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Quantization of Constrained Systems as Dirac First Class versus Second Class: A Toy Model and Its Implications

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Abstract: A toy model (suggested by Klauder) was analyzed from the perspective of first-class and second-class Dirac constrained systems. First-class constraints are often associated with the existence of important gauge symmetries in a system. A comparison was made by turning a first-class system into a second-class system with the introduction of suitable auxiliary conditions. The links between Dirac's system of constraints, the Faddeev–Popov canonical functional integral method and the Maskawa–Nakajima procedure for reducing the phase space are explicitly illustrated. The model reveals stark contrasts and physically distinguishable results between first and second-class routes. Physically relevant systems such as the relativistic point particle and electrodynamics are briefly recapped. Besides its pedagogical value, the article also advocates the route of rendering first-class systems into second-class systems prior to quantization. Second-class systems lead to a well-defined reduced phase space and physical observables; an absence of inconsistencies in the closure of quantum constraint algebra; and the consistent promotion of fundamental Dirac brackets to quantum commutators. As first-class systems can be turned into well-defined second-class ones, this has implications for the soundness of the "Dirac quantization" of first-class constrained systems by the simple promotion of Poisson brackets, rather than Dirac brackets, to commutators without proceeding through second-class procedures.

Keywords: Dirac; first class; second class; constraints

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1. Introduction

The Hamiltonian formalism provides a natural starting point for the development of classical dynamics, and, to quote Dirac, "For the purpose of setting up a quantum theory one must work from the Hamiltonian form" [1]. At the outset, one has an even-dimensional phase space (q_i, p_i) of dimension $2N$ with symplectic structure $\Omega = \sum_i^N dq_i \wedge dp_i$, and one can define functions on this phase space. The Poisson bracket forms a Lie algebra on this space of functions:

$$\{A(\vec{q},\vec{p}), B(\vec{q},\vec{p})\}_{\text{P.B.}} = \sum_{i=1}^{N}\Big(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\Big). \tag{1}$$

The Hamiltonian $H(\vec{q},\vec{p},t)$ is a special phase space function that generates evolution with respect to a parameter t, known as time, of any quantity $g(t)$ via

$$\dot{g} = \frac{\partial g}{\partial t} + \{q, H\}_{\text{P.B.}}. \tag{2}$$

For the choice $g(t) = (q_i(t), p_i(t))$, (2) corresponds to the Hamilton's equations for the phase space variables. The Hamilton's equations can be solved for the evolution of the system $(q_i(t), p_i(t))$ from initial conditions $(q_i(0), p_i(0))$. In its original formulation, quantization proceeds via the prescription of the promotion of dynamical variables to quantum operators $a \to \hat{a}$ and Poisson brackets to quantum commutators:

$$[\hat{a}, \hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a} = i\hbar\{a, b\}_{\text{P.B.}}. \tag{3}$$

A modulo operator orders ambiguities, a Hamiltonian operator is constructed, $\hat{H}(\vec{q}, \vec{p}, t) = H(\hat{\vec{q}}, \hat{\vec{p}}, t)$, and a wavefunction $\Psi[\vec{q}, t]$ describing the state of the system can be defined based on the configuration space $\vec{q}$, wherein the conjugate momenta act by differentiation. Then, one solves the Schrodinger equation for the system $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ governing the unitary evolution of the state with respect to time t in terms of the quantum mechanical axioms.

There are various ambiguities and subtleties inherent in the process of constructing a quantum theory from its classical starting point (see, for instance, Ref. [2]). For instance, using the Gronewald van Howe theorem, in general, one cannot consistently carry out (3) for all phase space functions $a(\vec{q}, \vec{p})$ and $b(\vec{q}, \vec{p})$, but rather only a limited subset [2]. In addition, the naive *Poisson Brackets to Quantum Commutators* prescription presumes that there are no constraints on the phase space variables. This fails for constrained systems and the whole quantization procedure must be carefully modified in order to take constraints into account. This brings us to the main topic of this paper, namely totally constrained first-class versus second-class systems and the interpretation of Dirac's formalism.

In Refs. [1,3] , Dirac addresses the issue of constrained systems, classifying constraints Φ_I into either first class or second class. First-class constraints Φ_I have $\{\Phi_I, \Phi_J\}_{\text{P.B.}} = C_{IJ}^K\Phi_K$, whereas, for second-class constraints, $\det(\{\Phi_I, \Phi_J\}_{\text{P.B.}}) \neq 0$. Dirac lays out the consistency conditions that must be satisfied for both types of constraints, both classically and upon quantization; however, these necessary conditions may not be sufficient to guarantee well-defined physical theories. This is a fundamentally important question regarding our universe, particularly in the case of totally constrained systems with a weakly vanishing Hamiltonian, as in the usual canonical formulation of Einstein's theory [4–6].

Given a choice between first-class and second-class constrained systems, it is a pertinent question as to which, if either, should be implemented in nature. It is the proposition of this paper that second-class constrained systems are superior to first-class constrained systems in possessing a well-defined even-dimensional phase space and in their avoidance of inconsistencies in the constraint algebra with Dirac brackets. Indeed, Dirac allows for the possibility (on pp. 33–36 of Ref. [7]) of physical states being annihilated by first-class quantum constraints. However, he does not specify whether the constraints should be quantized via Poisson brackets or otherwise, but only that $[\hat{\Phi}_I, \hat{\Phi}_J] = \hat{C}_{IJ}^K\hat{\Phi}_K$ with structure functions $\hat{C}_{IJ}^K(\hat{\vec{q}}, \hat{\vec{p}})$ appearing on the left as necessary in order to avoid inconsistency. On the other hand, second-class constraints must be solved and eliminated from the theory prior to quantization, which proceeds by replacing Dirac brackets with commutators.

We will show that (3) fails for first-class constrained systems. Some of the missing elements of first-class constrained Dirac quantization are that there is no clear phase space correspondence or direct link to perturbative quantum field theory via canonical functional integral formalism. Indeed, as we will show, there cannot be a canonical functional integral without Faddeev–Popov determinants [8] and the second-class Dirac scheme. For example, in the first-class constraint $q_1 = 0$ in a system with more than one variable, $p_1 = 0$ is obviously the proper subsidiary condition to convert to a second-class system, and hence the Dirac (not Poisson) bracket is zero. Then, one consistently promotes $q_1 = 0$, $p_1 = 0$ as operators without violation of canonical commutation relations (CCRs). On the other hand, the Poisson scheme will yield inconsistent $[q_1, p_1] = i\hbar$ and $p_1 = \frac{\hbar}{i}\frac{d}{dq_1}$. Suppose that the theory contains a Hamiltonian $H(p_1, p_2, \dots)$; then, Poisson–Dirac first-class quantization will result in a nonsensical theory lacking a sensible classical limit. Dirac himself advocated that second class be quantized via the Dirac-bracket-to-commutator route. A first-class system does not have access to the Dirac bracket unless it is first turned into second class with supplementary conditions.

The proper relation between phase space variables is paramount. A constraint C implemented via a Lagrange multiplier λ in the first-class approach can be deemed a phase space variable whose conjugate variable is $\dot{\lambda}$ and not λ. Thus, the natural question arises as to what is conjugate to C. Since this is part of the redundant variables, then the conjugacy as per the Dirac–Faddeev–Popov scheme is that a subsidiary condition $\chi = 0$ be taken to be equivalent to the new coordinate $\chi = Q = 0$ and its conjugate arises via $\delta(P)\delta(Q) = \delta(C)\det\{\chi, C\}\delta(\chi)$.

This is in contrast with the naive Poisson–CCR–first-class quantization, wherein one promotes $\{q_i, p_j\} = \frac{1}{i\hbar}[\hat{q}_i, \hat{p}_j]$ in any system. We will see that the correct prescription is to promote Dirac brackets $\{q_i, p_j\}_{\text{D.B.}}$ to quantum commutation relations. In a system with second-class constraints Φ_I, the Dirac bracket is given by

$$\{A, B\}_{\text{D.B.}} = \{A, B\}_{\text{P.B.}} - \{A, \Phi_I\}_{\text{P.B.}}(M^{-1})^{IJ}\{\Phi_J, B\}_{\text{P.B.}} \tag{4}$$

with $M_{IJ} = \{\Phi_I, \Phi_J\}_{\text{P.B.}}$, wherein constraints and auxiliary conditions are lumped together into the same even-dimensional phase subspace. In other words, for a constrained system, quantum CCR *cannot* be determined *before* Dirac brackets. Thus, given $\Phi(p_i, q_i) = 0$ above, one cannot know what $\hat{\Phi}(\hat{p}_i, \hat{q}_i) = 0$ is as a quantum equation. This is not an operator-ordering issue. One cannot even say that $\hat{p}_i = \frac{\hbar}{i}\frac{d}{dq_i}$ because that would violate the Dira-brackets-to-CCR rule. Thus, $[\hat{p}_i, \hat{q}_j]$ can only be determined from the promotion of Dirac brackets to commutators.

One of the main motivations of this paper is to highlight and address some of the difficulties associated with the treatment of general relativity as a constrained system. It is our view that the prescription whereby first-class constraints fix physical states $\hat{\Phi}|\psi\rangle_{Phys} = 0$ via quantization based on Poisson brackets (see, for instance, Refs. [6,9–11]) does not provide the correct interpretation of Dirac's work [7]. We utilized an interesting toy model for illustrative purposes. This model is known as the Klauder toy model for quantum gravity, due to John Klauder. It consists of a two-dimensional system with the potential energy of a spring that has a negative spring constant. As seen, this system accentuates the differences between first-class and second-class constrained systems, and suggests the latter as a more viable approach.

In this paper, we argue that nature has a preference for second-class constrained systems for several reasons. Every system is either (1) second class, in which case it cannot be made first class; (2) first class, in which case it can be made second class by introducing appropriate subsidiary conditions; or (3) partially first class and partially second class, in which case it can be made second class by (2). Using the Maskawa–Nakajima theorem [12], second-class systems inherit a natural phase space whereas first-class systems do not. It should be noted that all variables satisfying second-class constraints are observables in the sense that, by construction and through the definition of the Dirac bracket, all variables have trivial Dirac brackets with all of the constraints $\{A, \Phi_I\}_{\text{D.B.}} = 0\ \forall A, \Phi_I$; in particular, constraints themselves also have trivial Dirac brackets $\{\Phi_I, \Phi_J\}_{\text{D.B.}} = 0$ with each other. They can thus be promoted to quantum observables, where the concepts of Dirac observables weakly Poisson commuting with first-class constraints and the constraint algebra being weak at the Poisson bracket level make local Dirac observables extremely hard to find and the consistency of the quantum constraint algebra a formidable challenge [6,13].

In this paper, we focused on the Klauder problem, highlighting the differences between first-class and second-class constrained systems that it brings out, as well as its relation to general relativity.

2. Derivation of the Dirac Bracket and Reduced Phase Space

For a 2N dimensional phase space, given first-class constraints $\phi_i = 0$ and auxiliary conditions $\chi_j = 0$ with i and j ranging from 1 to $K < N$ such that:

$$\det(\{\chi_i, \phi_j\}_{\text{P.B.}}) \neq 0; \tag{5}$$

we can set $\{\chi_i\}$ as the first K variables of a new set of general coordinates of the same phase space. We denote this by * superscript $\chi_i =: q_i^* = 0$. Then, $\det(\{\chi_i, \phi_j\}_{\text{P.B.}}) \neq 0$ translates into invertible $\{q_i^*, \phi_j\}_{\text{P.B.}} = \frac{\partial \phi_j}{\partial p_i^*}$.

Maskawa and Nakajima proved that, for second-class systems, canonical variables exist for a given restricted submanifold, where the submanifold is specified by setting those canonical variables equal to zero. Thus, we may decompose the total canonical phase space $\{(q_I, p_I), I = 1, ..., N\}$ into the sum of the reduced physical phase space $\{(q_r, p_r),$ $r = K+1, ..., N\}$ and $\{(q_i^*, p_i^*), i = 1, ..., K\}$, and the constrained surface can be obtained by setting canonical $(q_i^*, p_i^*), i = 1, ..., K\}$ to zero (which is equivalent to satisfying $\phi_i = 0$, $\chi_i = 0, \forall i = 1, ...K$).

The Poisson bracket between A and B for the reduced phase space would then be

$$\begin{aligned}
\{A, B\}_{\text{reduced}} &= \sum_{I=1}^{N} \left(\frac{\partial A}{\partial q_I} \frac{\partial B}{\partial p_I} - \frac{\partial A}{\partial p_I} \frac{\partial B}{\partial q_I} \right) - \sum_{j=1}^{K} \left(\frac{\partial A}{\partial q_j^*} \frac{\partial B}{\partial p_j^*} - \frac{\partial A}{\partial p_j^*} \frac{\partial B}{\partial q_j^*} \right) \\
&= \{A, B\}_{\text{P.B.}} - \sum_{j=1}^{K} \Big(\{A, p_j^*\}_{\text{P.B.}} \{q_j^*, B\}_{\text{P.B.}} - \{q_j^*, A\}_{\text{P.B.}} \{B, p_j^*\}_{\text{P.B.}} \Big) \\
&= \{A, B\}_{\text{P.B.}} - \sum_{j=1}^{K} \sum_{L=1}^{N} \Big[\Big(\frac{\partial A}{\partial q_L} \frac{\partial p_j^*}{\partial p_L} - \frac{\partial A}{\partial p_L} \frac{\partial p_j^*}{\partial q_L} \Big) \{q_j^*, B\}_{\text{P.B.}} - \{q_j^*, A\}_{\text{P.B.}} \Big(\frac{\partial B}{\partial q_L} \frac{\partial p_j^*}{\partial p_L} - \frac{\partial B}{\partial p_L} \frac{\partial p_j^*}{\partial q_L} \Big) \Big] \\
&= \{A, B\}_{\text{P.B.}} - \sum_{j=1}^{K} \sum_{L=1}^{N} \Big[\Big(\frac{\partial A}{\partial q_L} \frac{\partial \phi_k}{\partial p_L} - \frac{\partial A}{\partial p_L} \frac{\partial \phi_k}{\partial q_L} \Big) \Big(\frac{\partial p_j^*}{\partial \phi_k} \Big) \{q_j^*, B\}_{\text{P.B.}} \\
&\quad + \{A, q_j^*\}_{\text{P.B.}} \Big(\frac{\partial p_j^*}{\partial \phi_k} \Big) \Big(\frac{\partial B}{\partial q_L} \frac{\partial \phi_k}{\partial p_L} - \frac{\partial B}{\partial p_L} \frac{\partial \phi_k}{\partial q_L} \Big) \Big] \\
&= \{A, B\}_{\text{P.B.}} - (\{A, \phi_k\}_{\text{P.B.}} \{\phi_k, \chi_j\}^{-1} \{\chi_j, B\}_{\text{P.B.}} + \{A, \chi_j\}_{\text{P.B.}} \{\chi_j, \phi_k\}^{-1} \{\phi_k, B\}_{\text{P.B.}}), \\
&= \{A, B\}_{\text{D.B.}}.
\end{aligned} \tag{6}$$

Thus, the Dirac bracket is equivalent to the Poisson bracket with regard to the total phase space subtracted by the unwanted phase space (q_i^*, p_i^*) eliminated by χ_i and ϕ_i. The correct evolution of an operator in a constrained system should be with regard to the physical reduced phase space, implying that the Dirac, rather than Poisson, bracket should be used to arrive at the right physical evolution, i.e., $\dot{A} = \{A, H\}_{\text{D.B.}}$, wherein H is the Hamiltonian of the system.

Interestingly, the following holds true:

$$\begin{aligned}
&\text{Let } A|_c := A|_{\chi=0,\phi=0},\ B|_c := B|_{\chi=0,\phi=0} \\
\{A|_c, B|_c\}_{\text{P.B.}} &= \sum_{j=1}^{K} \left(\frac{\partial (A|_c)}{\partial q_j^*} \frac{\partial (B|_c)}{\partial p_j^*} - \frac{\partial (A|_c)}{\partial p_j^*} \frac{\partial (B|_c)}{\partial q_j^*} \right) + \sum_{r=K+1}^{N} \left(\frac{\partial (A|_c)}{\partial q_r} \frac{\partial (B|_c)}{\partial p_r} - \frac{\partial (A|_c)}{\partial p_r} \frac{\partial (B|_c)}{\partial q_r} \right) \\
&= 0 + \sum_{r=K+1}^{N} \left(\frac{\partial A}{\partial q_r} \frac{\partial B}{\partial p_r} - \frac{\partial A}{\partial p_r} \frac{\partial B}{\partial q_r} \right) \Bigg|_{\chi=0,\phi=0} \\
&= \{A, B\}_{\text{D.B.}}|_{\chi=0,\phi=0},
\end{aligned} \tag{7}$$

The first entry of the intermediate step vanishes because, on the constrained surface, $\delta q_i^* = 0$, so $A|_c = A(q_r, p_r)$ and $B|_c = B(q_r, p_r)$ have no dependence on q_i^*. That the second entry is equivalent to the Dirac bracket is precisely a theorem of Maskawa and Nakajima: the Dirac bracket of a second-class system is equal to the Poisson bracket with regard to the reduced variables.

3. Symplectic Analysis: Relativistic Particle and Maxwell Theory

The treatment of second-class constrained systems resides in the existence of an invertible map between a subset of the starting phase space and the constraint and auxiliary condition that preserves the existence of an even-dimensional phase space. This can also be seen from the point of view of the symplectic two-form. We will start with two known examples, highlighting various attributes that each brings to the table.

The first example is the relativistic point particle in Minkowski spacetime. In the first-class approach, one starts from action $S = \int \left(p_\mu \frac{dx^\mu}{d\tau} - \lambda C \right) d\tau$, wherein the mass shell condition $\frac{1}{2}\left(p^\mu p_\mu - m^2 \right) = 0 = C$ becomes implemented as a constraint via Lagrange multiplier λ. The naive application of the Dirac procedure, wherein first-class quantum constraints annihilate physical states, leads to the Klein–Gordon equation (upon promotion $\hat{p}_\mu = -i\hbar\partial_\mu$)

$$\hat{C}\,\psi = \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} - \partial^2 - \frac{m^2}{\hbar^2} \right)\psi = 0; \tag{8}$$

$$\psi(x,t) = \frac{1}{(2\pi)^3}\int d^3k \Big(A(\vec{k})e^{(i/\hbar)(\sqrt{k^2+m^2}t - \vec{k}\cdot\vec{x})} + B(\vec{k})e^{-(i/\hbar)(\sqrt{k^2+m^2}t - \vec{k}\cdot\vec{x})} \Big) \tag{9}$$

whose solution consists of forward and backward propagation corresponding to positive and negative energy modes. This is inconsistent with the invariance of the state under transformations with respect to the parameter τ generated by the constraint through Poisson brackets:

$$\frac{dx^i}{d\tau} = \{x^i, \lambda C\}_{\text{P.B.}} = \lambda p^i;\ \frac{dp_i}{d\tau} = \{p_i, \lambda C\}_{\text{P.B.}} = 0 \tag{10}$$

since (1) the polarization of the state is not preserved on configuration space x^i as $\psi(x^i, \int \lambda d\tau) = \psi(x^i + p^i \int \lambda d\tau, \int \lambda d\tau)$ on account of the constraint's being quadratic in momentum, as well as (2) the arbitrariness in λ. The evolution generated in this case is fictitious, having nothing to do with physical time evolution. The analogue of this is the Wheeler–DeWitt equation for gravity, wherein one has the problem of time [11,13].

Conversely, in the second-class approach, one proceeds based on the premise of the preservation of an even-dimensional phase space With starting symplectic two-form

$$\Omega = dx^\mu \wedge dp_\mu = \Omega_{Phys} + \Omega_{constr.} = -dx_i \wedge dp_i + dx^0 \wedge dp_0 \tag{11}$$

wherein $\Omega_{constr.} = dx^0 \wedge dp_0$, we choose the following constraint and auxiliary condition:

$$C = \frac{1}{2}(p_\mu p^\mu - m^2) = \frac{1}{2}(p_0^2 - p_i p_i - m^2);\ \xi = x^0 - \tau. \tag{12}$$

The physical interpretation is the emergence of a genuine Hamiltonian $p_0 = \sqrt{p_i p_i + m^2}$ and a time $x^0 = \tau$ upon implementation. This is similar to the emergence of a reduced Hamiltonian in the full theory of gravitation [14].

One sees on the constraint surface that $\{\chi, C\} = p_0 = \sqrt{p_i p_i + m^2} \neq 0$, and that the constraint and auxiliary condition (which we will from now on lump into the same category as constraints for ease of exposition) produce a symplectic two-form $d\chi \wedge dC = dx^0 \wedge \left(p_0 dp_0 - p_i dp_i \right)$, yielding the desired invertible map

$$dx^0 \wedge dp_0 = \frac{d\chi \wedge dC}{\{\chi, C\}} + \frac{p_i}{p_0} dx^0 \wedge dp_i. \tag{13}$$

One sees in this case that there is a cross term $dx^0 \wedge dp_i$ between Ω_{Phys} and $\Omega_{constr.}$. Substituting (13) into (11), we have, for the physical part of the phase space, after subtracting out the constrained part,

$$\Omega_{Phys} = \Omega - \frac{d\chi \wedge dC}{\{\chi, C\}} = -\Big(dx_i - \frac{p_i}{p_0} dx^0 \Big) \wedge dp_i. \tag{14}$$

The vanishing of this symplectic two-form $\Omega_{Phys} = 0$ follows from Hamilton's equation from emergent Hamiltonian $H = \sqrt{p_i p_i + m^2}$, which generates evolution in time $t = \tau$, both determined from the constraints. The same time t governs the evolution of the wavefunction $\psi(x^i, t)$ via Schrodinger equation $i\hbar(\partial\psi/\partial t) = \hat{H}\psi$ with solution

$$x^i(\tau) = x^i(0) + \frac{p^i \tau}{\sqrt{p_i p_i + m^2}};\ \psi(\vec{x}, \tau) = \exp\left[-i\tau\sqrt{-\partial^2 + \frac{m^2}{\hbar^2}}\right]\psi(\vec{x}, 0) \tag{15}$$

which can be seen as time evolution within a single branch of (8). From a physical point of view, one must have positive energy solutions propagating forward in time via the unitary evolution of quantum mechanics. On the other hand, evolution via (10) must be interpreted as a gauge transformation, which is unphysical.

The second relevant example is the Maxwell theory, with starting symplectic two-form

$$\Omega = \int d^3x\, \delta A_i(x) \wedge \delta E_i(x) = \int d^3x\, \delta A^T{}_i(x) \wedge \delta E^T{}_i(x) + \int d^3x\, \delta A^L{}_i(x) \wedge \delta E^L{}_i(x) \tag{16}$$

with the association of Ω_{Phys} and $\Omega_{constr.}$ to the transverse and longitudinal parts, respectively, of the vector potential and electric field (A_i, E_i). We choose the constraint and auxiliary condition

$$C = \partial_i E_i;\ \chi = \partial_i A_i \tag{17}$$

whose interpretation is the imposition of transversality for physical fields. The constraint $C = \partial_i E_i = 0$ is the Gauss' law constraint. One sees that, for appropriately defined conditions, $\{\chi(x), C(y)\} = \partial^2\delta(x-y) \neq 0$, that the constraint and auxiliary condition imply $\delta\chi(x) \wedge \delta C(x) = \partial_i(\delta A_i(x)) \wedge \partial_i(\delta E_i(x))$ and that we have, for the longitudinal part of the phase space,

$$\delta A^L_i(x) \wedge \delta E^L_i(x) = \frac{\delta\chi(x) \wedge \delta C(x)}{\{\chi, C\}} = \frac{1}{\partial^2}\partial_i(\delta A_i(x)) \wedge \partial_j(\delta A_j(x)). \tag{18}$$

Substituting (18) back into (16), we have, for the transverse part of the phase space,

$$\Omega_{Phys} = \int d^3x\, \delta A^T{}_i(x) \wedge \delta E^T{}_i(x) = \int d^3x\, \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta A_i(x) \wedge \delta E_j(x) \tag{19}$$

with a transverse, albeit non-local, projector. There is no Hamiltonian and no time, as neither the Gauss' law constraint nor the auxiliary condition, unlike the case of the relativistic point particle, involve a Hamiltonian or time. To obtain nontrivial time evolution, both classically and quantum mechanically, we must choose a Hamiltonian involving the transverse physical fields. For the choice of Hamiltonian density $\frac{1}{2}\left(E^T_i E^T_i + \partial^2 A^T_i A^T_i\right)$, when we have a physical Hamiltonian and a physical time whose evolution is generated via Dirac brackets,

$$\dot{A}^T_i(x) = E^T_i(x);\ \dot{E}^T_i(x) = -\partial^2 A^T_i(x); \tag{20}$$

$$i\hbar\frac{\partial\psi[A^T_i, t]}{\partial t} = \int d^3x \frac{1}{2}\left(-\hbar^2\frac{\delta^2}{\delta A^T_i(x)\delta A^T_i(x)} + \partial^2 A^T(x) A^T_i(x)\right)\psi[A^T_i, t] \tag{21}$$

with a Schrodinger equation governing the evolution of the state with respect to the same time t. In this case, the analogue of (10), the Gauss' law constraint

$$\partial_i\frac{\delta\psi}{\delta A_i(x)} = 0 \longrightarrow \psi(A_i, t) = \psi(A_i + \partial_i\lambda, t) \tag{22}$$

can be correctly interpreted as invariance of the state under gauge transformations $\delta A_i(x) = \{A_i(x), \int d^3y \lambda(y)\partial_i E_i(y)\}_{\text{P.B.}}$ upon promotion to quantum commutators on account of the constraints being linear in momentum. Additionally, there is no conflict in the existence of a gauge symmetry with time evolution dictated by the reduced Hamiltonian.

4. The Klauder Toy Problem: Canonical Equivalence of the Second-Class Approach

We now move on to the Klauder toy problem for quantum gravity, which has various features in common with the previous two examples. Let us consider a two-dimensional starting phase space in polar coordinates $(q_i, p_i) = (r, \varphi, p_r, p_\varphi)$. In accordance with the procedure for handling second-class constraints, we need to impose, for each constraint $C = 0$ on this phase space, a suitable auxiliary condition $\chi = 0$. For the constraint $C = \frac{1}{2}(p_r^2 + \frac{L_\varphi^2}{r^2} - \alpha^2 r^2) = 0$, we can choose the auxiliary condition $\chi = rp_r - k = 0$, respectively. Here, k is constant with respect to the phase space variables but may in general have explicit time dependence. $L_\varphi := p_\varphi$ is actually the generator of "z-axis rotations" for all of the variables; moreover, L_φ commutes with C and also with χ since they are both rotationally invariant. Note that, other than at the point $(p, r) = (0, 0)$, the criterion

$$\{\chi, C\}_{\text{P.B.}} = p_r^2 + \frac{p_\varphi^2}{r^2} + \alpha^2 r^2 = p^2 + \alpha^2 r^2 \neq 0 \tag{23}$$

for Dirac's second-class system is met, where p is the total momentum of the system $p^2 = p_r^2 + \frac{p_\varphi^2}{r^2}$.

In order to see the implications for the canonical phase space path integral, it is instructive to examine the system in the formalism of differential forms. The symplectic two-form on the total unconstrained phase space is given by

$$\Omega = \sum_{I=1}^{2} \delta q_I \wedge \delta p_I = \delta\varphi \wedge \delta p_\varphi + \delta r \wedge \delta p_r. \tag{24}$$

The variations in the constraint and auxiliary condition are given by

$$\delta C = p_r \delta p_r + \frac{p_\varphi}{r^2}\delta p_\varphi - \Big(\frac{p_\varphi^2}{r^3} + \alpha^2 r\Big)\delta r; \; \delta\chi = r\delta p_r + p_r \delta r. \tag{25}$$

Thus, the constraint and auxiliary condition furnish a symplectic structure:

$$\begin{aligned}\delta\chi \wedge \delta C &= \frac{p_\varphi}{r^2}\Big(r\delta p_r + p_r\delta r\Big) \wedge \delta p_\varphi + \Big(p_r^2 + \frac{p_\varphi^2}{r^2} + \alpha^2 r^2\Big)\delta r \wedge \delta p_r \\ &= \frac{p_\varphi}{r^2}\delta(rp_r) \wedge \delta p_\varphi + \{\chi, C\}\delta r \wedge \delta p_r\end{aligned} \tag{26}$$

One sees that the transformation $(r, p_r) \to (\chi, C)$ saturates the constrained part of the phase space, whereupon the conditions $C = 0$, $\chi = 0$ can be imposed as strong equalities. Then, $rp_r = k$ implies that $\delta(rp_r) = \delta k = 0$ and $\{\chi, C\}_{\chi=0,C=0} = 2\alpha^2 r^2 = \frac{2(k^2+p_\varphi^2)}{r^2}$ and, subject to the constraint and auxiliary condition, we have

$$\delta\chi \wedge \delta C\Big|_{\chi=0,C=0} = \{\chi, C\}\delta r \wedge \delta p_r \longrightarrow \delta r \wedge \delta p_r = \frac{\delta\chi \wedge \delta C}{2\alpha^2 r^2}, \tag{27}$$

with a one-to-one correspondence to the relevant part of the canonical phase space implementing these conditions. In fact, there is a strong analogy to the Maxwell problem in that the angular phase space (φ, p_φ) plays an analogous role to the transverse field (A_i^T, E_i^T), as does the radial phase space (r, p_r) to the longitudinal (A_i^L, E_i^L), as the phase space variables will form an orthogonal decomposition with respect to some suitable inner product. The canonical functional integral measure with the constraint and auxiliary condition for a totally constrained system with a vanishing Hamiltonian is

$$\int dr\, dp_r\, d\varphi\, dp_\varphi \ \det(\{\chi, C\}_{\text{P.B.}})\delta(\chi)\delta(C) \exp\Big[\frac{i}{\hbar}\int\Big(p_r dr + p_\varphi d\varphi - H dt\Big)\Big] \quad (28)$$

$$= \int dr\, dp_r\, d\varphi\, dp_\varphi \ (p^2 + \alpha^2 r^2)\delta(\chi = r p_r - k)\delta(C) \exp\Big[\frac{i}{\hbar}\int\Big(p_r dr + p_\varphi d\varphi - H dt\Big)\Big]. \quad (29)$$

Subject to the constraint and auxiliary condition, there is a one-to-one mapping $(r, p_r) \to (\chi, \phi)$, with a contribution to the phase space measure given by

$$drdp_r = \det\begin{pmatrix} \frac{\partial\chi}{\partial r} & \frac{\partial\chi}{\partial p_r} \\ \frac{\partial C}{\partial r} & \frac{\partial C}{\partial p_r} \end{pmatrix}^{-1} d\chi dC = \det\begin{pmatrix} p_r & r \\ -\Big(\frac{p_\varphi^2}{r^3} + \alpha^2 r\Big) & p_r \end{pmatrix}^{-1} d\chi dC = \frac{d\chi dC}{p^2 + \alpha^2 r^2}$$

Substituting into the measure, we have

$$\int \frac{d\chi dC}{p^2 + \alpha^2 r^2} d\varphi dp_\varphi (p^2 + \alpha^2 r^2)\delta(\chi)\delta(C) \exp\Big[\frac{i}{\hbar}\int\Big(p_r dr + p_\varphi d\varphi - H dt\Big)\Big] \quad (30)$$

$$= \int d\varphi dp_\varphi dr dp_r \delta(p_r - p_r^*)\delta(r - r^*) \exp\Big[\frac{i}{\hbar}\int\Big(p_r dr + p_\varphi d\varphi - H dt\Big)\Big] \quad (31)$$

$$= \int d\varphi dp_\varphi e^{(i/\hbar)\int p^* dr^*} e^{(i/\hbar)\int p_\varphi d\varphi} e^{-(i/\hbar)\int H(r^*, p^*, \varphi, p_\varphi) dt} \quad (32)$$

The correctness of the measure is guaranteed by canonical equivalence

$$\int d\chi dC \delta(\chi)\delta(C) = \int dr dp_r \delta(r - r^*)\delta(p_r - p_r^*); \quad (33)$$

$$r^* = \Big(\frac{k^2 + p_\varphi^2}{\alpha^2}\Big)^{1/4}; \ p_r^* = \frac{k}{r^*} = k\Big(\frac{k^2 + p_\varphi^2}{\alpha^2}\Big)^{-1/4}. \quad (34)$$

wherein the reduced phase space can be identified as (φ, p_φ). It can be verified that the Dirac brackets are as follows:

$$\begin{aligned} \{\varphi, p_\varphi\}_{\text{D.B.}} &= \{\varphi, p_\varphi\}_{\text{P.B.}} - \{\varphi, \chi\}_{\text{P.B.}} \frac{-1}{\{\chi, C\}_{\text{P.B.}}}\{C, p_\varphi\}_{\text{P.B.}} - \{\varphi, C\}_{P.B} \frac{+1}{\{\chi, C\}_{\text{P.B.}}}\{\chi, p_\varphi\}_{\text{P.B.}} \quad (35) \\ &= \{\phi, p_\phi\}_{\text{P.B.}} = 1, \quad (36) \end{aligned}$$

since $\{\chi, p_\varphi = L_\varphi\}_{\text{P.B.}} = 0 = \{C, p_\varphi\}_{\text{P.B.}}$ due to rotational invariance under L_φ as noted above. The Dirac bracket of reduced resultant pair (φ, p_φ) equals the Poisson bracket, signifying that the pair is indeed canonical, even with the constraint solved and auxiliary condition imposed. On the other hand,

$$\begin{aligned} \{r, p_r\}_{\text{D.B.}} &= \{r, p_r\}_{\text{P.B.}} - \{r, \chi\}_{\text{P.B.}} \frac{-1}{\{\chi, C\}_{\text{P.B.}}}\{C, p_r\}_{\text{P.B.}} - \{r, C\}_{P.B} \frac{+1}{\{\chi, C\}_{\text{P.B.}}}\{\chi, p_r\}_{\text{P.B.}} \quad (37) \\ &= \{r, p_r\}_{\text{P.B.}} - (r)\frac{-1}{p^2 + \alpha^2 r^2}\Big(-\frac{p_\varphi^2}{r^3} - \alpha^2 r\Big) - (p_r)\frac{1}{p^2 + \alpha^2 r^2}(p_r) \quad (38) \\ &= 1 - \frac{1}{p^2 + \alpha^2 r^2}\Big(\frac{p_\varphi^2}{r^2} + \alpha^2 r^2 + p_r^2\Big) = 0, \quad (39) \end{aligned}$$

indicating that the pair (r, p_r) is no longer canonical!

The remaining Dirac brackets are given by

$$\{p_r, p_\varphi\}_{\text{D.B.}} = \{p_r, p_\varphi\}_{\text{P.B.}} - \{p_r, \chi\}_{\text{P.B.}} \frac{-1}{\{\chi, C\}_{\text{P.B.}}}\{C, p_\varphi\}_{\text{P.B.}} - \{p_r, C\}_{P.B} \frac{+1}{\{\chi, C\}_{\text{P.B.}}}\{\chi, p_\varphi\}_{\text{P.B.}} \quad (40)$$

$$-(-p_r)\Big(\frac{-1}{p^2 + \alpha^2 r^2}\Big)(0) - \Big(\frac{p_\varphi^2}{r^3} + \alpha^2 r\Big)\Big(\frac{1}{p^2 + \alpha^2 r^2}\Big)(0) = 0, \quad (41)$$

with cross-brackets

$$\{p_\varphi, r\}_{\text{D.B.}} = \{p_\varphi, r\}_{\text{P.B.}} - \{p_\varphi, \chi\}_{\text{P.B.}} \frac{-1}{\{\chi, C\}_{\text{P.B.}}} \{C, r\}_{\text{P.B.}} - \{p_\varphi, C\}_{P.B} \frac{+1}{\{\chi, C\}_{\text{P.B.}}} \{\chi, r\}_{\text{P.B.}} \tag{42}$$

$$= -(0)\Big(\frac{-1}{p^2+\alpha^2 r^2}\Big)(-p_r) - (0)\Big(\frac{1}{p^2+\alpha^2 r^2}\Big)(-r) = 0 \tag{43}$$

$$\{p_r, \varphi\}_{\text{D.B.}} = \{p_r, \varphi\}_{\text{P.B.}} - \{p_r, \chi\}_{\text{P.B.}} \frac{-1}{\{\chi, C\}_{\text{P.B.}}} \{C, \varphi\}_{\text{P.B.}} - \{p_r, C\}_{P.B} \frac{+1}{\{\chi, C\}_{\text{P.B.}}} \{\chi, \varphi\}_{\text{P.B.}} \tag{44}$$

$$= 0 - (-p_r)\Big(\frac{-1}{p^2+\alpha^2 r^2}\Big)\big(-\frac{p_\varphi}{r^2}\big) - \Big(\frac{p_\varphi^2}{r^3} - \alpha^2 r\Big)\Big(\frac{1}{p^2+\alpha^2 r^2}\Big)(0) = \frac{p_r p_\varphi}{r^2(p^2+\alpha^2 r^2)}, \tag{45}$$

The vanishing of any Dirac bracket with p_φ is consistent with rotational invariance. Finally, we have

$$\{r, \varphi\}_{\text{D.B.}} = \{r, \varphi\}_{\text{P.B.}} - \{r, \chi\}_{\text{P.B.}} \frac{-1}{\{\chi, C\}_{\text{P.B.}}} \{C, \varphi\}_{\text{P.B.}} - \{r, C\}_{P.B} \frac{+1}{\{\chi, C\}_{\text{P.B.}}} \{\chi, \varphi\}_{\text{P.B.}} \tag{46}$$

$$= 0 - (r)\Big(\frac{-1}{p^2+\alpha^2 r^2}\Big)\Big(-\frac{p_\varphi}{r^2}\Big) - (p_r)\Big(\frac{1}{p^2+\alpha^2 r^2}\Big)(0) = -\frac{p_\varphi}{r(p^2+\alpha^2 r^2)} \tag{47}$$

5. Dynamics on the Reduced Phase Space

Summarizing the previous results, the Dirac brackets of the variables give the following relations:

$$\begin{aligned}
\{r, p_r\}_{\text{D.B.}} &= 0 \\
\{r, p_\varphi\}_{\text{D.B.}} &= 0 \\
\{r, \varphi\}_{\text{D.B.}} &= -\frac{r p_\varphi}{p_\varphi^2 + r^2 p_r^2 + \alpha^2 r^4} \\
\{\varphi, p_r\}_{\text{D.B.}} &= -\frac{p_r p_\varphi}{p_\varphi^2 + r^2 p_r^2 + \alpha^2 r^4} \\
\{\varphi, p_\varphi\}_{\text{D.B.}} &= 1 \\
\{p_r, p_\varphi\}_{\text{D.B.}} &= 0
\end{aligned} \tag{48}$$

Solving for the variables from the χ, C restrictions, we obtain the following:

$$\begin{aligned}
r^2 p_r^2 + \alpha^2 r^4 &= p_\varphi^2 + 2k^2 \\
r &= \sqrt[4]{\frac{k^2+p_\varphi^2}{\alpha^2}};\ p_r = \frac{k}{r} \\
\{r, \varphi\}_{\text{D.B.}} &= -\frac{r p_\varphi}{2(p_\varphi^2+k^2)} = -\frac{p_\varphi}{2(p_\varphi^2+k^2)}\sqrt[4]{\frac{k^2+p_\varphi^2}{\alpha^2}} \xrightarrow{k\to 0} -\frac{1}{2p_\varphi^2}\sqrt[4]{\frac{p_\varphi^2}{m\alpha^2}} \\
\{\varphi, p_r\}_{\text{D.B.}} &= -\frac{k p_\varphi}{2r(p_\varphi^2+k^2)} = -\frac{k p_\varphi}{2(p_\varphi^2+k^2)}\sqrt[4]{\frac{\alpha^2}{k^2+p_\varphi^2}} \xrightarrow{k\to 0} 0
\end{aligned} \tag{49}$$

Had we substituted the solution of (r, p_r) in terms of (φ, p_φ) and computed the resultant P.B., we would have obtained the same results. This is a consequence of a theorem that we proved earlier.

Since all phase space functions $f(q_I, p_I)$ Dirac-commute with the constraints $\{f, C\}_{\text{D.B.}} = \{f, \chi\}_{\text{D.B.}} = 0$ for all f, all phase space functions in second-class constrained systems are said to be Dirac observables. As the system has been completely reduced without an emerging Hamiltonian, a physical Hamiltonian needs to be prescribed to discuss dynamics and also classical orbits. This is exactly analogous to the situation in the Maxwell problem considered earlier. To it, we introduce the general two-dimensional non-relativistic Hamiltonian with radial potential, which preserves the invariance under rotations generated by p_φ, which was present in C and χ. For any radial potential $V(r)$, the Hamiltonian $H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} + V(r)$ gives the following relation:

$$H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} - \frac{1}{2}\alpha^2 r^2 + U(r) = C + U(r) = \frac{1}{2}\alpha^2 r^2 + V(r) \tag{50}$$

wherein $C = 0$ has been imposed as a strong equality. In second-class constrained systems, constraints and auxiliary conditions may be a priori imposed on the system prior to the computation of evolution via Dirac brackets. This is in contrast to first-class systems, wherein constraints can be implemented only after Poisson brackets have been evaluated. In the former case, a genuine physical Hamiltonian serves as the generator of the evolution of the reduced variables with respect to some parameter t, which we will identify as time. Then, we have the following results, from the implementation of the constraints prior to the computation of brackets:

$$\begin{aligned}
\{r, H\}_{\text{D.B.}} &= \frac{\partial H}{\partial r}\{r, r\}_{\text{D.B.}} = 0 \\
\{p_r, H\}_{\text{D.B.}} &= \frac{\partial H}{\partial r}\{p_r, r\}_{\text{D.B.}} = 0 \\
\{\varphi, H\}_{\text{D.B.}} &= \frac{\partial H}{\partial r}\{\varphi, r\}_{\text{D.B.}} = \frac{r p_\varphi U'(r)}{p_\varphi^2 + r^2 p_r^2 + \alpha^2 r^4} \\
\{p_\varphi, H\}_{\text{D.B.}} &= \frac{\partial H}{\partial r}\{p_\varphi, r\}_{\text{D.B.}} = 0,
\end{aligned} \tag{51}$$

where $U'(r) = \dfrac{dU}{dr}$.

Dirac brackets, like Poisson brackets, obey the Leibniz rule from calculus. Substituting the above relation with $\chi = 0, \phi = 0$, we obtain:

$$\{\varphi, H\}_{\text{D.B.}} = -\frac{p_\varphi U'(r^*)}{2\alpha^2 r^{*3}};\ r^* = \left(\frac{k^2 + p_\varphi^2}{\alpha^2}\right)^{1/4} \tag{52}$$

Since the Dirac bracket of a variable with a Hamiltonian is the time-evolution of that variable under that Hamiltonian, we have

$$\dot{r} = \dot{p}_r = \dot{p}_\varphi = 0;\ \dot{\varphi} = -\frac{p_\varphi U'(r^*)}{2\alpha^2 r^{*3}}; \tag{53}$$

$$r(t) = \left(\frac{k^2 + p_\varphi^2}{\alpha^2}\right)^{1/4} = r^*;\ p_r(t) = k\left(\frac{k^2 + p_\varphi^2}{\alpha^2}\right)^{-1/4} = \frac{k}{r^*};\ \varphi(t) = \varphi(0) - \frac{p_\varphi U'(r^*)}{2\alpha^2 r^{*3}} t;\ p_\varphi(t) = p_\varphi \tag{54}$$

It is interesting to note that the system will only have circular orbits regardless of the value of k for constant k, since r is constant in time since p_φ is conserved as it Dirac-commutes with the Hamiltonian. With respect to observational applications, the results thus derived should be true just for experiments or observations that fulfill the auxiliary condition $p_r = \frac{k}{r}$.

6. Quantization

Canonical quantization proceeds by the promotion of all dynamical variables to operators $A \to \hat{A}$ and by promoting all Dirac brackets, *not* Poisson brackets, to quantum commutators $\{A, B\}_{\text{D.B.}} = \frac{1}{i\hbar}[\hat{A}, \hat{B}]$. Thus, we have

$$[\hat{r}, \hat{p}_r] = [\hat{r}, \hat{p}_\varphi] = [\hat{p}_r, \hat{p}_\varphi] = 0;\ [\hat{\varphi}, \hat{p}_\varphi] = i\hbar \tag{55}$$

Surprisingly, values of (r, p_r), which is albeit a non-commuting pair at the Poisson bracket level, can be determined with infinite precision quantum mechanically because

$[\hat{r}, \hat{p}_r] = i\hbar\{r, p_r\}_{\text{D.B.}} = 0$. For the remaining commutators, note that all variables may be reduced a priori by the constraint and auxiliary condition prior to quantization:

$$[\hat{\varphi}, \hat{r}] = \frac{i\hbar}{2\sqrt{\alpha}}\hat{p}_\varphi(\hat{p}_\varphi^2 + k^2)^{-3/4};\ [\hat{p}_r, \hat{\varphi}] = \frac{i\sqrt{\alpha}k\hbar}{2}\hat{p}_\varphi(\hat{p}_\varphi^2 + k^2)^{-5/4} \tag{56}$$

As all commutators have been reduced to dependence on one fundamental variable p_φ, there are no operator ordering ambiguities. Quantization of the theory gives eigenstates that obey $\hat{L}_\varphi \Psi_m(\varphi) = (m\hbar)\Psi_m(\varphi)$, and the relations have a well-defined action on eigenstates of p_φ. The physical Hamiltonian operator is also given by $\hat{H} = U(r^*(\hat{p}_\varphi))$.

A time t emerges from the classical equations via Dirac brackets, whose evolution is generated by a physical Hamiltonian $U(r) \neq 0$. We require, as a consistency condition on second-class constrained systems, that the same time t govern the unitary evolution of the quantum state ψ via the Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi;\ \psi(\varphi, t) = \exp\Big[-\frac{itU(\hat{p}_\varphi)}{\hbar}\Big]\psi(\varphi, 0) \tag{57}$$

Let us choose the initial wavefunction, which is polarized on a configuration space given by the unit circle S^1, as a normalizable superposition of eigenstates of angular momentum

$$\psi(\varphi, 0) = \sum_{m\in Z} c_m e^{(i/\hbar)m\varphi}; \tag{58}$$

$$\langle\psi|\psi\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\ \psi^*(\varphi, 0)\psi(\varphi, 0) = \sum_{m\in Z}|c_m|^2 = 1 \tag{59}$$

wherein c_m are the Fourier coefficients. For the totally constrained case, the state remains as in (58) for all time. However, under physical Hamiltonian $H_{Phys} = U(r)$, the state evolves via

$$\psi(\varphi, t) = \sum_{m\in Z} c_m e^{(i/\hbar)m\varphi} e^{-(i/\hbar)U(m)t} \tag{60}$$

which preserves its normalizability. In addition, the expectation values go through

$$\Big\langle A \Big\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\ \psi^*(\varphi, t)\hat{A}(\varphi, \hat{p}_\varphi)\psi(\varphi, t) \tag{61}$$

It is interesting to compute the expectation values of the dynamical variables in the state (60) and to compare them with the classical evolution via Dirac brackets. We have the following results:

$$\Big\langle r(t) \Big\rangle = \sum_{m\in Z}|c_m|^2\Big(\frac{k^2 + (m\hbar)^2}{\alpha^2}\Big)^{1/4}; \tag{62}$$

$$\Big\langle p_r(t) \Big\rangle = \sum_{m\in Z}|c_m|^2 k\Big(\frac{k^2 + (m\hbar)^2}{\alpha^2}\Big)^{-1/4};\ \Big\langle p_\varphi(t) \Big\rangle = \hbar\sum_{m\in Z} m|c_m|^2 \tag{63}$$

The variables that, in the classical theory, are time-independent, similar to in the quantum theory do not evolve in time, but now appear as superpositions labelled by the integers. The time independence is a consequence of k having no explicit time dependence, which results in a time-independent physical Hamiltonian (for the time-dependent case, we must construct solutions via time-ordered exponentials). For the angle φ, we have

$$\Big\langle \varphi(t) \Big\rangle = \sum_{m,n} e^{(i/\hbar)(U_m - U_n)t} c_m^* c_n \int_0^{2\pi} d\varphi\, \varphi\, e^{i(n-m)\varphi} \tag{64}$$

$$= \pi - \sum_{n \neq m} \frac{e^{(i/\hbar)(U_m - U_n)t} c_m^* c_n}{(m-n)^2}. \tag{65}$$

The quantum time evolution is more complicated than the linear evolution in t of the classical theory φ. An interesting question is whether canonical transformations on the phase space commute with quantization. While, as noted, there are no ordering ambiguities in polar coordinates, one sees that ordering ambiguities exist in Cartesian coordinates:

$$x = r\cos\varphi;\ y = r\sin\varphi; \tag{66}$$

$$p_x = p_r \cos\varphi - \frac{p_\varphi}{r}\sin\varphi;\ p_y = p_r\sin\varphi + \frac{p_\varphi}{r}\cos\varphi. \tag{67}$$

on account of the noncommutativity of φ and p_φ. Nevertheless, expectation values can be computed for a given ordering using the polar coordinate quantization. We have

$$\Big\langle (x(t))^M \Big\rangle = \sum_{m,n} \Big(\frac{k^2 + (m\hbar)^2}{\alpha^2} \Big)^{M/4} \delta(n - m + M\hbar) \cos\Big(\frac{U_m - U_n}{\hbar} \Big) t; \tag{68}$$

$$\Big\langle (y(t))^M \Big\rangle = \sum_{m,n} \Big(\frac{k^2 + (m\hbar)^2}{\alpha^2} \Big)^{M/4} \delta(n - m + M\hbar) \sin\Big(\frac{U_m - U_n}{\hbar} \Big) t \tag{69}$$

For $M = 1$, the delta functions have no support yielding $\langle x \rangle = \langle y \rangle = 0$ except in the classical limit $\hbar \to 0$, whereas the radius r has a nonzero value both classically and quantum mechanically that satisfies the $\{\chi, C\} \neq 0$ condition for second-class constrained systems. For the momentum components, we have

$$\Big\langle (p_x(t))^M \Big\rangle = \sum_{m,n} \Big(\frac{k^2 + (m\hbar)^2}{\alpha^2} \Big)^{-M/4} \delta(n - m + M\hbar) \Big[k\cos\Big(\frac{U_m - U_n}{\hbar} \Big) t + (m\hbar)\sin\Big(\frac{U_m - U_n}{\hbar} \Big) t \Big]; \tag{70}$$

$$\Big\langle (p_y(t))^M \Big\rangle = \sum_{m,n} \Big(\frac{k^2 + (m\hbar)^2}{\alpha^2} \Big)^{-M/4} \delta(n - m + M\hbar) \Big[(m\hbar)\cos\Big(\frac{U_m - U_n}{\hbar} \Big) t - k\sin\Big(\frac{U_m - U_n}{\hbar} \Big) t \Big] \tag{71}$$

7. First-Class Constraints Approach

The Klauder toy problem in the first-class approach with Lagrange multiplier λ, can be well illustrated in Cartesian coordinates via the starting action (using the summation convention):

$$S = \int \Big(p_i \dot{q}_i - \frac{\lambda}{2}(p_i p_i - \alpha^2 q_i q_i) \Big) dt. \tag{72}$$

Canonical equivalence with its representation in polar coordinates furnishes unconstrained canonical Poisson brackets $\{q_i, q_j\}_{\text{P.B.}} = \{p_i, p_j\}_{\text{P.B.}} = 0$ and $\{q_i, p_j\}_{\text{P.B.}} = \delta_{ij}$. The corresponding Hamilton's equations read as

$$\dot{q}_i = \lambda p_i;\ \dot{p}_i = \lambda\alpha^2 q_i;\ C = p_i p_i - \alpha^2 q_i q_i = 0. \tag{73}$$

The third equation is the constraint $C = 0$, which follows on from the equation of motion for the Lagrange multiplier λ, which, at this stage, can be an arbitrary function of time. It can also be viewed, within the context of the Dirac procedure, as a secondary constraint arising from the preservation of the primary constraint $p_\lambda = 0$, where p_λ is the conjugate momentum to the Lagrange multiplier λ. Then, we have $\dot{p}_\lambda = C = 0$.

The solution can be written down by inspection:

$$q_i(T) = q_i(0)\cosh(\alpha T) + \frac{p_i(0)}{\alpha}\sinh(\alpha T);\; p_i(T) = p_i(0)\cosh(\alpha T) + \alpha q_i(0)\sinh(\alpha T), \quad (74)$$

with $T = \int_0^t dt'\lambda(t')$ assuming the role of an evolution parameter. However, T is arbitrary on account of the arbitrariness of λ, and does not correspond to physical time evolution; indeed, this system is invariant under time reparametrizations $t \to T(t)$. This has the interpretation that the state $(q_i(t), p_i(t))$ at any time t can be obtained from the initial state $(q_i(0), p_i(0))$ by some sort of group action (the group is nonabelian and noncompact, and resembles the preservation of a hyperboloidal light-cone structure as in special relativity). Note that, throughout its evolution, the system respects the constraint $C(q_i(t), p_i(t)) = C(q_i(0), p_i(0))$.

That the constraint is preserved under evolution via Poisson brackets can also be seen directly from the Hamilton's Equation (73):

$$\frac{d}{dt}(p_i p_i - \alpha^2 q_i q_i) = 2p_i\dot{p}_i - 2\alpha^2 q_i\dot{q}_i = 2p_i(\lambda\alpha^2 q_i) - 2\alpha^2 q_i(\lambda p_i) = 0. \quad (75)$$

No further constraints are generated and the system is first class. Indeed, on page 21 of [7], Dirac states (and we paraphrase) that *Primary first-class constraints are generating functions of infinitesimal contact transformations, which lead to changes in the q's and p's that do not affect the physical state.* Within the context of first-class systems, one must interpret the auxiliary condition as a gauge-fixing condition. Let

$$\chi = \vec{q}\cdot\vec{p} - k(t), \quad (76)$$

where $k = k(t)$ has explicit time dependence, the reason for which we will see momentarily. In a gauge-fixed first-class system, we require the gauge-fixing condition to be preserved for all time. Thus,

$$\frac{d\chi}{dt} = \frac{d(q_i p_i) - k(t)}{dt} = \dot{q}_i p_i + q_i\dot{p}_i - \dot{k} = (\lambda p_i)p_i + q_i(\lambda\alpha^2 q_i) - \dot{k} = \lambda\big(p_i p_i + \alpha^2 q_i q_i\big) - \dot{k} = 0. \quad (77)$$

A gauge-fixing condition must be accessible, starting from any initial configuration of the system. Thus, we would like to examine whether $\chi = 0$ lies in the orbit generated by the constraint $C = 0$. Note that, for constant k, we have $\dot{k} = 0$ and $\dot{\chi} \neq 0$, and the auxiliary condition cannot be preserved. Thus, for $k = k(t)$, this enables the determination of the Lagrange multiplier $\lambda = \frac{\dot{k}(t)}{|\vec{p}|^2 + \alpha^2|\vec{q}|^2} = \frac{\dot{k}(t)}{2\alpha^2 r^2}$. The meaning of λ is unclear, especially since the second-class approach furnishes a well-defined time independently of any explicit t dependence in $k(t)$.

Another flaw in the methodology of first-class constrained systems is when one wants to quantize the system via the so-called Wheeler–DeWitt equation [11]. Let us first examine the prescription, in the quantum theory, wherein Poisson brackets get promoted to $\frac{i}{\hbar}$ times commutators. Since the relations are canonical, the momentum acts on wavefunctions $\Psi(q_1, q_2, t)$ by differentiation $\hat{p}_i\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial q_i}$ and the physical states of the system must satisfy a Schrodinger equation $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$. On account of the first-class constraint $\hat{C} = 0$, we obtain the following partial differential equation:

$$\hat{H}\Psi = \lambda\big(\hat{p}^2 - \alpha^2\hat{q}^2\big)\Psi = 0 \longrightarrow \Big(\frac{\partial^2}{\partial {q_1}^2} + \frac{\partial^2}{\partial {q_2}^2} + \frac{\alpha^2}{\hbar^2}(q_1^2 + q_2^2)\Big)\Psi = 0. \quad (78)$$

This means that $\frac{\partial\Psi}{\partial t} = 0$, namely that the physical state Ψ cannot evolve in time. This is precisely what the procedure for Dirac quantization would prescribe, namely that the physical states are annihilated by the quantum constraints. In general relativity, this leads to the problem of time in the analogous "Wheeler–DeWitt" equation.

Constraints linear in momenta admit an interpretation of generating gauge transformations that preserve the configuration space polarization of the wavefunction; for example, those found in Maxwell theory. Since the wavefunctions $\Psi(q_1, q_1)$ should be defined on configuration space and not phase space, the gauge transformation should admit a quantum interpretation wherein annihilation of the state by the constraint signifies its invariance under the transformation

$$\Psi(q_i + \delta q_i, t) = e^{(i/\hbar)\delta\vec{q}\cdot\vec{C}}\Psi = \Psi(q_i, t) + \sum_i \delta q_i \frac{\partial \Psi}{\partial q_i}. \tag{79}$$

However, $C = 0$ is quadratic in momenta. As can be seen from (74), this constraint generates transformations that mix coordinate space with momentum space variables. Thus, the analogous Wheeler–DeWitt Equation (78) does not have the same meaning as a quantum wavefunction invariant under a transformation generated by $C = 0$ since the polarization of such a wavefunction on configuration space cannot be preserved.

It is interesting to note that the "First Class Dirac Quantization" procedure of having the constraint annihilate physical states,

$$\hat{C}\Psi(r, \varphi) = \frac{1}{2}(\hat{p}_r^2 + \frac{\hat{p}_\varphi^2}{r^2} - \alpha^2 r^2)\Psi(r, \varphi) = 0, \tag{80}$$

can yield a correspondence if we were to replace $\hat{p}_r$ with k/r, where it would yield the restriction $r = \left(\frac{k^2 + p_\varphi^2}{\alpha^2}\right)^{1/4}$, and the results would correspond roughly to those presented in the earlier second-class analysis. However, $\hat{p}_r \to k/r$ is consistent with the Dirac bracket, and not Poisson bracket, to commutator rule, and this rule cannot be deduced a prioriwithin the context of first-class quantization without first turning the system into second class. Furthermore the Dirac bracket between r and φ does not vanish, so quantum states cannot be a function of both of these variables. In fact, this highlights another complication for first-class quantization: in a theory with K first-class constraints, the reduced phase space will only be $2N - 2K$-dimensional, so at most $(N - K)$ variables, and not N variables, can play the role of commuting configuration variables. It is far too optimistic in the first-class scheme to assume that the K constraints annihilating the wave function (which is still assumed to depend on the original commuting N configuration variables) will naturally reduce the system to one with $(N - K)$ *commuting configuration variables,* whereas examples show that some of the configuration variables may in fact be reduced to non-commuting variables.

8. Summary/Conclusions

This work illustrated the vast differences in the interpretation of totally constrained systems as seen from a second-class versus first-class perspective. The main motivation was to highlight some of the main problems that arise in general relativity and to provide a resolution within the simpler context of a toy model sharing some of the relevant features in common. We argued that nature should exhibit a preference for second-class systems over first-class systems with a view toward the preservation of an even-dimensional phase space. This is the most natural configuration on which to base a quantum theory.

There are several points that should be reiterated. (i) In the case of unconstrained systems, Poisson brackets and Dirac brakets coincide $\{A, B\}_{P.B} = \{A, B\}_{D.B.}$, and the standard quantization procedures carry through. However, where constraints exist, they should be supplemented with auxiliary conditions to make the system second class. The prescription that constraints annihilate physical states still holds—for example, as in (80)—upon the promotion of Dirac brackets, not Poisson brackets, to commutators. In this sense, unlike in the case of first-class systems, wherein the constraint fixes the state through a solution to a differential equation, for second-class systems, the condition $\hat{\Phi}|\psi\rangle_{Phys}$ is trivially satisfied, and a physical state $|\psi\rangle_{Phys}$ can be fixed only through unitary evolution under a genuine Hamiltonian. (ii) All phase space functions A Dirac-commute with the constraints

$\{A, \Phi_I\}_{D.B.} = 0$ by construction, which also make them observables in the quantum sense $[\hat{A}, \hat{\Phi}_I] = 0$. Nevertheless, transverse fields $(A_i^T(x), E_i^T(x))$, for example, which form the reduced phase space for Maxwell theory, are also observables in the first-class sense in that they are gauge-invariant. (iii) While first-class constraints generate transformations of the phase space variables, this is not time evolution, as shown in all examples. Time evolution can only arise from the existence of a physical Hamiltonian H_{Phys}, and is the fundamental quantity when constructed on the reduced phase space. (vi) In the process of formulating second-class systems, it may not be possible to satisfy the required condition $\det\{\Phi_I, \Phi_J\} \neq 0$ in the full phase space. In this case, when the condition fails on a set of phase space points of measure zero, these points can be excluded from the phase space in the construction of the reduced phase space. In the case of the Klauder problem, the origin $(x^i, p_i) = (0, 0)$ must be excluded. However, such exclusions should be considered as a small price to pay for the availability of a vastly more colossal phase space. In the case of the Maxwell field, analogous to gravity [14], the condition can be satisfied globally. (iv) The Klauder toy problem suggests a strategy for the study of physical systems. One could envision the process of this article in reverse. The Klauder problem can be seen as the particle in a plane wherein the radial momentum is placed on the same footing as angular momentum (as the condition $k = rp_r$ is dimensionally consistent with the interpretation of k as angular momentum). Then, the problem becomes that of a particle on a circle, which can be solved exactly, classically and quantum mechanically. Using this as the starting point, one may enlarge the phase space by introducing a radial degree of freedom r, along with its conjugate momentum p_r, to preserve an even-dimensional phase space. Then, one constructs a constraint with an auxiliary condition that can be invertibly mapped to these additional variables, and one has the Klauder problem and its variations for that system. On a final note, one may analyze the effect wherein $k = k(t)$ is no longer a numerical constant and has explicit time dependence. In this case, we no longer have circular orbits as in (52), (53), (62). Additionally, we now have a time-dependent Hamiltonian $U(\hat{p}_\varphi, t)$ resulting in a time-ordered unitary evolution of the state

$$\psi(\varphi, t) = \mathcal{T}\left[\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} U(\hat{p}_\varphi, t')dt'\right]\psi(\varphi, 0)\right. \tag{81}$$

Equation (81) is analogous to the case for full gravity [15] within the context of the relativistic point particle example considered, and the incorporation of transverse-traceless auxiliary conditions turning general relativity into a second-class system with a reduced Hamiltonian generating cosmic time translations has been addressed in Ref. [14].

From an observational standpoint, the theoretical results obtained through the techniques demonstrated in this article should only be true for experiments or observations that fulfill the associated constraints and auxiliary conditions. So far, the best observational evidence in support of second-class systems resides in the physicality of transverse electromagnetic fields, as well as the detection of transverse-traceless perturbative gravitational waves.

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