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Metric-Based Fractional Dimension of Rotationally-Symmetric Line Networks

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Abstract: The parameter of distance plays an important role in studying the properties symmetric networks such as connectedness, diameter, vertex centrality and complexity. Particularly different metric-based fractional models are used in diverse fields of computer science such as integer programming, pattern recognition, and in robot navigation. In this manuscript, we have computed all the local resolving neighborhood sets and established sharp bounds of a metric-based fractional dimension called by the local fractional metric dimension of the rotationally symmetric line networks of wheel and prism networks. Furthermore, the bounded and unboundedness of these networks is also checked under local fractional metric dimension when the order of these networks approaches to infinity. The lower and upper bounds of local fractional metric dimension of all the rotationally symmetric line networks is also analyzed by using 3D shapes.

Keywords: local fractional metric dimension; local resolving neighborhood sets; rotationally symmetric line networks

MSC: 05C12; 05C90; 05C15; 05C62

1. Introduction

The fundamental concepts of the metric dimension (MD) of connected networks were first revealed by Slater in 1975 [1] and the notion of MD was initiated by Melter and Hararay in 1976 [2]. Robot navigation in a network space was studied by Khuller et al. with the help of MD [3]. Gerey and Johnson proved that computing MD for any connected network is an NP-complete problem in general [4]. Melter and Tomescu studied metric basis in digital geometry and they also computed the MD of grid-related networks [5]. MD has applications in the processing of maps, pattern reorganization, robot navigation [6], network discovery and verifcation [7], hierarchical lattice [8], pharmaceutical chemistry, and in integer programming [9]. Since then, various types of MD such as edge MD [10], mixed MD [11], K MD [12], partition dimension [13] have been discovered. The general definition of cone metric spaces in the context of neutrosophic cone metric space theory was developed by Al-Omeri et al. and they have developed some fundamental results as well [14]; to study the common fixed point theorems in neutrosophic cone metric space, see [15]. The most recent development in this field of MD has been made by Bokhary et al., and they have computed the MD of subdivision of circulant networks [16].

The notion of fractional metric dimension (FMD) has been introduced by Currie and Hellerman, they proposed that the finding of the FMD of a network is formulated as a certain integer programming problem [17] and the idea of FMD in the field of networking theory was introduced by S. Arumugam and V. Mathew. They have developed different techniques to find the FMD of diversely connected networks and they have also obtained the exact value of FMD of famous networks such as Petersen, cycle, friendship, hypercubes,
wheel, and grid networks [18]. Feng et al. determined the FMD of distance regular networks and FMD of Hamming and Johnson networks. Moreover, they proposed an inequality for the MD and FMD [19] and FMD of trees, and unicyclic networks were obtained by Kismanto et al. [20]. Zafar et al. obtained the exact value of the FMD of prism and path-related networks [21].

The latest invariant of FMD called local fractional metric dimension (LFMD) was introduced by Asiyah et al., and they computed the exact value of the LFMD of the corona product of different types of networks [22]. Liu et al. derived some significant results on the upper bounds of LFMD of rotationally symmetric and planner networks [23] and Ali et al. recently extended the work of Liu et al. and also computed upper bounds of the LFMD of some rotationally symmetric planner networks [24]. Javaid et al. established the bounds of the LFMD of all the networks and they also obtained the exact value of the LFMD of path, cycle, bipartite, and complete networks [25]. The lower bound of the LFMD is improved by Javaid et al. and they also established the bounds of the LFMD of antiprism and sunflower networks [26]. Since discovering the bounds of the LFMD of generalized sunlet [27] and convex polytopes, [28,29], Sierpinski networks have been established [30].

Now, we are presenting some applications of MD in the field of chemical graph theory; the chemical graph theory applies in chemistry and focuses on the molecular topology. After converting a chemical structure into a specific network, a comprehensive structural analysis can be performed. Some of the chemical compounds are considered as functional groups, where atoms represented by nodes and bonds among them represented by edges. By using the idea of characteristic polynomials the different common substructures are characterized and the certain resolving sets are used to find the specific position when two chemical structures have the same functional group. This study has been used in pharmaceutical activities and in drug discovery [9,31].

This article is an extension of work done by Ali et al. [24], as they have established upper bounds of LFMDs for certain rotationally symmetric networks. In this manuscript, our aim is to compute both the upper and lower bounds of LFMDs of rotationally symmetric line networks of wheels and prism networks. The detail of line networks prism and wheel network is given from Figures 1 and 2, 3D representation of all the obtained results is given from Figures 3–9. The boundedness and unboundedness of all these networks is also obtained. Section 2 contains preliminary concepts, Section 3 deals with the main results, and Section 4 represents the conclusion of the manuscript.

![Figure 1. Wheel network \(W_t\) and its line network \(LW_t\).]
Figure 2. Prism network $D_t$ and its line network $LD_t$.

Figure 3. Graphical representation of lower bound of LFMD of $LW_t$, when $t \equiv 1 \pmod{2}$.

Figure 4. Graphical representation of upper bound of LFMD of $LW_t$, when $t \equiv 1 \pmod{2}$ and $t \equiv 0 \pmod{2}$. 
Figure 5. Graphical representation of lower bound of LFMD of LW, when $t \equiv 0 \pmod{2}$.

Figure 6. Graphical representation of lower bound of LFMD of LD, when $t \equiv 1 \pmod{2}$.

Figure 7. Graphical representation of upper bound of LFMD of LD, when $t \equiv 1 \pmod{2}$. 
2. Preliminaries

Let \( B \) be a connected network with vertex set \( V = V(B) \) and edge set \( E = E(B) \). A walk between two vertices \( u_i \) and \( u_j \) is the sequence of edges and vertices. A path between two vertices \( u_i \) and \( u_j \) is a walk in which neither vertex nor edge is repeated. The distance between any two vertices \( u_i \) and \( u_j \) \((d(u_i, u_j))\) is the length of the shortest path connecting them. For further study about the preliminary concepts of networking theory see [32]. A vertex \( u \in V(B) \) resolves a pair \((v, w)\) if the distance from \( u \) to \( v \) is not equal to the distance from \( u \) to \( w \) \((d(u, v) \neq d(u, w))\). Let \( L = \{u_1, u_2, u_3, ..., u_t\} \subseteq V(B) \), then \( t \) tuple representation of \( v \) with respect to \( L \) is \( d(u|L) = (d(v, u_1), d(v, u_2), d(v, u_3), ..., d(v, u_t)) \). If distinct elements of \( B \) have a unique representation with respect to \( L \), then \( L \) becomes a resolving set. The minimum cardinality of a resolving set is called \( MD \) of \( B \), thus \( MD \) of \( B \) is defined as follows:

\[
\dim(B) = \min\{|L| : L \text{ is a resolving set of } B\}.
\]

In a connected network \( B \) for \( uv \in E(B) \) a vertex \( x \in V(B) \) is said to resolve adjacent pairs of vertices as \( L_x(uv) = \{x \in V(B) : d(x, u) \neq d(x, u)\} \) and it is called a local resolving neighborhood (LRN) set of an edge \( uv \in E(B) \). A real-valued function \( \lambda : V(B) \to [0, 1] \) is
called a local resolving function (LRF) of \( B \) if \( \lambda(L_r(uv)) \geq 1 \) for an edge \( uv \in E(B) \), where 
\[
\lambda(L_r(uv)) = \sum_{x \in L_r(uv)} \lambda(x).
\] An LRF \( \lambda \) is called a minimal LRF if there exists another LRF \( \lambda' : V(B) \to [0,1] \) such that \( |\lambda'| < |\lambda| \) and \( \lambda'(x) \neq \lambda(x) \) for at least \( x \in V(B) \) that is not LRF of \( B \). Thus, the LFMD of \( B \) is defined as follows:

\[
\text{Ldim}_F(B) = \min \{|\lambda| : \lambda \text{ is a minimal local resolving function of } B\}.
\]

A line network \( L(B) \) of \( B \) is a network whose vertices are the edges of \( B \), and two vertices \( u,v \in L(B) \) are connected if they have a common end vertex in \( B \). For more results about line networks and their MD, we refer [33,34].

3. Main Results

In this section, we have computed the LRN sets of rotationally symmetric line networks of prism and wheel networks and the bounds of the LFMDs of these networks are also established. Furthermore, all the theorems are divided into two cases, the case 1 is particular and case 2 is general.

3.1. LRN Sets and LFMD of Line Network of Wheel Network

In this subsection, our aim to compute the LRN sets and the LFMDs of the line network of wheel networks. The network is defined as follows:

Let \( LW_t \) be a line network of a wheel network with a vertex set \( V(LW_t) = \{a_i,b_i : 1 \leq i \leq t\} \) and edge set \( E(LW_t) = \{b_ib_{i+1},a_ib_i,\ldots,a_{i+1}a_i : 1 \leq i \leq t\} \) with order \( 2t \) and size \( \frac{t^2 - t}{2} \). For more information about \( LW_t \), see Figure 1.

Lemma 1. Let \( LW_t \) be the line network of wheel network, where \( t \equiv 1 \pmod{2} \). Then

(a) \( |L_r(a_i,a_{i+1})| = 4 \) and \( \bigcup_{i=1}^{t} L_r(a_i,a_{i+1}) = |V(LW_t)| \).

(b) \( |L_r(a_i,a_{i+1})| \leq |L_r(x)| \) and \( |L_r(x) \cap \bigcup_{i=1}^{t} L_r(a_i,a_{i+1})| \geq |L_r(a_i,a_{i+1})| \forall x \in E(LW_t) \).

Proof. Consider \( a_i \) inner and \( b_i \) are outer vertices of \( LW_t \), where \( 1 \leq i \leq t \) and \( t+1 \equiv 1 \pmod{t} \).

(a) \( L_r(a_i,a_{i+1}) = \{a_i,a_{i+1},b_i,b_{i+1}\} \) therefore, \( |L_r(a_i,a_{i+1})| = 4 \) also \( \bigcup_{i=1}^{t} L_r(a_i,a_{i+1})| = 2t = |V(LW_t)| \).

(b) The LRN sets other than \( L_r(a_i,a_{i+1}) \) are \( L_r(b_1b_{i+1}) = V(LW_t) - \{b_1b_{2i+1},a_{i+1},b_1b_{i+2},b_1b_{i+3},\ldots,b_{i+2i-2},b_{i+2i-1}\} \), \( L_r(a_ib_i) = V(LW_t) - \{a_{i+1},b_{i+2},b_{i-1}\} \). Since \( \bigcup_{i=1}^{t} L_r(a_i,a_{i+1}) = V(LW_t), |L_r(x) \cap \bigcup_{i=1}^{t} L_r(a_i,a_{i+1})| \geq |L_r(a_i,a_{i+1})| \). The comparison among the cardinalities of all the LRN sets of \( LW_t \) is given in Table 1.

\[\begin{array}{|c|c|}
\hline
\text{LRN Set} & \text{Cardinality} \\
\hline
L_r(a_1b_1) & 2t - 3 > 4 \\
L_r(b_1b_{i+1}) & 2t - 4 > 4 \\
\hline
\end{array}\]

It is clear from Table 1 that \( |L_r(a_i,a_{i+1})| < |L_r(x)| \), where \( L_r(x) \) are the other LRN sets of \( LW_t \).
Theorem 1. Let LW₃ be a line network of generalized wheel network, then

\[ \text{Ldim}_F(LW_3) = \frac{3}{2}. \]

Proof. The LRN sets of LW₃ are as follows:

\[
\begin{align*}
L_r(b_1b_2) &= \{b_1, b_2, a_1, a_3\}, \\
L_r(b_2b_3) &= \{b_2, b_3, a_2, a_1\}, \\
L_r(b_3b_1) &= \{b_3, b_1, a_3, a_2\}, \\
L_r(a_1b_1) &= \{a_1, a_3, b_1, b_2\}, \\
L_r(a_2b_2) &= \{a_2, a_1, b_2, b_3\}, \\
L_r(a_3b_3) &= \{a_3, a_2, b_3, b_1\}, \\
L_r(a_1a_2) &= \{a_1, a_2, b_2, b_3\}, \\
L_r(a_2a_3) &= \{a_2, a_3, b_3, b_1\}, \\
L_r(a_3a_4) &= \{a_3, a_4, b_4, b_2\}, \\
L_r(a_4a_5) &= \{a_4, a_5, b_5, b_3\}, \\
L_r(a_5a_1) &= \{a_5, a_1, b_1, b_2\}.
\end{align*}
\]

From above, LRN sets the cardinality of all the LRN sets as 4, therefore, we define a constant mapping \(\lambda(V(LW_3)) \rightarrow [0, 1]\) as \(\lambda = \frac{1}{4}\) to each \(v \in V(LW_3)\), hence

\[ \text{Ldim}_F(LW_3) = \sum_{i=1}^{6} \frac{1}{4} = \frac{3}{2}. \]

\(\Box\)

Theorem 2. Let LWₙ be a line network of a wheel network, where \(t \equiv 1 \pmod{2}\). Then

\[ \frac{2t}{2t-3} \leq \text{Ldim}_F(LW_t) \leq \frac{t}{2}. \]

Proof. To prove the theorem, we have divided it in two cases:

Case 1:

For \(t = 5\), we have following LRN sets:

\[
\begin{align*}
L_r(b_1b_2) &= V(LW_5) - \{b_4, a_2, a_4, a_3\}, \\
L_r(b_2b_3) &= V(LW_5) - \{b_5, a_3, a_5, a_1\}, \\
L_r(b_3b_4) &= V(LW_5) - \{b_1, a_4, a_1, a_2\}, \\
L_r(b_4b_5) &= V(LW_5) - \{b_2, a_5, a_2, a_3\}, \\
L_r(b_5b_1) &= V(LW_5) - \{b_3, a_1, a_3, a_4\}, \\
L_r(a_1b_1) &= V(LW_5) - \{b_3, b_4, a_2\}, \\
L_r(a_2b_2) &= V(LW_5) - \{b_4, b_5, a_3\}, \\
L_r(a_3b_3) &= V(LW_5) - \{b_5, b_1, a_4\}, \\
L_r(a_4b_4) &= V(LW_5) - \{b_1, b_2, a_5\}, \\
L_r(a_5b_5) &= V(LW_5) - \{b_2, b_3, a_1\}, \\
L_r(a_1a_2) &= \{a_1, a_2, b_2, b_3\}, \\
L_r(a_2a_3) &= \{a_2, a_3, b_3, b_1\}, \\
L_r(a_3a_4) &= \{a_3, a_4, b_4, b_2\}, \\
L_r(a_4a_5) &= \{a_4, a_5, b_5, b_3\}, \\
L_r(a_5a_1) &= \{a_5, a_1, b_1, b_2\}.
\end{align*}
\]

From above, LRN sets the minimum cardinality of LRN set \(L_r(a_ia_{i+1})\) as 4, where \(1 \leq i \leq 5\); therefore, we define a minimal LRF \(\lambda(V(LW_5)) \rightarrow [0, 1]\) as \(\lambda = \frac{1}{4}\) to each \(v \in V(LW_5)\), hence \(\text{Ldim}_F(LW_5) \leq \sum_{i=1}^{10} \frac{1}{4} = \frac{5}{2}\). The maximum cardinality of (LRN) set \(L_r(a_ib_i)\) is 7; therefore, we define a maximal LRF \(\lambda'(V(LW_5)) \rightarrow [0, 1]\) as \(\lambda' = \frac{1}{7}\) to each \(v \in V(LW_5)\), hence

\[ \text{Ldim}_F(LW_5) \geq \sum_{i=1}^{10} \frac{1}{7} = \frac{10}{7}. \]

\[ \frac{10}{7} \leq \text{Ldim}_F(LW_5) \leq \frac{5}{2}. \]
Case 2:
For \( t \geq 7 \), in the view of Lemma 1 the cardinality of LRN set \( L_r(a_ia_{i+1}) \) is 4 and \( |L_r(a_ia_{i+1})| < |L_r(x)| \), where \( L_r(x) \) are other LRN sets of \( LW_t \). Therefore, we define a minimal, \( \lambda(V(LW_t)) \rightarrow [0,1] \) as \( \frac{1}{t} \) to each \( v \in V(LW_t) \), hence \( Ldim_F(LW_t) \leq \frac{2t}{t-3} \).
In the same context, by Lemma 1, the maximum cardinality of LRN set \( L_r(a_ib_i) \) is \( 2t-3 \) and \( |L_r(a_ib_i)| > |L_r(x)| \), where \( L_r(x) \) are other LRN sets of \( LW_t \). Therefore, we define a maximal \( \lambda'(V(LW_t)) \rightarrow [0,1] \) as \( \frac{1}{2t-3} \) to each \( v \in V(LW_t) \), hence \( Ldim_F(LW_t) \)
\[ \geq \frac{2t}{2t-3} = \frac{2t}{t-3}. \]
\[ \frac{2t}{2t-3} \leq Ldim_F(LW_t) \leq \frac{t}{2} . \]
\[ \Box \]

**Lemma 2.** Let \( LW_t \) be the line network of wheel network then, where \( t \equiv 0(\text{mod} \ 2) \). Then

(a) \( |L_r(a_ia_{i+1})| = 4 \) and \( \bigcup_{i=1}^{t} L_r(a_ia_{i+1}) = V(LW_t); \)

(b) \( |L_r(a_ia_{i+1})| \leq |L_r(x)| \) and \( |L_r(x) \cap \bigcup_{i=1}^{t} L_r(a_ia_{i+1})| \geq |L_r(a_ia_{i+1})|, \forall x \in E(LW_t). \)

**Proof.** Consider \( a_i \) inner and \( b_i \) are outer vertices of \( LW_t \), where \( 1 \leq i \leq t \) and \( t+1 \equiv 1(\text{mod} \ t) \).

(a) \( L_r(a_ia_{i+1}) = \{a_i, a_{i+1}, b_{i+1}, b_{i+1-1}\} \), therefore, \( |L_r(a_ia_{i+1})| = 4 \) also \( \bigcup_{i=1}^{t} L_r(a_ia_{i+1})| = 2t. \)

(b) The LRN sets other than \( L_r(a_ia_{i+1}) \) are \( L_r(b_ib_{i+1}) = \{b_i, b_{i+1}, b_{i+1}, b_{i+2}, ..., b_{i+1-1}\} \cup \{a_i, a_{i+2}\} \) and \( L_r(a_ib_i) = V(LW_t) \setminus \{a_{i+1}, b_{i+2}, b_{i+1-2}, b_{i+1-1}\} \). Since \( \bigcup_{i=1}^{t} L_r(a_ia_{i+1}) = V(LW_t) \), therefore \( |L_r(x) \cap \bigcup_{i=1}^{t} L_r(a_ia_{i+1})| \geq |L_r(a_ia_{i+1})| \). The comparison among the cardinalities of all the LRN sets is given in Table 2.

\[ \Box \]

**Table 2.** Comparison between the cardinalities of LRN sets of \( LW_t \).

<table>
<thead>
<tr>
<th>LRN Set</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_r(a_ib_i) )</td>
<td>( 2t - 4 &gt; 4 )</td>
</tr>
<tr>
<td>( L_r(b_ib_{i+1}) )</td>
<td>( t + 2 &gt; 4 )</td>
</tr>
</tbody>
</table>

It is clear from Table 2 that \( |L_r(a_ia_{i+1})| < |L_r(x)| \), where \( L_r(x) \) are the other LRN sets of \( LW_t \).

**Theorem 3.** Let \( LW_4 \) be the line network of generalized wheel network then

\[ \frac{4}{3} \leq Ldim_F(LW_4) \leq 2. \]

**Proof.** The LRN sets of \( LW_4 \) are given as follows:

\( L_r(b_1b_2) = V(LW_4) \setminus \{a_2, a_4\}, \)
\( L_r(b_2b_3) = V(LW_4) \setminus \{a_3, a_1\}, \)
\( L_r(b_3b_4) = V(LW_4) \setminus \{a_4, a_2\}, \)
\( L_r(b_4b_1) = V(LW_4) \setminus \{a_1, a_3\}, \)
\( L_r(a_1b_1) = V(LW_4) \setminus \{a_2, b_3, b_4\}, \)
\( L_r(a_2b_2) = V(LW_4) \setminus \{a_3, b_4, b_1\}, \)
\( L_r(a_3b_3) = V(LW_4) \setminus \{a_4, b_1, b_2\}, \)
\( L_r(a_4b_4) = V(LW_4) \setminus \{a_1, b_2, b_3\}. \)
\[ L_r(a_3b_3) = V(LW_4) - \{a_4, b_1, b_2\}, \]
\[ L_r(a_4b_4) = V(LW_4) - \{a_1, b_2, b_3\}, \]
\[ L_r(a_1a_2) = \{a_1, a_2, b_2, b_4\}, \]
\[ L_r(a_2a_3) = \{a_2, a_3, b_3, b_1\}, \]
\[ L_r(a_3a_4) = \{a_3, a_1, b_1, b_2\}, \]
\[ L_r(a_4a_1) = \{a_4, a_4, b_2, b_3\}. \]

From the above, LRN sets the minimum cardinality of LRN set \( L_r(a_i, a_{i+1}) \) as 4, where \( 1 \leq i \leq 4 \); therefore, we define a minimal LRF \( \lambda(V(LW_4)) \rightarrow [0, 1] \) as \( \frac{1}{4} \) to each \( v \in V(LW_4) \), hence \( Ldim_F(LW_4) \leq \sum_{i=1}^{8} \frac{1}{4} = 2 \). The maximum cardinality of \( L_r(b_ib_{i+1}) \) is 6; therefore, we define a maximal LRF \( \lambda(V(LW_4)) \rightarrow [0, 1] \) as \( \frac{1}{6} \) to each \( v \in V(LW_4) \), hence \( Ldim_F(LW_4) \geq \sum_{i=1}^{8} \frac{1}{6} = \frac{4}{3} \).

\[ \frac{4}{3} \leq Ldim_F(LW_4) \leq 2. \]

\[ \square \]

**Theorem 4.** Let \( LW \) be a line network of a generalized wheel network, where \( t \equiv 0 \pmod{2} \), then

\[ \frac{t}{t-2} \leq Ldim_F(LW) \leq \frac{t}{2}. \]

**Proof.** To prove the theorem, we have divided it in two cases:

Case 1:

For \( t = 6 \), we have the following possible LRN sets:

\[ L_r(b_1b_2) = V(LW_6) - \{a_2, a_4, a_5, a_6\}, \]
\[ L_r(b_2b_3) = V(LW_6) - \{a_3, a_5, a_6, a_1\}, \]
\[ L_r(b_3b_4) = V(LW_6) - \{a_4, a_6, a_1, a_2\}, \]
\[ L_r(b_4b_5) = V(LW_6) - \{a_5, a_1, a_2, a_3\}, \]
\[ L_r(b_5b_6) = V(LW_6) - \{a_6, a_2, a_3, a_4\}, \]
\[ L_r(b_6b_1) = V(LW_6) - \{a_1, a_3, a_4, a_5\}, \]
\[ L_r(a_1b_1) = V(LW_6) - \{a_2, a_3, b_3, b_6\}, \]
\[ L_r(a_2b_2) = V(LW_6) - \{a_3, b_4, b_5, b_1\}, \]
\[ L_r(a_3b_3) = V(LW_6) - \{a_4, b_5, b_1, b_2\}, \]
\[ L_r(a_4b_4) = V(LW_6) - \{a_5, b_6, b_2, b_3\}, \]
\[ L_r(a_5b_5) = V(LW_6) - \{a_6, b_1, b_3, b_4\}, \]
\[ L_r(a_6b_6) = V(LW_6) - \{a_1, b_2, b_4, b_5\}, \]
\[ L_r(a_1a_2) = \{a_1, a_2, b_2, b_6\}, \]
\[ L_r(a_2a_3) = \{a_2, a_3, b_3, b_1\}, \]
\[ L_r(a_3a_4) = \{a_3, a_4, b_4, b_2\}, \]
\[ L_r(a_4a_5) = \{a_4, a_5, b_5, b_3\}, \]
\[ L_r(a_5a_6) = \{a_5, a_6, b_6, b_4\}, \]
\[ L_r(a_6a_1) = \{a_6, a_3, b_1, b_5\}. \]

From above, LRN sets the cardinality of LRN set \( L_r(a_i, a_{i+1}) \) as 4, where \( 1 \leq i \leq 6 \); therefore, we define a minimal LRF \( \lambda(V(LW_6)) \rightarrow [0, 1] \) as \( \frac{1}{4} \) to each \( v \in V(LW_6) \), hence \( Ldim_F(LW_6) \leq \sum_{i=1}^{12} \frac{1}{4} = 3 \). The maximum cardinality of LRN set \( L_r(b_ib_{i+1}) \) is 8; therefore, we define a maximal LRF \( \lambda(V(LW_6)) \rightarrow [0, 1] \) as \( \frac{1}{8} \) to each \( v \in V(LW_6) \), hence \( Ldim_F(LW_6) \geq \sum_{i=1}^{12} \frac{1}{8} = \frac{3}{2} \).

\[ \frac{3}{2} \leq Ldim_F(LW_6) \leq 3. \]
Case 2: 
For \( t \geq 8 \), in the view of Lemma 2, the cardinality of LRN set \( L_r(a_i a_{i+1}) \) is 4 and \( |L_r(a_i a_{i+1})| < |L_r(x)| \), where \( L_r(x) \) are other LRN sets of \( LW_i \). Therefore, we define a minimal LRF \( \lambda(V(LW_i)) \rightarrow [0,1] \) as \( \frac{1}{t} \) to each \( v \in V(LW_i) \), hence \( Ldim_f(LW_i) \leq \sum_{i=1}^{2t} \frac{1}{t} = \frac{2t}{t} = \frac{2}{t} \). In the same context, by Lemma 2, the maximum cardinality of LRN set \( L_r(a_i b_i) \) is \( 2t - 4 \) and \( |L_r(a_i b_i)| > |L_r(x)| \), where \( L_r(x) \) are the other LRN sets of \( LW_i \). Therefore, we define a maximal LRF \( \lambda'(V(LW_i)) \rightarrow [0,1] \) as \( \frac{1}{2t-4} \) to each \( v \in V(LW_i) \), hence \( Ldim_f(LW_i) \geq \frac{2t}{t} \cdot \frac{1}{t-2} = \frac{2t}{t-2} \).

\[
\frac{t}{t-2} \leq Ldim_f(LW_i) \leq \frac{t}{2}.
\]

\[
\square
\]

3.2. Line Network of Prism Network \( LD_t \)
In this subsection, our aim is to compute LRN sets and LFMD of the line network of prism network. The line network of prism network is defined as follows:

Let \( LD_t \) be the line network of prism network with vertex set \( V(LD_t) = \{a_i, b_i, c_i : 1 \leq i \leq t\} \) and edge set \( E(LD_t) = \{a_i a_{i+1}, a_i b_i, b_i a_{i+1}, c_i b_i, c_i c_{i+1} : 1 \leq i \leq t\} \) with order \( 3t \) and size \( 6t \). For more information see Figure 2.

**Lemma 3.** Let \( LD_t \) be the line network of prism network, where \( t \equiv 1 \) (mod 2). Then

(a) \(|L_r(b_i a_{i+1})| = t + 3 \) and \(|\bigcup_{i=1}^{t} L_r(b_i a_{i+1})| = 3t.

(b) \(|L_r(b_i a_{i+1})| \leq |L_r(x)| \) and \(|L_r(x) \cap \bigcup_{i=1}^{t} L_r(b_i a_{i+1})| \geq |L_r(b_i a_{i+1})|, \forall x \in E(LD_t).

**Proof.** Consider \( a_i \) inner, \( b_i \) middle, and \( c_i \) are outer vertices of \( LD_t \), where \( 1 \leq i \leq t \) and \( t + 1 \equiv 1 \) (mod \( t \)).

(a) \( L_r(b_i a_{i+1}) = V(LD_t) - \{a_i, a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{t+1}, b_{i+1}, b_{i+2}, \ldots, b_{t+1}, c_{i+1}, c_{i+2}, \ldots, c_{t+1}\}; \) therefore, \(|L_r(b_i a_{i+1})| = t + 3 \) also \(|\bigcup_{i=1}^{t} L_r(b_i a_{i+1})| = 3t.

(b) The LRN sets other then \( L_r(b_i a_{i+1}) \) are \( L_r(a_i b_i) = V(LD_t) - \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{t+1}\} \cup \{b_{i+1}, b_{i+2}, b_{i+3}, \ldots, b_{t+1}\} \cup \{c_{i+1}, c_{i+2}, \ldots, c_{t+1}\} \}

\( L_r(a_i a_{i+1}) = V(LD_t) - \{a_{i+2}, a_{i+3}, \ldots, a_{t+1}\} \cup \{b_{i+2}, b_{i+3}, \ldots, b_{t+1}\} \cup \{c_{i+2}, c_{i+3}, \ldots, c_{t+1}\}. \)

Since \(|\bigcup_{i=1}^{t} L_r(b_i a_{i+1})| = V(LD_t), |L_r(x) | \cap \bigcup_{i=1}^{t} L_r(b_i a_{i+1})| \geq |L_r(b_i a_{i+1})|). \) The comparison among the cardinalities of all the LRN sets is given in Table 3.

\[
\square
\]
Table 3. Comparison between the cardinalities of LRN sets of $LD_1$.

<table>
<thead>
<tr>
<th>LRN Set</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_r(a_1a_{i+1})$</td>
<td>$3t - 5 &gt; t + 3$</td>
</tr>
<tr>
<td>$L_r(a_ib_i)$</td>
<td>$\frac{3t+3}{2} &gt; t + 3$</td>
</tr>
<tr>
<td>$L_r(b_ic_i)$</td>
<td>$t + 7 &gt; t + 3$</td>
</tr>
<tr>
<td>$L_r(c_ic_{i+1})$</td>
<td>$\frac{3t-1}{2} &gt; t + 3$</td>
</tr>
<tr>
<td>$L_r(b_ic_{i+1})$</td>
<td>$\frac{3t+3}{2} &gt; t + 3$</td>
</tr>
</tbody>
</table>

It is clear from Table 3 that $|L_r(b_ia_{i+1})| < |L_r(x)|$, where $L_r(x)$ are the other LRN sets of $LD_1$.

Theorem 5. Let $LD_1$ be a line network of prism network then

$$Ldim_F(LD_3) = \frac{3}{2}.$$  

Proof. The LRN sets of $LD_3$ are given by:

$L_r(a_1a_2) = V(LD_3) - \{a_4, b_1, c_1, c_2, c_4\}$,

$L_r(a_2a_3) = V(LD_3) - \{a_5, b_2, c_2, c_3, c_5\}$,

$L_r(a_3a_4) = V(LD_3) - \{a_4, b_3, c_3, c_4, c_1\}$,

$L_r(a_4a_5) = V(LD_3) - \{a_2, b_4, c_4, c_5, c_2\}$,

$L_r(a_5a_1) = V(LD_3) - \{a_3, b_5, c_5, c_1, c_3\}$,

$L_r(a_1b_1) = V(LD_3) - \{a_2, a_3, b_2, c_4, c_5\}$,

$L_r(a_2b_2) = V(LD_3) - \{a_3, a_4, b_3, c_5, c_1\}$,

$L_r(a_3b_3) = V(LD_3) - \{a_4, a_5, b_4, c_1, c_2\}$,

$L_r(a_4b_4) = V(LD_3) - \{a_5, a_1, b_5, c_2, c_3\}$,

$L_r(a_5b_5) = V(LD_3) - \{a_1, a_2, a_3, b_1, c_3, c_4\}$,

$L_r(b_1a_2) = V(LD_3) - \{a_1, a_5, b_4, c_5, c_4\}$,

$L_r(b_2a_3) = V(LD_3) - \{a_2, a_1, b_5, c_1, c_5\}$,

$L_r(b_3a_4) = V(LD_3) - \{a_3, a_2, b_1, c_2, c_1\}$,

$L_r(b_4a_5) = V(LD_3) - \{a_4, a_3, b_2, c_3, c_2\}$,

$L_r(b_5a_1) = V(LD_3) - \{a_5, a_4, b_3, c_4, c_3\}$,

$L_r(b_1c_1) = V(LD_3) - \{a_4, a_5, b_4, c_5, c_3\}$,

$L_r(b_2c_2) = V(LD_3) - \{a_5, a_1, b_5, c_1, c_4\}$,

$L_r(b_3c_3) = V(LD_3) - \{a_1, a_5, b_4, b_5, c_5\}$,

$L_r(b_4c_4) = V(LD_3) - \{a_2, a_1, b_5, b_1, c_1\}$,

$L_r(b_5c_5) = V(LD_3) - \{a_3, a_2, b_1, b_2, c_2\}$,

$L_r(b_1c_2) = V(LD_3) - \{a_3, a_4, b_4, b_5, c_1\}$,

$L_r(b_2c_3) = V(LD_3) - \{a_4, a_5, b_5, b_1, c_2\}$,

$L_r(b_3c_4) = V(LD_3) - \{a_5, a_1, b_1, b_2, c_3\}$,

$L_r(b_4c_5) = V(LD_3) - \{a_1, a_2, b_2, b_3, c_4\}$,

$L_r(b_5c_1) = V(LD_3) - \{a_2, a_3, b_3, b_4, c_5\}$,

$L_r(c_1c_2) = V(LD_3) - \{a_1, a_2, a_4, b_1, c_4\}$,

$L_r(c_2c_3) = V(LD_3) - \{a_2, a_3, a_5, b_2, c_5\}$,

$L_r(c_3c_4) = V(LD_3) - \{a_3, a_4, a_1, b_3, c_1\}$,

$L_r(c_4c_5) = V(LD_3) - \{a_4, a_5, a_2, b_4, c_2\}$,

$L_r(c_5c_1) = V(LD_3) - \{a_5, a_1, a_3, b_5, c_3\}$.

Since the cardinality of each LRN set of $LD_3$ is 10, we define a constant $LRF \lambda(V(LD_3)) \rightarrow [0, 1]$ as $\frac{1}{10}$ to each $v \in V(LD_3)$, hence

$$Ldim_F(LD_3) = \sum_{i=1}^{15} \frac{1}{10} = \frac{3}{2}.$$
Theorem 6. Let $LD_t$ be a line network of prism network, where $t \equiv 1 \pmod{2}$, then

$$\frac{3t}{3t-5} \leq Ldim_F(LD_t) \leq \frac{3t}{1+3}.$$ 

Proof. To prove the theorem, we have divided it in two cases:

Case I:

The LRN sets of $LD_7$ are given by:

\begin{align*}
L_r(b_1c_1) &= V(LD_7) - \{a_5, a_6, a_7, b_5, b_7, c_2, c_3, c_4\}, \\
L_r(b_2c_2) &= V(LD_7) - \{a_5, a_7, a_1, b_5, b_7, c_1, c_4, c_5\}, \\
L_r(b_3c_3) &= V(LD_7) - \{a_7, a_1, a_2, b_7, b_1, b_2, c_4, c_5, c_6\}, \\
L_r(b_4c_4) &= V(LD_7) - \{a_1, a_2, a_3, b_1, b_2, b_3, c_5, c_6, c_7\}, \\
L_r(b_5c_5) &= V(LD_7) - \{a_2, a_3, a_4, b_3, b_4, c_6, c_7, c_1\}, \\
L_r(b_6c_6) &= V(LD_7) - \{a_3, a_4, a_5, b_4, b_5, c_7, c_1, c_2\}, \\
L_r(b_7c_7) &= V(LD_7) - \{a_4, a_5, a_6, b_4, b_5, b_1, c_2, c_3\}, \\
L_r(b_1b_1) &= V(LD_7) - \{a_2, a_3, a_4, b_2, b_3, b_4, c_5, c_6, c_7\}, \\
L_r(b_2b_2) &= V(LD_7) - \{a_3, a_4, a_5, b_3, b_4, b_5, c_6, c_7, c_1\}, \\
L_r(b_3b_3) &= V(LD_7) - \{a_4, a_5, a_6, b_4, b_5, b_6, c_7, c_1, c_2\}, \\
L_r(b_4b_4) &= V(LD_7) - \{a_5, a_6, a_7, b_5, b_6, b_7, c_1, c_2, c_3\}, \\
L_r(b_5b_5) &= V(LD_7) - \{a_6, a_7, a_1, b_6, b_7, b_1, c_2, c_3, c_4\}, \\
L_r(b_6b_6) &= V(LD_7) - \{a_7, a_1, a_2, b_7, b_1, b_2, c_3, c_4, c_5\}, \\
L_r(b_7b_7) &= V(LD_7) - \{a_1, a_2, a_3, b_1, b_2, b_3, c_4, c_5, c_6\}, \\
L_r(b_1a_2) &= V(LD_7) - \{a_1, a_6, a_7, b_5, b_6, b_7, c_3, c_4, c_5\}, \\
L_r(b_2a_3) &= V(LD_7) - \{a_2, a_7, a_1, b_6, b_7, b_1, c_4, c_5, c_6\}, \\
L_r(b_3a_4) &= V(LD_7) - \{a_3, a_1, a_2, b_7, b_1, b_2, c_5, c_6, c_7\}, \\
L_r(b_4a_5) &= V(LD_7) - \{a_4, a_2, a_3, b_1, b_2, b_3, c_6, c_7, c_1\}, \\
L_r(b_5a_6) &= V(LD_7) - \{a_5, a_3, a_4, b_3, b_4, b_5, c_7, c_1, c_2\}, \\
L_r(b_6a_7) &= V(LD_7) - \{a_6, a_4, a_5, b_4, b_5, b_6, c_1, c_2, c_3\}, \\
L_r(a_1a_2) &= V(LD_7) - \{a_5, b_1, c_1, c_2, c_5\}, \\
L_r(a_2a_3) &= V(LD_7) - \{a_6, b_2, c_2, c_3, c_6\}, \\
L_r(a_3a_4) &= V(LD_7) - \{a_7, b_3, c_3, c_4, c_7\}, \\
L_r(a_4a_5) &= V(LD_7) - \{a_1, b_4, c_4, c_5, c_1\}, \\
L_r(a_5a_6) &= V(LD_7) - \{a_2, c_5, c_6, c_2\}, \\
L_r(a_6a_7) &= V(LD_7) - \{a_3, b_6, c_6, c_7, c_3\}, \\
L_r(a_7a_1) &= V(LD_7) - \{a_4, b_7, c_7, c_1, c_4\}, \\
L_r(c_1c_2) &= V(LD_7) - \{a_1, a_2, a_5, b_1, c_5\}, \\
L_r(c_2c_3) &= V(LD_7) - \{a_2, a_3, a_6, b_2, c_6\}, \\
L_r(c_3c_4) &= V(LD_7) - \{a_3, a_4, a_7, b_3, c_7\}, \\
L_r(c_4c_5) &= V(LD_7) - \{a_4, a_5, a_1, b_4, c_1\}, \\
L_r(c_5c_6) &= V(LD_7) - \{a_5, a_6, a_2, b_5, c_2\}, \\
L_r(c_6c_7) &= V(LD_7) - \{a_6, a_7, a_3, b_6, c_3\}, \\
L_r(c_7c_1) &= V(LD_7) - \{a_7, a_1, a_4, b_7, c_4\}.
\end{align*}

From the above LRN sets, the LRN sets having the minimum cardinalities are $L_r(b_2c_1), L_r(a_1b_1)$ and $L_r(b_1a_{i+1})$ and the cardinality of each of them is 12, where $1 \leq i \leq 7$ therefore, we define a minimal LRF $\lambda(V(LD_7)) \rightarrow [0, 1]$ as $\frac{1}{12}$ to each $v \in V(LW_7)$, hence $Ldim_F(LD_7) \leq \sum_{i=1}^{21} \frac{1}{12} = \frac{7}{4}$. The LRN sets having maximum cardinality are $L_r(a_1a_{i+1}), L_r(c_{i+1}),$ where $1 \leq i \leq 7$ and cardinality of each of them is 17; therefore, we define a maximal LRF $\lambda'(V(LD_7)) \rightarrow [0, 1]$ as $\frac{1}{17}$ to each $v \in V(LD_7)$, hence $Ldim_F(LD_7) \geq \sum_{i=1}^{21} \frac{1}{17} = \frac{21}{17}$. The bounds of LFMD of $LD_7$ are given as follows:

$$\frac{21}{17} \leq Ldim_F(LD_7) \leq \frac{7}{4}.$$
Case 2:
For \( t \geq 7 \), in the view of Lemma 3, the cardinality of LRN set \( L_r(b_ia_{i+1}) \) is \( t + 3 \) and \( |L_r(b_ia_{i+1})| < |L_r(x)| \), where \( L_r(x) \) are other LRN sets of \( LD_t \), where \( 1 \leq i \leq t \). Therefore, we define a minimal LRF \( \lambda(V(LD_t)) \rightarrow [0,1] \) as \( \frac{1}{3t+3} \) to each \( v \in V(LD_t) \), hence \( Ldim_F(LD_t) \leq \sum_{i=1}^{3t} \frac{1}{3t+3} = \frac{3t}{3t+3} \). In the same context by Lemma 3 the maximum cardinality of LRN set \( L_r(a_ia_{i+1}) \) is \( 3t - 5 \) and \( |L_r(a_ia_{i+1})| > |L_r(x)| \), where \( L_r(x) \) are other LRN sets of \( LW_t \), where \( 1 \leq i \leq t \). Therefore, we define a maximal LRF \( \lambda'(V(LW_t)) \rightarrow [0,1] \) as \( \frac{1}{3t-5} \) to each \( v \in V(LD_t) \), hence \( Ldim_F(LD_t) \geq \sum_{i=1}^{3t} \frac{3t}{3t-5} \). The bounds of LFMD of \( LD_t \) are given as follows:

\[
\frac{3t}{3t-5} \leq Ldim_F(LD_t) \leq \frac{3t}{t+3}.
\]

\( \square \)

**Lemma 4.** Let \( LD_t \) be the line network of prism network then, where \( t \equiv 0 \mod 2 \) then

(a) \( |L_r(b_ia_{i+1})| = \frac{3t+4}{2} \) and \( \bigcup_{i=1}^{t} L_r(b_ia_{i+1}) = 3t; \)
(b) \( |L_r(b_ia_{i+1})| \leq |L_r(x)| \) and \( |L_r(x) \cap \bigcup_{i=1}^{t} L_r(b_ia_{i+1})| \geq |L_r(b_ia_{i+1})|, \forall x \in E(LD_t). \)

**Proof.** Consider \( a_i \) inner, \( b_i \) middle, and \( c_i \) are outer vertices of \( LD_t \), where \( 1 \leq i \leq t \) and \( t + 1 \equiv 1 \mod t \).

(a) \( L_r(a_ib_t) = V(LD_t) - \{a_{i+1}, a_{i+2}, a_{i+3}, ..., a_{i+2t} \} \cup \{b_{t+1}, b_{t+2}, ..., b_{t+2i-2}, b_{t+2i-1} \} \cup \{c_{t+2}, c_{t+2i+4}, c_{t+2i+6}, ..., c_{t+2i+2t-1} \}, L_r(b_ia_{i+1}) = V(LD_t) - \{a_{i+1}, a_{i+2}, a_{i+3}, ..., a_{i+2t} \} \cup \{b_{t+1}, b_{t+2}, ..., b_{t+2i-2}, b_{t+2i-1} \} \cup \{c_{t+2}, c_{t+2i+4}, c_{t+2i+6}, ..., c_{t+2i+2t-1} \}, L_r(b_ia_{i+1}) = V(LD_t) - \{a_{i+1}, a_{i+2}, a_{i+3}, ..., a_{i+2t} \} \cup \{b_{t+1}, b_{t+2}, ..., b_{t+2i-2}, b_{t+2i-1} \} \cup \{c_{t+2}, c_{t+2i+4}, c_{t+2i+6}, ..., c_{t+2i+2t-1} \}, \)

Therefore, \( |L_r(b_ia_{i+1})| = |L_r(a_ib_t)| = |L_r(b_ic_t)| = \frac{3t+4}{2} \) also \( \bigcup_{i=1}^{t} L_r(b_ia_{i+1}) \).
(b) \( L_r(a_ia_{i+1}) = V(LD_t) - \{b_t, c_t, c_{i+1}\} \) and \( L_r(c_ia_{i+1}) = V(LD_t) - \{a_t, a_{i+1}, b_t\} \). The comparison among the cardinalities of all the LRN sets is given in Table 4.

\( \square \)

**Table 4.** Comparison among the cardinalities of LRN sets of \( LD_t \).

<table>
<thead>
<tr>
<th>LRN Set</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_r(a_ia_{i+1}) )</td>
<td>( 3t - 3 &gt; \frac{3t+4}{2} )</td>
</tr>
<tr>
<td>( L_r(c_ia_{i+1}) )</td>
<td>( 3t - 3 &gt; \frac{3t+4}{2} )</td>
</tr>
</tbody>
</table>

**Theorem 7.** Let \( LD_t \) be a line network of prism network, then

\[
Ldim_F(LD_t) = \frac{3}{2}.
\]

**Proof.** For \( t = 4 \), we have following LRN sets:

- \( L_r(a_1b_2) = V(LD_4) - \{b_1, b_3, c_1, c_2\} \),
- \( L_r(a_2b_3) = V(LD_4) - \{b_2, b_4, c_2, c_3\} \),
- \( L_r(a_3b_4) = V(LD_4) - \{b_3, b_5, c_3, c_4\} \),
- \( L_r(a_4b_5) = V(LD_4) - \{b_4, b_6, c_4, c_5\} \),
- \( L_r(a_2b_1) = V(LD_4) - \{a_2, a_3, b_2, c_2\} \),
- \( L_r(a_2b_2) = V(LD_4) - \{a_3, a_4, b_3, c_1\} \),
- \( L_r(a_3b_3) = V(LD_4) - \{a_4, a_1, b_4, c_2\} \),
Theorem 8. Proof. To prove the theorem, we have divided it in two cases:

Case 1: The LRN sets of LD₆ are:
- \(L_r(a_1a_2) = V(LD₆) - \{a_1, c_1, c_2\} \),
- \(L_r(a_2a_3) = V(LD₆) - \{b_2, c_2, c_3\} \),
- \(L_r(a_3a_4) = V(LD₆) - \{b_3, c_3, c_4\} \),
- \(L_r(a_4a_5) = V(LD₆) - \{b_4, c_4, c_5\} \),
- \(L_r(a_5a_6) = V(LD₆) - \{b_5, c_5, c_6\} \),
- \(L_r(a_6a_1) = V(LD₆) - \{b_6, c_6, c_1\} \),
- \(L_r(c_1c_2) = V(LD₆) - \{a_1, a_2, b_1, b_4\} \),
- \(L_r(c_2c_3) = V(LD₆) - \{a_2, a_3, b_2, b_5\} \),
- \(L_r(c_3c_4) = V(LD₆) - \{a_3, a_4, b_3, b_6\} \),
- \(L_r(c_4c_5) = V(LD₆) - \{a_4, a_5, b_4, b_1\} \),
- \(L_r(c_5c_6) = V(LD₆) - \{a_5, a_6, b_5, b_2\} \),
- \(L_r(c_6c_1) = V(LD₆) - \{a_6, a_1, b_6, b_3\} \),
- \(L_r(a_1b_1) = V(LD₆) - \{a_2, a_3, a_4, b_2, b_3, c_5, c_6\} \),
- \(L_r(a_2b_2) = V(LD₆) - \{a_3, a_4, a_5, b_3, b_4, c_6, c_1\} \),
- \(L_r(a_3b_3) = V(LD₆) - \{a_4, a_5, a_6, b_4, b_5, c_1, c_2\} \),
- \(L_r(a_4b_4) = V(LD₆) - \{a_5, a_6, a_1, b_5, b_6, c_2, c_3\} \),
- \(L_r(a_5b_5) = V(LD₆) - \{a_6, a_1, a_2, b_6, b_1, c_3, c_4\} \),
- \(L_r(a_6b_6) = V(LD₆) - \{a_1, a_2, a_3, b_1, b_2, c_4, c_5\} \).

Since the cardinality of each LRN set of LD₄ is 10, therefore, we define a constant LRF \(\lambda(V(LD₄)) \rightarrow [0, 1]\) as \(\frac{1}{8}\) to each \(v \in V(LD₄)\), hence

\[L_{dim}_F(LD₄) = \frac{12}{1} \times \frac{1}{8} = \frac{3}{2}\]

\(\Box\)

Theorem 8. Let LD₁ be a line network of prism network, where \(t \equiv 0 \pmod{2}\). Then

\[\frac{t}{t-1} \leq L_{dim}_F(LD₁) \leq \frac{6t}{3n+4}\]

Proof. To prove the theorem, we have divided it in two cases:

Case 1:

The LRN sets of LD₆ are:
\[ L_r(b_1a_2) = V(LD_6) - \{ a_1, a_5, a_6, b_5, b_6, c_3, c_4 \}, \]
\[ L_r(b_2a_3) = V(LD_6) - \{ a_2, a_6, a_1, b_6, b_1, c_4, c_3 \}, \]
\[ L_r(b_3a_4) = V(LD_6) - \{ a_3, a_4, a_2, b_2, c_5, c_6 \}, \]
\[ L_r(b_4a_5) = V(LD_6) - \{ a_4, a_5, a_3, b_3, c_4, c_6 \}, \]
\[ L_r(b_5a_6) = V(LD_6) - \{ a_4, a_5, a_6, b_4, b_5, c_1, c_2 \}, \]
\[ L_r(b_6a_1) = V(LD_6) - \{ a_5, a_6, a_1, b_5, b_2, c_3 \}, \]
\[ L_r(b_1c_1) = V(LD_6) - \{ a_6, a_6, b_2, b_3, c_2, c_4 \}, \]
\[ L_r(b_2c_2) = V(LD_6) - \{ a_6, a_1, b_3, b_4, c_3, c_4 \}, \]
\[ L_r(b_3c_3) = V(LD_6) - \{ a_1, a_2, b_4, b_5, c_4, c_5 \}, \]
\[ L_r(b_4c_4) = V(LD_6) - \{ a_2, a_3, b_5, b_6, c_6, c_1 \}, \]
\[ L_r(b_5c_5) = V(LD_6) - \{ a_3, a_4, b_6, b_1, c_6, c_1, c_2 \}, \]
\[ L_r(b_6c_6) = V(LD_6) - \{ a_4, a_5, b_1, b_2, c_1, c_2, c_3 \}, \]
\[ L_r(b_1c_2) = V(LD_6) - \{ a_3, a_4, b_5, b_6, c_1, c_5, c_6 \}, \]
\[ L_r(b_1c_2) = V(LD_6) - \{ a_3, a_4, b_5, b_6, c_1, c_5, c_6 \}, \]
\[ L_r(b_2c_3) = V(LD_6) - \{ a_4, a_5, b_6, b_1, c_2, c_6, c_1 \}, \]
\[ L_r(b_3c_4) = V(LD_6) - \{ a_6, a_6, b_1, b_2, c_3, c_1, c_2 \}, \]
\[ L_r(b_4c_5) = V(LD_6) - \{ a_6, a_1, b_2, b_3, c_4, c_2, c_3 \}, \]
\[ L_r(b_5c_6) = V(LD_6) - \{ a_1, a_2, b_3, b_4, c_5, c_3, c_4 \}, \]
\[ L_r(b_6c_1) = V(LD_6) - \{ a_2, a_3, b_4, b_5, c_6, c_4, c_5 \}. \]

The LRN sets with a minimum cardinality are \( L_r(b_1c_1), L_r(a_6b_1), L_r(b_1a_{i+1}), L_r(b_1c_{i+1}) \) and the cardinality of each of them is 11, where 1 ≤ i ≤ 6. Therefore, we define a minimal LRF \( \lambda(V(LD_6)) \) as \( \frac{1}{11} \) to each v ∈ V(LD_6), hence \( Ldim_F(LD_6) ≤ \sum_{i=1}^{18} \frac{1}{11} = \frac{18}{11} \). The LRN set having maximum cardinality is \( L_r(a_6b_1) \), and its cardinality is 15; therefore, we define a maximal LRN \( \lambda(V(LD_6)) \) as \( \frac{1}{11} \) to each v ∈ V(LD_6), hence \( Ldim_F(LD_6) ≥ \sum_{i=1}^{18} \frac{1}{11} = \frac{18}{11} \). The bounds of LFMD of LD_6 is given as follows:

\[ \frac{6}{5} ≤ Ldim_F(LD_6) ≤ \frac{18}{11}. \]

Case 2:

For \( t ≥ 6 \), in the view of Lemma 4, the cardinalities of the LRN sets \( L_r(a_6b_1), L_r(b_1a_{i+1}), L_r(b_1c_{i+1}), \) and \( L_r(b_1c_{i+1}) \) is \( \frac{3t+4}{3t+3} \) and \( |L_r(a_6b_1)| ≤ |L_r(x)| \) where \( L_r(x) \) are other LRN sets of LD_6, where 1 ≤ i ≤ t. Therefore, we define a minimal LRF \( \lambda(V(LD_t)) \) as \( \frac{2}{3t+4} \) to each v ∈ V(LD_t), hence \( Ldim_F(LD_t) ≤ \sum_{i=1}^{3t} \frac{2}{3t+4} = \frac{6t}{3t+4} \). In the same context by Lemma 4 the maximum cardinalities of the LRN sets are \( L_r(a_6b_1) \) and \( L_r(c_1c_{i+1}) \) is \( 3t - 3 \) and \( |L_r(a_6b_1)| ≥ |L_r(x)| \), where \( L_r(x) \) are other LRN sets of LD_t, where 1 ≤ i ≤ t. Therefore, we define a maximal LRF \( \lambda'(V(LD_t)) \) as \( \frac{1}{3t+3} \) to each v ∈ V(LD_t), hence

\[ Ldim_F(LD_t) ≥ \sum_{i=1}^{3t} \frac{1}{3t+3} = \frac{t}{3t+3}. \]

Hence, the bounds of LFMD of LD_t are given as follows:

\[ \frac{t}{t-1} ≤ Ldim_F(LD_t) ≤ \frac{6t}{3t+4}. \]

4. Conclusions

In this manuscript, we have established sharp bounds of the LFMD of the rotationally symmetric line networks of the wheel (LW_t) and prism (LD_t). It is proved that for \( t = 3 \), LW_3 attains the exact value of LFMD which is \( \frac{7}{3} \) and for \( t = 4,5 \) the LFMD of LD_t is \( \frac{3}{2} \) as well. It has been observed that the LW_t remains unbounded and LD_t remains bounded under LFMD, when the order of these networks approaches \( \infty \). The boundedness and
unboundedness of these networks is illustrated in Table 5. Furthermore, the results are more precise as the both lower and upper bounds LFMD of these line networks have been established. Now in the end of our discussion, we suggest an open problem that characterizes all the rotationally symmetric networks having the exact value of LFMD.

Table 5. Boundedness and unboundedness of $LW_t$ and $LD_t$ via LFMD.

<table>
<thead>
<tr>
<th>Network</th>
<th>LFMD</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$LW_t, t \equiv 1 ,(mod, 2)$</td>
<td>$\frac{2^t}{Ldim_F(LW_t)} \leq \frac{t}{4}$</td>
<td>$1$</td>
<td>$\infty$</td>
<td>Unbounded</td>
</tr>
<tr>
<td>$LW_t, t \equiv 0 ,(mod, 2)$</td>
<td>$\frac{1}{Ldim_F(LW_t)} \leq \frac{t}{4}$</td>
<td>$1$</td>
<td>$\infty$</td>
<td>Unbounded</td>
</tr>
<tr>
<td>$LD_t, t \equiv 1 ,(mod, 2)$</td>
<td>$\frac{3^t}{Ldim_F(LD_t)} \leq \frac{3^t}{t}$</td>
<td>$1$</td>
<td>$3$</td>
<td>Bounded</td>
</tr>
<tr>
<td>$LD_t, t \equiv 0 ,(mod, 2)$</td>
<td>$\frac{1}{Ldim_F(LD_t)} \leq \frac{3^t}{t}$</td>
<td>$1$</td>
<td>$2$</td>
<td>Bounded</td>
</tr>
</tbody>
</table>

3D representation of lower and upper bounds of LFMD of rotationally symmetric line networks

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References


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