Refinements of the Euclidean Operator Radius and Davis–Wielandt Radius-Type Inequalities

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Abstract: This paper proves several new inequalities for the Euclidean operator radius, which refine some recent results. It is shown that the new results are much more accurate than the related, recently published results. Moreover, inequalities for both symmetric and non-symmetric Hilbert space operators are studied.

Keywords: Euclidean operator radius; Davis–Wielandt radius; numerical radius

MSC: 47A12; 47A30; 47A63

1. Introduction

Let $\mathcal{A}(\mathcal{M})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{M}; \langle \cdot, \cdot \rangle)$ with the identity operator $1_\mathcal{M}$. For a bounded linear operator $M$ on a Hilbert space $\mathcal{M}$, the numerical range $W(P)$ is the image of the unit sphere of $\mathcal{M}$ under the quadratic form $c \rightarrow \langle Mc, c \rangle$ associated with the operator. More precisely,

$$W(M) = \{ \langle Mc, c \rangle : c \in \mathcal{M}, \|c\| = 1 \}.$$ 

Moreover, the numerical radius is defined to be

$$w(M) = \sup_{\lambda \in W(T)} |\lambda| = \sup_{\|c\|=1} |\langle Mc, c \rangle|.$$ 

We recall that the usual operator norm of an operator $T$ is defined to be

$$\|T\| = \sup\{ \|Mc\| : c \in \mathcal{M}, \|c\| = 1 \},$$

It is well known that $w(\cdot)$ defines an operator norm on $\mathcal{A}(\mathcal{M})$ that is equivalent to the operator norm $\| \cdot \|$. Moreover, we have

$$\frac{1}{2} \|M\| \leq w(M) \leq \|M\| \tag{1}$$

for any $M \in \mathcal{A}(\mathcal{M})$. 

Symmetry 2023, 15, 1061. https://doi.org/10.3390/sym15051061
The Euclidean operator radius of an \( n \)-tuple \( P = (P_1, \cdots, P_n) \in \mathcal{A} (\mathcal{M})^n := \mathcal{A} (\mathcal{M}) \times \cdots \times \mathcal{A} (\mathcal{M}) \) was introduced by Popescu in [1], where \( P_1, \cdots, P_n \in \mathcal{A} (\mathcal{M}) \). The Euclidean operator radius of \( P_1, \cdots, P_n \) is defined by

\[
\omega_e(P_1, \cdots, P_n) := \sup_{\|c\|=1} \left( \frac{1}{n} \sum_{i=1}^{n} |\langle P_i c, c \rangle|^2 \right)^{1/2}.
\]

Indeed, the Euclidean operator radius was generalized in [2] as follows:

\[
w_p(P_1, \cdots, P_n) := \sup_{\|c\|=1} \left( \frac{1}{n} \sum_{i=1}^{n} |\langle P_i c, c \rangle|^p \right)^{1/p}, \quad p \geq 1.
\]

If \( p = 1 \), then \( w_1(P_1, \cdots, P_n) \) (in addition, it is denoted by \( w_R(P_1, \cdots, P_n) \)) is called the Rhombic numerical radius, which has been studied in [3]. In particular, if \( P_1 = \cdots = P_n = P \), then it is interesting that \( w_1(P, \cdots, P) = n \, w(P) \), where \( w(P) \) is the numerical radius of \( P \).

We note that the inequality

\[
w_\infty(P_1, \cdots, P_n) \leq w_p(P_1, \cdots, P_n) \leq w_R(P_1, \cdots, P_n)
\]

(2)

holds for all \( p \in (1, \infty) \); see [4].

In addition, Popescu [1] proved that

\[
\frac{1}{2\sqrt{n}} \left\| \sum_{k=1}^{n} P_k P_k^* \right\|^\frac{1}{2} \leq \omega_e(P_1, \cdots, P_n) \leq \left\| \sum_{k=1}^{n} P_k P_k^* \right\|^\frac{1}{2}.
\]

(3)

As noted in [5], and as a special case of (3), if \( Y = F + iG \) is the Cartesian decomposition of \( A \), then

\[
w_e^2(F, G) = \sup_{\|c\|=1} \left\{ |\langle Fc, c \rangle|^2 + |\langle Gc, c \rangle|^2 \right\} = \sup_{\|c\|=1} |\langle Yc, c \rangle|^2 = w^2(Y).
\]

Since \( Y^*Y + YY^* = 2(F^2 + G^2) \), we have

\[
\frac{1}{16} ||Y^*Y + YY^*|| \leq \omega^2(Y) \leq \frac{1}{2} ||Y^*Y + YY^*||.
\]

(4)

Note that the case of \( n = 2 \) was studied by Dragomir in [6], and he obtained some interesting results regarding the Euclidean operator radius of two operators \( \omega_e(P_1, P_2) \).

The Euclidean operator radius was generalized in [5] as follows:

\[
\omega_p(P_1, \cdots, P_n) := \sup_{\|c\|=1} \left( \frac{1}{n} \sum_{i=1}^{n} |\langle P_i c, c \rangle|^p \right)^{\frac{1}{p}}, \quad p \geq 1.
\]

In [5], Moslehian, Sattari, and Shebrawi proved several inequalities regarding \( n \)-tuple operators \( P \in \mathcal{A} (\mathcal{M})^n \). In particular, they proved the following two results:

\[
\omega_p(P_1, \cdots, P_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |P_i|^{2p} + |P_i|^p |1 - p| \right)^p \right\|^\frac{1}{p}
\]

(5)
and
\[ \omega_p(P_1, \cdots, P_n) \leq \left\| \sum_{i=1}^{n} a|P_i|^p + (1-a)|P^*|^p \right\|^\frac{1}{p} \]  
(6)

for \( a \in [0,1] \) and \( p \geq 1 \). For the case \( p = 2 \), (5) and (6) studied upper bounds for the Euclidean operator radius \( \omega_{\ell_2}() \). It should be noted that in case \( n = 1 \) and \( p = 1 \), then (5) reduces to the main result in [7].

An inequality for a product of two Hilbert space operators was also deduced in [5], as follows:
\[ \omega^2_{\ell_2}(Q_1^* P_1, \cdots, Q_n^* P_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} |P_i|^4r + |Q_i|^4r \right\| \]  
(7)

for all \( P_i, Q_i \in \mathcal{A}(\mathcal{H}) \) and \( r \geq 1 \). This inequality generalizes and extends the result in [8].

In [2], Sheikhosseini, Moslehian, and Shebrawi refined the above two inequalities by proving the following two results, respectively,
\[ \omega_p(P_1, \cdots, P_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |P_i|^{2a} + |P^*_i|^{2(1-a)} \right)^\frac{1}{p} \right\| - \inf_{\|c\|=1} \xi(c) \]  
(8)

where
\[ \xi(c) = \frac{1}{2} \sum_{i=1}^{n} \left( \left| \langle |P_i|^{2a} c, c \rangle \right|^\frac{1}{2} - \left| \langle |P^*_i|^{2(1-a)} c, c \rangle \right|^\frac{1}{2} \right)^2 \]

and
\[ \omega^p_{\ell_2}(P_1, \cdots, P_n) \leq \left\| \sum_{i=1}^{n} (a|P_i|^{2a} + (1-a)|P_i|^{2(1-a)})^{\frac{1}{p}} \right\| - \inf_{\|c\|=1} \xi(c) \]  
(9)

where
\[ \xi(c) = \min\{a, 1-a\} \sum_{i=1}^{n} \left( \left| \langle |P_i|^{2a} c, c \rangle \right|^\frac{2}{p} - \left| \langle |P^*_i|^{2(1-a)} c, c \rangle \right|^\frac{2}{p} \right)^2 \]

For further inequalities of the Euclidean operator radius combined with several basic properties, the reader may refer to [3,4,6,9,10]. For more generalization, counterparts, and recent related results, the reader may refer to [11–19].

In [19], Alomari proved the following version of the Euclidean operator radius, which generalized the celebrated Kittaneh inequality [20].
\[ \frac{1}{2p+1} \left\| \sum_{k=1}^{n} P^*_k P_k + P_k P^*_k \right\|^p \leq \omega^2_{2p}(P_1, \cdots, P_n) \leq \frac{1}{2p} \left\| \sum_{k=1}^{n} (P^*_k P_k + P_k P^*_k)^p \right\| \]

for all \( P_k \in \mathcal{A}(\mathcal{H}) \) \((k = 1, \cdots, n)\) and \( p \geq 1 \). In particular, we have
\[ \frac{1}{4} \left\| \sum_{k=1}^{n} P^*_k P_k + P_k P^*_k \right\| \leq \omega^2_{2}(P_1, \cdots, P_n) \leq \frac{1}{2} \left\| \sum_{k=1}^{n} (P^*_k P_k + P_k P^*_k) \right\| \]

This article proves several new inequalities for the Euclidean operator radius \( \omega() \). More precisely, refinement inequalities of some old results are presented. Section 2 recalls some key inequalities used in the following section. Section 3 is focused on the diverse upper bounds for the Euclidean operator radius \( \omega() \), and this gives an extension and refinements of (5) and (7) when \( p = 2 \). Our new inequalities are devoted to refining the
Euclidean operator radius $\omega_e(\cdot)$. A similar approach could be used to refine several inequalities for $\omega_p(\cdot)$. Inequalities for symmetric (self-adjoint) and non-symmetric (arbitrary) Hilbert space operators are also covered.

2. Lemmas

To prove our results, we need a sequence of lemmas.

**Lemma 1** ([21]). The Power-Mean inequality states that

$$t^\alpha s^{1-\alpha} \leq \alpha t + (1 - \alpha)s \leq (\alpha t^p + (1 - \alpha)s^p)^{1/p}$$ (10)

for all $\alpha \in [0, 1]$, $s, t \geq 0$ and $p \geq 1$.

**Lemma 2** ([22]). [Theorem 1.4] Let $P \in \mathcal{A}(\mathcal{M})^+$, then

$$\langle Pc, c \rangle^p \leq \langle Pc, c \rangle, \quad p \geq 1$$ (11)

for any vector $c \in \mathcal{M}$. The inequality (11) is reversed if $0 \leq p \leq 1$.

The following result generalizes and refines Kato’s inequality or the so-called mixed Schwarz inequality [23].

**Lemma 3** ([24]). [Lemma 5] Let $P \in \mathcal{A}(\mathcal{M})$, $0 \leq \alpha \leq 1$ and $p \geq 1$. Then,

$$\|\langle Pc, d \rangle\|^{2p} \leq \beta \langle |P|^{2\alpha}c, c \rangle \langle |P^{*}|^{2(1-\alpha)}d, d \rangle$$

$$+ (1 - \beta)\|\langle Pc, d \rangle\|^p \sqrt{\langle |P|^{2\alpha}c, c \rangle \langle |P^{*}|^{2(1-\alpha)}d, d \rangle}$$

$$\leq \langle |P|^{2\alpha}c, c \rangle \langle |P^{*}|^{2(1-\alpha)}d, d \rangle.$$ (12)

for all $\beta \in [0, 1]$.

**Corollary 1.** Let $P \in \mathcal{A}(\mathcal{M})$, $0 \leq \alpha, \beta \leq 1$. Then,

$$|\langle Pc, d \rangle|^2 \leq \beta \langle |P|^{2\alpha}c, c \rangle \langle |P^{*}|^{2(1-\alpha)}d, d \rangle + (1 - \beta)\|\langle Pc, d \rangle\|^p \sqrt{\langle |P|^{2\alpha}c, c \rangle \langle |P^{*}|^{2(1-\alpha)}d, d \rangle}$$

$$\leq \langle |P|^{2\alpha}c, c \rangle \langle |P^{*}|^{2(1-\alpha)}d, d \rangle.$$ (13)

**Proof.** Setting $p = 1$ in (12). □

**Lemma 4** ([24]). Let $P, Q \in \mathcal{A}(\mathcal{M})$. Then,

$$|\langle Pc, Qd \rangle|^2 \leq \beta\langle |P|^{2\alpha}c, c \rangle \langle Q^2d, d \rangle + (1 - \beta)\|\langle Pc, Qd \rangle\|^p \sqrt{\langle |P|^{2\alpha}c, c \rangle \langle Q^2d, d \rangle}$$

$$\leq \|Pc\|^2\|Qd\|^2$$ (14)

for any vectors $c, d \in \mathcal{M}$ and all $\beta \in [0, 1]$.

**Lemma 5** ([25]). Let $P \in \mathcal{A}(\mathcal{M})$. Then,

$$|\langle Pc, c \rangle|^2 \leq \frac{1}{2} \langle |P^2c, c \rangle \rangle + \frac{1}{4} \langle \langle |P|^{2} + |P^*|^{2} \rangle c, c \rangle$$ (15)

for any vectors $c \in \mathcal{M}$. 
Lemma 6 ([26]). Theorem 2.3 Let $f$ be a non-negative convex function on $[0, \infty)$, and let $P, Q \in \mathcal{A}(\mathcal{M})$ be two positive operators. Then,
\[ \left\| f \left( \frac{P + Q}{2} \right) \right\| \leq \left\| \frac{f(P) + f(Q)}{2} \right\|. \] (16)

3. Applications to Numerical Radius Inequalities

We are in a position to state our first main result involving the numerical radius inequalities for a product of two Hilbert space operators.

Theorem 1. Let $P_k, Q_k \in \mathcal{A}(\mathcal{M}) \ (k = 1, 2, \cdots, n)$. Then,
\[ \omega^2_n(Q_k P_k, \cdots, Q_n P_n) \]
\[ \leq \frac{1}{2^n} \beta \left\| \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) \right\|^r + \frac{1}{2^n} (1 - \beta) \omega^2_n(Q_1 P_1, \cdots, Q_n P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right) \right\| \] \hspace{1cm} (17)

for all $\beta \in [0, 1]$ and $r \geq 1$.

Proof. Let $u \in \mathcal{M}$ be a unit vector. We set $c = P_k u$ and $d = Q_k u \ (k = 1, \cdots, n)$ in the first inequality in (14). Employing the AM–GM inequality, the convexity of $t^2 \ (t > 0)$, and using Lemma 2, we obtain
\[ |\langle Q_k^2 P_k u, u \rangle|^2 \]
\[ \leq (1 - \beta) |\langle Q_k^2 P_k u, u \rangle| \left( \langle |P_k|^2 u, u \rangle \right)^{1/2} \left( \langle |Q_k|^2 u, u \rangle \right)^{1/2} + \beta \left( \langle |P_k|^2 u, u \rangle \right)^{1/2} \left( \langle |Q_k|^2 u, u \rangle \right)^{1/2} \]
\[ \leq (1 - \beta) |\langle Q_k^2 P_k u, u \rangle| \left( \langle |P_k|^2 u, u \rangle \right)^{1/2} \left( \langle |Q_k|^2 u, u \rangle \right)^{1/2} + \frac{1}{2} \beta \left( \langle |P_k|^2 u, u \rangle \right)^{1/2} \left( \langle |Q_k|^2 u, u \rangle \right)^{1/2} \]
\[ \leq \frac{1}{2} (1 - \beta) |\langle Q_k^2 P_k u, u \rangle| \left( \langle |P_k|^2 u, u \rangle \right)^{1/2} \left( \langle |Q_k|^2 u, u \rangle \right)^{1/2} + \frac{1}{2} \beta \left( \langle |P_k|^4 u, u \rangle + \langle |Q_k|^4 u, u \rangle \right) \]
\[ = \frac{1}{2} (1 - \beta) |\langle Q_k P_k u, u \rangle| \left( \langle |P_k|^2 + |Q_k|^2 \rangle u, u \right) + \frac{1}{2} \beta \left( \langle |P_k|^4 + |Q_k|^4 \rangle u, u \right). \]

Taking the summation over $k = 1$ up to $n$ for both sides, we have
\[ \sum_{k=1}^{n} |\langle Q_k^2 P_k u, u \rangle|^2 \]
\[ \leq \frac{1}{2} (1 - \beta) \sum_{k=1}^{n} |\langle Q_k^2 P_k u, u \rangle| \left( \langle |P_k|^2 + |Q_k|^2 \rangle u, u \right) + \frac{1}{2} \beta \sum_{k=1}^{n} \left( \langle |P_k|^4 + |Q_k|^4 \rangle u, u \right) \]

Applying the Cauchy–Schwarz inequality to real numbers and then applying Lemma 2, we obtain
\[
\sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \\
\leq \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right)^2 \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \beta \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right) \\
\leq \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right)^2 u, u \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \beta \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right) \\
= \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right)^2 u, u \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \beta \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right)
\]

Again, by applying the convexity of \( t' (r \geq 1) \), we obtain

\[
\left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{r}{2}} \\
\leq \left( \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right)^2 \right)^{\frac{1}{2}} \right)^{\frac{r}{2}} \\
+ \frac{1}{2} \beta \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right)^{\frac{r}{2}} \\
\leq \frac{1}{2r} (1 - \beta) \left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right)^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right)^{\frac{r}{2}} \\
+ \frac{1}{2r} \beta \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right)^{\frac{r}{2}} \\
\leq \frac{1}{2r} (1 - \beta) \left( \sum_{k=1}^{n} |\langle Q_k^* P_k u, u \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^2 + |Q_k|^2 \right)^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right)^{\frac{r}{2}} \\
+ \frac{1}{2r} \beta \left( \sum_{k=1}^{n} \left( |P_k|^4 + |Q_k|^4 \right) u, u \right)^{\frac{r}{2}}
\]

Taking the supremum over all unit vectors \( u \in \mathcal{M} \), we obtain the desired result in (17). \( \blacksquare \)

**Corollary 2.** Let \( P_k, Q_k \in \mathcal{M} (\mathcal{M}) (k = 1, 2, \cdots, n) \). Then,

\[
\alpha_*^2 (Q_1^* P_1, \cdots, Q_n^* P_n) \leq \frac{1}{2} \left\| \sum_{k=1}^{n} (|P_k|^4 + |Q_k|^4) \right\| 
\]

(18)

for all \( \beta \in [0, 1] \).
Proof. Setting $\beta = 1$ in (17). \hfill \Box

**Theorem 2.** Let $P_k, Q_k \in \mathcal{A}(\mathcal{M})$ ($k = 1, 2, \cdots, n$). Then,

$$
\omega^{2r}(Q_1^*P_1, \cdots, Q_n^*P_n) \leq \frac{1}{2^r}(1-\beta)^r \omega^r(Q_1^*P_1, \cdots, Q_n^*P_n) \left\| \sum_{k=1}^n (|P_k|^2 + |Q_k|^2)^2 \right\|^\frac{r}{2} + \frac{1}{2^r} \beta^r \left\| \sum_{k=1}^n (|P_k|^4 + |Q_k|^4) \right\|^r
$$

(19)

for all $\beta, r \in [0, 1]$.

**Proof.** Form the proof of Theorem 1. Since $t \mapsto t^r$ ($t > 0$) for subadditive for all $r \in [0, 1]$, then we have

$$
\left( \sum_{k=1}^n |Q_k^*P_k u, u|^2 \right)^r
\leq \left( \frac{1}{2} (1-\beta) \left( \sum_{k=1}^n |Q_k^*P_k u, u|^2 \right) \right)^2 \left\langle \left[ \sum_{k=1}^n (|P_k|^2 + |Q_k|^2)^2 \right] u, u \right\rangle \frac{1}{2} + \frac{1}{2^r} \left\langle \left[ \sum_{k=1}^n (|P_k|^4 + |Q_k|^4) \right] u, u \right\rangle^r
$$

$$
\leq \left( \frac{1}{2} (1-\beta) \left( \sum_{k=1}^n |Q_k^*P_k u, u|^2 \right) \right)^2 \left\langle \left[ \sum_{k=1}^n (|P_k|^2 + |Q_k|^2)^2 \right] u, u \right\rangle \frac{1}{2} + \frac{1}{2^r} \left\langle \left[ \sum_{k=1}^n (|P_k|^4 + |Q_k|^4) \right] u, u \right\rangle^r
$$

Taking the supremum over all unit vectors $u \in \mathcal{M}$, we obtain the desired result. \hfill \Box

Another interesting inequality involving the product of two Hilbert space operators is elaborated in the following result that refines (7).

**Theorem 3.** Let $P_k, Q_k \in \mathcal{A}(\mathcal{M})$ ($k = 1, 2, \cdots, n$), $r \geq 1$ and $\beta \in [0, 1]$. Then,

$$
\omega^{2r}(Q_1^*P_1, \cdots, Q_n^*P_n)
\leq \frac{1}{2} \left\| \sum_{k=1}^n (|P_k|^4 + |Q_k|^4) \right\|_r + \frac{1}{\sqrt{2}} (1-\beta) \omega^r(Q_1^*P_1, \cdots, Q_n^*P_n) \left\| \sum_{k=1}^n (|P_k|^4 + |Q_k|^4) \right\|_r
$$

(20)

$$
\leq \frac{1}{2} \left\| \sum_{k=1}^n (|P_k|^4 + |Q_k|^4) \right\|.
$$
Theorem 4. Let \( P \in \mathcal{A}(\mathcal{H}) \) then, for all \( \beta \in [0, 1] \), we have

\[
\omega^2_\beta(Q_1^*P_1, \ldots, S_n^*P_n) = \beta\omega^2_\beta(Q_1^*P_1, \ldots, Q_n^*P_n) + (1 - \beta)\omega^2_\beta(Q_1^*P_1, \ldots, Q_n^*P_n) \\
= \beta\omega^2_\beta(Q_1^*P_1, \ldots, Q_n^*P_n) + (1 - \beta)\omega^2_\beta(Q_1^*P_1, \ldots, Q_n^*P_n) \omega_\beta(Q_1^*P_1, \ldots, S_n^*P_n) \\
\leq \frac{1}{2}\beta \left[ \sum_{k=1}^{n} (|P_k|^{4\beta} + |Q_k|^{4\beta}) \right] + \frac{1}{\sqrt{2}}(1 - \beta)\omega^2_\beta(Q_1^*P_1, \ldots, Q_n^*P_n) \left[ \sum_{k=1}^{n} (|P_k|^{4\beta} + |Q_k|^{4\beta}) \right]^{\frac{1}{2}} \\
\leq \frac{1}{2} \left[ \sum_{k=1}^{n} (|P_k|^{4\beta} + |Q_k|^{4\beta}) \right],
\]

where the first and second inequalities follow from (7), which proves (20). \( \square \)

Now, we present some inequalities concerning the numerical radius of Hilbert space operators beginning with generalizing (15).

Theorem 4. Let \( P_k \in \mathcal{A}(\mathcal{H}) \) \( (k = 1, \ldots, n) \). Then,

\[
w^2_{2p}(P_1, \ldots, P_k) = \frac{1}{2} w_p(P_1^2, \ldots, P_k^2) + \frac{1}{2p+1} \left[ \sum_{k=1}^{n} (|P_k|^2 + |P_k^*|^2)^\frac{p}{2} \right]. \tag{21}
\]

for all \( p \geq 1 \). In particular, we have

\[
w^2_\alpha(P_1, \ldots, P_k) \leq \frac{1}{2} w_k(P_1^2, \ldots, P_k^2) + \frac{1}{4} \left[ \sum_{k=1}^{n} (|P_k|^2 + |P_k^*|^2)^\frac{1}{2} \right]. \tag{22}
\]

Proof. Replacing \( P \) with \( P_k \) in (15), we obtain

\[
|\langle P_k c, c \rangle|^{2p} \leq \left( \frac{1}{2} \left| \langle P_k^2 c, c \rangle \right| + \frac{1}{2} \left( \frac{|P_k|^2 + |P_k^*|^2}{2} \right) c, c \right)^{p} \\
\leq \frac{1}{2} \left[ \left| \langle P_k^2 c, c \rangle \right| + \frac{1}{2p} \left( \left( |P_k|^2 + |P_k^*|^2 \right) c, c \right)^p \right] \\
\leq \frac{1}{2} \left[ \left| \langle P_k^2 c, c \rangle \right| + \frac{1}{2p} \left( \left( |P_k|^2 + |P_k^*|^2 \right) c, c \right)^p \right]
\]

Summing over \( k \), we obtain

\[
\sum_{k=1}^{n} |\langle P_k c, c \rangle|^{2p} \leq \frac{1}{2} \left[ \sum_{k=1}^{n} \left| \langle P_k^2 c, c \rangle \right| + \frac{1}{2p} \sum_{k=1}^{n} \left( \left( |P_k|^2 + |P_k^*|^2 \right) c, c \right)^p \right] \\
= \frac{1}{2} \left[ \sum_{k=1}^{n} \left| \langle P_k^2 c, c \rangle \right| + \frac{1}{2p} \sum_{k=1}^{n} \left( \left( |P_k|^2 + |P_k^*|^2 \right) c, c \right)^p \right]
\]

Taking the supremum over all unit vectors \( c \in \mathcal{H} \), we have

\[
w^2_{2p}(P_1, \ldots, P_k) \leq \frac{1}{2} w_p(P_1^2, \ldots, P_k^2) + \frac{1}{2p+1} \left[ \sum_{k=1}^{n} (|P_k|^2 + |P_k^*|^2)^\frac{p}{2} \right],
\]

and this yields (21); the particular case follows by setting \( p = 1 \) in (21). \( \square \)
Example 1. Let \( P_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \) and \( P_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \) be \( 2 \times 2 \)-matrices. Employing (20) with \( n = 2 \), and \( p = 1 \), we have

\[
\omega_2^e(P_1, P_2) = \sup_{\|z\| = 1} \left( |\langle P_1 z, z \rangle|^2 + |\langle P_2 z, z \rangle|^2 \right) = 3.25,
\]

\[
w_R(P_1^2, P_2^2) = 2
\]

\[
\left\| \sum_{k=1}^2 \left( |P_k|^2 + |P_k^*|^2 \right) \right\| = 9.
\]

Thus,

\[
1.802 = \omega_2(P_1, P_2) \leq \frac{1}{2} w_R(P_1, P_2) + \frac{1}{4} \left\| \sum_{k=1}^2 \left( |P_k|^2 + |P_k^*|^2 \right) \right\| = 1.802
\]

which gives the exact value for \( \omega_2(P_1, P_2) \) in this example. In fact, our bound improves both estimates given in (5) and (6).

Corollary 3. Let \( P_k, Q_k \in \mathcal{A}(\mathcal{M}) \) \((k = 1, \ldots, n)\). Then,

\[
w^2_{2p}(Q_1^* P_1, \ldots, Q_n^* P_n)
\]

\[
\leq \frac{1}{2} w_p \left( |Q_1^* P_1|^2, \ldots, |Q_n^* P_n|^2 \right) + \frac{1}{2p+1} \left\| \sum_{k=1}^n \left( |Q_k^* P_k|^2 + |P_k^* Q_k|^2 \right) \right\|^p. \tag{23}
\]

for all \( p \geq 1 \). In particular, we have

\[
w_2^2(Q_1^* P_1, \ldots, Q_n^* P_n)
\]

\[
\leq \frac{1}{2} w_R \left( |Q_1^* P_1|^2, \ldots, |Q_n^* P_n|^2 \right) + \frac{1}{4} \left\| \sum_{k=1}^n \left( |Q_k^* P_k|^2 + |P_k^* Q_k|^2 \right) \right\|. \tag{24}
\]

Theorem 5. Let \( P_k \in \mathcal{A}(\mathcal{M}) \) \((k = 1, \ldots, n)\). Then,

\[
\omega_2^e(P_1, \ldots, P_n) \leq \beta \left\| \sum_{k=1}^n \left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) \right\|
\]

\[
+ \frac{1}{2} (1 - \beta) \omega_2(P_1, \ldots, P_n) \left\| \sum_{k=1}^n \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) \right\|^2 \tag{25}
\]

for all \( 0 \leq \alpha, \beta \leq 1 \).
Proof. Let \( c \in \mathcal{M} \) be a unit vector. Setting \( d = c \) in (13), it follows that

\[
|\langle P_k c, c \rangle|^2 \leq \beta \left| \langle |P_k|^{2a} c, c \rangle \right| \left| \langle |P_k|^2 (1-a) c, c \rangle \right|
\]

\[
+ (1 - \beta) |\langle P_k c, c \rangle| \sqrt{\langle |P_k|^{2a} c, c \rangle \langle |P_k|^2 (1-a) c, c \rangle}
\]

\[
\leq \beta \left| \langle |P_k|^2 c, c \rangle \right| \left| \langle |P_k|^2 (1-a) c, c \rangle \right|^{1-a}
\]

\[
+ (1 - \beta) |\langle P_k c, c \rangle| \cdot \frac{1}{2} \left( \langle |P_k|^{2a} c, c \rangle + \langle |P_k|^2 (1-a) c, c \rangle \right)
\]

(by (11))

\[
\leq \beta \left[ a \langle |P_k|^2, c \rangle + (1 - a) \langle |P_k|^2, c \rangle \right]
\]

(by (10))

\[
+ \frac{1}{2} (1 - \beta) |\langle P_k c, c \rangle| \left( \langle |P_k|^{2a} + |P_k|^2 (1-a) \rangle c, c \rangle \right)
\]

\[
= \beta \left( a |P_k|^2 + (1 - a)|P_k|^2 \right) c, c \]

\[
+ \frac{1}{2} (1 - \beta) |\langle P_k c, c \rangle| \left( \langle |P_k|^{2a} + |P_k|^2 (1-a) \rangle c, c \rangle \right).
\]

Summing over \( k = 1 \) up to \( k = n \) and then applying the Cauchy–Schwarz inequality for real numbers, we obtain

\[
\sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \leq \beta \sum_{k=1}^{n} \left( \left| \langle a|P_k|^2 + (1-a)|P_k|^2 \rangle c, c \right| \right.
\]

\[
+ \frac{1}{2} (1 - \beta) \sum_{k=1}^{n} |\langle P_k c, c \rangle| \left( \langle |P_k|^{2a} + |P_k|^2 (1-a) \rangle c, c \rangle \right)
\]

\[
\leq \beta \left( \sum_{k=1}^{n} \left( a |P_k|^2 + (1 - a)|P_k|^2 \right) c, c \right)
\]

\[
+ \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{n} \left( \langle |P_k|^{2a} + |P_k|^2 (1-a) \rangle c, c \rangle \right)^2 \right)^{1/2}
\]

\[
\leq \beta \left( \sum_{k=1}^{n} \left( a |P_k|^2 + (1 - a)|P_k|^2 \right) c, c \right)
\]

\[
+ \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{n} \left( \langle |P_k|^{2a} + |P_k|^2 (1-a) \rangle c, c \rangle \right)^2 \right)^{1/2}
\]

Taking the supremum over all unit vectors \( c \in \mathcal{M} \), we obtain the required result in (25). \( \Box \)

The following result extends and generalizes the Kittaneh–Moradi inequality [5] for the Euclidean operator radius.

**Corollary 4.** Let \( P_k \in \mathcal{M}(\mathcal{M}) \) (\( k = 1, 2, \ldots, n \)). Then,

\[
\omega^2_e(P_1, \ldots, P_n) \leq \frac{1}{6} \left| \sum_{k=1}^{n} \left( |P_k|^2 + |P_k|^2 \right) \right| + \frac{1}{3} \omega^2_e(P_1, \ldots, P_n) \left| \sum_{k=1}^{n} \left( |P_k|^{2a} + |P_k|^2 (1-a) \right) \right|^{1/2}
\]

for all \( 0 \leq a, \beta \leq 1. \)
Example 2. Let \( P_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \) and \( P_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \) be \( 2 \times 2 \)-matrices. Employing (26) with \( n = 2, \alpha = \frac{1}{2} \) and \( p = 1 \), we obtain

\[
\omega^2_e(P_1, P_2) = \sup_{\|z\|=1} \left( |\langle P_1 z, z \rangle|^2 + |\langle P_2 z, z \rangle|^2 \right) = 3.25.
\]

However,

\[
1.8027 = \omega_e(P_1, P_2) 
\leq \left( \frac{1}{6} \right) \| P_1^2 + |P_1|^2 + |P_2|^2 + |P_2^*|^2 \| + \frac{1}{3} \omega_e(P_1, P_2) \left( |P_1|^2 + |P_1^*|^2 + (|P_2| + |P_2^*|)^2 \right)^{1/2} 
\leq 1.97577
\]

However, the upper bound in (5) gives 2.5495 and that in (6) gives 2.1213, and this verifies that our bound in (26) is better than both estimates given in (5) and (6).

A refinement of (6) with \( p = 2 \) is incorporated in the following result.

Theorem 6. Let \( P_k \in \mathcal{A}(\mathcal{M}) \). Then,

\[
\omega^2_e(P_1, \cdots, P_n)
\leq \beta \left\| \sum_{k=1}^n \left( \alpha |P_k|^2 + (1-\alpha) |P_k^*|^2 \right) \right\| + (1-\beta) \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^n \left( \alpha |P_k|^2 + (1-\alpha) |P_k^*|^2 \right) \right\|^{1/2}
\leq \left\| \sum_{k=1}^n \left( \alpha |P_k|^2 + (1-\alpha) |P_k^*|^2 \right) \right\|^{1/2}
\]

for all \( 0 \leq \alpha, \beta \leq 1 \).

Proof. Let \( c \in \mathcal{M} \) be a unit vector. Setting \( d = c \) in (13), it follows that

\[
|\langle P_k c, c \rangle|^2 \leq \beta \left( |P_k|^{2\alpha} c, c \right) \left( |P_k|^{2(1-\alpha)} c, c \right) + (1-\beta) |\langle P_k c, c \rangle| \left\| \left( |P_k|^{2\alpha} c, c \right) \left( |P_k|^{2(1-\alpha)} c, c \right) \right\|^{1/2}
\leq \beta \left( |P_k|^{2\alpha} c, c \right) \left| \left( |P_k|^{2(1-\alpha)} c, c \right) \right| + (1-\beta) |\langle P_k c, c \rangle| \left\| \left( |P_k|^{2\alpha} c, c \right) \left( |P_k|^{2(1-\alpha)} c, c \right) \right\|^{1/2}
\leq \beta \left( \alpha |P_k|^2 + (1-\alpha) |P_k^*|^2 \right) c, c \right) + (1-\beta) |\langle P_k c, c \rangle| \left\| \left( \alpha |P_k|^2 + (1-\alpha) |P_k^*|^2 \right) c, c \right\|.
\]

(by (11))

(by (10))
Summing over $k = 1$ up to $k = n$ and then applying the Cauchy–Schwarz inequality for real numbers, we obtain

$$\sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \leq \beta \sum_{k=1}^{n} \left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) c, c$$

$$+ (1 - \beta) \sum_{k=1}^{n} |\langle P_k c, c \rangle| \sqrt{\left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) c, c}$$

$$\leq \beta \left[ \sum_{k=1}^{n} \left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) c, c \right]$$

$$+ (1 - \beta) \left( \sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{n} \left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) c, c \right)^{1/2}$$

$$= \beta \left[ \sum_{k=1}^{n} \left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) c, c \right]$$

$$+ (1 - \beta) \left( \sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{n} \left( \alpha |P_k|^2 + (1 - \alpha) |P_k^*|^2 \right) c, c \right)^{1/2}$$

Taking the supremum over all unit vectors $c \in \mathcal{A}$, we obtain the first in (27).

To obtain the second inequality from the first inequality, we have

$$\omega^2(P_1, \cdots, P_n) \leq \frac{1}{2} \beta \left[ \sum_{k=1}^{n} \left( |P_k|^4 + |P_k^*|^4 \right) \right]$$

$$+ (1 - \beta) \omega_e(P_1, \cdots, P_n) \left[ \sum_{k=1}^{n} \left( |P_k|^4 + |P_k^*|^4 \right) \right]^{1/2}$$

$$\leq \left[ \sum_{k=1}^{n} \left( |P_k|^4 + |P_k^*|^4 \right) \right]$$

(by (6) with $p = 2$)

which proves the required result.

A refinement of (5) with $p = 2$ is incorporated in the following result.

**Theorem 7.** Let $P \in \mathcal{A}$. Then,

$$\omega^2_e(P_1, \cdots, P_n) \leq \frac{1}{2} \beta \left[ \sum_{k=1}^{n} \left( |P_k|^4 + |P_k^*|^4(1 - \alpha) \right) \right]$$

$$+ \frac{1}{\sqrt{2}} (1 - \beta) \omega_e(P_1, \cdots, P_n) \left[ \sum_{k=1}^{n} \left( |P_k|^4 + |P_k^*|^4(1 - \alpha) \right) \right]^{1/2}$$

(28)

for all $0 \leq \alpha, \beta \leq 1$. 
Proof. Let \( c \in \mathcal{M} \) be a unit vector. Setting \( d = c \) and \( p = 2 \) in (13), it follows that

\[
|\langle P_k, c \rangle|^2 \leq \beta \left( |\langle P_k^d, c \rangle|^2 + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle|^2 \right)^{\frac{1}{2}} \\
+ (1 - \beta) |\langle P_k, c \rangle| \sqrt{\left( |\langle P_k^d, c \rangle|^2 + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle|^2 \right)^{\frac{1}{2}}}
\]

\[
\leq \frac{1}{2} \beta \left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right) \\
+ \frac{1}{2} \sqrt{1 - \beta) |\langle P_k, c \rangle| \sqrt{\left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right)^{\frac{1}{2}}}} \tag{by (10)}
\]

Summing over \( k = 1 \) up to \( k = n \) and then applying the Cauchy–Schwarz inequality for real numbers, we have

\[
\sum_{k=1}^{n} |\langle P_k, c \rangle|^2 \leq \frac{1}{2} \beta \sum_{k=1}^{n} \left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right) \\
+ \frac{1}{2} \sqrt{1 - \beta) \sum_{k=1}^{n} \langle P_k, c \rangle| \sqrt{\left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right)^{\frac{1}{2}}}}
\]

\[
\leq \frac{1}{2} \beta \left[ \sum_{k=1}^{n} \left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right) c, c \right] \\
+ \frac{1}{2} \sqrt{1 - \beta) \sum_{k=1}^{n} \langle P_k, c \rangle| \sqrt{\left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right)^{\frac{1}{2}}}}
\]

\[
= \frac{1}{2} \beta \left[ \sum_{k=1}^{n} \left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right) c, c \right] \\
+ \frac{1}{2} \sqrt{1 - \beta) \left( \sum_{k=1}^{n} |\langle P_k, c \rangle|^2 \right)^{\frac{1}{2}} \left[ \sum_{k=1}^{n} \left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right) c, c \right]^{\frac{1}{2}}
\]

Taking the supremum over all unit vectors \( c \in \mathcal{M} \), we obtain the required result. \( \square \)

Corollary 5. Let \( P_k \in \mathcal{M}(\mathcal{M}) \). Then,

\[
\omega_k^2(P_1, \cdots, P_n) \leq \frac{1}{2} \beta \left[ \sum_{k=1}^{n} \left( |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right) c, c \right] \\
+ \frac{1}{2} \sqrt{1 - \beta) \omega_k(P_1, \cdots, P_n) \left( \sum_{k=1}^{n} |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right)^{\frac{1}{2}}}
\]

\[
\leq \frac{1}{2} \left[ |\langle P_k^d, c \rangle| + |\langle P_k^p, |^{4(1-\alpha)}c, c \rangle| \right]
\]

for all \( p \geq 1 \) and \( 0 \leq \alpha, \beta \leq 1 \).
Theorem 8. Let $P_k \in \mathcal{A}(\mathcal{M})$ ($k = 1, 2, \ldots, n$). Then,

$$\omega_2^2(P_1, \cdots, P_n) \leq \frac{1}{2} \beta \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\|$$

$$+ \frac{1}{2} (1 - \beta) \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\| \frac{1}{2}$$

$$\leq \frac{1}{2} \beta \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\|$$

$$+ \frac{1}{2} (1 - \beta) \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) \right\| \frac{1}{2}$$

(29)

as required.

The following two results extend the generalized Kittaneh–Moradi inequality (26).

Proof. From (28), we have

$$\omega_2^2(P_1, \cdots, P_n) \leq \frac{1}{2} \beta \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\|$$

$$+ \frac{1}{\sqrt{2}} (1 - \beta) \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\| \frac{1}{2}$$

$$\leq \frac{1}{2} \beta \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\|$$

$$+ \frac{1}{2} (1 - \beta) \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) \right\| \frac{1}{2}$$

(by (28) with $\beta = 0$)

for all $0 \leq \alpha, \beta \leq 1$.

Proof. Let $c \in \mathcal{A}$ be a unit vector. Setting $d = c$ in (12) with $p = 1$, it follows that

$$|\langle P_k c, c \rangle|^2 \leq \beta \langle |P_k|^{2\alpha} c, c \rangle \langle |P_k^*|^{2(1-\alpha)} c, c \rangle$$

$$+ (1 - \beta) |\langle P_k c, c \rangle| \sqrt{\langle |P_k|^{2\alpha} c, c \rangle \langle |P_k^*|^{2(1-\alpha)} c, c \rangle}$$

$$\leq \frac{1}{2} \beta \left( \langle |P_k|^{2\alpha} c, c \rangle \right)^2 + \left( \langle |P_k^*|^{2(1-\alpha)} c, c \rangle \right)^2$$

(by (10))

$$+ \frac{1}{2} (1 - \beta) |\langle P_k c, c \rangle| \langle \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) c, c \rangle$$

(by (10))

$$\leq \frac{1}{2} \beta \left( \langle |P_k|^{4\alpha} c, c \rangle + \langle |P_k^*|^{4(1-\alpha)} c, c \rangle \right)$$

(by (11))

$$+ \frac{1}{2} (1 - \beta) |\langle P_k c, c \rangle| \langle \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) c, c \rangle$$

$$= \frac{1}{2} \beta \left( \langle |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \rangle c, c \rangle$$

(by (11))

$$+ \frac{1}{2} (1 - \beta) |\langle P_k c, c \rangle| \langle \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) c, c \rangle$$
Summing over \( k = 1 \) up to \( k = n \), and then applying the Cauchy–Schwarz inequality for real numbers, we obtain
\[
\sum_{k=1}^{n} |\langle P_k c, c \rangle|^2 \leq \frac{1}{2} \beta \sum_{k=1}^{n} \left( |\langle P_k c, c \rangle| \langle |P_k|^{4(1-a)} + |P_k^*|^{4(1-a)} \rangle, c, c \right)
\]
\[
+ \frac{1}{2} (1 - \beta) \sum_{k=1}^{n} \left( |\langle P_k c, c \rangle| \langle |P_k|^{2(1-a)} + |P_k^*|^{2(1-a)} \rangle, c, c \right)
\]
\[
\leq \frac{1}{2} \beta \left( \sum_{k=1}^{n} \left( |\langle P_k c, c \rangle| \langle |P_k|^{4(1-a)} + |P_k^*|^{4(1-a)} \rangle, c, c \right) \right)^{\frac{1}{2}}
\]
\[
+ \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} \left( |\langle P_k c, c \rangle| \langle |P_k|^{2(1-a)} + |P_k^*|^{2(1-a)} \rangle, c, c \right) \right)^{\frac{1}{2}}
\]
\[
= \frac{1}{2} \beta \left( \sum_{k=1}^{n} \left( |\langle P_k c, c \rangle| \langle |P_k|^{4(1-a)} + |P_k^*|^{4(1-a)} \rangle, c, c \right) \right)^{\frac{1}{2}}
\]
\[
+ \frac{1}{2} (1 - \beta) \left( \sum_{k=1}^{n} \left( |\langle P_k c, c \rangle| \langle |P_k|^{2(1-a)} + |P_k^*|^{2(1-a)} \rangle, c, c \right) \right)^{\frac{1}{2}}
\]
We obtain the required result by taking the supremum over all unit vectors \( c \in \mathcal{M} \).

Alomari [24] proved a refinement of Kittaneh–Moradi [27], which is better than the result of Kittaneh and Moradi. An extension of Alomari’s inequality (3.9, Ref. [24]) to the Euclidean operator radius is considered in the following result.

**Theorem 9.** Let \( P_k \in \mathcal{A}(\mathcal{M}) \) \( (k = 1, 2, \cdots, n) \). Then,
\[
\omega_2^2(P_1, \cdots, P_n) \leq \frac{1}{4} \lambda \left\| \sum_{i=1}^{n} \left( |P_i|^{2a} + |P_i^*|^{2(1-a)} \right)^2 \right\|^{\frac{1}{2}}
\]
\[
+ \frac{1}{2} (1 - \lambda) \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{2a} + |P_k^*|^{2(1-a)} \right)^2 \right\|^{\frac{1}{2}}
\]
\[
(30)
\]
for all \( \lambda \in [0, 1] \). In particular, we have
\[
\omega_2^2(P_1, \cdots, P_n) \leq \frac{1}{12} \left\| \sum_{i=1}^{n} \left( |P_i|^{2a} + |P_i^*|^{2(1-a)} \right)^2 \right\|
\]
\[
+ \frac{1}{3} \omega_e(P_1, \cdots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{2a} + |P_k^*|^{2(1-a)} \right)^2 \right\|^{\frac{1}{2}}
\]
\[
(31)
\]
\[
\leq \frac{1}{2} \left\| \sum_{k=1}^{n} \left( |P_k|^{2a} + |P_k^*|^{2(1-a)} \right)^2 \right\|.
\]
Proof. Form (29) and (5), and for all \( \lambda \in [0,1] \), we have

\[
\omega_\epsilon^2(P_1, \ldots, P_n) = (1-\lambda)\omega_\epsilon^2(P_1, \ldots, P_n) + \lambda \omega_\epsilon^2(P_1, \ldots, P_n)
\]

\[
\leq \frac{1}{2}(1-\lambda)\beta \left\| \sum_{k=1}^{n} \left( |P_k|^{4\alpha} + |P_k^*|^{4(1-\alpha)} \right) \right\| + \frac{1}{2}(1-\lambda)(1-\beta)\omega_\epsilon(P_1, \ldots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) \right\|^2
\]

\[
+ \frac{1}{4}\lambda \left\| \sum_{i=1}^{n} \left( |P_i|^{2\alpha} + |P_i^*|^{2(1-\alpha)} \right) \right\|^2
\]

(by (5) with \( p = 2 \))

Setting \( \beta = 0 \), we obtain

\[
\omega_\epsilon^2(P_1, \ldots, P_n) \leq \frac{1}{4}\lambda \left\| \sum_{i=1}^{n} \left( |P_i|^{2\alpha} + |P_i^*|^{2(1-\alpha)} \right) \right\|^2 + \frac{1}{2}(1-\lambda)\omega_\epsilon(P_1, \ldots, P_n) \left\| \sum_{k=1}^{n} \left( |P_k|^{2\alpha} + |P_k^*|^{2(1-\alpha)} \right) \right\|^2
\]

which gives the required result. The particular case follows by choosing \( \lambda = \frac{1}{2} \). The second inequality in (31) follows directly from (5). \( \square \)

Hence, as pointed out above, (31) is stronger than (26), as well as (31) is much better than the inequalities (5) and (29).

Example 3. Let \( C_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \) and \( C_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \) be \( 2 \times 2 \)-matrices. Employing (31) with \( n = 2, \alpha = \frac{1}{2} \) and \( p = 2 \), we obtain

\[
\omega_\epsilon^2(C_1, C_2) = \sup_{||z||=1} \left( ||\langle C_1 z, z \rangle||^2 + ||\langle C_2 z, z \rangle||^2 \right) = 3.25,
\]

\[
1.802775638 = \omega_\epsilon(C_1, C_2) \leq \sqrt{\frac{1}{12} \left\| \sum_{i=1}^{2} \left( |C_i| + |C_i^*| \right) \right\|^2 + \frac{1}{3} \omega_\epsilon(C_1, C_2) \left\| \sum_{k=1}^{2} \left( |C_k| + |C_k^*| \right) \right\|^2}
\]

\[
= 1.802775638 \leq \left\| \sum_{k=1}^{2} \left( |C_k| + |C_k^*| \right) \right\|^2 = 2.549509757.
\]

Fortunately, our bound in (31) gives the exact value of \( \omega_\epsilon(C_1, C_2) \), which, as shown, is better than both estimates given by (5) and (6) (see Example 2). Moreover, this example proves that our inequality in (31) is better than (26).

Remark 1. All obtained results of Section 3 are valid for the generalized Euclidean operator radius \( \omega_p(\cdot) \) \( (p \geq 1) \) by using (12) instead of (13). We leave the rest of the generalizations to the interested reader.
4. The Davis–Wielandt Radius-Type Inequalities

One of the most recent and interesting generalizations of the numerical range of Hilbert space operators is the Davis–Wielandt shell, which is well known as

$$DW(Q) = \{ ((Qc, c), \langle Qc, Qc \rangle), c \in M, \|c\| = 1 \}$$

for any $Q \in \mathcal{A}(M)$. Clearly, the projection of the set $DW(Q)$ on the first coordinate is $W(Q)$.

The Davis–Wielandt shell and its radius were introduced and described firstly by Davis in [28,29] and Wielandt [30]. The Davis–Wielandt radius of $Q \in \mathcal{A}(M)$ is defined as

$$dw(Q) = \sup_{c \in M, \|c\| = 1} \left\{ \sqrt{\|Qc\|^2 + \|Qc\|^4} \right\}.$$  

One can easily check that $dw(Q)$ is unitarily invariant, but it does not define a norm on $\mathcal{A}(M)$.

It is shown that [25]

$$\max\{\omega(Q), \|Q\|^2\} \leq dw(Q) \leq \sqrt{\omega^2(Q) + \|Q\|^4} \quad (32)$$

for all $Q \in \mathcal{A}(M)$. The inequalities are sharp. For further results concerning Davis–Wielandt radius inequalities, the reader may refer to [12,13,15,18,31–40].

The Euclidean Davis–Wielandt radius has been introduced in [19]. In fact, for an $n$-tuple $M = (M_1, \cdots, M_n) \in \mathcal{A}(M)^n := \mathcal{A}(M) \times \cdots \times \mathcal{A}(M)$, i.e., for $M_1, \cdots, M_n \in \mathcal{A}(M)$, one of the most interesting generalizations of the Davis–Wielandt radius $dw(\cdot)$ is the Euclidean Davis–Wielandt radius, which is defined as

$$dw_e(M_1, \cdots, M_n) = \sup_{\|c\| = 1} \left( \sum_{i=1}^n \left| \langle M_i c, c \rangle \right|^2 + \|M_i c\|^4 \right)^{1/2}. \quad (33)$$

Indeed, a suitable relation between the Euclidean operator radius and the Euclidean Davis–Wielandt radius (33) can be constructed as follows.

For any positive integer $n$, let $G_i \in \mathcal{A}(M)$ ($i = 1, \cdots, 2n$). Therefore, we have

$$w_e(G_1, \cdots, G_{2n}) := \sup_{\|c\| = 1} \left( \sum_{i=1}^{2n} \left| \langle G_i c, c \rangle \right|^2 \right)^{1/2} \quad \text{for all } c \in M.$$

Let $M_i \in \mathcal{A}(M)$ ($i = 1, \cdots, n$). Define the sequence of operators $M_i$ in terms of $G_i$, such that

$$G_1 = M_1, \quad G_2 = M_1^* M_1; \quad G_3 = M_2, \quad G_4 = M_2^* M_2; \quad G_5 = M_3, \quad G_6 = M_3^* M_3; \quad \vdots \quad G_{2n-1} = M_n, \quad G_{2n} = M_n^* M_n.$$
Now, we have

\[
\begin{align*}
    w_e(G_1, \cdots, G_{2n}) &:= \sup_{\|c\|=1} \left( \sum_{k=1}^{2n} |(G_k c, c)|^2 \right)^{1/2} \\
    &= \sup_{\|c\|=1} \left( \sum_{j=1}^{n} \left( |(M_j c, c)|^2 + |(M_j^* M_j c, c)|^2 \right) \right)^{1/2} \\
    &= dw_e(M_1, \cdots, M_n).
\end{align*}
\]

which gives a very elegant relation between the Euclidean operator radius and the Euclidean Davis–Wielandt radius.

In light of the above construction, we have

**Theorem 10** ([19]). [Theorem 3.4] Let \( Q_i \in \mathcal{A}(\mathcal{M}) \) \((i = 1, \cdots, n)\). Then,

\[
\max \left\{ w_e(Q_1, \cdots, Q_n), w_e(|Q_1|^2, \cdots, |Q_n|^2) \right\} \leq dw_e(Q_1, \cdots, Q_n) \leq w_e(Q_1, \cdots, Q_n) + w_e(|Q_1|^2, \cdots, |Q_n|^2).
\]

One can generalize the results in Section 3 by following the same procedure above. A very powerful inequality has been proven recently by Alomari [19], as follows:

\[
\frac{1}{4} \left\| \sum_{k=1}^{n} \left( |Q_k|^2 + |Q_k^*|^2 + 2|Q_k|^4 \right) \right\| \leq dw_e(Q_1, \cdots, Q_n) \leq \frac{1}{2} \left\| \sum_{k=1}^{n} \left( |Q_k|^2 + |Q_k^*|^2 + 2|Q_k|^4 \right) \right\|. \tag{34}
\]

We finish our results by obtaining a new bound for the Davis–Wielandt radius \( dw(\cdot) \). To do so, we need the following observation.

**Lemma 7** ([19]). [Lemma 2] Let \( Q \in \mathcal{A}(\mathcal{M}) \). Then,

\[
w_e(Q, Q^* Q) = dw(Q). \tag{35}
\]

**Theorem 11.** Let \( Q \in \mathcal{A}(\mathcal{M}) \). Then,

\[
dw(Q) \leq \sqrt{\frac{w(Q^2) + \|Q\|^4}{2} + \frac{1}{4} \left\| |Q|^2 + |Q^*|^2 + 2|Q|^4 \right\|}. \tag{36}
\]

**Proof.** Replacing \( P \) with \( P_k \) \((k = 1, \cdots, n)\) in Lemma 5, we obtain

\[
|(P_k c, c)|^2 \leq \frac{1}{2} \left\langle P_k^2 c, c \right\rangle + \frac{1}{4} \left\langle |P_k|^2 + |P_k^*|^2 \right\rangle c, c \rangle.
\]

Summing over \( k \), we obtain

\[
\sum_{k=1}^{n} |(P_k c, c)|^2 \leq \frac{1}{2} \sum_{k=1}^{n} \left\langle P_k^2 c, c \right\rangle + \frac{1}{4} \sum_{k=1}^{n} \left\langle |P_k|^2 + |P_k^*|^2 \right\rangle c, c \rangle
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} \left\langle P_k^2 c, c \right\rangle + \frac{1}{4} \sum_{k=1}^{n} \left\langle |P_k|^2 + |P_k^*|^2 \right\rangle c, c \rangle
\]
Taking the supremum over all unit vectors $c \in \mathcal{M}$, we have
\[
    w_e^2(P_1, \ldots, P_k) \leq \frac{1}{2} w_R(P_1^2, \ldots, P_k^2) + \frac{1}{4} \sum_{k=1}^{n} \left( |P_k|^2 + |P_k^*|^2 \right).
\] (37)

For $n = 2$, we have
\[
    w_e^2(P_1, P_2) \leq \frac{1}{2} w_R(P_1^2, P_2^2) + \frac{1}{4} \left( |P_1|^2 + |P_1^*|^2 \right) + \left( |P_2|^2 + |P_2^*|^2 \right). \quad \text{(38)}
\]

Now, setting $P_1 = Q$ and $P_2 = Q^* Q$ in (38), by Lemma 7, we have
\[
    dw_e^2(Q) = w_e^2(Q, |Q|^2) \leq \frac{1}{2} w_R(Q^2, |Q|^4) + \frac{1}{4} \left( |Q|^2 + |Q^*|^2 + 2|Q|^4 \right).
\]

However, since
\[
    w_R(Q^2, |Q|^4) = \sup_{\|c\| = 1} \left\{ |\langle Q^2 c, c \rangle| + \left| \langle Q^4 c, c \rangle \right| \right\} \leq w(Q^2) + \|Q\|^4,
\]

Then, the inequality (36) follows from the previous inequality. \(\Box\)

**Example 4.** Let $Q = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ be a $2 \times 2$-matrix. Employing (31), we have
\[
    dw(Q) \leq \sqrt{w(Q^2) + \|Q\|^4} + \left( \frac{1}{4} \left( |Q|^2 + |Q^*|^2 + 2|Q|^4 \right) \right) = 4.123105626,
\]

which is better than both estimates given in (32) (=4.472135954) and in (34) (=4.242640686). It is convenient to note that, according to (32), the lower bound of $dw(Q) \geq 4$. Fortunately, the definition of the Davis–Wielandt radius gives
\[
    dw(Q) = \sup_{c \in \mathcal{M}, \|c\| = 1} \left\{ \sqrt{|\langle Qc, c \rangle|^2 + \|Qc\|^4} \right\} = 4.123105626
\]

which is exactly our estimate. This implies that our estimate in (36) is very close to the exact value, in general.

**Remark 2.** All obtained results of Section 3 are valid for the generalized Euclidean Davis–Wielandt radius by noting that the number of operators should be $2n$ instead of $n$ and the previously mentioned sequence of operators. We leave the rest of the generalizations to the interested reader.

5. Conclusions
This work brings together, with several refinements, inequalities for the Euclidean operator radius $\omega_e(\cdot)$. Namely, it is shown that the inequalities (17)–(31) are much better than (5)–(7). This is shown mathematically and supported with several examples. In fact, some of the obtained results are sharper than other inequalities. Among others, (26), (31), (34), and (36) are the most interesting improved refinements of the obtained inequalities. Nevertheless, the other presented inequalities are still better than (5) and (6) and all amplify their inequalities. Supporting our assertions with various examples, we show that our results are much better than all older and earlier inequalities. Finally, an interesting new bound for the Davis–Wielandt radius (36) is established. We note that our result could be generalized for the generalized operator radius $\omega_p(\cdot)$; we leave the details to the interested reader.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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