r. Moreover, γ_{AB} is the metric on the two-sphere described by

$$\gamma_{\zeta\bar{\zeta}} = \frac{2}{(1 + \zeta\bar{\zeta})^2}. \tag{30}$$

By using the falloff conditions (28), one finds in Bondi gauge

$$(\mathcal{L}_\varepsilon g)_{rr} = 2 g_{ur} \partial_r {}^{(u)}\varepsilon, \tag{31}$$

which implies that $^{(u)}\varepsilon$ must be independent of r. In addition, to the leading order in the asymptotic expansion

$$(\mathcal{L}_\varepsilon g)_{ur} = g_{ur} \partial_u {}^{(u)}\varepsilon + \mathcal{O}(r^{-1}). \tag{32}$$

Hence, $^{(u)}\varepsilon(\zeta, \bar{\zeta})$ is a function on the two-sphere which we fix as being equal to $\alpha(\zeta, \bar{\zeta})$. Then, requiring that the Bondi conditions (27) and the falloffs (29) are preserved implies that at large r

$$\varepsilon = \alpha(\zeta, \bar{\zeta})\, \partial_u + D^\zeta D_\zeta\, \alpha(\zeta, \bar{\zeta})\, \partial_r - \frac{1}{r} D^\zeta\, \alpha(\zeta, \bar{\zeta})\, \partial_\zeta + c.c + \ldots, \tag{33}$$

where $\alpha(\zeta, \bar{\zeta})$ can be any function of $(\zeta, \bar{\zeta})$ on the two-sphere. The function $\alpha(\zeta, \bar{\zeta})$ can be expanded in spherical harmonics on the two-sphere. The modes $l = 0$ and $l = 1$ correspond to the standard global translations in Minkowski space-time. The vector field $\varepsilon(\alpha(\zeta, \bar{\zeta}))$ on the $l = 0$ and $l = 1$ spherical harmonics can be evaluated as

$$\begin{aligned} \varepsilon(Y_0^0) &= Y_0^0\, \partial_u, \\ \varepsilon(Y_1^m) &= Y_1^m\, \partial_u + \frac{1}{2} D^2 Y_1^m \partial_r - \frac{\gamma^{AB}\, \partial_B Y_1^m}{r}\, \partial_A, \end{aligned} \tag{34}$$

with the following normalization for spherical harmonics:

$$Y_0^0 = 1, \;\; Y_1^1 = \frac{\zeta}{(1 + \zeta\bar{\zeta})}, \;\; Y_1^0 = \frac{(1 - \zeta\bar{\zeta})}{(1 + \zeta\bar{\zeta})}, \;\; Y_1^{-1} = \frac{\bar{\zeta}}{(1 + \zeta\bar{\zeta})}. \tag{35}$$

Then, the standard global translations in Minkowski space-time are defined as

$$\begin{aligned}
\varepsilon(Y_0^0) &= \partial_u, \\
\varepsilon(Y_1^1) &= \frac{\zeta}{(1+\zeta\overline{\zeta})}\,(\partial_u - \partial_r) + \frac{\zeta^2}{2r}\partial_\zeta - \frac{1}{2r}\partial_{\overline{\zeta}}, \\
\varepsilon(Y_1^0) &= \frac{(1-\zeta\overline{\zeta})}{(1+\zeta\overline{\zeta})}\,(\partial_u - \partial_r) + \frac{\zeta}{r}\partial_\zeta - \frac{\overline{\zeta}}{r}\partial_{\overline{\zeta}}, \\
\varepsilon(Y_1^{-1}) &= \frac{\overline{\zeta}}{(1+\zeta\overline{\zeta})}\,(\partial_u - \partial_r) + \frac{1}{2r}\partial_\zeta - \frac{\overline{\zeta}^2}{2r}\partial_{\overline{\zeta}}.
\end{aligned} \tag{36}$$

Other choices of l engender all supertranslations. □

4. Behavior of the Conformal Factor

The conformal factors (20), (22) and (23) have the limiting behaviors

$$\lim_{|\zeta|\to 0} K_P = \frac{1}{(1+|\beta|^2)}, \ \lim_{|\zeta|\to\infty} K_P = 1, \tag{37}$$

$$\lim_{|\zeta|\to 0} K_H = |\kappa|, \ \lim_{|\zeta|\to\infty} K_H = \frac{1}{|\kappa|}, \tag{38}$$

$$\lim_{|\zeta|\to 0} K_L = \rho, \ \lim_{|\zeta|\to\infty} \ = \frac{1}{\rho}. \tag{39}$$

Moreover, since the independent variable $x = |\zeta|$ is always ≥ 0, both K_H and K_L can be studied by considering the function

$$F : x \in [0,\infty] \to F(x) = \frac{\Xi(1+x^2)}{(1+\Xi^2 x^2)}, \tag{40}$$

where $\Xi = |\kappa|$ or $\Xi = \rho$ in the hyperbolic and loxodromic cases, respectively. Since the first two derivatives of F are given by

$$F'(x) = \frac{2\Xi(1-\Xi^2)x}{(1+\Xi^2x^2)^2}, \ F''(x) = \frac{2\Xi(1-\Xi^2)}{(1+\Xi^2x^2)^3}(1-3\Xi^2x^2), \tag{41}$$

we find that, if $\Xi \in]0,1[$, the function F is monotonically increasing $\forall x \in [0,\infty]$, displays an upwards concavity and takes its absolute minimum at $x=0$. Figure 1 plots the graph of F when Ξ is either less than 1 or bigger than 1.

In the parabolic case, the conformal factor given in Equation (20) can be re-expressed in the form

$$K_P(\zeta) = \frac{1+|\zeta|^2}{\left[1+|\beta|^2+\left(1\pm 2\mathrm{Re}\left(\frac{\beta}{\zeta}\right)\right)|\zeta|^2\right]}, \tag{42}$$

and hence, we cannot exploit the theory of functions of a real variable for the parabolic conformal factor. Figure 2 plots the graph of the conformal factor $K_P(\zeta)$ in the $(\zeta,\overline{\zeta})$ plane. Of course, using either Equation (20) or Equation (42) leads to the same plot (K_{P+} and K_{P-} are devoted to the plus–minus in the denominator of the conformal factor $K_P(\zeta)$).

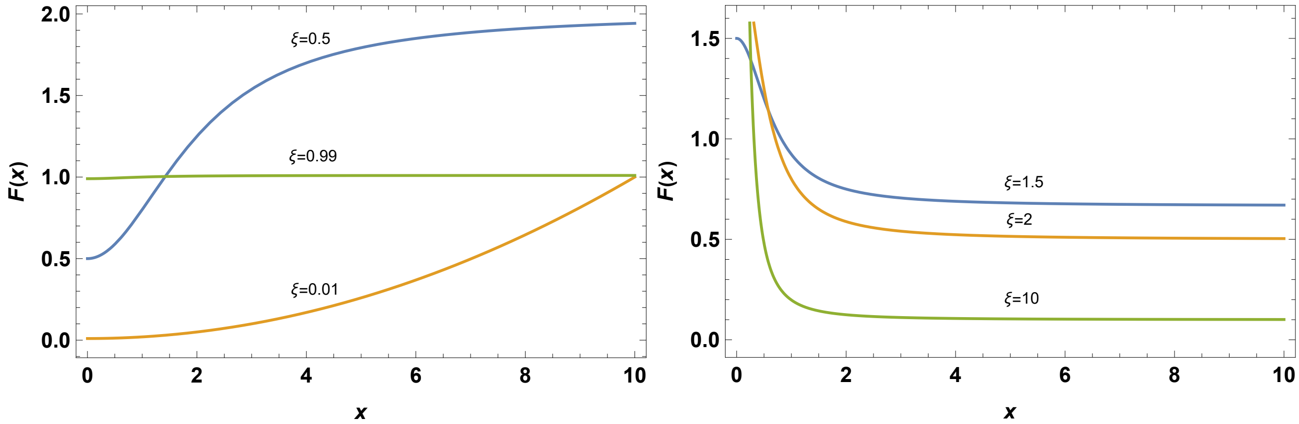

Figure 1. The conformal factor is monotonically increasing if $\Xi \in]0,1[$ and monotonically decreasing if $\Xi > 1$ in the hyperbolic and loxodromic cases.

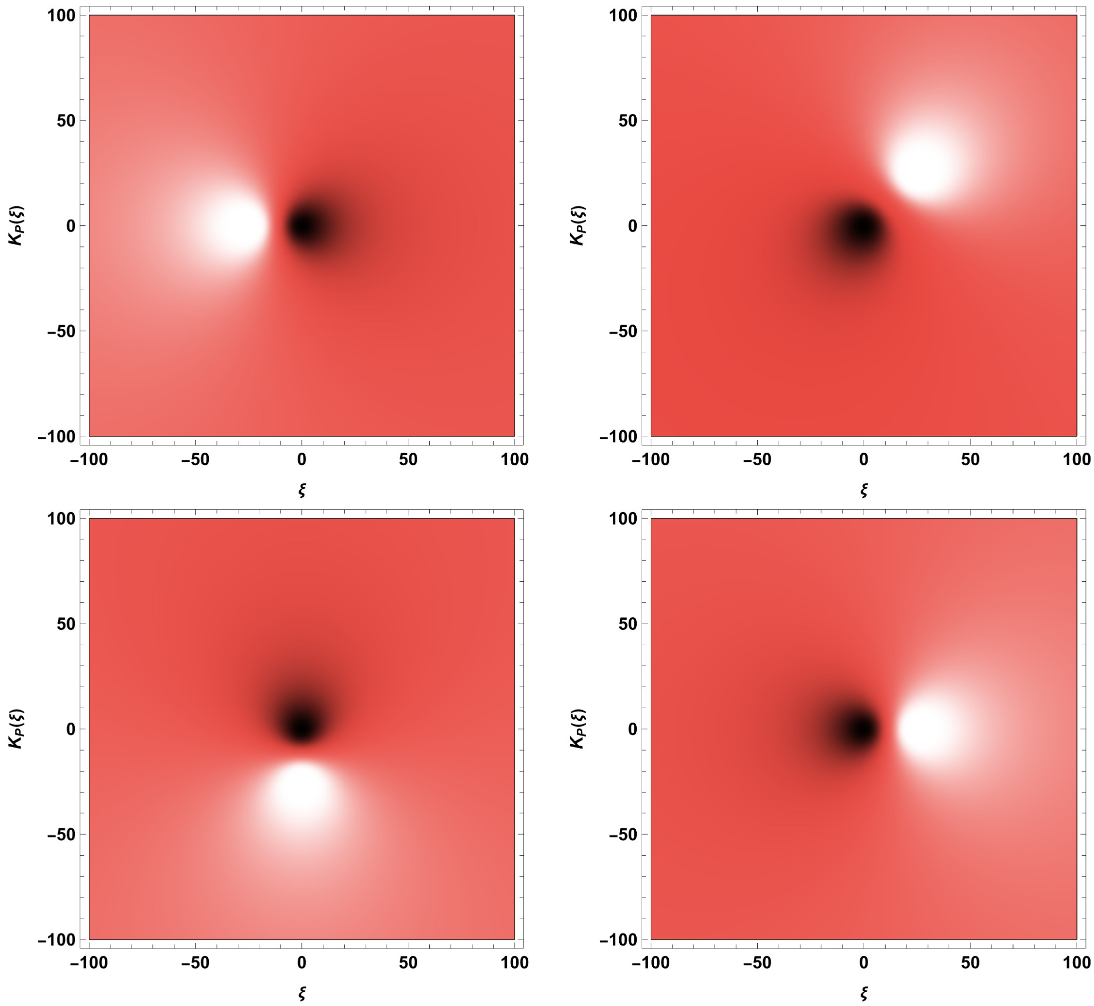

Figure 2. *Cont.*

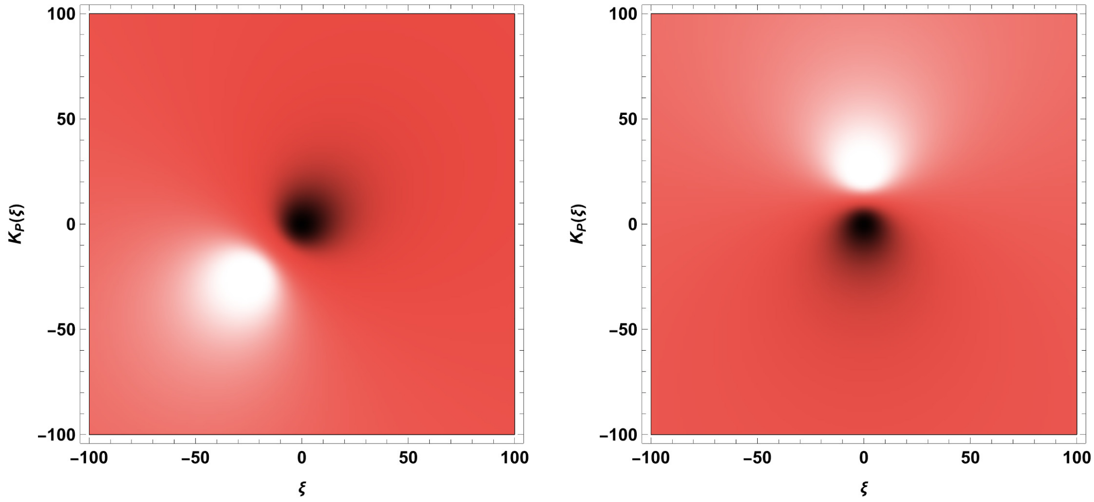

Figure 2. $K_P(\zeta)$ in the $(\zeta, \overline{\zeta})$ plane. First row from left to right: K_{P+} with β as a real parameter, K_{P+} with β as a complex parameter. Second row: K_{P+} with β as a purely imaginary parameter, K_{P-} with β as a real parameter. Third row: K_{P-} with β as a complex parameter, K_{P-} with β as a purely imaginary parameter.

5. Behavior of Killing Vector Fields under BMS Transformations

It is interesting to derive the most general form of the diffeomorphism associated with the four branches of the BMS group. We look for a diffeomorphism ϵ which satisfies the asymptotic falloff condition defined in Equation (28) together with the asymptotic symmetries preserving the falloff conditions described in Equation (29) in Bondi gauge. As already mentioned in Equations (9) and (10), the group of BMS transformations is defined as

$$T(\zeta) = f_\Lambda(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)}$$
$$T(u) = K_\Lambda(\zeta)\Big[u + \alpha(\zeta, \overline{\zeta})\Big]. \tag{43}$$

We recall that the first line of Equation (43) can be always reduced to one of the forms (13), (15), (17) or (19), where $f_\Lambda(\zeta)$ reads eventually

$$f_\Lambda(\zeta) = \mathcal{N}_\gamma \zeta + \gamma.$$

Moreover, we consider the asymptotic expansion of the vector field ε

$$\varepsilon = {}^{(u)}\varepsilon \partial_u + \sum_{n=0}^{\infty} \frac{{}^{(r)}\varepsilon_n}{r^n}\partial_r + \sum_{n=1}^{\infty} \frac{{}^{(\zeta)}\varepsilon_n}{r^n}\partial_\zeta + \sum_{n=1}^{\infty} \frac{{}^{(\overline{\zeta})}\varepsilon_n}{r^n}\partial_{\overline{\zeta}}. \tag{44}$$

The variation of the metric under a diffeomorphism is given by

$$(\mathcal{L}_\varepsilon g)_{\mu\nu} = \varepsilon^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \varepsilon^\rho + g_{\nu\rho}\partial_\mu \varepsilon^\rho. \tag{45}$$

In Bondi gauge,

$$(\mathcal{L}_\varepsilon g)_{rr} = 2 g_{ur} \partial_r {}^{(u)}\varepsilon, \tag{46}$$

which implies that ${}^{(u)}\varepsilon$ must be independent of r:

$${}^{(u)}\varepsilon = {}^{(u)}\varepsilon(u, \zeta, \overline{\zeta}). \tag{47}$$

By using Equation (7) together with the falloff conditions for the metric, the leading order in the asymptotic expansion gives

$$(\mathcal{L}_\varepsilon g)_{ur} = -\partial_u{}^{(u)}\varepsilon = -(K_\Lambda(\zeta) - 1), \tag{48}$$

where the last equality is obtained by evaluating the difference between the metric when du is conformally rescaled according to (7) and the original metric with no rescaling of du. Equation (48) suggests the following form for ${}^{(u)}\varepsilon$

$${}^{(u)}\varepsilon = F(\zeta, \overline{\zeta}) + u\ (K_\Lambda(\zeta) - 1). \tag{49}$$

From $(\mathcal{L}_\varepsilon g)_{r\zeta}$ at order $\mathcal{O}(r^0)$, one finds

$${}^{(\zeta)}\varepsilon_1 = -\mathcal{N}_\gamma\ D^{\zeta(u)}\varepsilon. \tag{50}$$

Moreover, $(\mathcal{L}_\varepsilon g)_{r\zeta}$ at order $\mathcal{O}(r^{-1})$ gives us

$${}^{(\zeta)}\varepsilon_2 = \frac{\mathcal{N}_\gamma}{2} C^{\zeta\zeta} D_\zeta{}^{(u)}\varepsilon. \tag{51}$$

The leading order of $\mathcal{O}(r)$ term of $(\mathcal{L}_\varepsilon g)_{uu}$ requires

$${}^{(r)}\varepsilon_0 = G(\zeta, \overline{\zeta}) + \frac{u}{2}(K_\Lambda^2 - 1). \tag{52}$$

The function $G(\zeta, \overline{\zeta})$ can be defined from the traceless nature of the $\mathcal{O}(r)$ term of $(\mathcal{L}_\varepsilon g)_{\zeta\zeta}$ as

$${}^{(r)}\varepsilon_0 = \frac{\mathcal{N}_\gamma^2}{2} D^{2(u)}\varepsilon + \frac{u}{2}(K_\Lambda^2 - 1). \tag{53}$$

Thus, the Killing vector field of the Bondi metric in correspondence with the four branches of the BMS transformations reads as

$$\begin{aligned} \varepsilon = &\ \big(F(\zeta, \overline{\zeta}) + u(K_\Lambda(\zeta) - 1)\big)\partial_u + \left(\frac{\mathcal{N}_\gamma^2}{2} D^{2(u)}\varepsilon + \frac{u}{2}(K_\Lambda^2 - 1)\right)\partial_r \\ &- \frac{1}{r}\big(\mathcal{N}_\gamma\ D^\zeta(F(\zeta, \overline{\zeta}) + u\ (K_\Lambda(\zeta) - 1))\big)\partial_\zeta + c.c + \ldots \end{aligned} \tag{54}$$

Hence, four families of diffeomorphisms in correspondence with the BMS transformations exist:

(i) **Parabolic.** In the case of a parabolic fractional linear map for ζ

$$K_P(\zeta) = \frac{1 + |\zeta|^2}{\Big(1 + |\pm\beta + \zeta|^2\Big)}, \tag{55}$$

$$\mathcal{N}_\gamma = \mathcal{N}_P = \pm 1, \quad \gamma = \gamma_P = \beta. \tag{56}$$

(ii) **Elliptic.** For an elliptic fractional linear map,

$$K_E(\zeta) = 1, \tag{57}$$

$$\mathcal{N}_\gamma = \mathcal{N}_E = 1, \quad \gamma = \gamma_E = 0. \tag{58}$$

Therefore, the Killing vector field (54) coincides with the form obtained in Equation (33).

(iii) **Hyperbolic.**

$$K_H(\zeta) = \frac{|\kappa|(1+|\zeta|^2)}{\left(1+|\kappa|^2|\zeta|^2\right)}, \tag{59}$$

$$\mathcal{N}_\gamma = \mathcal{N}_H = |k|, \quad \gamma = \gamma_H = 0. \tag{60}$$

(iv) **Loxodromic.**

$$K_L(\zeta) = \frac{\rho(1+|\zeta|^2)}{\left(1+\rho^2|\zeta|^2\right)}, \tag{61}$$

$$\mathcal{N}_\gamma = \mathcal{N}_L = \rho e^{i\sigma}, \quad \gamma = \gamma_L = 0. \tag{62}$$

Thus, the Killing vector fields associated with the four branches of the BMS transformations have been here derived for the first time in the literature by substituting $K_\Lambda(\zeta)$, $\mathcal{N}_\gamma$ and γ from Equations (55)–(62) into Equation (54).

6. BMS Transformations in Homogeneous Coordinates

The material at the beginning of Appendix B suggests expressing our complex variable $\zeta = e^{i\phi}\cot\frac{\theta}{2}$ in the form $\zeta = \frac{z_0}{\zeta_1}$. This is easily accomplished by defining

$$z_0 \equiv e^{i\frac{\phi}{2}}\cos\frac{\theta}{2},\ z_1 \equiv e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}, \tag{63}$$

and hence writing the first half of BMS transformations, our Equation (1), in the form (A20):

$$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \tag{64}$$

where the $\mathrm{SL}(2,\mathbb{C})$ matrix can only be either (12), or (14), or (16), or (18).

The second half of BMS transformations, our Equation (8), now reads as

$$u' = K_\Lambda(z_0,z_1)\left[u + \alpha(z_0,z_1;\bar{z}_0,\bar{z}_1)\right], \tag{65}$$

where the conformal factor can only take one of the four forms (20)–(23), upon setting $\zeta = \frac{z_0}{z_1}$ therein.

In Equations (64) and (65), the complex variables are defined as in (63) and obey therefore the restrictions

$$|z_0| \le 1,\ |z_1| \le 1,$$

i.e., they correspond to a pair of unit circles Γ_0 and Γ_1. Thus, we may recognize that the BMS transformations are the restrictions to these circles of a more general set of transformations, i.e.,

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{66}$$

$$u' = K_\Lambda(w_1,w_2)\left[u + \alpha(w_1,w_2;\bar{w}_1,\bar{w}_2)\right], \tag{67}$$

where both w_1 and w_2 belong to $\mathbb{C}\cup\{\infty\}$, and they are such that

$$w_1|_{\Gamma_0} = z_0,\ w_2|_{\Gamma_1} = z_1.$$

Within this broader framework, one can consider two complex projective planes. Let P be a point of the first plane with coordinates (w_0,w_1,w_2), and let P' be a point of the second plane, with coordinates (u_0, u_1, u_2). We can now take all nine products between a complex coordinate of P and a complex coordinate of P', i.e.,

$$Z_{hk} = w_h u_k, \ h, k = 0, 1, 2. \quad (68)$$

These nine complex quantities are defined up to a proportionality factor, since this is the case for both w_h and u_k. They can be therefore interpreted as the coordinates of a point Z of eight-dimensional complex projective space S_8. To the pair of points P and P', there corresponds a point Z of S_8 by means of Equation (68). As P and P' are varying in their own plane, the point Z describes in S_8 a four-complex-dimensional manifold, since both P and P' are varying on a plane, i.e., a two-complex-dimensional geometric object. Equation (68) represents therefore a four-complex-dimensional manifold V_4 in the complex projective space S_8. Such a manifold is the *Segre manifold* [25,26].

If in the first plane we fix the point $P = (w_0, w_1, w_2)$, Equation (68) becomes linear homogeneous in the u_k coordinates and, as such, they represent a plane in S_8. Thus, to every point of the first plane, there corresponds a plane on the Segre manifold V_4. The Segre manifold contains therefore a complex double infinitude of planes. In a completely analogous way, another double infinitude of planes of V_4 corresponds to the double infinitude of points of the second plane. A plane of this second infinitude is obtained by fixing a point P' in the second plane and then letting P vary in the first plane. Each of these ∞^1 systems of planes is an array in light of the correspondence between elements of the system and points of a plane. Hence, the Segre manifold contains two arrays of planes. Two planes of the same array do not have common points, whereas two planes belonging to different arrays have one and only one common point [25].

One can also fix the point P and let the point P' vary not over the whole plane but only on a line in such a plane. In correspondence, one obtains on the Segre manifold V_4 a V_1 subset, i.e., a curve. If both P and P' describe a line in their own plane, one obtains on the Segre manifold V_4 a V_2 subset, i.e., a quadric. Hence, to every pair of lines, there corresponds a quadric. Since there exist ∞^2 lines in a plane, the quadrics of a Segre manifold are ∞^4. In other words, the Segre manifold contains a complex fourfold infinity of quadrics.

At a deeper level, we can say that the Segre manifold is the projective image of the product of projective spaces, and it is a natural tool for studying the framework where we can accommodate the transformations that reduce to the BMS transformations upon restriction to the pair of unit circles Γ_0 and Γ_1.

7. Concluding Remarks

Since asymptotic flatness is a limiting case of classical general relativity, in our opinion, our work is relevant for the scope of this special issue on Extreme Regimes of Classical and Quantum Gravity Models, bearing also in mind the relevance of the BMS group for modern studies of black holes [1,3,4]. Moreover, we possibly fill a gap in the literature, because we have not found previous papers on the BMS group among those published in Symmetry. The original contributions of our paper are as follows.

(i) Proof that to each normal elliptic transformation of the complex variable ζ used in the metric for cuts of null infinity, there corresponds a BMS supertranslation. Although this might be seen as a corollary of the work initiated in Ref. [17], it has prepared the ground for the items below.

(ii) Study of the conformal factor in the BMS transformation of the u variable as a function of the squared modulus of ζ. In the loxodromic and hyperbolic cases, such a conformal factor turns out to be either monotonically increasing or monotonically decreasing as a function of the real variable given by the absolute value of ζ. In the parabolic case, the conformal factor is instead a real-valued function of complex variable, and one needs the plots of Figure 2.

(iii) A classification of Killing vector fields of the Bondi metric has been obtained in Section 5.

(iv) In Section 6, we have found that BMS transformations are the restriction to a pair of unit circles of a more general set of transformations. Within this broader framework, the geometry of such transformations is studied by means of its Segre manifold. This provides an unforeseen bridge between the language of algebraic geometry and the analysis of BMS transformations in general relativity.
(v) Our remarks at the end of Section 5 might lead to a systematic application of projective geometry techniques for the definition of points at infinity in general relativity.
(vi) Our results in Section 5 suggest four sets of Killing fields associated with the four branches of BMS transformations. As discussed in Section 3, the elliptic transformations (the case with $K_\Lambda(\zeta) = 1$) define the Abelian subgroup of supertranslations. The linearized action of supertranslations in the Schwarzschild case is already studed in [3], which results in a black hole with linearized supertranslation hair. It would be interesting to study the action of parabolic, hyperbolic and loxodromic transformations defined by the Killing fields (54) on a black hole metric.

To sum up, we have addressed the physical problem of obtaining a more complete understanding of BMS diffeomorphisms of an asymptotically flat space-time. The tools we have developed might therefore lead to new developments in black hole physics (see item (vi) above) and in the area of geometric methods in theories of gravity, especially in light of the original framework described in Section 6.

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Appendix A. Composition of BMS Transformations

It is helpful to derive, with the notation in our Section 2, the composition rule of two BMS transformations. For this purpose, we note that since a BMS transformation yields

$$T(\zeta, u) = (\zeta', u') = (T(\zeta), T(u)), \tag{A1}$$

where

$$\zeta' = T(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = f_\Lambda(\zeta), \tag{A2}$$

$$u' = T(u) = K_\Lambda(\zeta)\Big[u + \alpha(\zeta, \overline{\zeta})\Big], \tag{A3}$$

the subsequent BMS map leads to

$$T(\zeta', u') = (\zeta'', u'') = (T(\zeta'), T(u')), \tag{A4}$$

where, by virtue of Equation (A2), one obtains

$$\begin{aligned} \zeta'' &= T(\zeta') = \frac{(a'\zeta' + b')}{(c'\zeta' + d')} = f_{\Lambda'}(\zeta') \\ &= \frac{(a'a + b'c)\zeta + (a'b + b'd)}{(c'a + d'c)\zeta + (c'b + d'd)} = \frac{(A\zeta + B)}{(C\zeta + D)} = f_{\Lambda''}(\zeta), \end{aligned} \tag{A5}$$

having defined

$$\Lambda'' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \Lambda'\Lambda, \tag{A6}$$

which is the product of the PSL$(2,\mathbb{C})$ matrices

$$\Lambda' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \; \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Moreover, one finds

$$u'' = T(u') = K_{\Lambda'}(\zeta')\Big[u' + \alpha(\zeta', \overline{\zeta}')\Big] = \widetilde{K}_{\Lambda'\Lambda}\Big[u + \widetilde{\alpha}(\zeta, \overline{\zeta})\Big], \tag{A7}$$

having defined

$$\widetilde{K}_{\Lambda'\Lambda}(\zeta) \equiv \Big[K_{\Lambda'}(f_\Lambda(\zeta))\Big]K_\Lambda(\zeta), \tag{A8}$$

$$\widetilde{\alpha}(\zeta, \overline{\zeta}) \equiv \alpha(\zeta, \overline{\zeta}) + \frac{\alpha\Big(f_\Lambda(\zeta), \overline{f}_\Lambda(\zeta)\Big)}{K_\Lambda(\zeta)}. \tag{A9}$$

Appendix B. Origin and Properties of Fractional Linear Maps

Suppose that the pair of complex coordinates (z_0, z_1) is mapped into the pair (z'_0, z'_1) by the linear transformation

$$\begin{pmatrix} z'_0 \\ z'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \; ad - bc \neq 0. \tag{A10}$$

This means that the ratio $\zeta = \frac{z_0}{z_1}$ is mapped into

$$\zeta' = \frac{z'_0}{z'_1} = \frac{(az_0 + bz_1)}{(cz_0 + dz_1)} = \frac{(a\zeta + b)}{(c\zeta + d)}. \tag{A11}$$

Thus, a fractional linear map arises from a linear transformation acting on the homogeneous coordinates z_0, z_1. For further insight, we refer the reader to the lecture notes in Ref. [27].

If the matrix on the right-hand side of Equation (A10) pertains to PSL$(2,\mathbb{C})$, the condition of unit determinant yields

$$b = \frac{(ad-1)}{c}, \tag{A12}$$

and hence, one finds [28]

$$\zeta' = \frac{(a\zeta + b)}{(c\zeta + d)} = \frac{\frac{a}{c}(c\zeta + d) - \frac{1}{c}}{(c\zeta + d)} = \frac{a}{c} - \frac{1}{|c|^2}\left(\frac{|c|}{c}\right)^2 \frac{1}{\left(\zeta + \frac{d}{c}\right)}. \tag{A13}$$

Thus, half of the BMS transformations as in Equation (A2) arise by composition of the following maps [28]:

$$\text{Translation} \;\; \zeta \rightarrow \zeta + \frac{d}{c}, \tag{A14}$$

$$\text{Inversion} \;\; \zeta + \frac{d}{c} \rightarrow \frac{1}{\left(\zeta + \frac{d}{c}\right)}, \tag{A15}$$

$$\text{Rotation} \;\; \zeta \rightarrow -\left(\frac{|c|}{c}\right)^2 z, \tag{A16}$$

$$\text{Dilation} \;\; \zeta \rightarrow \frac{1}{|c|^2}\zeta, \tag{A17}$$

and eventually a further translation

$$\zeta \rightarrow \zeta + \frac{a}{c}. \tag{A18}$$

The interplay of homogeneous and non-homogeneous coordinates has not been fully exploited in general relativity so far, to the best of our knowledge. For example, linear transformations among real homogeneous coordinates may be a powerful tool for studying points at infinity. In particular, one could imagine that the coordinates x^1, x^2, x^3, x^4 used for a real four-dimensional Lorentzian space-time manifold arise from five homogeneous coordinates $(y^0, y^1, y^2, y^3, y^4)$ according to the rule

$$x^1 = \frac{y^1}{y^0},\ x^2 = \frac{y^2}{y^0},\ x^3 = \frac{y^3}{y^0},\ x^4 = \frac{y^4}{y^0}, \tag{A19}$$

the y values being subject to the linear transformations

$$y'^{\mu} = \sum_{\nu=0}^{4} A^{\mu}_{\ \nu}\, y^{\nu},\ \det\left(A^{\mu}_{\ \nu}\right) \neq 0, \tag{A20}$$

which imply the following transformation rules for space-time coordinates:

$$x'^{\mu} = \frac{\sum_{\lambda=0}^{4} A^{\mu}_{\ \lambda}\, y^{\lambda}}{\sum_{\nu=0}^{4} A^{0}_{\ \nu}\, y^{\nu}}\ \forall \mu = 1, 2, 3, 4. \tag{A21}$$

Equation (A21) might provide a fully projective way of studying the concept of infinity (cf. Ref. [29]).

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