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# New Applications of the Sălăgean Quantum Differential Operator for New Subclasses of *q*-Starlike and *q*-Convex Functions Associated with the Cardioid Domain

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**Abstract:** In this paper, we define a new family of $q$-starlike and $q$-convex functions related to the cardioid domain utilizing the ideas of subordination and the Sălăgean quantum differential operator. The primary contribution of this article is the derivation of a sharp inequality for the newly established subclasses of $q$-starlike and $q$-convex functions in the open unit disc $\mathcal{U}$. For this novel family, bounds of the first two Taylor-Maclaurin coefficients, the Fekete-Szegö-type functional, and coefficient inequalities are studied. Furthermore, we also investigate some new results for the inverse function belonging to the classes of $q$-starlike and $q$-convex functions. The results presented in this article are sharp. To draw connections between the early and present findings, several well-known corollaries are also highlighted. Symmetric quantum calculus operator theory can be used to investigate the symmetry properties of this new family of functions.

**Keywords:** analytic functions; univalent functions; quantum-calculus; $q$-convex and $q$-starlike functions; subordination; Sălăgean $q$-differential operator shell-like curve; cardioid domain

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## 1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all analytic functions $g(z)$ in the open unit disc

$$\mathcal{U} = \{z : |z| < 1\},$$

which are normalized by

$$g(0) = 0 \text{ and } g'(0) = 1.$$

Any function $g \in \mathcal{A}$, has the following series expansion:

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A complex value function $g \in \mathcal{A}$, is univalent if

$$z_1 = z_2 \Rightarrow g(z_1) = g(z_2), \quad \text{for all } z_1, z_2 \in \mathcal{U}.$$

The symbol $\mathcal{S}$ stands for a set of functions from $\mathcal{A}$ that are univalent in the open unit disc $\mathcal{U}$.

Function theory was first proposed in 1851. When Bieberbach [1] examined the coefficient conjecture in 1916, this field first came into focus as an interesting field for future study. In 1985, De Branges [2] elucidated this concept. A number of leading scientists sought to support or disprove the Bieberbach hypothesis between 1916 and 1985. The theories of analytic and univalent functions, as well as how they estimate function growth in their stated domains, are of great importance. This includes Taylor series representation, coefficients of functions, and their associated functional inequalities. One of the most significant and practical functional inequalities is the Fekete-Szegö inequality. The Fekete-Szegö inequality [3] was identified by Fekete and Szegö in 1933. It is a mathematical inequality that is connected to the Bieberbach conjecture and concerns the coefficients of univalent analytic functions. It is known as the Fekete-Szegö problem to find comparable estimates for different types of functions. The maximization of the non-linear functional $\left|a_3 - \mu a_2^2\right|$ and other subclasses of univalent functions has been shown to produce a variety of results; this type of problem is known as a Fekete-Szegö problem.

If $g \in \mathcal{S}$, and it is of the form (1), then

$$\left|a_3 - \mu a_2^2\right| \leq \left\{ \begin{array}{ccc} 3-4\mu & \text{if} & \mu \leq 0 \\ 1+2\exp\left(\frac{2\mu}{\mu-1}\right) & \text{if} & 0 \leq \mu < 1 \\ 4\mu - 3 & \text{if} & \mu \geq 1 \end{array} \right\}$$

and the result $\left|a_3 - \mu a_2^2\right|$ is sharp (see [3]).

The subordination of two analytic functions $g_1$ and $g_2$ can be written as

$$g_1(z) \prec g_2(z), \ \ z \in \mathcal{U},$$

if there exists a Schawarz function $u$, such that $|u(z)| < 1$ and $u(0) = 0$ and

$$g_1(z) = g_2(u(z)), \ z \in \mathcal{U}.$$

Furthermore, if the function $g_2$ is univalent in $\mathcal{U}$, then

$$g_1(0) = g_2(0) \ \text{ and } \ g_1(\mathcal{U}) \subset g_2(\mathcal{U}).$$

The families of starlike and convex functions, denoted by the letters $\mathcal{S}^*$ and $\mathcal{C}$, respectively, are the most fundamental and significant subclasses of the set $\mathcal{S}$.

The familiar class of starlike ($\mathcal{S}^*$), consists of functions $g \in \mathcal{S}^*$, that satisfy the following condition

$$Re\left(\frac{zg'(z)}{g(z)}\right) > 0, \ z \in \mathcal{U}.$$

Or, in terms of subordination

$$\mathcal{S}^* = \left\{g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec \frac{1+z}{1-z}\right\}.$$

The class of convex functions ($\mathcal{C}$), consists of functions $g \in \mathcal{C}$, that satisfy the following condition

$$1 + Re\left(\frac{zg''(z)}{g'(z)}\right) > 0, \ z \in \mathcal{U}.$$

Or, in terms of subordination

$$\mathcal{C} = \left\{g \in \mathcal{A} : 1 + \frac{zg''(z)}{g'(z)} \prec \frac{1+z}{1-z}\right\}.$$

Ma and Minda [4] gave generalizations of $\mathcal{S}^*$ and $\mathcal{C}$ for analytic functions and defined new classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ in terms of subordination as follows:

$$f \in \mathcal{S}^*(\varphi) \Longleftrightarrow \frac{zg'(z)}{g(z)} \prec \varphi(z)$$

and

$$f \in \mathcal{C}(\varphi) \Longleftrightarrow 1 + \frac{zg^{''}(z)}{g^{'}(z)} \prec \varphi(z),$$

where $\varphi(z)$ have a positive real part and are normalized by

$$\varphi(0) = 1, \text{ and } \varphi^{'}(0) > 0.$$

Note that $\varphi$ maps $\mathcal{U}$ onto a region that is starlike with respect to 1 and symmetric with respect to the real axis. Many subfamilies of the class $\mathcal{A}$ of normalized analytic functions have been studied recently as a particular case of the class $\mathcal{S}^*(\varphi)$. For example, Sokól and Stankiewicz investigated the class $\mathcal{S}^*_L$ in [5], the class of starlike functions $\mathcal{S}^*(Q, R)$ associated with the Janowski function was studied in [6], and Cho et al. [7] examined the class $\mathcal{S}^*_{\sin}$. The class $\mathcal{S}^*_{\tan}$ and the class $\mathcal{S}^*(e^z)$ were investigated in [8,9], respectively. For further information on sharp coefficient estimations, please refer to [10–15].

The image of $\mathcal{U}$ under every $g \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$ and every $g \in \mathcal{S}$ has an inverse defined by:

$$g^{-1}(g(z)) = z, \;\; z \in \mathcal{U}$$

and

$$g(g^{-1}(w)) = w, \; |w| < r_0(g), \; r_0(g) \geq \frac{1}{4}.$$

The series of $g^{-1}$ is

$$g^{-1}(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 \ldots, \tag{2}$$

where

$$A_2 = -a_2, \quad A_3 = (2a_2^2 - a_3) \tag{3}$$

and

$$A_4 = -(5a_2^3 - 5a_2 a_3 + a_4).$$

Using the idea of subordination, many subclasses of analytic functions have been defined based on the geometrical interpretation of their image domains. These include the right-half plane [16], the circular disc [17], the oval- and petal-type domains [18], the conic domain [19,20], the leaf-like domain [21], and the generalized conic domains [22]; the most important is the shell-like curve [23–26]. The function

$$p(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad \tau = \frac{1 - \sqrt{5}}{2} \tag{4}$$

is necessary for forming the shell-like curve. The conchoid of Maclaurin is produced by the image of the unit circle under the function $p$; that is,

$$p\left(e^{i\varphi}\right) = \frac{\sqrt{5}}{2(3 - 2\cos\varphi)} + i\frac{\sin\varphi(4\cos\varphi - 1)}{2(3 - 2\cos\varphi)(1 + \cos\varphi)}, \;\; 0 \leq \varphi < 2\pi.$$

The series representation for the function given in (4) is as follows:

$$p(z) = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1})\tau^n z^n,$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}},$$

and $u_n$ generate a series of coefficients of Fibonacci numbers.

Malik et al. [27] created a new class $CP[Q, R]$ of analytic functions associated with $\widetilde{p}(Q, R, z)$.

**Definition 1** ([27]). *Let the function $p \in CP[Q,R]$ and*

$$p(z) \prec \widetilde{p}(Q,R,z),$$

*where $\widetilde{p}(Q,R,z)$ is given by*

$$\widetilde{p}(Q,R,z) = \frac{2Q\tau^2z^2 + (Q-1)\tau z + 2}{2R\tau^2z^2 + (R-1)\tau z + 2} \tag{5}$$

*and*

$$-1 < R < Q \leq 1,\ \tau = \frac{1-\sqrt{5}}{2}, z \in \mathcal{U}.$$

In this case, a geometric description of the function $\widetilde{p}(Q,R,z)$ could be useful in studying the class $CP[Q,R]$ in detail. If we denote

$$R_{\widetilde{p}}\left(Q,R;e^{i\theta}\right) = u$$

and

$$I_{\widetilde{p}}\left(Q,R;e^{i\theta}\right) = v$$

then, the image $\widetilde{p}(Q,R,e^{i\theta})$ of the unit circle is a cardioid-like curve defined by

$$\begin{aligned} u &= \frac{4+(Q-1)(R-1)\tau^2+4QR\tau^4+2\lambda\cos\theta+4(Q+R)\tau^2\cos 2\theta}{4+(R-1)^2\tau^2+4R^2\tau^4+4(R-1)(\tau+R\tau^3)\cos\theta+8R\tau^2\cos 2\theta}, \\ v &= \frac{(Q-R)(\tau-\tau^3)\sin\theta+2\tau^2\sin 2\theta}{4+(R-1)^2\tau^2+4R^2\tau^4+4(R-1)(\tau+R\tau^3)\cos\theta+8R\tau^2\cos 2\theta}, \end{aligned} \tag{6}$$

where

$$\lambda = (Q+R-2)\tau + (2QR-Q-R)\tau^3.$$

We note that

$$\widetilde{p}(Q,R,0) = 1 \text{ and } \widetilde{p}(Q,R,1) = \frac{QR+9(Q+R)+1+4(R-Q)\sqrt{5}}{R^2+18R+1}.$$

According to (6), the cusp of the cardioid-like curve is provided by

$$\gamma(Q,R) = \widetilde{p}\left(Q,R;e^{\pm i\arccos\left(\frac{1}{4}\right)}\right) = \frac{2QR-3(Q+R)+2+(Q-R)\sqrt{5}}{2(R^2-3R+1)}.$$

If the open unit disc $\mathcal{U}$ is considered to be a collection of concentric circles with the center at the origin, then, the image of each inner circle is a nested cardioid-like curve. Thus, using the function $\widetilde{p}(Q,R,z)$, the open unit disc $\mathcal{U}$ is mapped onto a cardioid region. As a result, the domain $\widetilde{p}(Q,R;\mathcal{U})$ is a cardioid domain.

In the area of geometric function theory (GFT), researchers have constructed and explored a number of novel subclasses of analytic, univalent, and bi-univalent functions using quantum calculus and fractional quantum calculus. In 1909, Jackson [28,29] presented the concept of the $q$-calculus operator and gave the definition of the $q$-difference operator $D_q$. Ismail et al. [30], for instance, were the first to develop a class of $q$-starlike functions in $\mathcal{U}$ by using $D_q$. The most important uses of the $q$-calculus from the perspective of GFT were essentially provided by Srivastava in [31]. They used the fundamental (or $q$-) hypergeometric functions for the first time in GFT in a book chapter (see, for details, [31]). Very recently, Attiya et al. [32] studied new applications of differential operators associated with the $q$-raina function, while Raza et al. [33] defined a class of starlike functions related to symmetric booth lemniscate and determined the sharp estimates of the functions that belong to this class.

Jackson [28] introduced the $q$-difference operator for analytic functions as follows:

**Definition 2** ([28]). *For $f \in \mathcal{A}$, the q-difference operator is defined as:*

$$D_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in \mathcal{U}. \tag{7}$$

*Using (1) in (7) and, after some simple steps, we get*

$$D_q f(z) = 1 + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

*For $n \in \mathbb{N}$ and*

$$D_q(z^n) = [n]_q z^{n-1}, \; D_q\left(\sum_{n=1}^{\infty} a_n z^n\right) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

**Definition 3** ([34]). *For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The definition of the Sălăgean q-differential operator for $g \in \mathcal{A}$ is given as:*

$$\begin{aligned} \mathcal{S}_q^0 g(z) &= g(z), \; \mathcal{S}_q^1 g(z) = zD_q g(z) = \frac{g(qz) - g(z)}{(q-1)}, \cdots, \\ \mathcal{S}_q^m g(z) &= zD_q\left(\mathcal{S}_q^{m-1} g(z)\right) = g(z) * \left(z + \sum_{n=2}^{\infty} \left([n]_q\right)^m z^n\right). \end{aligned}$$

*Using the definition of convolution, we have*

$$\mathcal{S}_q^m g(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n.$$

**Remark 1.** *Then, the series of the Sălăgean q-differential operator for $g^{-1} \in \mathcal{A}$ can be written as:*

$$\begin{aligned} \mathcal{S}_q^m\left(g^{-1}(w)\right) &= w - [2]_q^m a_2 w^2 + [3]_q^m (2a_2^2 - a_3) w^3 - \\ &\quad [4]_q^m (5a_2^3 - 5a_2 a_3 + a_4) w^4 .... \end{aligned} \tag{8}$$

To further refine our understanding of the cardioid domain, we used the approach taken in the previous study [27] to define new subclasses of $q$-starlike and $q$-convex functions.

**Definition 4.** *A function $g$ of the form (1) belongs to the class $\mathcal{S}^*(q, m, Q, R)$, if*

$$\frac{S_q^{m+1} g(z)}{g(z)} \prec \widetilde{p}(Q, R; z),$$

*where $\widetilde{p}(Q, R; z)$ is given by (5).*

Or, $g \in \mathcal{S}^*(q, m, Q, R)$, when the function $\frac{S_q^{m+1} g(z)}{g(z)}$ takes its values from the cardioid domain $\widetilde{p}(Q, R; z)$.

**Definition 5.** *A function $g$ of the form (1) belongs to the class $\mathcal{C}(q, m, Q, R)$, if*

$$\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} \prec \widetilde{p}(Q, R; z),$$

*where $\widetilde{p}(Q, R; z)$ is given by (5).*

**Remark 2.** *For $q \to 1-$ and $m = 0$ in Definition (MDPI: We removed the unnecessary bracket outside of the Definition citation number, please confirm.) 4, then, $\mathcal{S}^*(q, m, Q, R) = \mathcal{S}^*(Q, R)$, as studied by Malik et al. in [35].*

**Remark 3.** *For $Q = 1$, $R = -1$, $q \to 1-$ and $m = 0$ in Definition 4, then, $\mathcal{S}^*(q, m, Q, R) = SL$, as introduced and studied by Sokół in [26].*

**Remark 4.** *For $q \to 1-$ and $m = 0$ in Definition 5, then, $\mathcal{C}(q, m, Q, R) = \mathcal{S}^*(Q, R)$, as studied by Malik et al. in [35].*

**Remark 5.** *For $q \to 1-$ and $m = 1$ in Definition 5, then, $\mathcal{C}(q, m, Q, R) = \mathcal{C}(Q, R)$ is the class of convex functions connected with the cardioid domain.*

## 2. A Set of Lemmas

We will demonstrate our findings by utilizing the following lemmas:

**Lemma 1** ([27])**.** *Let the function $\widetilde{p}(Q, R; z)$, defined by (5), and, if $p(z) \prec \widetilde{p}(Q, R; z)$. then,*

$$Rep(z) > \alpha,$$

*where*

$$\begin{aligned} \alpha &= \frac{2(Q + R - 2)\tau + 2(2QR - Q - R)\tau^3 + 16(Q + R)\tau^2\eta}{4(R - 1)(\tau + R\tau^3) + 32R\tau^2\eta}, \\ \eta &= \frac{4 + \tau^2 - R^2\tau^2 - 4R^2\tau^4 - (1 - R\tau^2)\chi(R)}{4\tau(1 + R^2\tau^2)}, \\ \chi(R) &= \sqrt{5(2R\tau^2 - (R - 1)\tau + 2)(2R\tau^2 + (R - 1)\tau + 2)} \end{aligned}$$

*and*

$$-1 < R < Q \leq 1, \text{ and } \tau = \frac{1 - \sqrt{5}}{2}.$$

Note that the function $\widetilde{p}(Q, R; z)$ is univalent for the disc $|z| < \tau^2$.

**Lemma 2** ([27])**.** *Let the function $\widetilde{p}(Q, R; z)$, given by (5), and $\widetilde{p}(Q, R; z) = 1 + \sum_{n=1}^{\infty} \widetilde{p}_n z^n$. Then,*

$$\widetilde{p}_n = \begin{cases} (Q - R)\frac{\tau}{2} & \text{for } n = 1, \\ (Q - R)(5 - R)\frac{\tau^2}{2^2} & \text{for } n = 2, \\ \frac{1-R}{2}\tau p_{n-1} - R\tau^2 p_{n-2} & \text{for } n = 3, 4, 5, \ldots \end{cases} \tag{9}$$

*where $-1 < R < Q \leq 1$.*

**Lemma 3** ([27])**.** *Let the function $\widetilde{p}(Q, R; z)$, given by (5). Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \prec \widetilde{p}(Q, R; z)$ $= 1 + \sum_{n=1}^{\infty} \widetilde{p}_n z^n$. Then,*

$$\left|p_2 - v p_1^2\right| \leq \frac{(Q - R)|\tau|}{4} \max\{2, |\tau(v(Q - R) + R - 5)|\}, \quad v \in \mathbb{C}. \tag{10}$$

**Lemma 4** ([36])**.** *Let $p \in \mathcal{P}$, and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then,*

$$\left|c_2 - \frac{v}{2}c_1^2\right| \leq \max\{2, 2|v - 1|\} = \left\{ \begin{array}{cc} 2 & \text{if } 0 \leq v \leq 2, \\ 2|v - 1|, & \text{elsewhere} \end{array} \right\} \tag{11}$$

*and*

$$|c_n| \leq 2, \ \ \textit{for} \ \ n \geq 1. \tag{12}$$

**Lemma 5** ([37]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b_n z^n$$

*be convex in* $\mathcal{U}$ *and let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

*be analytic in* $\mathcal{U}$. *If*

$$f(z) \prec g(z),$$

*then,*

$$|a_n| < |b_1|, \quad k = 1, 2, 3....$$

In this section, we show sharp coefficient estimates for the Taylor series, sharp Fekete-Szeg problems, and coefficient inequalities for the functions belonging to newly defined subclasses of $q$-starlike and $q$-convex functions. In addition, we also consider the same type of problem for the inverse functions in this study.

## 3. Main Results

In the following theorem, we find sharp coefficient estimates for the functions belonging to $\mathcal{S}^*(q, m, Q, R)$:

**Theorem 1.** *Let* $g \in \mathcal{S}^*(q, m, Q, R)$ *be given by* (1), $-1 \leq R < Q \leq 1$. *Then,*

$$\begin{aligned} |a_2| &\leq \frac{(Q-R)|\tau|}{2\left([2]_q^{m+1} - 1\right)}, \\ |a_3| &\leq \frac{(Q-R)|\tau|^2}{4\left([3]_q^{m+1} - 1\right)} \left\{ (5-R) - \frac{(Q-R)}{\left([2]_q^{m+1} - 1\right)} \right\}. \end{aligned}$$

These inequalities are sharp.

**Proof.** Let $g \in \mathcal{S}^*(q, m, Q, R)$, and of the form (1). Then,

$$\frac{S_q^{m+1} g(z)}{g(z)} \prec \widetilde{p}(Q, R; z), \tag{13}$$

where

$$\widetilde{p}(Q, R, z) = \frac{2Q\tau^2 z^2 + (Q-1)\tau z + 2}{2R\tau^2 z^2 + (R-1)\tau z + 2}.$$

Utilizing the idea of subordination, then, we have a function $u$ with

$$u(0) = 0 \text{ and } |u(z)| < 1$$

such that

$$\frac{S_q^{m+1} g(z)}{g(z)} = \widetilde{p}(Q, R; u(z)). \tag{14}$$

Let

$$\begin{aligned} u(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + \dots}{2 + c_1 z + c_2 z^2 + \dots} \\ &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{1}{4}c_1^3\right)z^3 + \cdots . \end{aligned} \tag{15}$$

Since $\widetilde{p}(Q,R;z) = 1 + \sum_{n=1}^{\infty} \widetilde{p}_n z^n$, then,

$$\begin{aligned} &\widetilde{p}(Q,R;u(z)) \\ &= 1 + \widetilde{p}_1\left\{\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 \dots\right\} + \widetilde{p}_2\left\{\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 \dots\right\} + \dots \\ &= 1 + \frac{\widetilde{p}_1 c_1}{2}z + \left(\frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)\widetilde{p}_1 + \frac{\widetilde{p}_2 c_1^2}{4}\right)z^2 + \dots . \end{aligned} \tag{16}$$

Moreover, considering the function

$$\widetilde{p}(Q,R;z) = \frac{2Q\tau^2 z^2 + (Q-1)\tau z + 2}{2R\tau^2 z^2 + (R-1)\tau z + 2}.$$

Letting $\tau z = \beta$. Then,

$$\begin{aligned} &\widetilde{p}(Q,R,z) \\ &= \frac{2Q\beta^2 + (Q-1)\beta + 2}{2R\beta^2 + (R-1)\beta + 2} \\ &= \frac{Q\beta^2 + \frac{(Q-1)}{2}\beta + 1}{R\beta^2 + \frac{(R-1)}{2}\beta + 1} \\ &= \left(Q\beta^2 + \frac{(Q-1)}{2}\beta + 1\right)\left(R\beta^2 + \frac{(R-1)}{2}\beta + 1\right)^{-1} \\ &= \left(Q\beta^2 + \frac{(Q-1)}{2}\beta + 1\right)\left[1 + \frac{1}{2}(1-R)\beta + \left(\frac{R^2 - 6R + 1}{4}\right)\beta^2 + \dots\right] \\ &= 1 + \frac{1}{2}(Q-R)\beta + \frac{1}{4}(Q-R)(5-R)\beta^2 + \dots . \end{aligned} \tag{17}$$

Putting back $\tau z = \beta$ in (17), we have

$$\widetilde{p}(Q,R;z) = 1 + \frac{1}{2}(Q-R)\tau z + \frac{1}{4}(Q-R)(5-R)\tau^2 z^2 + \dots . \tag{18}$$

From (16), it is clear to see that

$$\begin{aligned} &\widetilde{p}(Q,R;u(z)) \\ &= 1 + \frac{1}{4}(Q-R)\tau c_1 z + \begin{pmatrix} \frac{1}{4}(Q-R)\tau\left(c_2 - \frac{1}{2}c_1^2\right) \\ + \frac{(Q-R)(5-R)\tau^2 c_1^2}{16} \end{pmatrix} z^2 + \dots . \end{aligned} \tag{19}$$

Since $g \in \mathcal{S}^*(q, m, Q, R)$, then,

$$\frac{S_q^{m+1} g(z)}{g(z)} = 1 + \left([2]_q^{m+1} - 1\right) a_2 z + \left(\left([3]_q^{m+1} - 1\right) a_3 - \left([2]_q^{m+1} - 1\right) a_2^2\right) z^2 + \ldots. \tag{20}$$

Comparing the coefficients from (19) and (20), we get

$$a_2 = \frac{(Q-R)\tau c_1}{4\left([2]_q^{m+1} - 1\right)}. \tag{21}$$

Applying the modulus, we have

$$|a_2| \leq \frac{(Q-R)|\tau|}{2\left([2]_q^{m+1} - 1\right)}.$$

Comparing the coefficients once more of (19) and (20), we have

$$\begin{aligned} &\left([3]_q^{m+1} - 1\right) a_3 \\ = \;& \frac{1}{4}(Q-R)\tau\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{(Q-R)(5-R)\tau^2}{16} c_1^2 + \left([2]_q^{m+1} - 1\right) a_2^2, \\ = \;& \frac{(Q-R)\tau c_2}{4} - \frac{(Q-R)\tau}{4}\frac{c_1^2}{2} + \frac{(Q-R)(5-R)\tau^2}{16} c_1^2 + \frac{(Q-R)^2\tau^2}{16\left([2]_q^{m+1} - 1\right)} c_1^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} a_3 &= \frac{(Q-R)\tau}{4\left([3]_q^{m+1} - 1\right)} \left(c_2 - \frac{c_1^2}{2} + \frac{(5-R)\tau}{4} c_1^2 + \frac{(Q-R)\tau}{4\left([2]_q^{m+1} - 1\right)} c_1^2\right) \\ &= \frac{(Q-R)\tau}{4\left([3]_q^{m+1} - 1\right)} \left(c_2 - \frac{1}{2}\left(1 - \frac{(5-R)\tau}{2} - \frac{(Q-R)\tau}{2\left([2]_q^{m+1} - 1\right)}\right) c_1^2\right) \\ &= \frac{(Q-R)\tau}{4\left([3]_q^{m+1} - 1\right)} \left\{c_2 - \frac{v}{2} c_1^2\right\}, \end{aligned} \tag{22}$$

where

$$v = \left(1 - \frac{(5-R)\tau}{2} - \frac{(Q-R)\tau}{2\left([2]_q^{m+1} - 1\right)}\right).$$

Which shows that $v > 2$ for relation $Q > R$. Hence, by applying Lemma 4, the desired result is attained. The result is sharp.

$$\frac{S_q^{m+1} g(z)}{g(z)} = 1 + \frac{(Q-R)\tau}{2} z + \frac{(Q-R)(5-R)\tau^2}{4} z^2 + \ldots.$$

□

Taking $q \to 1-$ and $m = 0$ in Theorem 1, we have the known corollary proven in [38].

**Corollary 1** ([38]). *Let $g \in \mathcal{S}^*(Q,R)$ be given by (1), $-1 \leq R < Q \leq 1$. Then,*

$$\begin{aligned} |a_2| &\leq \frac{(Q-R)|\tau|}{2}, \\ |a_3| &\leq \frac{(Q-R)|\tau|^2}{8}\{Q-2R+5\}. \end{aligned}$$

**Theorem 2.** *Let $g \in \mathcal{S}^*(q,m,Q,R)$, and of the form (1). Then,*

$$\begin{aligned} &\left|a_3-\mu a_2^2\right| \\ &\leq \frac{(Q-R)|\tau|}{4\left([3]_q^{m+1}-1\right)} \max\left\{2, \left|\tau\left(\begin{array}{c} -\frac{1}{[2]_q^{m+1}-1}(Q-R)+R-5 \\ +\frac{1}{[2]_q^{m+1}-1}\left(\frac{(Q-R)\left([3]_q^{m+1}-1\right)\mu}{[2]_q^{m+1}-1}\right) \end{array}\right)\right|\right\}. \end{aligned}$$

*This result is sharp.*

**Proof.** Since $g \in \mathcal{S}^*(q,m,Q,R)$, we have

$$\frac{S_q^{m+1}g(z)}{g(z)} = \widetilde{p}(Q,R;u(z)),\ z \in \mathcal{U},$$

where the Schwarz function $u$, such that $u(0)$ and $|u(z)| < 1$ in $\mathcal{U}$. Therefore,

$$z+[2]_q^{m+1}a_2z^2+[3]_q^{m+1}a_3z^3+\ldots = \left\{z+a_2z^2+a_3z^3+\ldots\right\}\left\{1+p_1z+p_2z^2+\ldots\right\}.$$

Comparing the coefficients of both sides, we get

$$a_2 = \frac{p_1}{[2]_q^{m+1}-1}, \quad \text{and} \quad \left([3]_q^{m+1}-1\right)a_3 = (p_1a_2+p_2).$$

This implies that

$$\begin{aligned} \left|a_3-\mu a_2^2\right| &= \frac{1}{[3]_q^{m+1}-1}\left|p_2-\frac{1}{[2]_q^{m+1}-1}\left(\mu\frac{[3]_q^{m+1}-1}{[2]_q^{m+1}-1}-1\right)p_1^2\right| \\ &= \frac{1}{[3]_q^{m+1}-1}\left|p_2-vp_1^2\right|, \end{aligned}$$

where

$$v = \frac{1}{[2]_q^{m+1}-1}\left(\mu\frac{[3]_q^{m+1}-1}{[2]_q^{m+1}-1}-1\right).$$

By using the Lemma 3 for $v = \frac{1}{[2]_q^{m+1}-1}\left(\mu\frac{[3]_q^{m+1}-1}{[2]_q^{m+1}-1}-1\right)$, we have the required result. The equality

$$\left|a_3-\mu a_2^2\right| = \frac{(Q-R)|\tau|^2}{4\left([3]_q^{m+1}-1\right)}\left|\begin{array}{c} -\frac{1}{[2]_q^{m+1}-1}(Q-R)+R-5 \\ +\frac{1}{[2]_q^{m+1}-1}\left(\frac{(Q-R)\left([3]_q^{m+1}-1\right)\mu}{[2]_q^{m+1}-1}\right) \end{array}\right|$$

holds for

$$g_*(z) = z+\frac{\tau}{2}(Q-R)z^2+\frac{\tau^2}{8}(Q-R)(Q-2R+5)z^3+\ldots. \tag{23}$$

Let the function $g_0 : \mathcal{U} \to \mathbb{C}$ be given as:

$$g_0(z) = z \exp \int_0^z \frac{\widetilde{p}(Q, R; t^2) - 1}{t} dt = z + \frac{\tau}{2}(Q - R)z^3 + ..., \quad (24)$$

where $\widetilde{p}(Q, R; z)$ is defined in (5). Hence, it is clear that $g_0(0) = 0$ and $g_0'(0) = 1$ and

$$\frac{S_q^{m+1} g(z)}{g(z)} = \widetilde{p}\Big(Q, R; t^2\Big).$$

This shows that $g_0 \in \mathcal{S}^*(q, m, Q, R)$. Hence, the equality $\left|a_3 - \mu a_2^2\right| = \frac{(Q-R)|\tau|}{2\left([2]_q^{m+1}-1\right)}$ holds for the function $g_0$ given in (24). □

Taking $q \to 1-$ and $m = 0$ in Theorem 1, we get the known result.

**Corollary 2** ([38])**.** *Let $g \in \mathcal{S}^*(Q, R)$, and of the form (1). Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{(Q - R)|\tau|}{8} \max\{2, |\tau(-(Q - 2R + 5) + 2(Q - R)\mu)|\}.$$

**Theorem 3.** *For a function $g \in \mathcal{A}$, defined by (1). If $g \in \mathcal{S}^*(q, m, Q, R)$, then,*

$$|a_2| \le \frac{|\widetilde{p}_1|}{[2]_q^{m+1} - 1}$$

*and*

$$|a_n| \le \frac{|\widetilde{p}_1|}{[n]_q^{m+1} - 1} \prod_{k=2}^{n-1} \left(1 + \frac{|\widetilde{p}_1|}{[k]_q^{m+1} - 1}\right), \text{ for } n \ge 3.$$

**Proof.** Let $g \in \mathcal{S}^*(q, m, Q, R)$ and suppose

$$K(z) = \frac{S_q^{m+1} g(z)}{g(z)}. \quad (25)$$

Then, by Definition 4, we have

$$K(z) \prec \widetilde{p}(Q, R; z),$$

where $\widetilde{p}(L, R; z)$ is defined by (5). Hence, applying the Lemma 5, we get

$$\left|\frac{K^{(j)}(0)}{j!}\right| = \left|c_j\right| \le |\widetilde{p}_1|, \quad j \in \mathbb{N}, \quad (26)$$

where

$$K(z) = 1 + c_1 z + c_2 z^2 + ....$$

Since $a_1 = 1$, in view of (25), we have

$$\left([n]_q^{m+1} - 1\right) a_n = \{c_{n-1} + c_{n-2} a_2 + ... + c_1 a_{n-1}\} = \sum_{i=1}^{n-1} c_i a_{n-i}. \quad (27)$$

Using (26) into (27), we get

$$\left([n]_q^{m+1} - 1\right) |a_n| \le |\widetilde{p}_1| \sum_{i=1}^{n-1} |a_{n-i}|, \ n \in \mathbb{N}.$$

For $n = 2, 3, 4$, we have

$$\begin{aligned}|a_2| &\leq \frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1},\\ |a_3| &\leq \frac{|\widetilde{p}_1|}{[3]_q^{m+1}-1}(1+|a_2|)\\ &\leq \frac{|\widetilde{p}_1|}{[3]_q^{m+1}-1}\left(1+\frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1}\right)\end{aligned}$$

and

$$\begin{aligned}|a_4| &\leq \frac{|\widetilde{p}_1|}{[4]_q^{m+1}-1}(1+|a_2|+|a_3|)\\ &\leq \frac{|\widetilde{p}_1|}{[4]_q^{m+1}-1}\left(1+\frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1}+\frac{|\widetilde{p}_1|}{[3]_q^{m+1}-1}\left(1+\frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1}\right)\right)\\ &\quad \frac{|\widetilde{p}_1|}{[4]_q^{m+1}-1}\left(1+\frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1}+\frac{|\widetilde{p}_1|}{[3]_q^{m+1}-1}+\frac{|\widetilde{p}_1|}{[3]_q^{m+1}-1}\left(\frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1}\right)\right)\\ &= \frac{|\widetilde{p}_1|}{[4]_q^{m+1}-1}\left(1+\frac{|\widetilde{p}_1|}{[3]_q^{m+1}-1}\right)\left(1+\frac{|\widetilde{p}_1|}{[2]_q^{m+1}-1}\right).\end{aligned}$$

Applying the mathematical induction, we have

$$|a_n| \leq \frac{|\widetilde{p}_1|}{[n]_q^{m+1}-1}\prod_{k=2}^{n-1}\left(1+\frac{|\widetilde{p}_1|}{[k]_q^{m+1}-1}\right), \text{ for } n \geq 3.$$

This completes the proof of Theorem 3. □

In Theorem 4, we get the new result for a class $\mathcal{C}(q, m, Q, R)$.

**Theorem 4.** *Let $g \in \mathcal{C}(q, m, Q, R)$ be given by (1), $-1 \leq R < Q \leq 1$. Then,*

$$\begin{aligned}|a_2| &\leq \frac{(Q-R)|\tau|}{2[2]_q^m\left([2]_q-1\right)},\\ |a_3| &\leq \frac{(Q-R)|\tau|^2}{4[3]_q^m\left([3]_q-1\right)}\left\{(5-R)-\frac{(Q-R)}{\left([2]_q-1\right)}\right\}.\end{aligned}$$

**Proof.** Let $g \in \mathcal{C}(q, m, Q, R)$, and of the form (1). Then,

$$\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} \prec \widetilde{p}(Q, R; z), \tag{28}$$

where

$$\widetilde{p}(Q, R, z) = \frac{2Q\tau^2 z^2 + (Q-1)\tau z + 2}{2R\tau^2 z^2 + (R-1)\tau z + 2}.$$

Using the definition of subordination, there exists a function $u$ with

$$u(0) = 0 \text{ and } |u(z)| < 1$$

such that

$$\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} = \widetilde{p}(Q, R; u(z)). \tag{29}$$

It is simple to observe from (15), (18) and (16) that

$$\begin{aligned} &\widetilde{p}(Q, R; u(z)) \\ = \; & 1 + \frac{1}{4}(Q-R)\tau c_1 z + \begin{pmatrix} \frac{1}{4}(Q-R)\tau\left(c_2 - \frac{1}{2}c_1^2\right) \\ + \frac{(Q-R)(5-R)\tau^2 c_1^2}{16} \end{pmatrix} z^2 + \ldots \end{aligned} \tag{30}$$

and

$$\begin{aligned} &\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} \\ = \; & 1 + [2]_q^m\left([2]_q - 1\right)a_2 z + \left([3]_q^m\left([3]_q - 1\right)a_3 - [2]_q^m[2]_q^m\left([2]_q - 1\right)a_2^2\right)z^2 + \ldots. \end{aligned} \tag{31}$$

It is simple to show that, by using (29) and comparing the coefficients from (30) and (31), we get

$$a_2 = \frac{(Q-R)\tau c_1}{4[2]_q^m\left([2]_q - 1\right)}.$$

Taking the mod on both sides, we have

$$|a_2| \leq \frac{(Q-R)|\tau|}{2[2]_q^m\left([2]_q - 1\right)}. \tag{32}$$

Now, again comparing the coefficients from (30) and (31), we have

$$\begin{aligned} &[3]_q^m\left([3]_q - 1\right)a_3 \\ = \; & \frac{1}{4}(Q-R)\tau\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{(Q-R)(5-R)\tau^2}{16}c_1^2 + [2]_q^m[2]_q^m\left([2]_q - 1\right)a_2^2 \\ = \; & \frac{(Q-R)\tau c_2}{4} - \frac{(Q-R)\tau}{4}\frac{c_1^2}{2} + \frac{(Q-R)(5-R)\tau^2}{16}c_1^2 + \frac{(Q-R)^2\tau^2}{16\left([2]_q - 1\right)}c_1^2 \\ = \; & \frac{(Q-R)\tau}{4}\left(c_2 - \frac{c_1^2}{2} + \frac{(5-R)\tau}{4}c_1^2 + \frac{(Q-R)\tau}{4\left([2]_q - 1\right)}c_1^2\right) \end{aligned}$$

and

$$\begin{aligned} a_3 &= \frac{(Q-R)\tau}{4[3]_q^m\left([3]_q - 1\right)}\left(c_2 - \frac{1}{2}\left(1 - \frac{(5-R)\tau}{2} - \frac{(Q-R)\tau}{2\left([2]_q - 1\right)}\right)c_1^2\right) \\ &= \frac{(Q-R)\tau}{4[3]_q^m\left([3]_q - 1\right)}\left\{c_2 - \frac{v}{2}c_1^2\right\}, \end{aligned} \tag{33}$$

where

$$v = \left(1 - \frac{(5-R)\tau}{2} - \frac{(Q-R)\tau}{2\left([2]_q - 1\right)}\right).$$

which shows that $v > 2$. Hence, by using the Lemma 4, we get the required result. The sharpness can be calculated by using

$$\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} = 1 + \frac{(Q-R)\tau}{2}z + \frac{(Q-R)(5-R)\tau^2}{4}z^2 + \ldots.$$

□

**Theorem 5.** *Let $g \in \mathcal{C}(q,m,Q,R)$, and of the form (1). Then,*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ \leq \quad & \frac{(Q-R)|\tau|}{4[3]_q^m\left([3]_q - 1\right)} \max\left\{2, \left|\tau\left(\begin{array}{c} -\frac{1}{[2]_q-1}(Q-R) + (R-5) \\ \\ +\frac{1}{[2]_q-1}\left(\frac{(Q-R)[3]_q^m\left([3]_q-1\right)\mu}{\left([2]_q^m\right)^2\left([2]_q-1\right)}\right) \end{array}\right)\right|\right\}. \end{aligned}$$

*This result is sharp.*

**Proof.** Since $g \in \mathcal{C}(q,m,Q,R)$, so

$$\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} = \widetilde{p}(Q,R;u(z)),\ z \in \mathcal{U},$$

Therefore,

$$\begin{aligned} & z + [2]_q^m[2]_q a_2 z^2 + [3]_q^m[3]_q a_3 z^3 + \ldots \\ = \quad & \left\{z + [2]_q^m a_2 z^2 + [3]_q^m a_3 z^3 + \ldots\right\}\left\{1 + p_1 z + p_2 z^2 + \ldots\right\}. \end{aligned}$$

Comparing the coefficients of both sides, we get

$$a_2 = \frac{p_1}{[2]_q^m\left([2]_q - 1\right)}, \quad \text{and} \quad [3]_q^m\left([3]_q - 1\right)a_3 = \left([2]_q^m p_1 a_2 + p_2\right).$$

This implies that

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &= \frac{1}{[3]_q^m\left([3]_q - 1\right)}\left|p_2 - \frac{1}{[2]_q - 1}\left(\mu\frac{[3]_q^m\left([3]_q - 1\right)}{[2]_q^m[2]_q^m\left([2]_q - 1\right)} - 1\right)p_1^2\right| \\ &= \frac{1}{[3]_q^m\left([3]_q - 1\right)}\left|p_2 - v p_1^2\right|, \end{aligned}$$

where

$$v = \frac{1}{[2]_q - 1}\left(\mu\frac{[3]_q^m\left([3]_q - 1\right)}{\left([2]_q^m\right)^2\left([2]_q - 1\right)} - 1\right).$$

By using Lemma 3 for $v = \frac{1}{[2]_q - 1}\left(\mu \frac{[3]_q^m \left([3]_q - 1\right)}{\left([2]_q^m\right)^2 \left([2]_q - 1\right)} - 1\right)$, we have the required result. The equality

$$\left|a_3 - \mu a_2^2\right| = \frac{(Q-R)|\tau|^2}{4[3]_q^m\left([3]_q - 1\right)}\left|\begin{array}{c} -\frac{1}{[2]_q - 1}(Q-R) + (R-5) \\ -\frac{1}{[2]_q - 1}\left(\frac{(Q-R)[3]_q^m\left([3]_q - 1\right)\mu}{\left([2]_q^m\right)^2\left([2]_q - 1\right)}\right) \end{array}\right|$$

holds for

$$g_*(z) = z + \frac{\tau}{2}(Q-R)z^2 + \frac{\tau^2}{8}(Q-R)(Q-2R+5)z^3 + \ldots.$$

Now, consider the function $g_0 : \mathcal{U} \to \mathbb{C}$ be defined as:

$$g_0(z) = z \exp \int_0^z \frac{\widetilde{p}(Q,R;t^2) - 1}{t} dt = z + \frac{\tau}{2}(Q-R)z^3 + \ldots,$$

where $\widetilde{p}(Q,R;z)$ is defined in (5). Hence, it is clear that $g_0(0) = 0$ and $g_0'(0) = 1$ and

$$\frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} = \widetilde{p}\left(Q,R;t^2\right).$$

This demonstrates $g_0 \in \mathcal{C}(q,m,Q,R)$. Hence, the equality

$$\left|a_3 - \mu a_2^2\right| = \frac{(Q-R)|\tau|}{2[2]_q^m\left([2]_q - 1\right)}$$

holds for the function $g_0$ given in (24). □

**Theorem 6.** *For function $g \in \mathcal{A}$, given by (1). If $g \in \mathcal{C}(q,m,Q,R)$, then,*

$$|a_2| \leq \frac{|\widetilde{p}_1|}{[2]_q^m\left([2]_q^m - 1\right)}$$

*and*

$$|a_n| \leq \frac{|\widetilde{p}_1|}{[n]_q^m\left([n]_q - 1\right)} \prod_{k=2}^{n-1}\left(1 + \frac{|\widetilde{p}_1|}{\left([k]_q - 1\right)}\right), \quad \textit{for } n \geq 3.$$

**Proof.** Suppose $g \in \mathcal{C}(q,m,Q,R)$ and let

$$K(z) = \frac{zD_q\left(S_q^m g(z)\right)}{S_q^m g(z)} \tag{34}$$

Then, by Definition 4, we have

$$K(z) \prec \widetilde{p}(L,R;z).$$

Using the Lemma 5, we get

$$\left|\frac{K^{(j)}(0)}{j!}\right| = |c_j| \leq |\widetilde{p}_1|, \quad m \in \mathbb{N}, \tag{35}$$

where

$$K(z) = 1 + c_1 z + c_2 z^2 + \ldots.$$

Since $a_1 = 1$, in view of (34), we get

$$\begin{aligned} [n]_q^m \left([n]_q - 1\right) a_n &= \left\{c_{n-1} + [2]_q^m c_{n-2} a_2 + ... + [n-1]_q^m c_1 a_{n-1}\right\} \\ &= \sum_{i=1}^{n-1} [n-i]_q^m c_i a_{n-i}. \end{aligned} \tag{36}$$

Applying (35) into (36), we get

$$[n]_q^m \left([n]_q - 1\right) |a_n| \le |\widetilde{p}_1| \sum_{i=1}^{n-1} [n-i]_q^m |a_{n-i}|, \ n \in \mathbb{N}.$$

For $n = 2, 3, 4$, then

$$\begin{aligned} |a_2| &\le \frac{|\widetilde{p}_1|}{[2]_q^m \left([2]_q - 1\right)}, \\ |a_3| &\le \frac{|\widetilde{p}_1|}{[3]_q^m \left([3]_q - 1\right)} (1 + |a_2|) \\ &\le \frac{|\widetilde{p}_1|}{[3]_q^m \left([3]_q - 1\right)} \left(1 + \frac{|\widetilde{p}_1|}{\left([2]_q - 1\right)}\right) \end{aligned}$$

and

$$\begin{aligned} |a_4| &\le \frac{|\widetilde{p}_1|}{[4]_q^m \left([4]_q - 1\right)} (1 + |a_2| + |a_3|) \\ &\le \frac{|\widetilde{p}_1|}{[4]_q^m \left([4]_q - 1\right)} \left(1 + \frac{|\widetilde{p}_1|}{\left([2]_q - 1\right)} + \frac{|\widetilde{p}_1|}{\left([3]_q - 1\right)} \left(1 + \frac{|\widetilde{p}_1|}{\left([2]_q - 1\right)}\right)\right) \\ &= \frac{|\widetilde{p}_1|}{[4]_q^m \left([4]_q - 1\right)} \left(1 + \frac{|\widetilde{p}_1|}{\left([3]_q - 1\right)}\right) \left(1 + \frac{|\widetilde{p}_1|}{\left([2]_q - 1\right)}\right). \end{aligned}$$

Hence, by mathematical induction, we arrive at

$$|a_n| \le \frac{|\widetilde{p}_1|}{[n]_q^m \left([n]_q - 1\right)} \prod_{k=2}^{n-1} \left(1 + \frac{|\widetilde{p}_1|}{\left([k]_q - 1\right)}\right), \text{ for } n \ge 3.$$

This completes the proof of Theorem 3. □

*Inverse Coefficients*

**Theorem 7.** *Let $g \in \mathcal{S}^*(q, m, Q, R)$ be given by (1), and $g^{-1}$ have coefficients of the form (2) $-1 \le R < Q \le 1$. Then,*

$$|A_2| \le \frac{(Q-R)|\tau|}{2\left([2]_q^m - 1\right)} \tag{37}$$

*and*

$$|A_3| \le \frac{(Q-R)|\tau|}{4\left([3]_q^m - 1\right)} \max\left\{2, \left|\tau\left((5-R) + \frac{(Q-R)}{[2]_q^m - 1} - \frac{2(Q-R)\left([3]_q^m - 1\right)}{\left([2]_q^m - 1\right)^2}\right)\right|\right\}. \tag{38}$$

The result is sharp.

**Proof.** Let $g \in \mathcal{S}^*(q, m, Q, R)$, then, using (21) and (22), we have

$$a_2 = \frac{(Q-R)\tau c_1}{4\left([2]_q^m - 1\right)} \tag{39}$$

and

$$a_3 = \frac{(Q-R)\tau}{4\left([3]_q^m - 1\right)}\left(c_2 - \frac{c_1^2}{2} + \frac{(5-R)\tau}{4}c_1^2 + \frac{(Q-R)\tau}{4\left([2]_q^m - 1\right)}c_1^2\right). \tag{40}$$

Since $g(g^{-1})(w) = w$, and from (2), we have

$$A_2 = -a_2. \tag{41}$$

By solving (39) and (41), we have

$$|A_2| \leq \frac{(Q-R)|\tau|}{2\left([2]_q^m - 1\right)}$$

and, from (3), we have

$$A_3 = 2a_2^2 - a_3. \tag{42}$$

Putting (39) and (40) in (42), we get

$$|A_3| = \frac{(Q-R)|\tau|}{4\left([3]_q^m - 1\right)}\left|c_2 - \frac{1}{2}Vc_1^2\right|,$$

where

$$V = \left(1 - \frac{\tau}{2}\left((5-R) + \frac{(Q-R)}{\left([2]_q^m - 1\right)} - \frac{2(Q-R)\left([3]_q^m - 1\right)}{\left([2]_q^m - 1\right)^2}\right)\right).$$

Hence, by applications of the Lemma 4, we have

$$|A_3| \leq \frac{(Q-R)|\tau|}{4\left([3]_q^m - 1\right)}\max\left\{2, \left|\tau\left((5-R) + \frac{(Q-R)}{[2]_q^m - 1} - \frac{2(Q-R)\left([3]_q^m - 1\right)}{\left([2]_q^m - 1\right)^2}\right)\right|\right\}.$$

Hence, the required result is proved.
The equality holds for the functions given in (23) and (24). □

Taking $q \to 1-$ and $m = 1$ in Theorem 7, we get the known corollary proved in [38].

**Corollary 3** ([38])**.** *Let $g \in \mathcal{S}^*(Q, R)$ be given by (1), and $g^{-1}$ have coefficients of the form (2), $-1 \leq R < Q \leq 1$. Then,*

$$|A_2| \leq \frac{(Q-R)|\tau|}{2}$$

*and*

$$|A_3| \leq \frac{(Q-R)|\tau|}{8}\max\{2, \tau|3Q - 2R - 5|\}.$$

**Theorem 8.** *Let $g \in \mathcal{S}^*(q,m,Q,R)$, and of the form (1), and $g^{-1}$ have coefficients of the form (2) $-1 \le R < Q \le 1$. Then,*

$$\begin{aligned} &\left|A_3 - \mu A_2^2\right| \\ &\le \frac{(Q-R)|\tau|}{4\left([3]_q^m - 1\right)} \max\left\{2, \left|\tau\left(\begin{array}{c} \frac{1}{[2]_q^m - 1}\left(\frac{(2-\mu)\left([3]_q^m-1\right)}{\left([2]_q^m-1\right)^2}(Q-R)\right) \\ -\frac{1}{[2]_q^m-1}(Q-R) + R - 5 \end{array}\right)\right|\right\}. \end{aligned}$$

The result is sharp.

**Proof.** Since

$$A_2 = -a_2 \tag{43}$$

and

$$A_3 = 2a_2^2 - a_3. \tag{44}$$

Therefore, by using $a_2 = \frac{p_1}{[2]_q^m - 1}$ and $a_3 = \frac{1}{[3]_q^m - 1}(p_1 a_2 + p_2)$, we can write

$$\begin{aligned} \left|A_3 - \mu A_2^2\right| &= \left|2a_2^2 - a_3 - \mu a_2^2\right| \\ &= \left|a_3 - (2-\mu)a_2^2\right| \\ &= \frac{1}{[3]_q^m - 1}\left|p_2 - \frac{1}{[2]_q^m - 1}\left(\frac{(2-\mu)\left([3]_q^m - 1\right)}{[2]_q^m - 1} - 1\right)p_1^2\right| \\ &= \frac{1}{[3]_q^m - 1}\left|p_2 - v p_1^2\right|, \end{aligned}$$

where

$$v = \frac{1}{[2]_q^m - 1}\left(\frac{(2-\mu)\left([3]_q^m - 1\right)}{[2]_q^m - 1} - 1\right).$$

Hence, by application of Lemma 3, we obtain the required result

$$\left|A_3 - \mu A_2^2\right| \le \frac{(Q-R)|\tau|}{4\left([3]_q^m - 1\right)} \max\left\{2, \left|\tau\left(\begin{array}{c} \frac{1}{[2]_q^m - 1}\left(\frac{(2-\mu)\left([3]_q^m-1\right)}{[2]_q^m-1}(Q-R)\right) \\ -\frac{1}{[2]_q^m-1}(Q-R) + R - 5 \end{array}\right)\right|\right\}.$$

The equality holds for the functions given in (23) and (24). □

Taking $m = 1$ and $q \to 1-$ in Theorem 8, we get the known result.

**Corollary 4** ([38])**.** *Let $g \in \mathcal{S}^*(Q,R)$, and of the form (1), and $g^{-1}$ of the form (2) $-1 \le R < Q \le 1$. Then, $\mu \in \mathbb{C}$ and $|z| < \tau^2$.*

$$\begin{aligned} &\left|A_3 - \mu A_2^2\right| \\ &\le \frac{(Q-R)|\tau|}{8} \max\{2, |\tau(3Q - 2R - 5 - 2\mu(Q-R))|\}. \end{aligned}$$

**Theorem 9.** *Let $g \in \mathcal{C}(q, m, Q, R)$ be defined in (1), and $g^{-1}$ of the form (2) $-1 \leq R < Q \leq 1$. Then,*

$$|A_2| \leq \frac{(Q-R)|\tau|}{2\left([2]_q^m - 1\right)}$$

*and*

$$|A_3| \leq \frac{(Q-R)|\tau|}{4[3]_q^m\left([3]_q - 1\right)} \max\left\{2, \tau\left|\left(\begin{array}{c}(5-R) + \frac{(Q-R)}{[2]_q - 1} - \\ \frac{2(Q-R)[3]_q^m\left([3]_q - 1\right)}{[2]_q^{2m}\left([2]_q - 1\right)^2}\end{array}\right)\right|\right\}$$

The result is sharp.

**Proof.** Let $g \in \mathcal{C}(q, m, Q, R)$, then, using (32) and (33), we have

$$a_2 = \frac{(Q-R)\tau c_1}{4[2]_q^m\left([2]_q - 1\right)} \tag{45}$$

and

$$a_3 = \frac{(Q-R)\tau}{4[3]_q^m\left([3]_q - 1\right)}\left(c_2 - \frac{1}{2}\left(1 - \frac{(5-R)\tau}{2} - \frac{(Q-R)\tau}{2\left([2]_q - 1\right)}\right)c_1^2\right). \tag{46}$$

Since $g(g^{-1})(w) = w$, and from (2),

$$A_2 = -a_2. \tag{47}$$

By solving (45) and (47), we have

$$|A_2| \leq \frac{(Q-R)|\tau|}{2[2]_q^m\left([2]_q - 1\right)}.$$

From (3), we have

$$A_3 = 2a_2^2 - a_3. \tag{48}$$

Putting (45) and (46) in (48), we get

$$|A_3| = \frac{(Q-R)\tau}{4[3]_q^m\left([3]_q - 1\right)}\left|c_2 - \frac{1}{2}V_0 c_1^2\right|,$$

where

$$V_0 = \left(1 - \frac{\tau}{2}\left((5-R) + \frac{(Q-R)}{\left([2]_q - 1\right)} - \frac{2(Q-R)[3]_q^m\left([3]_q - 1\right)}{[2]_q^{2m}\left([2]_q - 1\right)^2}\right)\right).$$

Hence, by using the Lemma 4, we get

$$|A_3| \leq \frac{(Q-R)|\tau|}{4[3]_q^m\left([3]_q - 1\right)} \max\left\{2, \tau\left|\left(\begin{array}{c}(5-R) + \frac{(Q-R)}{[2]_q - 1} \\ -\frac{2(Q-R)[3]_q^m\left([3]_q - 1\right)}{[2]_q^{2m}\left([2]_q - 1\right)^2}\end{array}\right)\right|\right\}.$$

Hence, the required result is proved.
The equality holds for the functions given in (23) and (24). □

**Theorem 10.** *Let $g \in \mathcal{C}(q,m,Q,R)$ be of the form (1) and $g^{-1}$ from (2) $-1 \leq R < Q \leq 1$. Then, $\mu \in \mathbb{C}$ and $|z| < \tau^2$.*

$$\left|A_3 - \mu A_2^2\right| \leq \frac{(Q-R)|\tau|}{4[3]_q^m\left([3]_q - 1\right)} \max\left\{2, \left|\tau\left(\begin{array}{c}\frac{1}{[2]_q-1}\left(\frac{(2-\mu)[3]_q^m\left([3]_q-1\right)}{[2]_q^{2m}\left([2]_q-1\right)}\right) \\ -\frac{1}{[2]_q-1}(Q-R)+R-5\end{array}\right)\right|\right\}.$$

The result is sharp.

**Proof.** Since

$$A_2 = -a_2, \quad A_3 = 2a_2^2 - a_3$$

and

$$a_2 = \frac{p_1}{[2]_q^m\left([2]_q - 1\right)}, \text{ and } [3]_q^m\left([3]_q - 1\right)a_3 = \left([2]_q^m p_1 a_2 + p_2\right).$$

Therefore, by using $a_2 = \frac{p_1}{[2]_q^m\left([2]_q-1\right)}$ and $a_3 = \frac{1}{[3]_q^m\left([3]_q-1\right)}\left([2]_q^m p_1 a_2 + p_2\right)$, we can write

$$\begin{aligned}\left|A_3 - \mu A_2^2\right| &= \left|a_3 - (2-\mu)a_2^2\right| \\ &= \frac{1}{[3]_q^m\left([3]_q - 1\right)}\left|p_2 - \frac{1}{[2]_q - 1}\left(\frac{(2-\mu)[3]_q^m\left([3]_q - 1\right)}{[2]_q^{2m}\left([2]_q - 1\right)} - 1\right)p_1^2\right| \\ &= \frac{1}{[3]_q^m\left([3]_q - 1\right)}\left|p_2 - vp_1^2\right|,\end{aligned}$$

where

$$v = \frac{1}{[2]_q - 1}\left(\frac{(2-\mu)[3]_q^m\left([3]_q - 1\right)}{[2]_q^{2m}\left([2]_q - 1\right)} - 1\right).$$

Hence, by applications of the Lemma 3, we obtain the required result:

$$\left|A_3 - \mu A_2^2\right| \leq \frac{(Q-R)|\tau|}{4[3]_q^m\left([3]_q - 1\right)} \max\left\{2, \left|\tau\left(\begin{array}{c}\frac{1}{[2]_q-1}\left(\frac{(2-\mu)[3]_q^m\left([3]_q-1\right)}{[2]_q^{2m}\left([2]_q-1\right)}\right) \\ -\frac{1}{[2]_q-1}(Q-R)+R-5\end{array}\right)\right|\right\}.$$

The equality holds for the functions given in (23) and (24). □

## 4. Conclusions

In this article, two new subclasses of $q$-starlike and $q$-convex functions are defined by the use of the Sălăgean $q$-differential operator and the definition of subordination. This article is organized in three sections. In Section 1, a brief introduction and definitions are discussed, and, in Section 2, some known lemmas are presented. The first two Taylor-Maclaurin coefficients, coefficient inequalities and estimates for the Fekete-Szegö-type functional are only some of the fascinating problems we examine in Section 3 for functions belonging to the subclasses of $q$-starlike and $q$-convex functions. All of the bounds that we looked at in this article have been shown to be sharp. The inverse functions were also examined for the same kind of results. Our study also highlighted some of the primary effects that are already known to exist. We anticipate that study of this article will motivate researchers to extend this idea for meromorphic functions and the class of bi-univalent functions.

Additionally, we would like to point out that the concept conveyed in this article can be expanded using symmetric $q$-calculus, which can be used to replace the original article's use of the Sălăgean quantum differential operator with the symmetric $q$-derivative operator [39] and the Sălăgean quantum differential operator [40]. Using this operator, new subclasses of starlike and convex functions connected to the cardioid domain can then be defined, and the results of this article can be examined by connecting to a symmetric $q$-calculus. Furthermore, based on particular probability distributions with particular functions, new classes can be defined and investigated.

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