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A One-Dimensional Time-Fractional Damped Wave Equation with a Convection Term

Ibtisam Aldawish [1], Mohamed Jleli [2] and Bessem Samet [2,*]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11566, Saudi Arabia; imaldawish@imamu.edu.sa
[2] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; jleli@ksu.edu.sa
* Correspondence: bsamet@ksu.edu.sa

Abstract: We investigate a semilinear time-fractional damped wave equation in one dimension, posed in a bounded interval. The considered equation involves a convection term and singular potentials on one extremity of the interval. A Dirichlet boundary condition depending on the time-variable is imposed. Using nonlinear capacity estimates, we establish sufficient conditions for the nonexistence of weak solutions to the considered problem. In particular, when the boundary condition is independent of time, we show the existence of a Fujita-type critical exponent.

Keywords: damped wave equation; Caputo fractional derivative; convection term; nonexistence

MSC: 35L05; 35A01; 26A33

1. Introduction

We investigate the nonexistence of weak solutions to the one-dimensional time-fractional damped wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\beta u}{\partial t^\beta}-\frac{\partial^2 u}{\partial x^2}+\frac{\lambda}{x}\frac{\partial u}{\partial x}=x^{-a}|u|^p \quad \text{in } (0,\infty)\times]0,1], \tag{1}$$

where $0<\alpha<1$, $1<\beta<2$, $\frac{\partial^\kappa u}{\partial t^\kappa}$ ($\kappa=\alpha,\beta$) is the time-Caputo fractional derivative of order κ, $\lambda\in\mathbb{R}$, $a\geq 0$ and $p>1$. Problem (1) is considered under the initial conditions

$$u(0,x)=u_0(x),\ \frac{\partial u}{\partial t}(0,x)=u_1(x) \quad \text{in }]0,1] \tag{2}$$

and the Dirichlet boundary condition

$$u(t,1)=\delta(t+1)^\rho \quad \text{in } (0,\infty), \tag{3}$$

where $\delta>0$ and $\rho\in\mathbb{R}$.

Wave-type equations are frequently used to recast several propagation phenomena and develop numerical methods for solving physics problems. Several papers in the literature have dealt with symmetries of wave-type equations and their solutions. For instance, by means of the symmetry's properties, the orthogonality's criteria for the existence of solutions in elastic and anisotropic media have been derived. For more details, we refer to [1–3]. Concerning the numerical approaches for the study of wave type equations, we point out that by means of symmetry transformations, a nonlinear wave equation can be linked to a linear wave equation. Namely, it is possible to linearize a nonlinear wave equation by a nonlocal symmetry analysis, see e.g., [4]. For more details about the advantages of this technique in numerical computations, we refer to [5,6].

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Several works in the literature have dealt with the investigation of the blow-up of solutions to semilinear wave equations. For instance, in [7], the author considered the nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + b\frac{\partial u}{\partial t} = F(u)$$

under Dirichlet boundary conditions, where the parameter b is nonnegative, and the nonlinear term F satisfies a certain condition. Using the energy method, the blow-up results were derived. Levine [8] considered the problem

$$M\frac{\partial^2 u}{\partial t^2} + Lu + Q\frac{\partial u}{\partial t} = F(u),$$

posed in a Hilbert space, where M and L are positive adjoint operators. Using the concavity approach, the blow-up of solutions was investigated. We also refer to [9–15], where the large-time behavior of solutions to nonlinear wave equations has been studied by the energy and concavity methods.

The applications of fractional calculus are broad. Some of the interesting applications are the modeling of the heat flow in a porous medium (see e.g., [16]) and the identification of the fractional orders in anomalous diffusion models (see e.g., [17]). Other applications in physics, chemistry, engineering, biology, geophysics, and hydrology can be found in [18–21] (see also the references therein). In recent years, evolution equations with time-fractional derivatives have been investigated extensively, see for instance [22–30] and the references therein.

Kirane and Tatar [27] studied the time-fractional damped wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}} = b|u|^{p-1}u \quad \text{in } (0,\infty)\times\Omega$$

under the Dirichlet boundary condition

$$u(t,x) = 0 \quad \text{on}(0,\infty)\times\partial\Omega,$$

where $p > 1$, $\alpha \in (-1,1)$, and Ω is a bounded domain of $\mathbb{R}^N$. Namely, it was proven that the energy grows exponentially, when the initial values are sufficiently large. Later, Tatar [28] improved this result by showing that the solutions to the above problem blow up in finite time for sufficiently large initial data.

In [30], the authors considered a one-dimensional time-fractional damped wave inequality without a convection term ($\lambda = 0$). Namely, they studied the problem

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} + \dfrac{\partial^\beta u}{\partial t^\beta} - \dfrac{\partial^2 u}{\partial x^2} \geq x^\sigma |u|^p, & t > 0,\ x \in (0,L), \\ (u(t,0), u(t,L)) = (f(t), g(t)), & t > 0, \\ \left(u(0,x), \frac{\partial u}{\partial t}(0,x)\right) = (u_0(x), u_1(x)), & x \in (0,L), \end{cases} \tag{4}$$

where $\sigma \in \mathbb{R}$, $p > 1$, $0 < \alpha < 1$, $1 < \beta < 2$, $u_0, u_1 \in L^1([0,L])$, $f \in L^1_{\text{loc}}([0,\infty))$, and $g(t) = C_g t^\gamma$, with $\gamma > -1$ and $C_g \geq 0$ as constants. Two cases were investigated. In the case of $C_g = 0$ ($g = 0$), it was shown that if $\sigma < -(p+1)$ and one of the following conditions is satisfied:

$$\beta < \alpha + 1,\ \int_0^L u_1(x)(L-x)\,dx > 0;$$

$$\beta = \alpha + 1,\ \int_0^L (u_0(x) + u_1(x))(L-x)\,dx > 0;$$

$$\beta > \alpha + 1, \ \int_0^L u_0(x)(L-x)\,dx > 0,$$

then (4) admits no weak solution. In the inhomogeneous case, i.e., $C_g > 0$, it was proven that if $\alpha > \max\{-\gamma, 0\}$, $\beta > \max\{1-\gamma, 1\}$ and

$$\sigma < -(p+1); \text{ or } \sigma \geq -(p+1), \ \gamma > 0,$$

then the same conclusion holds as above.

In this paper, our aim is to study the influence of the convection term (namely the parameter λ) on the large-time behavior of solutions. As in [30], the approach used in this paper is based on suitable test functions and nonlinear capacity estimates. However, some key choices are completely different. For instance, due to the presence of the singular potential term $\frac{\lambda}{x}$ in (1), the used test function is different to that considered in [30]. On the other hand, unlike in [27,28], no restriction on the "size" of the initial data is imposed (the initial data are not assumed to be sufficiently large).

In Section 2, we recall some notions and results from fractional calculus. The definition of weak solutions to (1)-(2)-(3) and the statement of the main results are presented in Section 3. Section 4 is devoted to some preliminaries. Finally, we prove our results in Section 5.

Throughout this paper, by C or C_i, we mean generic positive constants whose values are not necessarily the same.

2. Some Notions on Fractional Calculus

We recall below some notions and results from fractional calculus (see [31] for more details) and fix some notations.

Let $T > 0$ be fixed. Given $f \in L^1([0,T])$ and $\sigma > 0$, the left-sided and right-sided Riemann–Liouville fractional integrals of order σ of f are defined, respectively, by

$$(I_0^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} f(s)\,ds$$

and

$$(I_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T (s-t)^{\sigma-1} f(s)\,ds$$

for almost everywhere $t \in [0,T]$, where $\Gamma(\cdot)$ denotes the Gamma function; that is,

$$\Gamma(\sigma) = \int_0^\infty t^{\sigma-1} e^{-t}\,dt.$$

If $f \in C([0,T])$, then

$$|(I_0^\sigma f)(t)| \leq \frac{\|f\|_\infty}{\Gamma(\sigma)} t^\sigma,$$

where $\|f\|_\infty = \max_{t\in[0,T]} |f(t)|$, which yields

$$\lim_{t\to 0^+} (I_0^\sigma f)(t) = 0.$$

Similarly, one has

$$|(I_T^\sigma f)(t)| \leq \frac{\|f\|_\infty}{\Gamma(\sigma)} (T-t)^\sigma,$$

which yields

$$\lim_{t\to T^-} (I_T^\sigma f)(t) = 0. \tag{5}$$

For the following property, see the Corollary in [32], p. 67.

Lemma 1. *Let $\sigma > 0$ and $f, g \in C([0,T])$. It holds that*

$$\int_0^T (I_0^\sigma f)(t) g(t)\, dt = \int_0^T f(t)(I_T^\sigma g)(t)\, dt.$$

For a positive integer k, let

$$AC^k([0,T]) = \left\{ f \in C^{k-1}([0,T] : \frac{d^{k-1}f}{dt^{k-1}} \in AC([0,T]) \right\},$$

where $AC([0,T])$ denotes the space of the absolutely continuous functions in $[0,T]$. Clearly, one has $AC^1([0,T]) = AC([0,T])$. For $\sigma \in (k-1,k)$, the Caputo fractional derivative of the order σ of $f \in AC^k([0,T])$ is defined by

$${}^C D_0^\sigma f(t) = \left(I_0^{k-\sigma} \frac{d^k f}{dt^k} \right)(t) = \frac{1}{\Gamma(k-\sigma)} \int_0^t (t-s)^{k-\sigma-1} \frac{d^k f}{dt^k}(s)\, ds,$$

for almost everywhere $t \in [0,T]$.

Let $F : [0,T] \times J \to \mathbb{R}$ be a given function, where J is an interval of $\mathbb{R}$. The left-sided and right-sided Riemann–Liouville fractional integrals of the order σ of F with respect to the time-variable t, are denoted respectively by $I_0^\sigma F$ and $I_T^\sigma F$; that is,

$$I_0^\sigma F(t,x) = (I_0^\sigma F(\cdot,x))(t),$$

and

$$I_T^\sigma F(t,x) = (I_I^\sigma F(\cdot,x))(t).$$

The time-Caputo fractional derivative (the Caputo fractional derivative with respect to the time-variable t) of the order $\sigma \in (k-1,k)$ of F, is denoted by $\dfrac{\partial^\sigma F}{\partial t^\sigma}$; that is,

$$\frac{\partial^\sigma F}{\partial t^\sigma}(t,x) = {}^C D_0^\sigma F(\cdot,x)(t) = I_0^{k-\sigma} \frac{\partial^k F}{\partial t^k}(t,x).$$

3. Main Results

First, let us define weak solutions to (1)-(2)-(3). Let

$$Q = [0,\infty) \times]0,1], \quad Q_T = [0,T] \times]0,1], \; T > 0.$$

Definition 1. *Let $T > 0$. We say that a function $\varphi = \varphi(t,x)$ belongs to Φ_T, if*

(i) $\varphi \in C^2(Q_T)$, $\varphi \geq 0$;

(ii) $\mathrm{supp}_x(\varphi) \subset\subset Q$ *(the support of φ with respect to the variable x is a compact subset of Q);*

(iiii) $\varphi(\cdot,1) = 0$, $\dfrac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(T,\cdot) = 0$.

Definition 2. *Let $u_i \in L^1_{\mathrm{loc}}(]0,1])$, $i=0,1$. A weak solution to (1)-(2)-(3) is a function $u \in L^p_{\mathrm{loc}}(Q)$ ($u \in L^p(K)$ for any compact $K \subset Q$), satisfying*

$$\begin{aligned}&\int_{Q_T} x^{-a}|u|^p\varphi\,dx\,dt+\int_0^1 u_0(x)\left(I_T^{1-\alpha}\varphi(0,x)-\frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x)\right)dx\\&+\int_0^1 u_1(x)I_T^{2-\beta}\varphi(0,x)\,dx-\delta\int_0^T(t+1)^\rho\frac{\partial\varphi}{\partial x}(t,1)\,dt\\&\le-\int_{Q_T}u\left(\frac{\partial(I_T^{1-\alpha}\varphi)}{\partial t}-\frac{\partial^2(I_T^{2-\beta}\varphi)}{\partial t^2}+\frac{\partial^2\varphi}{\partial x^2}+\lambda\frac{\partial}{\partial x}\left(\frac{\varphi}{x}\right)\right)dx\,dt\end{aligned}, \tag{6}$$

for all $T>0$ and $\varphi \in \Phi_T$.

Using standard integrations by parts, Lemma 1, and (5), it can be easily seen that, if u is a smooth solution to (1)-(2)-(3), then u is a weak solution, in the sense of Definition 2.

For $a \ge 0$ and $p>1$, let

$$\widetilde{\lambda} = \min\{-\lambda,1\}, \tag{7}$$

$$\zeta_1 = -\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right), \tag{8}$$

$$\zeta_2 = \frac{(1-\widetilde{\lambda})p+\widetilde{\lambda}-a+1}{p-1}. \tag{9}$$

Our main result is stated below.

Theorem 1. *Let $0<\alpha<1$, $1<\beta<2$, $\lambda,\rho \in \mathbb{R}$, $a \ge 0$, $\delta>0$, $p>1$, and $u_i \in L^1_{\mathrm{loc}}(]0,1])$, $u_i \ge 0$, $i=0,1$. Assume that*

$$(\rho+\alpha)p>\rho \tag{10}$$

and

$$\zeta_1-\theta\left(\rho+\frac{\alpha p}{p-1}\right)<0,\ \zeta_2-\rho\theta<0, \tag{11}$$

for some $\theta>0$. Then, (1)-(2)-(3) admits no weak solution.

In the proof of Theorem 1, we use nonlinear capacity estimates specifically adapted to the nonlocal operators $\frac{\partial^\alpha}{\partial t^\alpha}$ and $\frac{\partial^\beta}{\partial t^\alpha}$, the differential operator $-\frac{\partial^2}{\partial x^2}+\frac{\lambda}{x}\frac{\partial}{\partial x}$, and the boundary condition (3).

Let us discuss some special cases of Theorem 1. We first consider the case $\lambda \le -1$.

Corollary 1. *Let $0<\alpha<1$, $1<\beta<2$, $\delta>0$, and $u_i \in L^1_{\mathrm{loc}}(]0,1])$, $u_i \ge 0$, $i=0,1$. Assume that $\lambda \le -1$ and $a>2$.*

(I) *If $\rho \ge -\alpha$, then for all $p>1$, (1)-(2)-(3) admits no weak solution.*

(II) *If $\rho < -\alpha$, then for all*

$$1<p<1-\frac{\alpha}{\alpha+\rho},$$

(1)-(2)-(3) *admits no weak solution.*

We next study the case $\lambda > -1$.

Corollary 2. *Let $0<\alpha<1$, $1<\beta<2$, $\delta>0$, $a \in \mathbb{R}$, and $u_i \in L^1_{\mathrm{loc}}(]0,1])$, $u_i \ge 0$, $i=0,1$. Assume that $\lambda > -1$.*

(I) *If $\rho>0$, then for all $p>1$, (1)-(2)-(3) admits no weak solution.*

(II) *If $-\alpha \leq \rho \leq 0$ and $a > 2$, then for all*

$$1 < p < 1 + \frac{a-2}{\lambda+1},$$

(1)-(2)-(3) *admits no weak solution.*

(III) *If $\rho < -\alpha$ and $a > 2$, then for all*

$$1 < p < 1 + \min\left\{\frac{a-2}{\lambda+1}, \frac{-\alpha}{\alpha+\rho}\right\},$$

(1)-(2)-(3) *admits no weak solution.*

Remark 1. *Let us consider problem* (1) *under the initial condition* (2) *and the boundary condition*

$$u(t,1) = \delta, \tag{12}$$

where $\delta > 0$. Notice that (12) *is a special case of* (3) *with $\rho = 0$. Let $\lambda > -1$, $a > 2$, and*

$$p > 1 + \frac{a-2}{\lambda+1}. \tag{13}$$

Let us consider the function

$$u(x) = \varepsilon x^{\sigma}, \quad x \in]0,1],$$

where

$$\sigma = \frac{a-2}{p-1}, \ \varepsilon = [\sigma(1-\sigma+\lambda)]^{\frac{1}{p-1}}.$$

Notice that due to (13) *and since $a > 2$, one has*

$$\sigma(1-\sigma+\lambda) > 0,$$

which shows that $\varepsilon > 0$ is well-defined. Differentiating u, we obtain

$$\begin{aligned} -u''(x) + \frac{\lambda}{x}u'(x) &= \varepsilon\sigma(1-\sigma+\lambda)x^{\sigma-2} \\ &= \varepsilon\varepsilon^{p-1}x^{-a+\sigma p}x^{a+\sigma-2-\sigma p} \\ &= \varepsilon^p x^{-a} x^{\sigma p} \\ &= x^{-a}u^p(x). \end{aligned}$$

Hence, u is a stationary solution to (1)-(2)-(12) *with $\delta = \varepsilon > 0$, $u_0(x) = \varepsilon x^{\sigma} \geq 0$, and $u_1(x) = 0$. On the other hand, by Corollary 1 (I) and Corollary 2 (II), we deduce that, when $a > 2$, $\delta > 0$, and $u_i \in L^1_{\text{loc}}(]0,1])$, $u_i \geq 0$, $i = 0,1$, then*

(i) *if $\lambda \leq -1$, then for all $p > 1$,* (1)-(2)-(12) *admits no weak solution;*

(ii) *if $\lambda > -1$, then for all*

$$1 < p < 1 + \frac{a-2}{\lambda+1},$$

(1)-(2)-(12) *admits no weak solution.*

Therefore, we deduce that (1)-(2)-(12) *admits a critical exponent (Fujita-type critical exponent) given by*

$$p_c = \begin{cases} \infty & \text{if} \quad \lambda \leq -1, \\ 1 + \frac{a-2}{\lambda+1} & \text{if} \quad \lambda > -1. \end{cases}$$

Namely,

- *if $\delta > 0$ and $u_i \in L^1_{\text{loc}}(]0,1])$, $u_i \geq 0$, $i = 0,1$, then for all*

$$1 < p < p_c,$$

(1)-(2)-(12) admits no weak solution;

- *if*

$$p > p_c,$$

then (1)-(2)-(12) admits solutions for some $\delta > 0$ and $u_0, u_1 \geq 0$.

It is interesting to observe that p_c depends on λ and a but is independent of the fractional orders α and β.

4. Preliminaries

Let $0 < \alpha < 1$, $1 < \beta < 2$, $\lambda, \rho \in \mathbb{R}$, $a \geq 0$, $\delta > 0$, $p > 1$, and $u_i \in L^1_{\text{loc}}(]0,1])$, $i = 0, 1$. We denote by L_λ the differential operator defined by

$$L_\lambda = \frac{\partial^2 \cdot}{\partial x^2} + \lambda \frac{\partial}{\partial x}\left(\frac{\cdot}{x}\right).$$

4.1. A Priori Estimate

For $T > 0$ and $\varphi \in \Phi_T$, let

$$K_1(\varphi) = \int_{\text{supp}(\varphi)} x^{\frac{a}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial (I_T^{1-\alpha}\varphi)}{\partial t} \right|^{\frac{p}{p-1}} dx\,dt, \tag{14}$$

$$K_2(\varphi) = \int_{\text{supp}(\varphi)} x^{\frac{a}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 (I_T^{2-\beta}\varphi)}{\partial t^2} \right|^{\frac{p}{p-1}} dx\,dt, \tag{15}$$

$$K_3(\varphi) = \int_{\text{supp}(\varphi)} x^{\frac{a}{p-1}} \varphi^{\frac{-1}{p-1}} |L_\lambda \varphi|^{\frac{p}{p-1}} dx\,dt. \tag{16}$$

The following a priori estimate holds.

Lemma 2. *If $u \in L^p_{\text{loc}}(Q)$ is a weak solution to (1)-(2)-(3), then*

$$\begin{aligned}
&\int_0^1 u_0(x)\left(I_T^{1-\alpha}\varphi(0,x) - \frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x)\right)dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx \\
&- \delta \int_0^T (t+1)^\rho \frac{\partial \varphi}{\partial x}(t,1)\,dt \\
&\leq C \sum_{i=1}^3 K_i(\varphi)
\end{aligned} \tag{17}$$

for every $T > 0$ and $\varphi \in \Phi_T$, provided that $K_i(\varphi) < \infty$, $i = 1, 2, 3$.

Proof. Let $u \in L^p_{\text{loc}}(Q)$ be a weak solution to (1)-(2)-(3). Let $T > 0$ and $\varphi \in \Phi_T$ be such that $K_i(\varphi) < \infty$, $i = 1, 2, 3$. By (6), there holds

$$\begin{aligned}
&\int_{Q_T} x^{-a}|u|^p \varphi\,dx\,dt + \int_0^1 u_0(x)\left(I_T^{1-\alpha}\varphi(0,x) - \frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x)\right)dx \\
&+ \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx - \delta\int_0^T (t+1)^\rho \frac{\partial\varphi}{\partial x}(t,1)\,dt \\
&\leq \int_{Q_T} |u| \left|\frac{\partial(I_T^{1-\alpha}\varphi)}{\partial t}\right| dx\,dt + \int_{Q_T} |u| \left|\frac{\partial^2(I_T^{2-\beta}\varphi)}{\partial t^2}\right| dx\,dt + \int_{Q_T} |u||L_\lambda\varphi|\,dx\,dt.
\end{aligned} \tag{18}$$

By means of Young's inequality, we obtain

$$\begin{aligned}\int_{Q_T} |u| \left|\frac{\partial (I_T^{1-\alpha}\varphi)}{\partial t}\right| dx\,dt &= \int_{Q_T} \left(x^{\frac{-a}{p}} |u| \varphi^{\frac{1}{p}}\right) \left(x^{\frac{a}{p}} \varphi^{\frac{-1}{p}} \left|\frac{\partial (I_T^{1-\alpha}\varphi)}{\partial t}\right|\right) dx\,dt \\ &\leq \frac{1}{3}\int_{Q_T} x^{-a}|u|^p \varphi\, dx\,dt + CK_1(\varphi). \end{aligned} \tag{19}$$

Similarly, we obtain

$$\int_{Q_T} |u| \left|\frac{\partial^2 (I_T^{2-\beta}\varphi)}{\partial t^2}\right| dx\,dt \leq \frac{1}{3}\int_{Q_T} x^{-a}|u|^p \varphi\, dx\,dt + CK_2(\varphi), \tag{20}$$

and

$$\int_{Q_T} |u||L_\lambda \varphi|\, dx\,dt \leq \frac{1}{3}\int_{Q_T} x^{-a}|u|^p \varphi\, dx\,dt + CK_3(\varphi). \tag{21}$$

Hence, in view of (18)–(21), we obtain (17). □

4.2. Test Functions

Let

$$D(x) = x^{\widetilde{\lambda}}\left(1 - x^{|\lambda+1|}\right), \quad x \in]0,1], \tag{22}$$

where $\widetilde{\lambda}$ is given by (7). It can be easily seen that

$$D \in C^2(]0,1]),\ D \geq 0,\ L_\lambda D = 0,\ D(1) = 0. \tag{23}$$

Let $\xi \in C^\infty([0,\infty))$ be a cut-off function satisfying

$$0 \leq \xi \leq 1,\ \xi(s) = 0 \text{ if } 0 \leq s \leq \frac{1}{2},\ \xi(s) = 1 \text{ if } s \geq 1. \tag{24}$$

For sufficiently large R and ℓ, let

$$\xi_R(x) = D(x)\xi^\ell(Rx), \quad x \in]0,1];$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if} \quad 0 < x \leq (2R)^{-1}, \\ D(x)\xi^\ell(Rx) & \text{if} \quad (2R)^{-1} \leq x \leq R^{-1}, \\ D(x) & \text{if} \quad R^{-1} \leq x \leq 1. \end{cases} \tag{25}$$

For $T > 0$, let

$$\vartheta_T(t) = T^{-\ell}(T-t)^\ell, \quad 0 \leq t \leq T. \tag{26}$$

Let

$$\varphi(t,x) = \vartheta_T(t)\xi_R(x), \quad (t,x) \in Q_T. \tag{27}$$

The proof of the following lemma can be found in [30].

Lemma 3. *Let $T, \sigma > 0$. For every $t \in [0,T]$, we have*

$$\begin{aligned} (I_T^\sigma \vartheta_T)(t) &= \frac{\Gamma(\ell+1)}{\Gamma(\sigma+\ell+1)} T^{-\ell}(T-t)^{\sigma+\ell}, \\ (I_T^\sigma \vartheta_T)'(t) &= -\frac{\Gamma(\ell+1)}{\Gamma(\sigma+\ell)} T^{-\ell}(T-t)^{\sigma+\ell-1}, && (28) \\ (I_T^\sigma \vartheta_T)''(t) &= \frac{\Gamma(\ell+1)}{\Gamma(\sigma+\ell-1)} T^{-\ell}(T-t)^{\sigma+\ell-2}. && (29) \end{aligned}$$

Lemma 4. *For $T > 0$ and sufficiently large R and ℓ, the function φ defined by (27) belongs to Φ_T.*

Proof. By (23)–(28), we observe that the function φ satisfies the properties (i)–(iii) of Definition 1. □

4.3. Preliminary Estimates

For $T > 0$ and sufficiently large R and ℓ, let φ be the function defined by (27).

Lemma 5. *We have*

$$K_1(\varphi) \le CT^{1-\frac{\alpha p}{p-1}}\left(\ln R + R^{-\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right)}\right). \tag{30}$$

Proof. By (14) and (27), we obtain

$$K_1(\varphi) = \left(\int_0^T \vartheta_T^{\frac{-1}{p-1}}\left|(I_T^{1-\alpha}\vartheta_T)'(t)\right|^{\frac{p}{p-1}}\,dt\right)\left(\int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}}\xi_R(x)\,dx\right). \tag{31}$$

On the other hand, by (26) and (28) (with $\sigma = 1-\alpha$) , for all $0 < t < T$, we have

$$\vartheta_T^{\frac{-1}{p-1}}\left|(I_T^{1-\alpha}\vartheta_T)'(t)\right|^{\frac{p}{p-1}} = CT^{-\ell}(T-t)^{\ell-\frac{\alpha p}{p-1}}.$$

Integrating over $(0, T)$, we obtain

$$\int_0^T \vartheta_T^{\frac{-1}{p-1}}\left|(I_T^{1-\alpha}\vartheta_T)'(t)\right|^{\frac{p}{p-1}}\,dt \le CT^{1-\frac{\alpha p}{p-1}}. \tag{32}$$

Moreover, by (22), (24), and (25), we obtain

$$\begin{aligned}
\int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}}\xi_R(x)\,dx &= \int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}}D(x)\xi^\ell(Rx)\,dx \\
&\le \int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}+\widetilde{\lambda}}\,dx \\
&\le C\begin{cases} 1 & \text{if} \quad \frac{a}{p-1}+\widetilde{\lambda}+1 > 0, \\ \ln R & \text{if} \quad \frac{a}{p-1}+\widetilde{\lambda}+1 = 0, \\ R^{-\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right)} & \text{if} \quad \frac{a}{p-1}+\widetilde{\lambda}+1 < 0, \end{cases}
\end{aligned}$$

which implies that

$$\int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}}\xi_R(x)\,dx \le C\left(\ln R + R^{-\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right)}\right). \tag{33}$$

Therefore, (30) follows from (31)–(33). □

Lemma 6. *The following estimate holds:*

$$K_2(\varphi) \le CT^{1-\frac{\beta p}{p-1}}\left(\ln R + R^{-\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right)}\right). \tag{34}$$

Proof. By (15) and (27), we obtain

$$K_2(\varphi) = \left(\int_0^T \vartheta_T^{\frac{-1}{p-1}}\left|(I_T^{2-\beta}\vartheta_T)''(t)\right|^{\frac{p}{p-1}}\,dt\right)\left(\int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}}\xi_R(x)\,dx\right). \tag{35}$$

Using (26) and (29) (with $\sigma = 2-\beta$), for all $0<t<T$, we obtain

$$\begin{aligned}
&\vartheta_T^{\frac{-1}{p-1}}\left|(I_T^{2-\beta}\vartheta_T)''(t)\right|^{\frac{p}{p-1}}\\
&=\left[T^{-\ell}(T-t)^{\ell}\right]^{\frac{-1}{p-1}}\left[\frac{\Gamma(\ell+1)}{\Gamma(1-\beta+\ell)}T^{-\ell}(T-t)^{\ell-\beta}\right]^{\frac{p}{p-1}}\\
&=\left[\frac{\Gamma(\ell+1)}{\Gamma(1-\beta+\ell)}\right]^{\frac{p}{p-1}}T^{-\ell}(T-t)^{\ell-\frac{\beta p}{p-1}}\\
&=CT^{-\ell}(T-t)^{\ell-\frac{\beta p}{p-1}}.
\end{aligned}$$

Integrating over $(0,T)$, we obtain

$$\int_0^T \vartheta_T^{\frac{-1}{p-1}}\left|(I_T^{2-\beta}\vartheta_T)''(t)\right|^{\frac{p}{p-1}}\,dt \le CT^{1-\frac{\beta p}{p-1}}. \tag{36}$$

Therefore, using (33), (35), and (36), we obtain (34). □

Lemma 7. *The following estimate holds:*

$$K_3(\varphi) \le CTR^{\frac{(1-\widetilde{\lambda})p+\widetilde{\lambda}-a+1}{p-1}}. \tag{37}$$

Proof. By (16) and (27), we obtain

$$K_3(\varphi) = \left(\int_0^T \vartheta_T(t)\,dt\right)\left(\int_{\frac{1}{2R}}^1 x^{\frac{a}{p-1}}\xi_R^{\frac{-1}{p-1}}|L_\lambda \xi_R|^{\frac{p}{p-1}}\,dx\right). \tag{38}$$

On the other hand, by (26), we have

$$\begin{aligned}
\int_0^T \vartheta_T(t)\,dt &= T^{-\ell}\int_0^T (T-t)^{\ell}\,dt\\
&= CT.
\end{aligned}\tag{39}$$

Moreover, by (25), for all $\frac{1}{2R}<x<1$, one has

$$\begin{aligned}
L_\lambda\xi_R(x) &= L_\lambda\Big(D(x)\xi^{\ell}(Rx)\Big)\\
&= \Big(D(x)\xi^{\ell}(Rx)\Big)''+\lambda\Big(x^{-1}D(x)\xi^{\ell}(Rx)\Big)'\\
&= D''(x)\xi^{\ell}(Rx)+(\xi^{\ell}(Rx))''D(x)+2\Big(\xi^{\ell}(Rx)\Big)'D'(x)+\lambda\Big(x^{-1}D(x)\Big)'\xi^{\ell}(Rx)\\
&\quad +\lambda x^{-1}D(x)\Big(\xi^{\ell}(Rx)\Big)'\\
&= \Big(D''(x)+\lambda\Big(x^{-1}D(x)\Big)'\Big)\xi^{\ell}(Rx)+(\xi^{\ell}(Rx))''D(x)+2\Big(\xi^{\ell}(Rx)\Big)'D'(x)\\
&\quad +\lambda x^{-1}D(x)\Big(\xi^{\ell}(Rx)\Big)'\\
&= L_\lambda D(x)\xi^{\ell}(Rx)+(\xi^{\ell}(Rx))''D(x)+2\Big(\xi^{\ell}(Rx)\Big)'D'(x)\\
&\quad +\lambda x^{-1}D(x)\Big(\xi^{\ell}(Rx)\Big)'.
\end{aligned}$$

Therefore, by (23) ($L_\lambda D=0$), we obtain

$$L_\lambda\xi_R(x) = (\xi^{\ell}(Rx))''D(x)+2\Big(\xi^{\ell}(Rx)\Big)'D'(x)+\lambda x^{-1}D(x)\Big(\xi^{\ell}(Rx)\Big)', \tag{40}$$

which implies by (24) that

$$\int_{\frac{1}{2R}}^{1} x^{\frac{a}{p-1}} \xi_R^{\frac{-1}{p-1}} |L_\lambda \xi_R|^{\frac{p}{p-1}}\, dx = \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{a}{p-1}} \xi_R^{\frac{-1}{p-1}} |L_\lambda \xi_R|^{\frac{p}{p-1}}\, dx. \tag{41}$$

On the other hand, by (22) and (24), for all $\frac{1}{2R} < x < \frac{1}{R}$, one has

$$C_1 R^{-\widetilde{\lambda}} \leq D(x) \leq C_2 R^{-\widetilde{\lambda}},\ |D'(x)| \leq C R^{1-\widetilde{\lambda}} \tag{42}$$

and

$$|(\xi^\ell(Rx))''| \leq C R^2 \xi^{\ell-2}(Rx),\ |(\xi^\ell(Rx))'| \leq C R \xi^{\ell-2}(Rx). \tag{43}$$

Hence, in view of (41), (42), and (43), we obtain

$$|L_\lambda \xi_R(x)| \leq C R^{2-\widetilde{\lambda}} \xi^{\ell-2}(Rx), \quad \frac{1}{2R} < x < \frac{1}{R}. \tag{44}$$

Therefore, using (25), (41), (42), and (44), we obtain

$$\begin{aligned} \int_{\frac{1}{2R}}^{1} x^{\frac{a}{p-1}} \xi_R^{\frac{-1}{p-1}} |L_\lambda \xi_R|^{\frac{p}{p-1}}\, dx &\leq C R^{\frac{(2-\widetilde{\lambda})p+\widetilde{\lambda}}{p-1}} \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{a}{p-1}} \xi^{\ell-\frac{2p}{p-1}}(Rx)\, dx \\ &\leq C R^{\frac{(2-\widetilde{\lambda})p+\widetilde{\lambda}}{p-1}} \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{a}{p-1}}\, dx \\ &\leq C R^{\frac{(1-\widetilde{\lambda})p+\widetilde{\lambda}-a+1}{p-1}}. \end{aligned} \tag{45}$$

Thus, (37) follows from (38), (39), and (45). □

5. Proofs of the Obtained Results

We need the following result.

Lemma 8. *Let* $0 < \alpha < 1$, $1 < \beta < 2$, $\lambda \in \mathbb{R}$, $a \geq 0$, $\delta > 0$, $\rho \in \mathbb{R}$, $p > 1$, *and* $u_i \in L^1_{\text{loc}}(]0,1])$, $u_i \geq 0$, $i = 0,1$. *Assume that* $u \in L^p_{\text{loc}}(Q)$ *is a weak solution to* (1)-(2)-(3). *Then, for sufficiently large T and R, there holds*

$$\delta \leq C\left(T^{-\rho-\frac{\alpha p}{p-1}} \ln R + T^{-\rho-\frac{\alpha p}{p-1}} R^{\zeta_1} + T^{-\rho} R^{\zeta_2}\right), \tag{46}$$

where ζ_1 *and* ζ_2 *are given by* (8) *and* (9), *respectively.*

Proof. Let $u \in L^p_{\text{loc}}(Q)$ be a weak solution to (1)-(2)-(3). Then, by Lemmas 2 and 4, for sufficiently large T, R, and ℓ, there holds

$$\begin{aligned} &\int_0^1 u_0(x)\left(I_T^{1-\alpha}\varphi(0,x) - \frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x)\right) dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\, dx \\ &- \delta \int_0^T (t+1)^\rho \frac{\partial \varphi}{\partial x}(t,1)\, dt \\ &\leq C \sum_{i=1}^{3} K_i(\varphi). \end{aligned} \tag{47}$$

where φ is the function defined by (27). On the other hand, one has

$$I_T^{1-\alpha}\varphi(0,x) \geq 0,\ \frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x) \leq 0,\ I_T^{2-\beta}\varphi(0,x) \geq 0, \quad x \in]0,1]. \tag{48}$$

Namely, by the definition of φ (see (27)), we have $\varphi \geq 0$, which implies that $I_T^{1-\alpha}\varphi(0,x) \geq 0$ and $I_T^{2-\beta}\varphi(0,x) \geq 0$. Moreover, one has

$$I_T^{2-\beta}\varphi(t,x) = \xi_R(x) I_T^{2-\beta}\vartheta_T(t),$$

which implies that

$$\frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(t,x) = \xi_R(x)(I_T^{2-\beta}\vartheta_T)'(t).$$

Using (28) with $\sigma = 2-\beta$, we obtain

$$(I_T^{2-\beta}\vartheta_T)'(t) = -\frac{\Gamma(\ell+1)}{\Gamma(2-\beta+\ell)} T^{-\ell}(T-t)^{1-\beta+\ell}.$$

Then, it holds that

$$\frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(t,x) = -\frac{\Gamma(\ell+1)}{\Gamma(2-\beta+\ell)} T^{-\ell}(T-t)^{1-\beta+\ell}\xi_R(x),$$

and

$$\frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x) = -\frac{\Gamma(\ell+1)}{\Gamma(2-\beta+\ell)} T^{1-\beta}\xi_R(x) \leq 0.$$

This proves (48). Now, since $u_i \geq 0$, $i = 0,1$, due to (48), we have

$$u_0(x)\left(I_T^{1-\alpha}\varphi(0,x) - \frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x)\right) \geq 0, \;\; u_1(x) I_T^{2-\beta}\varphi(0,x) \geq 0,$$

which yields

$$\int_0^1 u_0(x)\left(I_T^{1-\alpha}\varphi(0,x) - \frac{\partial(I_T^{2-\beta}\varphi)}{\partial t}(0,x)\right) dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\, dx \geq 0. \tag{49}$$

Moreover, by (22), (25), (26), and (27), we obtain

$$\begin{aligned} -\delta\int_0^T (t+1)^\rho \frac{\partial\varphi}{\partial x}(t,1)\,dt &= -\delta\xi_R'(1)\int_0^T (t+1)^\rho \vartheta_T(t)\,dt \\ &= -\delta D'(1) T^{-\ell}\int_0^T (t+1)^\rho (T-t)^\ell\,dt \\ &= C\delta T^{-\ell}\int_0^T (t+1)^\rho (T-t)^\ell\,dt \\ &\geq C\delta T^{-\ell}\int_{\frac{T}{2}}^T (t+1)^\rho (T-t)^\ell\,dt \\ &\geq C\delta T^{\rho+1}. \end{aligned} \tag{50}$$

Hence, using Lemma 5, Lemma 6, Lemma 7, (51), (49), and (50), we obtain

$$\begin{aligned} &\delta T^{\rho+1} \\ &\leq C\left[T^{1-\frac{\alpha p}{p-1}}\left(\ln R + R^{-\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right)}\right) + T^{1-\frac{\beta p}{p-1}}\left(\ln R + R^{-\left(\frac{a}{p-1}+\widetilde{\lambda}+1\right)}\right) + TR^{\frac{(1-\widetilde{\lambda})p+\widetilde{\lambda}-a+1}{p-1}}\right]; \end{aligned}$$

that is,

$$\delta \leq C\left(T^{-\rho-\frac{\alpha p}{p-1}}\ln R + T^{-\rho-\frac{\beta p}{p-1}}\ln R + T^{-\rho-\frac{\alpha p}{p-1}}R^{\zeta_1} + T^{-\rho-\frac{\beta p}{p-1}}R^{\zeta_1} + T^{-\rho}R^{\zeta_2}\right).$$

Finally, since $\alpha < \beta$, the above estimate yields (46). $\square$

Now, we prove Theorem 1.

Proof of Theorem 1. Suppose that $u \in L^p_{\text{loc}}(Q)$ is a weak solution to (1)-(2)-(3). Then, by Lemma 8, (46) holds for sufficiently large T and R. Taking $T = R^\theta$, where $\theta > 0$ satisfies (11), (46) reduces to

$$\delta \leq C\left(R^{-\theta\left(\rho+\frac{\alpha p}{p-1}\right)} \ln R + R^{\zeta_1-\theta\left(\rho+\frac{\alpha p}{p-1}\right)} + R^{\zeta_2-\rho\theta}\right). \tag{51}$$

Hence, in view of (10) and (11), passing to the limit as $R \to \infty$ in (51), we obtain $\delta \leq 0$, which contradicts the positivity of δ. Consequently, (1)-(2)-(3) admits no weak solution. The proof of Theorem 1 is then completed. $\square$

We now prove Corollary 1.

Proof of Corollary 1. Let $\lambda \leq -1$. In this case, one has $\widetilde{\lambda} = \min\{-\lambda, 1\} = 1$, which implies that

$$\zeta_1 = -\left(\frac{a}{p-1} + 2\right) < 0, \tag{52}$$

and (since $a > 2$)

$$\zeta_2 = \frac{2-a}{p-1} < 0. \tag{53}$$

(I) For the case $\rho \geq -\alpha$, we discuss two sub-cases.
(i) If $-\alpha \leq \rho < 0$, in this case, one has

$$(\rho + \alpha)p \geq 0 > \rho, \tag{54}$$

which implies that (10) is satisfied. On the other hand, due to (52) and (54), for all $\theta > 0$, one has

$$\zeta_1 - \theta\left(\rho + \frac{\alpha p}{p-1}\right) < 0.$$

In particular, for

$$0 < \theta < \frac{\zeta_2}{\rho}, \tag{55}$$

we obtain

$$\zeta_2 - \rho\theta < 0.$$

Notice that due to (53) and since $\rho < 0$, the set of θ satisfying (55) is nonempty. Hence, for θ satisfying (55), (11) is satisfied. Then, Theorem 1 applies.
(ii) If $\rho \geq 0$, in this case, one has

$$(\rho + \alpha)p > \rho + \alpha > \rho$$

which implies that (10) is satisfied. Moreover, in view of (52) and (53), for all $\theta > 0$, we obtain

$$\zeta_1 - \theta\left(\rho + \frac{\alpha p}{p-1}\right) \leq \zeta_1 < 0$$

and

$$\zeta_2 - \rho\theta \leq \zeta_2 < 0.$$

Hence, (11) is satisfied for every $\theta > 0$. Then, Theorem 1 applies.
(II) For the case $\rho < -\alpha$, let

$$1 < p < 1 - \frac{\alpha}{\rho + \alpha}. \tag{56}$$

Then, (10) is satisfied. Moreover, due to (56), for all $\theta > 0$, one has

$$\zeta_1 - \theta\left(\rho + \frac{\alpha p}{p-1}\right) \le \zeta_1 < 0.$$

In particular, for θ satisfying (55), one has

$$\zeta_2 - \rho\theta < 0.$$

Therefore, Theorem 1 applies. □

Next, we prove Corollary 2.

Proof of Corollary 2. Let $\lambda > -1$. In this case, one has $\widetilde{\lambda} = \min\{-\lambda, 1\} = -\lambda$. Then,

$$\zeta_1 = -\left(\frac{a}{p-1} + 1 - \lambda\right) = -\frac{(1-\lambda)p + (\lambda + a - 1)}{p-1}$$

and

$$\zeta_2 = \frac{(\lambda+1)p - (\lambda + a - 1)}{p-1}.$$

(I) For the case $\rho > 0$, in this case, one has

$$(\rho + \alpha)p > \rho + \alpha > \rho,$$

which shows that (10) is satisfied. Moreover, for

$$\theta > \max\left\{0, \frac{\zeta_2}{\rho}, \frac{\zeta_1}{\rho + \frac{\alpha p}{p-1}}\right\},$$

(11) is satisfied. Theorem 1 applies.
(II) For the case $-\alpha \le \rho \le 0$ and $a > 2$, let

$$1 < p < 1 + \frac{a-2}{\lambda+1}. \tag{57}$$

In this case, one has

$$(\rho + \alpha)p \ge \rho + \alpha > \rho, \tag{58}$$

which shows that (10) is satisfied. On the other hand, by (57), we obtain

$$\lambda + a - 1 > (\lambda+1)p > (\lambda - 1)p,$$

which implies that $\zeta_1 < 0$. Then, by (58), for all $\theta > 0$, there holds

$$\zeta_1 - \theta\left(\rho + \frac{\alpha p}{p-1}\right) < 0.$$

Moreover, by (57), one has $\zeta_2 < 0$. Hence, if $\rho = 0$, then $\zeta_2 - \rho\theta < 0$ for every $\theta > 0$; otherwise, for θ satisfying (55), we obtain $\zeta_2 - \rho\theta < 0$. Hence, (11) is satisfied. Therefore, Theorem 1 applies.
(III) For the case $\rho < -\alpha$ and $a > 2$, let

$$1 < p < 1 + \min\left\{\frac{a-2}{\lambda+1}, \frac{-\alpha}{\alpha+\rho}\right\}. \tag{59}$$

By (59), we deduce that

$$p < 1 - \frac{\alpha}{\alpha + \rho},$$

which yields (10). Again, by (59), we obtain (57), which implies that $\zeta_1 < 0$ and $\zeta_2 < 0$. Thus, by (10), for all $\theta > 0$, there holds

$$\zeta_1 - \theta\left(\rho + \frac{\alpha p}{p-1}\right) < 0,$$

and for θ satisfying (55), we obtain $\zeta_2 - \rho\theta < 0$. Hence, for θ satisfying (55), (11) is satisfied. Then, Theorem 1 applies. $\square$

6. Conclusions

The one-dimensional time-fractional damped wave equation (1) under the initial conditions (2) and the Dirichlet boundary condition (3) was investigated. Namely, we obtained sufficient conditions under which the considered problem admits no weak solution in the sense of Definition 2 (see Theorem 1). Next, we discussed separately the cases $\lambda \le -1$ and $\lambda > -1$. Namely, we proved the existence of a critical exponent given by

$$p_c = \begin{cases} \infty & \text{if} \quad \lambda \le -1, \\ 1 + \frac{a-2}{\lambda+1} & \text{if} \quad \lambda > -1, \end{cases}$$

in the following sense:

- if $\delta > 0$ and $u_i \in L^1_{\mathrm{loc}}(]0,1])$, $u_i \ge 0$, $i = 0, 1$, then for all
$$1 < p < p_c,$$
(1)-(2)-(12) admits no weak solution;
- if
$$p > p_c,$$
then (1)-(2)-(12) admits solutions for some $\delta > 0$ and $u_0, u_1 \ge 0$.

This topic can be of some importance for the investigation of the controllability of solutions to certain nonlinear time-fractional models of physics systems, together with the symmetry analysis.

It would be also interesting to extend the present study to (1) with a variable exponent $p(x)$; that is,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^2 u}{\partial x^2} + \frac{\lambda}{x}\frac{\partial u}{\partial x} = x^{-a}|u|^{p(x)} \quad \text{in } (0,\infty)\times]0,1],$$

where $p(x) > 1$.

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