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C^* -Algebra-Valued Partial Modular Metric Spaces and Some Fixed Point Results

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Abstract: In the present paper, we introduce the notion of C^* -algebra-valued partial modular metric space satisfying the symmetry property that generalizes partial modular metric space, C^* -algebra-valued partial metric space, and C^* -algebra-valued modular metric space and discuss it with examples. Some fixed point results using $(\phi, \mathcal{MF})$ -contraction mapping are discussed in such space. In addition, we study the stability of obtained results in the spirit of Ulam and Hyers. As an application, we also provide the existence and uniqueness of the solution for a system of Fredholm integral equations.

Keywords: C^* -algebra-valued partial metric spaces; partial modular metric space; $\mathcal{F}$ -contraction mappings

MSC: 47H10; 54H25; 39B72



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1. Introduction

C^* -algebra is attractive due to its importance in many scientific fields such as mathematical analysis, quantum field theory, statistical mechanics, string theory, and non-commutative geometry. In 2014, Ma et al. [1] initiated the concept of C^* -algebra-valued metric space (in short, $C_{av}^* - MS$) and established some fixed point results with applications in integral and operator equations. On the other hand, Bakhtin [2] introduced b -metric space as a generalization of metric space (see also Czerwik [3]). In 2015, Ma and Jiang [4] introduced C^* -algebra-valued b -metric space (in short, $C_{av}^* - bMS$) and established some fixed point results with applications. In 2016, Alsulami et al. [5] and Kadelburg et al. [6] claimed that the fixed point results obtained in $C_{av}^* - MS$ and $C_{av}^* - bMS$ can be obtained easily from their metric counterparts. In 2021, Tomar and Joshi [7] demonstrated that the results in $C_{av}^* - MS$ cannot be obtained directly from their metric counterparts unless C^* -algebra $\mathbb{A}$ is $\mathbb{R}$, the set of real numbers. In 2010, Chistyakov [8] initiated modular metric space (in short, MMS), which generalizes metric space. In 2011, Chistyakov [9] established fixed point results in MMS (for more details see also [10–12]). Recently, Shateri [13] introduced the concept of C^* -algebra-valued modular space (in short, $C_{av}^* - MS$), which is a generalization of a modular space and proves some fixed point theorems for self-maps with contractive or expansive conditions in such spaces. In 2017, Moeini et al. [14–16] initiated the notion of C^* -algebra-valued modular metric space (in short, $C_{av}^* - MMS$) and developed some fixed point results with contractive conditions. Das et al. in [17–19], initiated the notions of C^* -algebra-valued modular b -metric space (in short, $C_{av}^* - MbMS$),

C^* -algebra-valued modular S -metric space (in short, $C_{av}^* - MSMS$), and C^* -algebra-valued modular G -metric space (in short, $C_{av}^* - MGMS$), respectively, and established some fixed point results in such respective spaces with applications. In 1994, Matthews [20] introduced the concept of partial metric space (in short, pMS). In 2019, Chandok et al. [21] initiated the notion of C^* -algebra-valued partial metric space (in short, $C_{av}^* - pMS$) and established some fixed point results. Recently, in 2020, Mlaiki et al. [22] enlarged $C_{av}^* - pMS$ by introducing the notion of C^* -algebra-valued partial b -metric space (in short, $C_{av}^* - pbMS$) and proved some results related to fixed points. Very recently, in 2022, Das et al. [23] redefined the concept of partial modular metric space (in short, $pMMS$) [24] and proved some fixed point results with an application to an integral equation. For the past decade, due to the significance of C^* -algebra, many researchers have been studying fixed point theory in $C_{av}^* - MS$ and its extended class of spaces (for more detail one may refer to [21,25–35]).

This paper consists of seven sections, wherein Sections 1 and 2 are the introduction and a preliminary, respectively. In Section 3, we introduce the notion of C^* -algebra-valued partial modular metric space and discuss some basic properties with suitable examples. Section 4 includes the results related to fixed points. In Section 5, we discuss the stability of the obtained results using the Ulam–Hyers technique. Section 6 is an application for existence and uniqueness results for a system of Fredholm integral equations, and the last section presents some concluding remarks on C^* -algebra-valued partial modular metric space.

2. Preliminaries

Recall that a Banach algebra $\mathbb{A}$ (over the field $\mathbb{C}$, the set of complex numbers) is said to be a C^* -algebra if there exists an involution $*$ in $\mathbb{A}$ (i.e., $*$: $\mathbb{A} \rightarrow \mathbb{A}$ satisfying $l^{**} = l$ for each $l \in \mathbb{A}$) such that for all $l_1, l_2 \in \mathbb{A}$ and for all $\alpha, \mu \in \mathbb{C}$, the following axiom holds:

$$(l_1 l_2)^* = l_2^* l_1^*; (l_1^*)^* = l_1, (\alpha l_1 + \mu l_2)^* = \bar{\alpha} l_1^* + \bar{\mu} l_2^* \text{ and } \|l_1^* l_1\| = \|l_1\|^2.$$

It is easy to see from the above that $\|l\| = \|l^*\|$ for all $l \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called unital C^* -algebra with unity $I_{\mathbb{A}}$. An element $l \in \mathbb{A}$ is called a positive transform; in denote it by $\theta \preceq l$, if $l = l^*$ and its spectrum $\sigma(l) = \{\lambda \in \mathbb{R} : \lambda I_{\mathbb{A}} - l \text{ is not invertible}\} \subseteq \mathbb{R}^+$, where θ is the zero of the element in $\mathbb{A}$. We denote $\mathbb{A}^+$ as the set of all positive elements of $\mathbb{A}$. Define a partial ordering on $\mathbb{A}$ such that $l_1 \preceq l_2$ if, and only if, $\theta \preceq l_2 - l_1$, i.e., $l_2 - l_1 \in \mathbb{A}^+$. (For more details refer to [21,22]).

Definition 1 ([14]). Let L be a non-empty set. A function $\omega^s : (0, +\infty) \times L \times L \rightarrow \mathbb{A}^+$ is called a C^* -algebra-valued modular metric ($C_{av}^* - MM$) on L if it satisfies the following axioms:

- (A₁): $\omega_{\xi}^s(l_1, l_2) = \theta$ if, and only if, $l_1 = l_2$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (A₂): $\omega_{\xi}^s(l_1, l_2) = \omega_{\xi}^s(l_2, l_1)$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (A₃): $\omega_{\xi+\mu}^s(l_1, l_2) \preceq \omega_{\xi}^s(l_1, l_3) + \omega_{\mu}^s(l_3, l_2)$, for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$.

A $C_{av}^* - MM$, i.e., ω^s on L , is said to be convex if the triangular inequality (A₃) is replaced by

$$\omega_{\xi+\mu}^s(l_1, l_2) \preceq \frac{\xi}{\xi+\mu} \omega_{\xi}^s(l_1, l_3) + \frac{\mu}{\xi+\mu} \omega_{\mu}^s(l_3, l_2), \text{ for all } l_1, l_2, l_3 \in L \text{ and } \mu, \xi > 0.$$

For any $l_0 \in L$, the set $L_{\omega^s} = \{l_1 \in L : \lim_{\xi \rightarrow \infty} \omega_{\xi}^s(l_1, l_0) = \theta\}$ is called a C^* -algebra-valued modular space with metric $d_{\omega^s}^0 : L_{\omega^s} \times L_{\omega^s} \rightarrow \mathbb{A}$ given by

$$d_{\omega^s}^0 = \inf \left\{ \xi > 0 : \|\omega_{\xi}^s(l_1, l_2)\| \leq \xi \right\}, \forall l_1, l_2 \in L_{\omega^s}.$$

Moreover, if ω^s is convex then the set will be

$$L_{\omega^s}^* = \left\{ l_1 \in L : \exists \xi = \xi(l_1) > 0 \text{ such that } \|\omega_{\xi}^s(l_1, l_0)\| < \infty \right\},$$

with the metric $d_{\omega^s}^* : L_{\omega^s}^* \times L_{\omega^s}^* \rightarrow \mathbb{A}$ given by

$$d_{\omega^s}^* = \inf \left\{ \xi > 0 : \|\omega_{\xi}(l_1, l_2)\| \leq 1 \right\}, \forall l_1, l_2 \in L_{\omega^s}^*.$$

Definition 2 ([21]). Let L be a non-empty set. A function $p : L \times L \rightarrow \mathbb{A}$ is called a C^* -algebra-valued partial metric on L if it satisfies:

- (B₁): $\theta \preceq p(l_1, l_2)$ for all $l_1, l_2 \in L$ and $p(l_1, l_1) = p(l_2, l_2) = p(l_1, l_2)$ if, and only if, $l_1 = l_2$;
- (B₂): $p(l_1, l_1) \preceq p(l_1, l_2)$, for all $l_1, l_2 \in L$;
- (B₃): $p(l_1, l_2) = p(l_2, l_1)$, for all $l_1, l_2 \in L$;
- (B₄): $p(l_1, l_2) \preceq p(l_1, l_3) + p(l_3, l_2) - p(l_3, l_3)$, for all $l_1, l_2, l_3 \in L$.

Then the triplet $(L, \mathbb{A}, p)$ is called a C^* -algebra-valued partial metric space (in short, $C_{av}^* - pMS$).

$C_{av}^* - pM$, i.e., p on L , generates a T_0 -topology τ_p with a base, defined by the family of p -open balls, $\{\mathfrak{B}_p(x, \epsilon) : l_1 \in L, \theta \prec \epsilon \in \mathbb{A}^+\}$, where $\mathfrak{B}_p(x, \epsilon) = \{l_2 \in L, p(l_1, l_2) \prec p(x, x) + \epsilon\}$.

Definition 3 ([24]). A function $p : (0, +\infty) \times L \times L \rightarrow [0, +\infty)$ is called a partial modular metric (pMM) on L if it satisfies:

- (C₁): $p_{\xi}(l_1, l_1) = p_{\xi}(l_2, l_2) = p_{\xi}(l_1, l_2)$ if, and only if, $l_1 = l_2$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (C₂): $p_{\xi}(l_1, l_1) \leq p_{\xi}(l_1, l_2)$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (C₃): $p_{\xi}(l_1, l_2) = p_{\xi}(l_2, l_1)$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (C₄): $p_{\xi+\mu}(l_1, l_2) \leq p_{\xi}(l_1, l_3) + p_{\mu}(l_3, l_2) - \frac{p_{\xi}(l_1, l_1) + p_{\xi}(l_3, l_3) + p_{\mu}(l_3, l_3) + p_{\mu}(l_2, l_2)}{2}$, for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$.

If $l_1 = l_2 = l_3$ in the triangular inequality (C₄), then a discrepancy occurs in the non-zero self distance for the partial modular metric.

Recently, Das et al. [23] redefined the notion of partial modular metric space as follows:

Definition 4. A function $\omega^p : (0, +\infty) \times L \times L \rightarrow [0, +\infty)$ is called a pMM on L if it satisfies:

- (D₁): $\omega_{\xi}^p(l_1, l_1) = \omega_{\mu}^p(l_1, l_1)$, and $\omega_{\xi}^p(l_1, l_1) = \omega_{\xi}^p(l_2, l_2) = \omega_{\xi}^p(l_1, l_2)$ if, and only if, $l_1 = l_2$, for all $l_1, l_2 \in L$ and $\xi, \mu > 0$;
- (D₂): $\omega_{\xi}^p(l_1, l_1) \leq \omega_{\xi}^p(l_1, l_2)$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (D₃): $\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^p(l_2, l_1)$, for all $l_1, l_2 \in L$ and $\xi > 0$;
- (D₄): $\omega_{\xi+\mu}^p(l_1, l_2) \leq \omega_{\xi}^p(l_1, l_3) + \omega_{\mu}^p(l_3, l_2) - \omega_{\xi}^p(l_3, l_3)$, for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$.

A pMM , i.e., ω^p on L , is said to be convex if (D₄) is replaced by

$$\omega_{\xi+\mu}^p(l_1, l_2) \leq \frac{\xi}{\xi+\mu} \omega_{\xi}^p(l_1, l_3) + \frac{\mu}{\xi+\mu} \omega_{\mu}^p(l_3, l_2) - \frac{\xi}{\xi+\mu} \omega_{\xi}^p(l_3, l_3),$$

for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$.

For any $l_0 \in L$, the set

$$L_{\omega^p} = \left\{ l_1 \in L : \lim_{\xi \rightarrow \infty} \omega_{\xi}^p(l_1, l_0) = c \right\},$$

for some $c \geq 0$ is a partial modular space. Moreover, if ω is convex then the set will be

$$L_{\omega^p}^* = \left\{ l_1 \in L : \exists \xi = \xi(l_1) > 0 \text{ such that } \|\omega_{\xi}^p(l_1, l_0)\| < \infty \right\}.$$

Definition 5 ([23]). A p MM, ω^p on L , is said to be weakly convex if it satisfies (D_1) , (D_2) , (D_3) , and the following:

$$\omega_{\xi+\mu}^p(l_1, l_2) \leq \alpha(\xi, \mu)\omega_{\xi}^p(l_1, l_3) + (1 - \alpha(\xi, \mu))\omega_{\mu}^p(l_3, l_2) - \alpha(\xi, \mu)\omega_{\xi}^p(l_3, l_3),$$

for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$, where $\alpha : (0, \infty) \times (0, \infty) \rightarrow (0, 1)$.

Lemma 1 ([23]). Let ω^p be a p MM on L . Then $\omega^s : (0, \infty) \times L \times L \rightarrow [0, \infty)$ is a modular metric on L , where

$$\omega_{\xi}^s(l_1, l_2) = 2\omega_{\xi}^p(l_1, l_2) - \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_2, l_2),$$

for all $\xi > 0$ and $l_1, l_2 \in L$.

Example 1 ([23]). For a non-empty set L , let ω^p be a p MM and ω^s be a MM. Then for any arbitrary $\xi > 0$ and $l_1, l_2 \in L$, $\omega_{\xi}^p(l_1, l_2) = \frac{\omega_{\xi}^s(l_1, l_2) + |l_1| + |l_2|}{2}$ defines a p MM.

Definition 6 ([36,37]). Let $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}^+$ be positive if it holds the following axioms:

- (i) ϕ is continuous and increasing;
- (ii) $\phi(l_1) = \theta$ if, and only if, $l_1 = \theta$;
- (iii) $\lim_{n \rightarrow \infty} \phi^n(l_1) = \theta$.

Definition 7 ([36,37]). Let A and B be two C^* -algebras, and $\phi : A \rightarrow B$. It is said to be a C^* -homomorphism if it satisfies the following axioms:

- (i) $\phi(al_1 + bl_2) = a\phi(l_1) + b\phi(l_2)$;
- (ii) $\phi(l_1 l_2) = \phi(l_1)\phi(l_2)$;
- (iii) $\phi(l_1^*) = \phi(l_1)^*$;
- (iv) ϕ maps the unit in A to the unit in B .

In [38], Wardowski introduced a new type of contraction named as $\mathcal{F}$ -contraction. Since then, development of fixed point theory can be seen with such type of contraction mapping (for further details, refer to [36,39–43]). Inspired by Wardowski in [36], Rossafi et al. [42] introduced $(\phi, \mathcal{MF})$ -contraction as follows:

Definition 8. Let $(L, \mathbb{A}, d)$ be a C^* -algebra-valued metric space and T be a self mapping on L . It is said to be a $(\phi, \mathcal{MF})$ -contraction if

$$\mathcal{M}(Tl_1, Tl_2) \succeq \theta \Rightarrow \mathcal{F}(\mathcal{M}(Tl_1, Tl_2)) + \phi(\mathcal{M}(l_1, l_2)) \preceq \mathcal{F}(\mathcal{M}(l_1, l_2)),$$

for all $l_1, l_2 \in L$, where

$$\mathcal{M}(l_1, l_2) = \max \left\{ d(l_1, l_2), d(l_1, Tl_1), d(l_2, Tl_2), \frac{d(l_1, Tl_2) + d(l_2, Tl_1)}{2} \right\},$$

mapping $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}$ is an $*$ -homomorphism and $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$ be a continuous and non-decreasing function such that $\mathcal{F}(l_1) = \theta$ if, and only if, $l_1 = \theta$.

3. C^* -Algebra-Valued Partial Modular Metric Space

In this section, we introduce the notion of C^* -algebra-valued partial modular metric space (in short, $C_{av}^* - pMMS$) satisfying the symmetry property and discuss it with examples T.

Definition 9. Let L be a non-empty set. A function $\omega^p : (0, +\infty) \times L \times L \rightarrow \mathbb{A}$ is called a C^* -algebra-valued partial modular metric on L if it satisfies:

- (P₁^{*}) : $\omega_{\xi}^p(l_1, l_1) = \omega_{\mu}^p(l_1, l_1)$, for all $l_1, l_2 \in L$ and $\xi, \mu > 0$; $\omega_{\xi}^p(l_1, l_1) = \omega_{\xi}^p(l_2, l_2) = \omega_{\xi}^p(l_1, l_2)$ if, and only if, $l_1 = l_2$;
 (P₂^{*}) : $\omega_{\xi}^p(l_1, l_1) \preceq \omega_{\xi}^p(l_1, l_2)$, for all $l_1, l_2 \in L$ and $\xi > 0$;
 (P₃^{*}) : $\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^p(l_2, l_1)$, for all $l_1 \in L$ and $\xi, \mu > 0$;
 (P₄^{*}) : $\omega_{\xi+\mu}^p(l_1, l_2) \preceq \omega_{\xi}^p(l_1, l_3) + \omega_{\mu}^p(l_3, l_2) - \omega_{\xi}^p(l_3, l_3)$, for all $\xi > 0, \mu > 0$ and $l_1, l_2, l_3 \in L$.

A $C_{av}^* - pMM$, i.e., ω^p on L , is said to be convex if (P₄^{*}) is replaced by the following:

$$(P_5^*) : \omega_{\xi+\mu}^p(l_1, l_2) \preceq \frac{\xi}{\xi+\mu} \omega_{\xi}^p(l_1, l_3) + \frac{\mu}{\xi+\mu} \omega_{\mu}^p(l_3, l_2) - \frac{\xi}{\xi+\mu} \omega_{\xi}^p(l_3, l_3),$$

for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$.

Definition 10. A $C_{av}^* - pMM$, i.e., ω^p on L , is said to be weakly convex if it satisfies (P₁^{*}), (P₂^{*}), (P₃^{*}), and the following:

$$(P_6^*) : \omega_{\xi+\mu}^p(l_1, l_2) \preceq \alpha(\xi, \mu) \omega_{\xi}^p(l_1, l_3) + (1 - \alpha(\xi, \mu)) \omega_{\mu}^p(l_3, l_2) - \alpha(\xi, \mu) \omega_{\xi}^p(l_3, l_3),$$

for all $l_1, l_2, l_3 \in L$ and $\mu, \xi > 0$, where $\alpha : (0, \infty) \times (0, \infty) \rightarrow (0, 1)$.

Definition 11. Let ω^p be a $C_{av}^* - pMM$ on L . For any $l_0 \in L$, the set

$$L_{\omega^p} = \left\{ l_1 \in L : \lim_{\xi \rightarrow \infty} \omega_{\xi}^p(l_1, l_0) = c \right\},$$

for some $\theta \preceq c \in \mathbb{A}^+$ is called C^* -algebra-valued partial modular space. Moreover, if ω^p is convex, then the set will be

$$L_{\omega^p}^* = \left\{ l_1 \in L : \exists \xi = \xi(l_1) > 0 \text{ such that } \|\omega_{\xi}^p(l_1, l_0)\| < \infty \right\}.$$

Remark 1. (i) For every $l_1, l_2 \in L$, the function $\xi \mapsto \omega^p(l_1, l_2) \in \mathbb{A}$ is non-increasing. In fact, for all $l_1, l_2 \in L$ and $0 < \mu < \xi$:

$$\omega_{\xi}^p(p_1, p_2) \preceq \omega_{\mu}^p(p_1, p_2).$$

(ii) For simplicity, one may consider the triplicate $(L_{\omega^p}, \mathbb{A}, \omega^p)$ as a $C_{av}^* - pMMS$, instead of writing "Let L_{ω^p} be a C_{av}^* -modular space and ω^p be a C_{av}^* -partial modular metric."

Example 2. Let ω^s be a $C_{av}^* - MM$ on a non-empty set L . Define $\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^s(l_1, l_2) + |c|$, where $c \succeq \theta$. Then for any arbitrary $\theta \preceq c \in \mathbb{A}^+$, ω^p is a $C_{av}^* - pMM$. Moreover, it is convex if ω^s is a C_{av}^* -convex modular metric with $c = \theta$.

Example 3. Let $L = \mathbb{C}$ and $\mathbb{A} = M_2(\mathbb{C})$, the class of linear and bounded operators on Hilbert space $\mathbb{C}^2$. Define $\omega^p : (0, \infty) \times L \times L \rightarrow \mathbb{A}$ by

$$\omega_{\xi}^p(l_1, l_2) = \frac{1}{\xi} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + I_{\mathbb{A}},$$

for all $\xi > 0, l_1, l_2 \in L$, and I is the identity matrix. Then ω^p is a $C_{av}^* - pMM$, but neither convex $C_{av}^* - pMM$ nor weakly convex $C_{av}^* - pMM$.

Proof. (P₁^{*}), (P₂^{*}), and (P₃^{*}) hold immediately. Now we check (P₄^{*}), (P₅^{*}), and (P₆^{*}).
 (P₄^{*}) : For any $\xi > 0$ and $\mu > 0$, $\frac{1}{\xi+\mu} < \frac{1}{\xi}$ or $\frac{1}{\mu}$.

$$\begin{aligned}
\omega_{\xi+\mu}^p(l_1, l_2) &= \frac{1}{\xi+\mu} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + I_{\mathbb{A}} \\
&\preceq \frac{1}{\xi+\mu} (\text{diag}(|l_1 - l_3|, |l_1 - l_3|) + \text{diag}(|l_3 - l_2|, |l_3 - l_2|)) + I_{\mathbb{A}} \\
&\preceq \frac{1}{\xi} \text{diag}(|l_1 - l_3|, |l_1 - l_3|) + I_{\mathbb{A}} + \frac{1}{\mu} \text{diag}(|l_3 - l_2|, |l_3 - l_2|) + I_{\mathbb{A}} \\
&\quad - [\frac{1}{\xi} \text{diag}(|l_3 - l_3|, |l_3 - l_3|) + I_{\mathbb{A}}] \\
&= \omega_{\xi}^p(l_1, l_3) + \omega_{\mu}^p(l_3, l_2) - \omega_{\mu}^p(l_3, l_3).
\end{aligned}$$

Hence, ω^p is a $C_{av}^* - pMM$.

$$\begin{aligned}
(P_5^*) : \omega_{\xi+\mu}^p(l_1, l_2) &= \frac{1}{\xi+\mu} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + I_{\mathbb{A}} \\
&\preceq \frac{1}{\xi+\mu} (\text{diag}(|l_1 - l_3|, |l_1 - l_3|) + \text{diag}(|l_3 - l_2|, |l_3 - l_2|)) + I_{\mathbb{A}} \\
&= \frac{\xi}{\xi+\mu} [\frac{1}{\xi} \text{diag}(|l_1 - l_3|, |l_1 - l_3|) + I_{\mathbb{A}}] \\
&\quad + \frac{\mu}{\xi+\mu} [\frac{1}{\mu} \text{diag}(|l_3 - l_2|, |l_3 - l_2|) + I_{\mathbb{A}}] \\
&= \frac{\xi}{\xi+\mu} \omega_{\xi}^p(l_1, l_3) + \frac{\mu}{\xi+\mu} \omega_{\mu}^p(l_3, l_2).
\end{aligned}$$

Hence, ω^p is not a convex $C_{av}^* - pMM$. Similarly, it can be shown that ω^p is also not a weakly convex $C_{av}^* - pMM$. $\square$

Example 4. Let $L = \mathbb{C}$ and $\mathbb{A} = M_2(\mathbb{C})$, the class of linear and bounded operators on Hilbert space $\mathbb{C}^2$. Define $\omega^p : (0, \infty) \times L \times L \rightarrow \mathbb{A}$ by

$$\omega_{\xi}^p(l_1, l_2) = e^{-\xi} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + I_{\mathbb{A}},$$

for all $\xi > 0$, $l_1, l_2 \in \mathbb{C}$, and $I_{\mathbb{A}}$ is the identity matrix. Then ω^p is a $C_{av}^* - pMM$ on L_{ω^p} .

Example 5. Let $L = \mathbb{C}$ and $\mathbb{A} = M_2(\mathbb{C})$, the class of linear and bounded operators on Hilbert space $\mathbb{C}^2$. Define $\omega^p : (0, \infty) \times L \times L \rightarrow \mathbb{A}$ by

$$\omega_{\xi}^p(l_1, l_2) = \frac{1}{\xi} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + \text{diag}(\max\{l_1, l_2\}, \max\{l_1, l_2\}),$$

for all $\xi > 0$, $l_1, l_2 \in \mathbb{C}$. Then ω^p is a $C_{av}^* - pMM$ on L_{ω^p} .

Lemma 2. Let ω^p be a $C_{av}^* - pMM$ on L such that

$$\omega_{\xi}^s(l_1, l_2) = 2\omega_{\xi}^p(l_1, l_2) - \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_2, l_2).$$

Then, ω^s is a $C_{av}^* - MM$.

Proof. (i) By definition:

$$\begin{aligned}
\omega_{\xi}^p(l_1, l_1) &\preceq \omega_{\xi}^p(l_1, l_2) \\
\omega_{\xi}^p(l_2, l_2) &\preceq \omega_{\xi}^p(l_2, l_1) = \omega_{\xi}^p(l_1, l_2).
\end{aligned} \tag{1}$$

So, we have $\omega_{\xi}^s(l_1, l_2) \succeq \theta$.

(ii) If $l_1 = l_2$, then $w_{\xi}^s(l_1, l_1) = 2\omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_1, l_1) = \theta$.

Suppose $\omega_{\xi}^s(l_1, l_2) = \theta$, then $2\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_2, l_2)$. Using (1), we have

$$\begin{aligned} 2\omega_{\xi}^p(l_1, l_1) &\preceq 2\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^p(l_1, l_1) + \omega_{\xi}^p(l_2, l_2) \\ 2\omega_{\xi}^p(l_2, l_2) &\preceq 2\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^p(l_1, l_1) + \omega_{\xi}^p(l_2, l_2). \end{aligned}$$

Consequently, we have

$\omega_{\xi}^p(l_1, l_2) \preceq \omega_{\xi}^p(l_1, l_1)$ and $\omega_{\xi}^p(l_2, l_2) \preceq \omega_{\xi}^p(l_1, l_2)$. Thus, we get

$$\omega_{\xi}^p(l_1, l_2) = \omega_{\xi}^p(l_1, l_1) = \omega_{\xi}^p(l_2, l_2) \Rightarrow l_1 = l_2.$$

(iii) Similarity:

$$\begin{aligned} \omega_{\xi}^s(l_1, l_2) &= 2\omega_{\xi}^p(l_1, l_2) - \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_2, l_2) \\ &= 2\omega_{\xi}^p(l_2, l_1) - \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_2, l_2) = \omega_{\xi}^s(l_2, l_1). \end{aligned}$$

(iv) Triangular inequality:

$$\begin{aligned} \omega_{\xi+\mu}^s(l_1, l_3) &= 2\omega_{\xi+\mu}^p(l_1, l_3) - \omega_{\xi+\mu}^p(l_1, l_1) - \omega_{\xi+\mu}^p(l_3, l_3) \\ &\preceq 2[\omega_{\xi}^p(l_1, l_2) + \omega_{\mu}^p(l_2, l_3) - \omega_{\xi}^p(l_2, l_2)] - \omega_{\xi}^p(l_1, l_1) - \omega_{\mu}^p(l_3, l_3) \\ &= 2\omega_{\xi}^p(l_1, l_2) - \omega_{\xi}^p(l_1, l_1) - \omega_{\xi}^p(l_2, l_2) + 2\omega_{\mu}^p(l_2, l_3) - \omega_{\mu}^p(l_2, l_2) - \omega_{\mu}^p(l_3, l_3) \\ &= \omega_{\xi}^s(l_1, l_2) + \omega_{\mu}^s(l_2, l_3). \end{aligned}$$

Hence, ω^s is a C_{av}^* - MM. $\square$

Definition 12. Let ω^p be a C_{av}^* - pMM on L , and $l, \{l_n\} \in L$ with $\xi > 0$. Then

(i) $\{l_n\}$ converges to l with respect to $\mathbb{A}$, whenever, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\omega_{\xi}^p(l_n, l) - \omega_{\xi}^p(l, l)\| \leq \epsilon$, for all $n \geq N$, i.e.,

$$\lim_{n \rightarrow \infty} (\omega_{\xi}^p(l_n, l) - \omega_{\xi}^p(l, l)) = \theta.$$

(ii) $\{l_n\}$ be Cauchy sequence with respect to $\mathbb{A}$, whenever, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$, we have

$$\begin{aligned} &(\omega_{\xi}^p(l_n, l_m) - \frac{1}{2}\omega_{\xi}^p(l_n, l_n) - \frac{1}{2}\omega_{\xi}^p(l_m, l_m))(\omega_{\xi}^p(l_n, l_m) - \frac{1}{2}\omega_{\xi}^p(l_n, l_n) - \frac{1}{2}\omega_{\xi}^p(l_m, l_m))^* \\ &\preceq \epsilon^2. \end{aligned}$$

(iii) C_{av}^* -partial modular space L_{ω^p} be complete with respect to $\mathbb{A}$ if every Cauchy sequence with respect to $\mathbb{A}$ converges to a point $l \in L$ such that

$$\lim_{n \rightarrow \infty} (\omega_{\xi}^p(l_n, l) - \frac{1}{2}\omega_{\xi}^p(l_n, l_n) - \frac{1}{2}\omega_{\xi}^p(l, l)) = \theta.$$

Lemma 3. Let ω^p be a C_{av}^* - pMM on L_{ω^p} .

- (i) $\{x_n\}$ is a partial modular Cauchy sequence in C_{av}^* -partial modular space L_{ω^p} with ω^p if, and only if, it is Cauchy sequence in C_{av}^* -modular space L_{ω^s} with ω^s .
- (ii) A C_{av}^* -partial modular space L_{ω^p} with modular ω^p , $(L_{\omega^p}, \mathbb{A}, \omega^p)$ is complete if, and only if, C_{av}^* -modular space L_{ω^s} with modular ω^s , $(L_{\omega^s}, \mathbb{A}, \omega^s)$ is complete. Furthermore,

$$\lim_{n \rightarrow \infty} \omega_{\xi}^s(l_n, l) = \theta \Leftrightarrow \lim_{n \rightarrow \infty} (2\omega_{\xi}^p(l_n, l) - \omega_{\xi}^p(l_n, l_n) - \omega_{\xi}^p(l, l)) = \theta$$

or

$$\lim_{n \rightarrow \infty} \omega_{\xi}^s(l_n, l) = \theta \Leftrightarrow \lim_{n \rightarrow \infty} (\omega_{\xi}^p(l_n, l) - \omega_{\xi}^p(l_n, l_n)) = \theta \text{ and} \\ \lim_{n \rightarrow \infty} (\omega_{\xi}^p(l_n, l) - \omega_{\xi}^p(l, l)) = \theta.$$

(iii) Assume that $l_n \rightarrow \infty$ and $y_n \rightarrow \infty$ as $n \rightarrow \infty$ in a C_{av}^* -pMM on L with modular ω^p . Then,

$$\lim_{n \rightarrow \infty} (\omega_{\xi}^p(l_n, y_n) - \omega_{\xi}^p(l_n, l_n)) = \omega_{\xi}^p(l, y) - \omega_{\xi}^p(l, l) \text{ and} \\ \lim_{n \rightarrow \infty} (\omega_{\xi}^p(l_n, y_n) - \omega_{\xi}^p(y_n, y_n)) = \omega_{\xi}^p(l, y) - \omega_{\xi}^p(y, y).$$

4. Some Fixed Point Results in C_{av}^* -pMMS

Theorem 1. Let L_{ω^p} be a complete C_{av}^* -partial modular space and ω^p be a C_{av}^* -pMM. Define two self mappings $P, Q : L_{\omega^p} \rightarrow L_{\omega^p}$ satisfying the following conditions:

- (i) $P(L_{\omega^p}) \subseteq Q(L_{\omega^p})$;
- (ii) for all $l_1, l_2 \in L_{\omega^p}$,

$$\omega_{\xi}^p(Pl_1, Ql_2) \succeq \theta \Rightarrow \mathcal{F}(\omega_{\xi}^p(Pl_1, Ql_2)) + \phi(\mathcal{M}(l_1, l_2)) \preceq \mathcal{F}(\mathcal{M}(l_1, l_2)),$$

where $\mathcal{M}(l_1, l_2) = \max \left\{ \omega_{\xi}^p(l_1, l_2), \omega_{\xi}^p(l_1, Pl_1), \omega_{\xi}^p(l_2, Ql_2), \frac{[\omega_{2\xi}^p(l_1, Ql_2) + \omega_{2\xi}^p(l_2, Pl_1)]}{2} \right\}$, $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}$ is a $*$ -homomorphism and $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$, be a continuous and non-decreasing function such that $\mathcal{F}(l_1) = \theta \Leftrightarrow l_1 = \theta$. If P has fixed point $u \in L_{\omega^p}$, then u is a unique common fixed point of P and Q .

Proof. Let $Pu = u$, $u \in L_{\omega^p}$. Assume, $\omega_{\xi}^p(Pu, Qu) \succ \theta$, we have

$$\mathcal{F}(\omega_{\xi}^p(Pu, Qu)) \preceq \mathcal{F}(\mathcal{M}(u, u)) - \phi(\mathcal{M}(u, u)),$$

where

$$\mathcal{M}(u, u) = \max \left\{ \omega_{\xi}^p(u, u), \omega_{\xi}^p(u, Pu), \omega_{\xi}^p(u, Qu), \right. \\ \left. \frac{[\omega_{2\xi}^p(u, Qu) + \omega_{2\xi}^p(u, Pu)]}{2} \right\} \\ = \omega_{\xi}^p(u, Qu).$$

Clearly,

$$\mathcal{F}(\omega_{\xi}^p(u, Qu)) \preceq \mathcal{F}(\omega_{\xi}^p(u, Qu)) - \phi(\omega_{\xi}^p(u, Qu)),$$

which is a contradiction, as ϕ is positive. So, $\omega_{\xi}^p(u, Qu) = \theta$. Similarly, we can show that $\omega_{\xi}^p(Qu, u) = \theta$. Hence, u is a common fixed point of P and Q .

Let there exist another $u^* \in L_{\omega^p}$ such that $Qu^* = Pu^* = u^*$, then we have

$$\mathcal{F}(\omega_{\xi}^p(u, u^*)) \preceq \mathcal{F}(\mathcal{M}(u, u^*)) - \phi(\mathcal{M}(u, u^*)),$$

where

$$\mathcal{M}(u, u^*) = \max \left\{ \omega_{\xi}^p(u, u^*), \omega_{\xi}^p(u, Pu), \omega_{\xi}^p(u^*, Qu^*), \right. \\ \left. \frac{[\omega_{2\xi}^p(u, Qu^*) + \omega_{2\xi}^p(u^*, Pu)]}{2} \right\} \\ = \omega_{\xi}^p(u, u^*).$$

Therefore, we obtain

$$\mathcal{F}(\omega_{\xi}^p(u, u^*)) \preceq \mathcal{F}(\omega_{\xi}^p(u, u^*)) - \phi(\omega_{\xi}^p(u, u^*)),$$

which is a contradiction, as ϕ is positive. So, $u = u^*$. $\square$

Theorem 2. Let L_{ω^p} be a complete C_{av}^* -modular space and ω^p be a $C_{av}^* - p$ MM. Define two self mappings $P, Q : L_{\omega^p} \rightarrow L_{\omega^p}$ satisfying the following conditions:

- (i) $P(L_{\omega^p}) \subseteq Q(L_{\omega^p})$;
- (ii) for all $l_1, l_2 \in L_{\omega^p}$,

$$\omega_{\xi}^p(Pl_1, Ql_2) \succeq \theta \Rightarrow \mathcal{F}(\omega_{\xi}^p(Pl_1, Ql_2)) + \phi(\mathcal{M}(l_1, l_2)) \preceq \mathcal{F}(\mathcal{M}(l_1, l_2)),$$

where $\mathcal{M}(l_1, l_2) = \max \left\{ \omega_{\xi}^p(l_1, l_2), \omega_{\xi}^p(l_1, Pl_1), \omega_{\xi}^p(l_2, Ql_2), \frac{[\omega_{2\xi}^p(l_1, Ql_2) + \omega_{2\xi}^p(l_2, Pl_1)]}{2} \right\}$, $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}$ is a $*$ -homomorphism and $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$ be a continuous and non-decreasing function such that $\mathcal{F}(l_1) = \theta$ if, and only if, $l_1 = \theta$. Then P and Q have a unique common fixed point in L_{ω^p} .

Proof. Let $l_0 \in L_{\omega^p}$ be any point. Define a sequence $\{l_n\}$, where $n \in \mathbb{N} \cup \{0\}$ such that $Pl_{2n} = l_{2n+1}$ and $Ql_{2n+1} = l_{2n+2}$. If $l_m = l_{m+1}$, for all $m \in \mathbb{N} \cup \{0\}$. Setting $m = 2n$, then we have

$$\begin{aligned} \mathcal{M}(l_{2n}, l_{2n+1}) &= \max \left\{ \omega_{\xi}^p(l_{2n}, l_{2n+1}), \omega_{\xi}^p(l_{2n}, Pl_{2n}), \omega_{\xi}^p(l_{2n+1}, Ql_{2n+1}), \right. \\ &\quad \left. \frac{[\omega_{2\xi}^p(l_{2n}, Ql_{2n+1}) + \omega_{2\xi}^p(l_{2n+1}, Pl_{2n})]}{2} \right\} \\ &= \max \left\{ \omega_{\xi}^p(l_{2n}, l_{2n}), \omega_{\xi}^p(l_{2n}, l_{2n}), \omega_{\xi}^p(l_{2n}, l_{2n+2}), \right. \\ &\quad \left. \frac{[\omega_{2\xi}^p(l_{2n}, l_{2n+2}) + \omega_{2\xi}^p(l_{2n}, l_{2n})]}{2} \right\} \\ &= \omega_{\xi}^p(l_{2n}, l_{2n+2}). \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{F}(\omega_{\xi}^p(Pl_{2n}, Ql_{2n+1})) &\preceq \mathcal{F}(\mathcal{M}(l_{2n}, l_{2n+1})) - \phi(\mathcal{M}(l_{2n}, l_{2n+1})) \\ \Rightarrow \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n+2})) &\preceq \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n+2})) - \phi(\omega_{\xi}^p(l_{2n}, l_{2n+2})). \end{aligned}$$

Hence, a contradiction arises, as ϕ is positive. So, $\omega_{\xi}^p(l_{2n}, l_{2n+2}) = \theta$. Clearly, we get $l_{2n} = l_{2n+1} = l_{2n+2}$. Similarly, we have $l_{2n} = l_{2n+1} = l_{2n+2} = l_{2n+3} = \dots$, and so $l_{2n} = Pl_{2n} = Ql_{2n+1}$, i.e., l_{2n} is the common fixed point. Now, let $l_m \neq l_{m+1}$, for all $m = 0, 1, 2, \dots$. Assuming, $\omega_{\xi}^p(l_{2n}, l_{2n-1}) \succ \theta$ we have

$$\begin{aligned} \mathcal{M}(l_{2n}, l_{2n-1}) &= \max \left\{ \omega_{\xi}^p(l_{2n}, l_{2n-1}), \omega_{\xi}^p(l_{2n}, Pl_{2n}), \omega_{\xi}^p(l_{2n-1}, Ql_{2n-1}), \right. \\ &\quad \left. \frac{[\omega_{2\xi}^p(l_{2n}, Ql_{2n-1}) + \omega_{2\xi}^p(l_{2n-1}, Pl_{2n})]}{2} \right\} \\ &= \max \left\{ \omega_{\xi}^p(l_{2n}, l_{2n-1}), \omega_{\xi}^p(l_{2n}, l_{2n+1}), \omega_{\xi}^p(l_{2n-1}, l_{2n}), \right. \\ &\quad \left. \frac{[\omega_{2\xi}^p(l_{2n}, l_{2n}) + \omega_{2\xi}^p(l_{2n-1}, l_{2n+1})]}{2} \right\} \\ &= \max \left\{ \omega_{\xi}^p(l_{2n}, l_{2n-1}), \omega_{\xi}^p(l_{2n}, l_{2n+1}), \omega_{\xi}^p(l_{2n-1}, l_{2n}), \right. \\ &\quad \left. \frac{[\omega_{\xi}^p(l_{2n}, l_{2n}) + \omega_{\xi}^p(l_{2n-1}, l_{2n}) + \omega_{\xi}^p(l_{2n}, l_{2n+1}) - \omega_{\xi}^p(l_{2n}, l_{2n})]}{2} \right\} \end{aligned}$$

$$= \max \left\{ \omega_{\xi}^p(l_{2n}, l_{2n-1}), \omega_{\xi}^p(l_{2n}, l_{2n+1}) \right\}.$$

Let $\mathcal{M}(l_{2n}, l_{2n-1}) = \omega_{\xi}^p(l_{2n}, l_{2n+1})$, then

$$\begin{aligned} \mathcal{F}(\omega_{\xi}^p(Pl_{2n}, Ql_{2n-1})) &\preceq \mathcal{F}(\mathcal{M}(l_{2n}, l_{2n-1})) - \phi(\mathcal{M}(l_{2n}, l_{2n-1})) \\ \Rightarrow \mathcal{F}(\omega_{\xi}^p(l_{2n+1}, l_{2n})) &\preceq \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n+1})) - \phi(\omega_{\xi}^p(l_{2n}, l_{2n+1})). \end{aligned}$$

Hence, a contradiction arises, as ϕ is positive. So, $\mathcal{M}(l_{2n}, l_{2n-1}) = \omega_{\xi}^p(l_{2n}, l_{2n-1})$, and we get

$$\begin{aligned} \mathcal{F}(\omega_{\xi}^p(Pl_{2n}, Ql_{2n-1})) &\preceq \mathcal{F}(\mathcal{M}(l_{2n}, l_{2n-1})) - \phi(\mathcal{M}(l_{2n}, l_{2n-1})) \\ \Rightarrow \mathcal{F}(\omega_{\xi}^p(l_{2n+1}, l_{2n})) &\preceq \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n-1})) - \phi(\omega_{\xi}^p(l_{2n}, l_{2n-1})) \\ &\prec \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n-1})). \end{aligned} \quad (2)$$

In a similar manner, we can also show that

$$\begin{aligned} \mathcal{F}(\omega_{\xi}^p(Pl_{2n}, Ql_{2n+1})) &\preceq \mathcal{F}(\mathcal{M}(l_{2n}, l_{2n+1})) - \phi(\mathcal{M}(l_{2n}, l_{2n+1})) \\ \Rightarrow \mathcal{F}(\omega_{\xi}^p(l_{2n+1}, l_{2n+2})) &\preceq \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n+1})) - \phi(\omega_{\xi}^p(l_{2n}, l_{2n+1})) \\ &\prec \mathcal{F}(\omega_{\xi}^p(l_{2n}, l_{2n+1})). \end{aligned} \quad (3)$$

Since $\mathcal{F}$ is non-decreasing, so from (2) and (3), $\omega_{\xi}^p(l_m, l_{m+1})$ is monotonically decreasing in $\mathbb{A}$. Let there exist $\theta \preceq \beta \in \mathbb{A}^+$ such that $\lim_{n \rightarrow \infty} \omega_{\xi}^p(l_m, l_{m+1}) = \beta$. Hence, from (2) and (3), we have

$$\lim_{n \rightarrow \infty} \mathcal{F}(\omega_{\xi}^p(l_m, l_{m+1})) = \theta, \text{ i.e., } \lim_{n \rightarrow \infty} \omega_{\xi}^p(l_m, l_{m+1}) = \theta. \quad (4)$$

Now we show that $\{l_m\}$ is a C^* -algebra-valued partial modular Cauchy sequence in L_{ω^p} . By Lemma 3, it is sufficient to prove that $\{Ql_{n+1}\}$ is a Cauchy sequence in L_{ω^s} . For the fact $\theta \preceq \omega_{\xi}^p(l_m, l_m) \preceq \omega_{\xi}^p(l_m, l_{m+1})$, we have

$$\lim_{n \rightarrow \infty} \omega_{\xi}^p(l_m, l_m) = \theta. \quad (5)$$

Additionally, $\theta \preceq \omega_{\xi}^p(l_{m+1}, l_{m+1}) \preceq \omega_{\xi}^p(l_{m+1}, l_{m+2})$ implies

$$\lim_{n \rightarrow \infty} \omega_{\xi}^p(l_{m+1}, l_{m+1}) = \theta. \quad (6)$$

Let $\{l_m\}$ not be a Cauchy sequence in L_{ω^s} if possible. Then there exist $\epsilon > 0$ and subsequences $\{l_{m(k)}\}$ and $\{l_{n(k)}\}$ with $n(k) > m(k) > k$ such that

$$\|w_{\xi}^s(l_{m(k)}, l_{n(k)})\| > \epsilon; \forall \xi > 0.$$

Now, corresponding to $m(k)$, we can choose $n(k)$ such that it is the smallest integer with $n(k) > m(k)$ and satisfying the above inequality. Hence,

$$\|w_{\xi}^s(l_{m(k)}, l_{n(k)-1})\| \leq \epsilon; \forall \xi > 0.$$

We have

$$\begin{aligned} \|w_{\xi}^s(l_{m(k)}, l_{n(k)})\| &\leq \|w_{\frac{\xi}{2}}^s(l_{m(k)}, l_{n(k)-1})\| + \|w_{\frac{\xi}{2}}^s(l_{n(k)-1}, l_{n(k)})\| \\ &\leq \epsilon + \|w_{\frac{\xi}{2}}^s(l_{n(k)-1}, l_{n(k)})\|. \end{aligned} \quad (7)$$

Again,

$$w_{\xi}^s(l_{n(k)-1}, l_{n(k)}) = 2\omega_{\xi}^p(l_{n(k)-1}, l_{n(k)}) - \omega_{\xi}^p(l_{n(k)-1}, l_{n(k)-1}) - \omega_{\xi}^p(l_{n(k)}, l_{n(k)}). \quad (8)$$

Using (3), (4), (5) and (7), we have

$$\lim_{k \rightarrow \infty} \|w_{\xi}^s(l_{n(k)-1}, l_{n(k)})\| = \theta. \quad (9)$$

Using (6) and (8), we have

$$\epsilon < \lim_{k \rightarrow \infty} \|w_{\xi}^s(l_{m(k)}, l_{n(k)})\| < \epsilon + \theta.$$

This implies

$$\lim_{k \rightarrow \infty} \|w_{\xi}^s(l_{m(k)}, l_{n(k)})\| = \epsilon. \quad (10)$$

In a similar manner, we can show that

$$\lim_{k \rightarrow \infty} \|w_{\xi}^s(l_{m(k)-1}, l_{n(k)-1})\| = \epsilon.$$

Thus, we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|\omega_{\xi}^p(l_{m(k)-1}, l_{n(k)-1})\| \\ &= \frac{1}{2} \lim_{k \rightarrow \infty} \|2\omega_{\xi}^p(l_{m(k)-1}, l_{n(k)-1}) - \omega_{\xi}^p(l_{n(k)-1}, l_{n(k)-1}) - \omega_{\xi}^p(l_{m(k)-1}, l_{m(k)-1})\| \\ &= \frac{1}{2} \lim_{k \rightarrow \infty} \|w_{\xi}^s(l_{m(k)-1}, l_{n(k)-1})\| \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Since $\omega_{\xi}^p(l_{m(k)-1}, l_{n(k)-1}), \omega_{\xi}^p(l_{m(k)}, l_{n(k)}) \in \mathbb{A}^+$ and

$$\lim_{k \rightarrow \infty} \|\omega_{\xi}^p(l_{m(k)-1}, l_{n(k)-1})\| = \lim_{k \rightarrow \infty} \|\omega_{\xi}^p(l_{m(k)}, l_{n(k)})\| = \frac{\epsilon}{2},$$

there exists $b \in \mathbb{A}^+$ with $\|b\| = \frac{\epsilon}{2}$ such that

$$\lim_{k \rightarrow \infty} \omega_{\xi}^p(l_{m(k)-1}, l_{n(k)-1}) = \lim_{k \rightarrow \infty} \omega_{\xi}^p(l_{m(k)}, l_{n(k)}) = b.$$

Now, from triangular inequality, we have

$$\begin{aligned} \omega_{\xi}^p(l_{m(k)}, l_{n(k)}) &\preceq \omega_{\xi}^p(l_{m(k)}, l_{m(k)+1}) + \omega_{\xi}^p(l_{m(k)+1}, l_{m(k)+2}) + \cdots + \omega_{\xi}^p(l_{n(k)-1}, l_{n(k)}) \\ &\quad - \omega_{\xi}^p(l_{m(k)+1}, l_{m(k)+1}) + \omega_{\xi}^p(l_{m(k)+2}, l_{m(k)+2}) + \cdots + \omega_{\xi}^p(l_{n(k)-1}, l_{n(k)-1}). \end{aligned}$$

Using (4)–(6), we have

$$\lim_{n \rightarrow \infty} \omega_{\xi}^p(l_{m(k)}, l_{n(k)}) \preceq \theta \Rightarrow b = \theta,$$

which is a contradiction. Thus, $\{l_m\}$ is a Cauchy sequence in L_{ω^s} . By Lemma 3, $\{l_m\}$ is a Cauchy sequence in complete C^* -algebra-valued partial modular metric space L_{ω^p} . Hence, there exists some $v \in L_{\omega^p}$ such that

$$\begin{aligned} \lim_{n, m \rightarrow \infty} \omega_{\xi}^p(l_m, l_n) &= \lim_{m \rightarrow \infty} \omega_{\xi}^p(l_m, v) = \omega_{\xi}^p(v, v) = \theta \text{ and} \\ \lim_{n \rightarrow \infty} \omega_{\xi}^p(Ql_{2n+1}, v) &= \lim_{n \rightarrow \infty} \omega_{\xi}^p(Pl_{2n}, v) = \theta. \end{aligned}$$

Now, letting $\omega_{\xi}^p(Pv, l_{2n+2}) \succeq \theta$, we have

$$\mathcal{F}(\omega_{\xi}^p(Pv, Ql_{2n+1})) \preceq \mathcal{F}(\mathcal{M}(v, l_{2n+1})) - \phi(\mathcal{M}(v, l_{2n+1})), \quad (11)$$

where

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{M}(v, l_{2n+1}) &= \lim_{n \rightarrow \infty} \max \left\{ \omega_{\xi}^p(v, l_{2n+1}), \omega_{\xi}^p(v, Pv), \omega_{\xi}^p(l_{2n+1}, Ql_{2n+1}), \right. \\ &\quad \left. \frac{[\omega_{2\xi}^p(v, Ql_{2n+1}) + \omega_{2\xi}^p(l_{2n+1}, Pv)]}{2} \right\} \\ &= \omega_{\xi}^p(Pv, v). \end{aligned}$$

As $n \rightarrow \infty$, from (11) we obtain

$$\mathcal{F}(\omega_{\xi}^p(Pv, v)) \preceq \mathcal{F}(\omega_{\xi}^p(Pv, v)) - \phi(\omega_{\xi}^p(Pv, v)),$$

which is a contradiction, as ϕ is positive. So, $\omega_{\xi}^p(Pv, v) = \theta$.

Similarly, we can show that $\omega_{\xi}^p(Qv, v) = \theta$. Hence, v is a common fixed point of P and Q .

As in Theorem 1, the uniqueness of a common fixed point can be shown. $\square$

We set $P = Q$ in Theorem 2 and have the following corollary.

Corollary 1. Let L_{ω^p} be a complete C_{av}^* -modular space and ω^p be a C_{av}^* - p MM. Define a self mapping $P : L_{\omega^p} \rightarrow L_{\omega^p}$ satisfying the following conditions: for all $l_1, l_2 \in L_{\omega^p}$,

$$\omega_{\xi}^p(Pl_1, Pl_2) \succeq \theta \Rightarrow \mathcal{F}(\omega_{\xi}^p(Pl_1, Pl_2)) + \phi(\mathcal{M}(l_1, l_2)) \preceq \mathcal{F}(\mathcal{M}(l_1, l_2)),$$

where, $\mathcal{M}(l_1, l_2) = \max \left\{ \omega_{\xi}^p(l_1, l_2), \omega_{\xi}^p(l_1, Pl_1), \omega_{\xi}^p(l_2, Pl_2), \frac{[\omega_{2\xi}^p(l_1, Pl_2) + \omega_{2\xi}^p(l_2, Pl_1)]}{2} \right\}$, $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}$ is a $*$ -homomorphism, and $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$ be a continuous and non-decreasing function such that $\mathcal{F}(l_1) = \theta \Leftrightarrow l_1 = \theta$. Then P has a unique fixed point in L_{ω^p} .

Theorem 3. Let L_{ω^p} be a complete C_{av}^* -modular space and ω^p be a C_{av}^* - p MM. Define a mapping $P : L_{\omega^p} \rightarrow L_{\omega^p}$ satisfying the following:

$$\mathcal{M}(Pl_1, Pl_2) \succeq \theta \Rightarrow \mathcal{F}(\mathcal{M}(Pl_1, Pl_2)) + \phi(\mathcal{M}(l_1, l_2)) \preceq \mathcal{F}(\mathcal{M}(l_1, l_2)),$$

for all $l_1, l_2 \in L$, where

$$\mathcal{M}(l_1, l_2) = \max \left\{ d(l_1, l_2), d(l_1, Pl_1), d(l_2, Pl_2), \frac{d(l_1, Pl_2) + d(l_2, Pl_1)}{2} \right\},$$

mapping $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}$ is a $*$ -homomorphism, and $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$ be a continuous and non-decreasing function such that $\mathcal{F}(l_1) = \theta$ if, and only if, $l_1 = \theta$. Then P has a unique fixed point in L_{ω^p} .

Theorem 4. Let L_{ω^p} be a complete C_{av}^* -modular space and ω^p be a C_{av}^* - p MM. Define a self mapping $P : L_{\omega^p} \rightarrow L_{\omega^p}$ satisfying

$$\omega_{\xi}^p(Pl_1, Pl_2) \preceq q^2 \omega_{2\xi}^p(l_1, l_2), \text{ for all } l_1, l_2 \in L_{\omega^p},$$

where $\|q\|^2 < 1$. Then P has a unique fixed point.

Example 6. Let $(L_{\omega^p}, \mathbb{A}, \omega^p)$ be a $C_{av}^* - pMMS$ defined as in Example 3, where $L = \mathbb{C}$, $\mathbb{A} = M_2(\mathbb{C})$ and $\omega_\xi^p(l_1, l_2) = \frac{1}{\xi} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + \text{diag}(\max\{l_1, l_2\}, \max\{l_1, l_2\})$, for all $\xi > 0$, $l_1, l_2 \in L_{\omega^p}$, and I be the identity matrix.

Let P and Q be self mappings on L_{ω^p} such that $Pl_1 = \frac{l_1}{4}$ and $Ql_1 = \frac{l_1}{2}$. Let $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$ define by $\mathcal{F}(T) = T^2$; and $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}^+$ define by $\phi(T) = \frac{T^2}{2}$.

- (i) Clearly, $P(L_{\omega^p}) \subseteq Q(L_{\omega^p})$.
- (ii) Moreover, $\omega_\xi^p(Pl_1, Ql_2) \preceq \theta$ and

$$\begin{aligned} \mathcal{F}(\omega_\xi^p(Pl_1, Ql_2)) &= [\omega_\xi^p(Pl_1, Ql_2)]^2 = \left[\omega_\xi^p\left(\frac{l_1}{4}, \frac{l_2}{2}\right) \right]^2 \\ &= \left[\frac{1}{\xi} \text{diag}\left(\left|\frac{l_1}{4} - \frac{l_2}{2}\right|, \left|\frac{l_1}{4} - \frac{l_2}{2}\right|\right) + \text{diag}\left(\max\left\{\frac{l_1}{4}, \frac{l_2}{2}\right\}, \max\left\{\frac{l_1}{4}, \frac{l_2}{2}\right\}\right) \right]^2 \\ &\preceq \frac{1}{2} \left[\frac{1}{\xi} \text{diag}(|l_1 - l_2|, |l_1 - l_2|) + \text{diag}(\max\{l_1, l_2\}, \max\{l_1, l_2\}) \right]^2 \\ &= \frac{1}{2} \mathcal{F}(\omega_\xi^p(M(l_1, l_2))) \\ &= \mathcal{F}(\omega_\xi^p(M(l_1, l_2))) - \frac{1}{2} \mathcal{F}(\omega_\xi^p(M(l_1, l_2))) \\ &= \mathcal{F}(\omega_\xi^p(M(l_1, l_2))) - \frac{(\omega_\xi^p(M(l_1, l_2)))^2}{2}. \end{aligned}$$

This implies that

$$\mathcal{F}(\omega_\xi^p(Pl_1, Ql_2)) + \phi(\omega_\xi^p(M(l_1, l_2))) \preceq \mathcal{F}(\omega_\xi^p(M(l_1, l_2))),$$

where $M(l_1, l_2) = \max\left\{\omega_\xi^p(l_1, l_2), \omega_\xi^p(l_1, Pl_1), \omega_\xi^p(l_2, Ql_2), \frac{[\omega_{2\xi}^p(l_2, Pl_1) + \omega_{2\xi}^p(l_1, Ql_2)]}{2}\right\}$. Thus, all the conditions of Theorem 2 are satisfied. So, 0 is the unique common fixed point of P and Q .

5. Ulam–Hyers Stability Results in $C_{av}^* - pMMS$

Let $(L_{\omega^p}, \mathbb{A}, \omega^p)$ be a $C_{av}^* - pMMS$ and $P : L_{\omega^p} \rightarrow L_{\omega^p}$ a mapping. The fixed point

$$Pz = z \tag{12}$$

is called generalised Ulam–Hyers stability [44] if, and only if, there exists increasing and continuous (at 0) mapping, $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lambda(0) = 0$ such that for every $\epsilon > 0$ and $z^* \in L_{\omega^p}$, an ϵ -solution of the fixed point Equation (12), i.e., z^* , satisfies the inequality

$$\omega_\xi^p(z^*, Pz^*) \preceq \epsilon. \tag{13}$$

There exists a solution $x^* \in L_{\omega^p}$ of (12) such that $\omega_\xi^p(z^*, x^*) \preceq \lambda(\epsilon)$. If there exists $k > 0$ such that $\lambda(s) = k.s$ for each $s \in \mathbb{R}^+$, then the fixed point Equation (12) is said to be an Ulam–Hyers stability.

Theorem 5. Let $(L_{\omega^p}, \mathbb{A}, \omega^p)$ be a complete $C_{av}^* - pMMS$ satisfying all the conditions of Theorem 4, with $\omega_\xi^p(Pz^*, z^*) \preceq \gamma$, $\gamma > 0$. Then Equation (12) is Ulam–Hyers stable. Moreover, the function $\pi : [0, \infty) \rightarrow [0, \infty)$ such that $\pi(r) = \alpha r$, is strictly increasing and onto.

- (a) Hence, Equation (12) is generalised Ulam–Hyers stable.
- (b) $\text{Fix}(P) = \{x^*\}$ and if $\{x_n\} \in (L_{\omega^p}, \mathbb{A}, \omega_\xi^p)$, $n \in \mathbb{N}$ are such that $\omega_\xi^p(l_n, Pl_n) \rightarrow 0$ as $n \rightarrow \infty$, then $l_n \rightarrow x^*$ as $n \rightarrow \infty$.
- (c) If $Q : L_{\omega^p} \rightarrow L_{\omega^p}$ such that $\omega_\xi^p(Pl, Ql) \preceq \eta$, for all $l \in L_{\omega^p}$, $\eta \in [0, \infty)$, then $p^* \in \text{Fix}(Q) \Rightarrow \omega_\xi^p(x^*, p^*) \preceq \pi^{-1}(\frac{\alpha}{1-q^2}\eta)$

Proof. (a) Let $\text{Fix}(P) = \{x^*\}$. Let $\epsilon > 0$ and $z^* \in L_{\omega^p}$ be a solution of Equation (13):

$$\begin{aligned}\omega_{\xi}^p(x^*, z^*) &= \omega_{\xi}^p(Px^*, z^*) \\ &\leq \omega_{\frac{\xi}{2}}^p(Px^*, Pz^*) + \omega_{\frac{\xi}{2}}^p(Pz^*, z^*) - \omega_{\frac{\xi}{2}}^p(Pz^*, Pz^*) \\ &\leq \omega_{\frac{\xi}{2}}^p(Px^*, Pz^*) + \gamma \\ &\leq q^2 \omega_{\xi}^p(x^*, z^*) + \gamma \\ \Rightarrow \omega_{\xi}^p(x^*, z^*) &\leq \frac{1}{1-q^2} \gamma, \text{ where } \frac{1}{1-q^2} > 0.\end{aligned}$$

So,

$$\begin{aligned}\pi(\omega_{\xi}^p(x^*, z^*)) &= \alpha \omega_{\xi}^p(x^*, z^*) \leq \frac{\alpha}{1-q^2} \epsilon \\ \Rightarrow \omega_{\xi}^p(x^*, z^*) &\leq \pi^{-1}\left(\frac{\alpha}{1-q^2} \epsilon\right).\end{aligned}$$

This shows that Equation (12) is generalised Ulam–Hyers stable.

(b) Let $\text{Fix}(P) = \{x^*\}$, $\epsilon > 0$ and $\{l_n\} \in L_{\omega^p}$.

$$\begin{aligned}\omega_{\xi}^p(l_n, x^*) &\leq \omega_{\frac{\xi}{2}}^p(l_n, Pl_n) + \omega_{\frac{\xi}{2}}^p(Pl_n, Px^*) - \omega_{\frac{\xi}{2}}^p(Pl_n, Pl_n) \\ &\leq q^2 \omega_{\xi}^p(l_n, x^*) + \omega_{\frac{\xi}{2}}^p(l_n, Pl_n) \\ &\leq \frac{1}{1-q^2} \omega_{\xi}^p(l_n, Pl_n) \rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

Therefore, $l_n \rightarrow x^*$ as $n \rightarrow \infty$.

(c) Let $\text{Fix}(P) = \{x^*\}$ and $p^* \in \text{Fix}(Q)$.

$$\begin{aligned}\omega_{\xi}^p(x^*, p^*) &= \omega_{\xi}^p(Px^*, p^*) \\ &\leq \omega_{\frac{\xi}{2}}^p(Px^*, Pp^*) + \omega_{\frac{\xi}{2}}^p(Pp^*, p^*) - \omega_{\frac{\xi}{2}}^p(Pp^*, Pp^*) \\ &\leq q^2 \omega_{\xi}^p(x^*, p^*) + \eta \\ \Rightarrow \omega_{\xi}^p(x^*, p^*) &\leq \frac{1}{1-q^2} \eta.\end{aligned}$$

Therefore,

$$\begin{aligned}\pi(\omega_{\xi}^p(x^*, p^*)) &= \alpha \omega_{\xi}^p(x^*, p^*) \leq \frac{\alpha}{1-q^2} \eta \\ \Rightarrow \omega_{\xi}^p(x^*, p^*) &\leq \pi^{-1}\left(\frac{\alpha}{1-q^2} \eta\right).\end{aligned}$$

□

6. Application

Consider the following integral equation (see [22,36]):

$$x(\gamma) = \int_E B(\gamma, v, x(v)) dv + h(\gamma), \gamma, v \in E, \quad (14)$$

where E is a measurable set, $B : E \times E \times \mathbb{R} \rightarrow \mathbb{R}$ and $h \in L^\infty(E)$. Let $L = L^\infty(E)$, $H = L^2(E)$ and $L(H) = \mathbb{A}$. Define $\omega^p : (0, \infty) \times X \times X \rightarrow \mathbb{A}^+$ by

$$\omega_{\xi}^p(l_1, l_2) = \beta_{\left|\frac{l_1-l_2}{\xi}\right|+I} \text{ for all } l_1, l_2, I \in L_{\omega^p},$$

where $\beta_u : H \rightarrow H$ is a multiplicative operator defined by $\beta_u(\gamma) = u\gamma$.

Theorem 6. Let $x, y \in L_{\omega^p}$, and $P : L_{\omega^p} \rightarrow L_{\omega^p}$ be two self mapping such that

- (i) There exist a continuous function $\psi : E \times E \rightarrow \mathbb{R}$ such that
 $|B(\gamma, v, x(v)) - B(\gamma, v, y(v))| \leq |\psi(\gamma, v)|(|x(v) - y(v)|)$, for all $\gamma, v \in E$;
- (ii) $\sup_{\gamma \in E} \int_E |\psi(\gamma, v)| dv \leq 1$.

Then the integral Equation (14) has a unique solution in L_{ω^p} .

Proof. Define

$$Px(\gamma) = \int_E B(\gamma, v, x(v)) dv + h(\gamma), \quad \forall \gamma, v \in E.$$

For any $u \in H$, we have

$$\begin{aligned} \|\omega_{\xi}^p(Px, Py)\| &= \sup_{\|u\|=1} (\beta_{|\frac{Px-Py}{\xi}|+I} u, u) \\ &= \sup_{\|u\|=1} \int_E \left[\frac{1}{\xi} \left| \int_E B(\gamma, v, x(v)) - B(\gamma, v, y(v)) dv \right| \right] u(\gamma) \overline{u(\gamma)} d\gamma \\ &\quad + \sup_{\|u\|=1} \int_E u(\gamma) \overline{u(\gamma)} d\gamma I_{\mathbb{A}} \\ &\leq \sup_{\|u\|=1} \int_E \left[\frac{1}{\xi} \left| \int_E |B(\gamma, v, x(v)) - B(\gamma, v, y(v))| dv \right| \right] |u(\gamma)|^2 d\gamma \\ &\quad + \sup_{\|u\|=1} \int_E |u(\gamma)|^2 d\gamma I_{\mathbb{A}} \\ &\leq \frac{1}{\xi} \sup_{\|u\|=1} \int_E \left[\int_E |\psi(\gamma, v)| (|x(v) - y(v)| + I_{\mathbb{A}} - I_{\mathbb{A}}) dv \right] |u(\gamma)|^2 d\gamma + I_{\mathbb{A}} \\ &\leq \frac{1}{\xi} \sup_{\|u\|=1} \int_E \left[\int_E |\psi(\gamma, v)| dv \right] |u(\gamma)|^2 d\gamma \|x - y\|_{\infty} \\ &\leq \frac{1}{\xi} \sup_{\gamma \in E} \int_E |\psi(\gamma, v)| dv \sup_{\|u\|=1} \int_E |u(\gamma)|^2 d\gamma \|x - y\|_{\infty} \\ &\leq \frac{1}{\xi} \|x - y\|_{\infty} \\ &= \|\omega_{\xi}^p(x, y)\|. \end{aligned}$$

This implies that $\omega_{\xi}^p(Px, Py) \preceq \omega_{\xi}^p(x, y)$. Let $\mathcal{F} : \mathbb{A}^+ \rightarrow \mathbb{A}$ define by $\mathcal{F}(T) = T^2$ and $\phi : \mathbb{A}^+ \rightarrow \mathbb{A}^+$ define by $\phi(T) = \frac{T^2}{2}$. Moreover, $\omega_{\xi}^p(Pl_1, Pl_2) \succeq \theta$ and

$$\mathcal{F}(\omega_{\xi}^p(Pl_1, Pl_2)) + \phi(\omega_{\xi}^p(M(l_1, l_2))) \preceq \mathcal{F}(\omega_{\xi}^p(M(l_1, l_2))),$$

where $M(l_1, l_2) = \max \left\{ \omega_{\xi}^p(l_1, l_2), \omega_{\xi}^p(l_1, Pl_1), \omega_{\xi}^p(l_2, Pl_2), \frac{[\omega_{2\xi}^p(l_2, Pl_1) + \omega_{2\xi}^p(l_1, Pl_2)]}{2} \right\}$. So, by Corollary 1, we have a unique fixed point of P , i.e., the integral Equation (14) has a unique solution. $\square$

7. Conclusions

We introduce the notion of $C_{av}^* - pMMS$ and prove some fixed point results using $(\phi, \mathcal{MF})$ -contraction mappings, which extend/generalize many results given in the literature. Some properties and examples related to $C_{av}^* - pMMS$ are discussed. $C_{av}^* - pMMS$ is a generalization of $pMMS$, $C_{av}^* - pMS$, and $C_{av}^* - MMS$.

Limitations: Fixed point results in $C_{av}^* - pMMS$ can be obtained from their partial metric counterparts when C^* -algebra, $\mathbb{A}$ is the real set $\mathbb{R}$. There might be a more generalized space than $C_{av}^* - pMMS$.

Future Scope: $pMMS$ and $C_{av}^* - pMMS$ play a crucial role in the study of generalized partial metric spaces. With the amalgamation of $C_{av}^* - pMMS$ and other existing spaces, several generalized spaces can be studied. Moreover, $C_{av}^* - pMMS$ paves the way for the possible generalization of existing concepts and applications in different branches of mathematical sciences.

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Abbreviations

The following abbreviations are used in this manuscript:

pMS	Partial modular metric spaces
MMS	Modular metric spaces
$pMMS$	Partial modular metric spaces
$C_{av}^* - MS$	C^* -algebra-valued metric spaces
$C_{av}^* - MMS$	C^* -algebra-valued modular metric spaces
$C_{av}^* - pMS$	C^* -algebra-valued partial metric spaces
$C_{av}^* - pMMS$	C^* -algebra-valued partial modular metric spaces

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