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A Vulnerability Measure of k-Uniform Linear Hypergraphs

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Abstract: Vulnerability refers to the ability of a network to continue functioning when part of the network is either naturally damaged or targeted for attack. In this paper, the rupture degree of graphs is employed to measure the vulnerability of uniform linear hypergraphs. First, we discuss the bounds of the rupture degrees of k-uniform linear hypergraphs. Then, we give a recursive algorithm for computing the rupture degree of k-uniform hypertrees.

Keywords: rupture degree; k-uniform linear hypergraph; k-uniform hypertree; recursive algorithm

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1. Introduction

Many complex structures resembling a network in real life are modeled as hypergraphs, which is a generalization of graphs, see [1–3].

Symmetry is a significant feature in hypergraph theory, especially in uniform hypergraph theory. Recent research on the open support of hypergraphs [4], the symmetric Lagrangian function of linear three-uniform hypergraphs [5], the embeddability of hypertrees and unicyclic hypergraphs [6], and the Laplacian energy of r-uniform hypergraphs [7] have been extensively studied. These results of hypergraphs are deeply dependent on the symmetric structure.

For a hypergraph $H = (V, E)$, $V = \{v_1, v_2, \cdots, v_n\}$ is a set of elements called vertices, and $E = \{e_1, e_2, \cdots, e_m\}$ is a set of nonempty subsets of V called edges. A hypergraph with only one vertex is called a trivial hypergraph. If $|e_i| = k$ for $i = 1, 2, \cdots, m$, then H is called a k-uniform hypergraph. Clearly, ordinary graphs are referred to as two-uniform hypergraphs. A vertex $v_i \in V$ is said to be incident to an edge $e_j \in E$ if $v_i \in e_j$. Two vertices are said to be adjacent if they are contained in one edge, and two edges are said to be adjacent if their intersection is not empty. For a vertex $v_i \in V$, its degree $d(v_i)$ is defined as $d(v_i) = |\{e_j : v_i \in e_j \in E\}|$. An edge $e_j \in E$ is called a pendant edge of a k-uniform hypergraph if e_j contains exactly $k-1$ vertices of degree one. Otherwise, it is called a nonpendant edge. A hypergraph $H' = (V', E')$ is called a sub-hypergraph of $H = (V, E)$, denoted as $H' \subseteq H$, if $V' \subseteq V$ and $E' \subseteq E$. For $X \subseteq V$, we use $H[X]$ to denote the hypergraph induced by X, where $V(H[X]) = X$ and $E(H[X]) = \{e \in E(H) : e \subseteq X\}$. $H - X$ is the hypergraph induced by $V(H) \setminus X$. For $A \subseteq E$, we use $H - A$ to denote the sub-hypergraph of H which is obtained by deleting all edges in A and keeping vertices. Given a hypergraph H, a walk W in H is a finite alternating sequence $v_1 e_1 v_2 \cdots e_q v_{q+1}$ of vertices and edges of H such that: $v_i \in V(H)$ for $i = 1, 2, \cdots, q+1$; $e_i \in E(H)$ for $i = 1, 2, \cdots, q$; and $v_i, v_{i+1} \in e_i$ for $i = 1, 2, \cdots, q$. If $q > 1$ and $v_{q+1} = v_1$, then W is called a circuit. A walk of H is called a path if no vertices and edges are repeated. A circuit W is called a cycle if no vertices and edges are repeated except $v_1 = v_{q+1}$. A hypergraph H is connected if for any pair of vertices $u, v \in V(H)$, there is a path connecting u and v, otherwise H is disconnected. A component of hypergraph H is a maximal connected sub-hypergraph of H. A subset $X \subseteq V(H)$ is called a cut set of H if $H - X$ is disconnected. A hypergraph H is a hypertree if H is connected and acyclic. A k-uniform hypergraph H is call a k-uniform linear hypergraph if every two edges have at most one common vertex in H. Clearly, a k-uniform hypertree is a k-uniform linear hypergraph. Let $G = (V, E)$ be an ordinary graph and $k(\geq 3)$ be an integer. For any edge $e = \{v_{e,1}, v_{e,k}\} \in E(G)$, by adding $k-2$ new vertices $v_{e,2}, v_{e,3}, \cdots, v_{e,k-1}$ into edge e, we obtain a hypergraph in which each edge possesses k vertices, which is called a k-uniform hypergraph that underlies graph G (k-power of graph G) and denoted by $G^k = (V^k, E^k)$. Clearly, $V^k = V \cup (\cup_{e \in E}\{v_{e,2}, v_{e,3}, \cdots, v_{e,k-1}\})$ and edge set $E^k = \{e \cup \{v_{e,2}, v_{e,3}, \cdots, v_{e,k-1}\} : e \in E\}$. Clearly, G^k is a k-uniform linear hypergraph.

In real life, we are becoming more and more dependent on networks, which makes it easy for us to fall into a crisis caused by a vulnerability of the network. For example, assume we have ten electronic components as vertices. If several electronic components work together to form an electronic module, then this electronic module is considered as a hyperedge with these electronic components as vertices. Based on their structural relationships, an integrated module network has been formed. The integrated module network can be seen as a hypergraph $H = (V, E)$ with vertex set V and hyperedge set E. Let $V = \{v_1, v_2, \cdots, v_{10}\}$ be a set of electronic components, $E = \{e_1, e_2, \cdots, e_5\}$ be a set of electronic modules; see Figure 1. Obviously, if one of electronic components $v_i \in V$ is damaged, the electronic modules which contain $v_i \in V$ would be damaged and thus affect the overall function of the whole integrated module network H. In addition, we also find that if there are different electronic components or different numbers of electronic components damaged, the damage to network H is different. For example, if only one electronic component (vertex) v_1 is damaged, it will cause damage to electronic module (hyperedge) e_1, but the other electronic modules e_i $(i = 2, 3, 4, 5)$ remain normal. If electronic component (vertex) v_3 is damaged, then electronic modules (hyperedges) e_1 and e_2 are damaged and electronic modules e_j $(j = 3, 4, 5)$ are normal. In order to make all the electronic modules e_i $(i = 1, 2, \cdots, 5)$ damaged, it is easiest to understand that all the electronic components v_j $(j = 1, 2, \cdots, 10)$ are damaged. In fact, for network H, as long as the electronic components v_1, v_7, and v_{10} are damaged, then all the electronic modules e_i $(i = 1, 2 \cdots, 5)$ are damaged. That is, the whole network H is completely paralyzed in this case. Therefore, how and at which cost can one restructure the network such that it becomes more robust against malicious attacks? For this purpose, many network vulnerability parameters have been proposed and studied, such as toughness [8], integrity [9], tenacity [10–12], scattering number [13], rupture degree, etc. These parameters are composed of some or all of the following three quantities: the number of elements that are not functioning, the number of remaining connected subnetworks, and the size of the largest remaining group within which mutual communication can still occur. Indeed, the above quantities play a key role in the robustness of networks. Although a large number of significant research works have been carried out on these parameters, all of them focus on the networks modeled by ordinary graphs, and there is still a gap in the research on the vulnerability of hypernetworks.

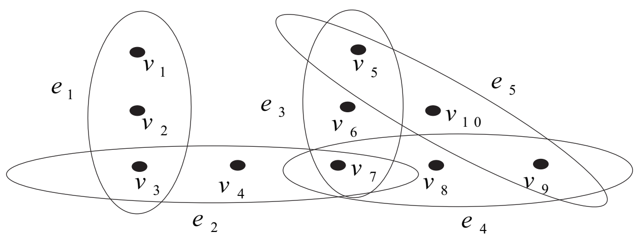

Figure 1. The integrated module network H.

The concept of the rupture degree of a graph was introduced in [14], which has been well used to measure the vulnerability of networks (see [15–19]). In this paper, we employ this parameter to measure the vulnerability of hypergraph.

Let $H = (V, E)$ be a connected hypergraph, the rupture degree of hypergraph H is defined as

$$r(H) = \max\{\omega(H - X) - |X| - \tau(H - X) : X \subset V(H), \omega(H - X) > 1\},$$

where $X \subset V(H)$ is a cut set of H, and $\omega(H - X)$ and $\tau(H - X)$ denote the number of components and the order of the largest component in $H - X$, respectively. The score of X is defined as $sc(X) = \omega(H - X) - |X| - \tau(H - X)$. A set $X \subset V(H)$ is called an r-set of H if $sc(X) = r(H)$.

By the definition of the rupture degree, we focus on a graph G and its four-uniform hypergraph G^4, which is obtained by adding two vertices to each edge of G. A example is shown in Figure 2. Obviously, $\{v\}$ is an r-set of G, but not an r-set of G^4, which has an r-set that is $\{u_1, u_2, u_3, u_4\}$. This means that it is interesting and meaningful to discuss the vulnerability by determining the rupture degree of a hypergraph.

In this paper, we first give the bounds of the rupture degree of k-uniform linear hypergraph G^k that underlies graph G in Section 2. In Section 3, we discuss the problem for computing the rupture degree of k-uniform hypertree H. In Section 4, we propose a recursive algorithm for computing the r-set of k-uniform hypertree H.

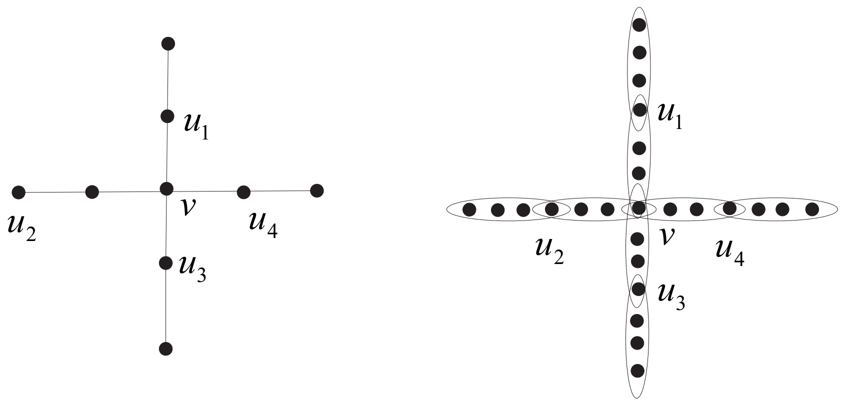

Figure 2. Graph G and 4-uniform hypergraph G^4 that underlies graph G.

Throughout this paper, by $\lfloor x \rfloor$ we denote the largest integer not larger than x and by $\lceil x \rceil$, the smallest integer not smaller than x. Any undefined terminology and notations can be found in [20,21].

2. The Rupture Degree of k-Uniform Linear Hypergraphs

In this section, we bound the rupture degree of k-uniform linear hypergraph G^k that underlies graph G with n vertices and m edges.

Lemma 1. *Let X be an r-set of k-uniform linear hypergraph G^k that underlies a connected graph G. Then, $\tau(G^k - X) = 1$.*

Proof. Let X be an r-set of G^k, we show that $\tau(G^k - X) = 1$. If not, by the structure of G^k, then $\tau(G^k - X) \geq k$. Suppose $C_1, C_2, \cdots, C_p$ are components of $G^k - X$ such that $|C_i| = \tau(G^k - X) \geq k$. Now, let $v_i \in C_i$ and $X^* = X \cup \{v_i\}$ for $1 \leq i \leq p$. Clearly, $\omega(G^k - X^*) \geq \omega(G^k - X) + p(k-2)$ and $\tau(G^k - X^*) \leq \tau(G^k - X) - (k-1)$. Consider $k \geq 3$ and $p \geq 1$; then,

$$\begin{aligned}
&Sc(X^*) - Sc(X) \\
&= \omega(T^k - X^*) - \tau(G^k - X^*) - |X^*| - [\omega(T^k - X) - \tau(G^k - X) - |X|] \\
&\geq p(k-2) - p + (k-1) = p(k-3) + (k-1) > 0.
\end{aligned}$$

This contradicts the choice of X. Thus, $\tau(G^k - X) = 1$. □

Given a graph $G = (V, E)$, an independent set of G is a subset of vertices which contains no pair of neighbors. The independence number $\alpha(G)$ of graph G is the size of the largest independent set in G. A set of vertices $S \subseteq V$ is a vertex cover of G if every edge has at least one vertex in S. The vertex cover number of G is the minimum cardinality of a vertex cover set of G, denote by $\beta(G)$.

Lemma 2. *Let X be an r-set of k-uniform linear hypergraph G^k that underlies connected graph G. Then, $|X| = \beta(G)$.*

Proof. Let X be an r-set of k-uniform linear hypergraph G^k that underlies a connected graph G. By Lemma 1, we obtain $\tau(G^k - X) = 1$. This means X is a vertex cover set of graph G, and thus $|X| \geq \beta(G)$. Vice versa, if X is a vertex cover set of graph G, then each edge of G has at least one end vertex belonging to X. Thus, $\tau(G^k - X) = 1$. By the definition of $\beta(G)$, consider $\beta(G) = |X|$. It follows that X is an r-set of G^k. □

By Lemmas 1 and 2, considering $|V(G^k)| = m(k-2) + n$, we directly obtain the rupture degree of G^k in terms of the vertex cover number $\beta(G)$ of G.

Theorem 1. *Let G^k be a k-uniform linear hypergraph that underlies connected graph G with n vertices and m edges. Then,*

$$r(G^k) = m(k-2) + n - 2\beta(G) - 1.$$

Unfortunately, the problem of computing the minimum vertex cover (MVC) is NPC [22]. In [23,24], the authors gave some exact algorithms for the MVC. However, all of them took an exponential time and so were not suited for practical use in large graphs. Therefore, it is interesting to discuss the bound of the $r(G^k)$ of k-uniform linear hypergraph G^k.

In [25], Harant bounded the independence number $\alpha(G)$ of G with n vertices and m edges.

Proposition 1. *Let G be a connected graph on n vertices with m edges [25]. Then,*

$$\frac{(2m+n+1) - \sqrt{(2m+n+1)^2 - 4n^2}}{2} \leq \alpha(G) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + n(n-1) - 2m}.$$

Combine the famous formula $\alpha(G) + \beta(G) = n$ for a connected graph G with order n, we obtain the following corollary.

Corollary 1. *Let G be a connected graph on n vertices with m edges. Then,*

$$n - \frac{1}{2} - \sqrt{\frac{1}{4} + n(n-1) - 2m} \leq \beta(G) \leq \frac{\sqrt{(2m+n+1)^2 - 4n^2} - 2m + n - 1}{2}.$$

By Theorem 1 and Corollary 1, we directly give a bound for the rupture degree of G^k as follows.

Theorem 2. *Let G^k be a k-uniform linear hypergraph that underlies connected graph G with n vertices and m edges. Then,*

$$mk - \sqrt{(2m+n+1)^2 - 4n^2} \leq r(G^k) \leq m(k-2) - n + 2\sqrt{\frac{1}{4} + n(n-1) - 2m}.$$

Remark 1. *The bounds in Theorem 2 are the best possible ones. The upper bound can meet at k-uniform linear hypergraph $K^k_{1,n-1}$ that underlies star graph $K_{1,n-1}$. The lower bound can meet at k-uniform linear hypergraph K^k_n that underlies complete graph K_n.*

Clearly, if we let $m = n-1$ in Theorem 2, then we can obtain the bounds for the rupture degree of k-uniform hypertree T^k.

Corollary 2. *Let T^k be a k-uniform hypertree that underlies a tree T with order n. Then,*

$$(n-1)k - \sqrt{(5n-1)(n-1)} \leq r(T^k) \leq (n-1)(k-2) + n - 1.$$

Remark 2. *The bounds in Corollary 2 are also best possible. The upper bound can meet at k-uniform linear hypergraph $K^k_{1,n-1}$ that underlies star graph $K_{1,n-1}$. The lower bound can meet at k-uniform linear hypergraph K^k_2 that underlies K_2.*

Notice that $n-1 \leq m \leq \frac{n(n-1)}{2}$ for any connected graph G, we also obtain the following result.

Corollary 3. *Let G^k be a k-uniform linear hypergraph that underlies connected graph G with order n. Then,*

$$(n-1)(k-2) - (n-1)^2 \leq r(G^k) \leq \frac{n(n-1)}{2}(k-2) + n - 3.$$

3. The Rupture Degree of k-Uniform Hypertrees

In this section, we discuss the rupture degree of k-uniform hypertree H. For k-uniform hypertree H, if X is an r-set of H, then we easily obtain the same result as in Lemma 1.

Lemma 3. *Let $H = (V, E)$ be a k-uniform hypertree. If X is an r-set of H, then $\tau(H - X) = 1$.*

Lemma 4. *Let $H = (V, E)$ be a k-uniform hypertree with n vertices and m edges. Then, $n = m(k-1) + 1$.*

Proof. We show how to proceed by induction on m. It is clear that the conclusion holds for $m = 0, 1$. Assume the conclusion holds for $m \leq p$. Now, we consider the case for $m = p + 1$; suppose $e \in E(H)$ and $H_1, H_2, \cdots, H_k$ are components of $H - e$. By the induction hypothesis, for every component $H_i = (V_i, E_i)$ and $n_i = |V_i|$, $m_i = |E_i|$ for $1 \leq i \leq k$, $i \cdot e \cdot$, $V_i = |E_i|(k-1) + 1$. Thus, we have

$$\begin{aligned}
|V| &= \sum_{i=1}^{k} |V_i| = \sum_{i=1}^{k} (|E_i|(k-1) + 1) \\
&= m_1(k-1) + 1 + m_2(k-1) + 1 + \cdots + m_k(k-1) + 1 \\
&= (m-1)(k-1) + k = m(k-1) + 1.
\end{aligned}$$

The proof is completed. □

Lemma 5. *Let $H = (V, E)$ be a hypertree with $|E| \geq 2$ [26]. Then, H has at least two pendant edges.*

Theorem 3. *Let $H = (V, E)$ be a k-uniform hypertree and X be the r-set of H. Then, $r(H) = n - 2|X| - 1$.*

Proof. Suppose X is an r-set of k-uniform hypertree H; by Lemma 3, we have $\tau(H - X) = 1$. Denote $|E| = m$ and $|V| = n$; by the structure properties of a k-uniform hypertree, we have

$$\omega(H - X) = mk - \sum_{v \in V \setminus X} (d(v) - 1) - \sum_{e \in E} (|e \cap X|).$$

Thus,

$$\begin{aligned} r(H) &= mk - \sum_{v \in V \setminus X} (d(v) - 1) - \sum_{e \in E} (|e \cap X|) - |X| - 1 \\ &= mk - [mk - \sum_{v \in X} d(v) - n + |X|] - \sum_{e \in E} (|e \cap X|) - |X| - 1 \\ &= \sum_{v \in X} d(v) + n - \sum_{e \in E} (|e \cap X|) - 2|X| - 1. \end{aligned}$$

Note that $\sum_{v \in X} d(v) = \sum_{e \in E} (|e \cap X|)$, and we obtain $r(H) = n - 2|X| - 1$. □

Lemma 6. *Let $H = (V, E)$ be a k-uniform hypertree with $|V| = n, |E| = m$, and X be the r-set of H. Then,*

$$1 \leq |X| \leq m - \lceil \frac{m-1}{k} \rceil.$$

Proof. Let X be an r-set of k-uniform hypertree H. It is easy to know that $|X| \geq 1$. On the other hand, by Theorem 3, we know that $r(H) = \omega(H - X) - |X| - \tau(H - X) = n - 2|X| - 1$. In order to let the value of $|X|$ be as large as possible, it suffices that the number of components of $H - X$ increases at most $k - 1$ when $|X|$ adds one. This means $|X|(k-1) + |X| \leq n$. Consider $n = m(k-1) + 1$; we have

$$m(k-1) + 1 = n \geq |X|k.$$

Thus,

$$|X| \leq m - \frac{m-1}{k}.$$

Because $|X|$ is a positive integer,

$$|X| \leq m - \lceil \frac{m-1}{k} \rceil.$$

The proof is completed. □

Base on Theorem 3 and Lemma 6, we give the bounds of the rupture degree of k-uniform hypertree H as follows.

Theorem 4. *Let $H = (V, E)$ be a k-uniform hypertree with $|V| = n, |E| = m$. Then,*

$$m(k-3) + 2\lceil \frac{m-1}{k} \rceil \leq r(H) \leq m(k-1) - 2.$$

Remark 3. *The bounds in Theorem 3 are best possible. The upper bound can meet at k-uniform linear hypertree $K^k_{1,n-1}$ that underlies star graph $K_{1,n-1}$. The lower bound can meet at k-uniform linear hypertree H, which is shown in Figure 3.*

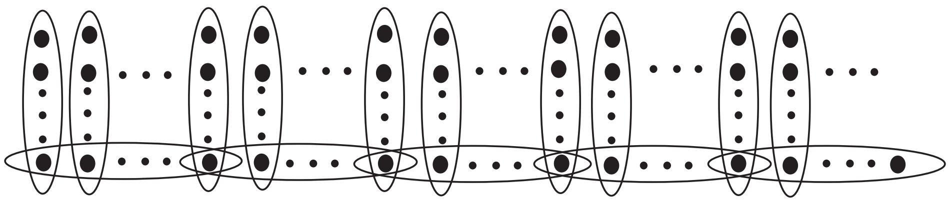

Figure 3. A k-uniform hypertree with a minimum rupture degree.

4. A Recursive Algorithm for Computing the r-Set of k-Uniform Hypertrees

Lemma 3 and Theorem 3 show that we can determine the rupture degree of k-uniform hypertree H by finding minimal cut set X to let $\tau(H-X)=1$. Here, we provide a method to obtain such vertex cut set X.

Let H be a k-uniform hypertree and $e=\{v_1,v_2,\cdots,v_k\}$ be an arbitrary edge of H. By $\pi(e)=\{d_1,d_2,\ldots,d_k\}$, we denote the nondecreasing degree sequence of edge e with $d_1 \le d_2 \le \cdots \le d_k$. For convenience, $\{\underbrace{1,1,\ldots,1}_{k-1},d\}$ is often simplified as $\{1^{(k-1)},d\}$ for $d \ge 2$.

A contraction of an edge $e=\{v_1,v_2,\cdots,v_k\}$ is an operation that identifies that vertices $v_1,v_2,\cdots,v_{k-1}$ are merged into the vertex v_k and the edges incident to these vertices are transformed into the edges incident to the vertex v_k in H. The resulting graph is denoted as $H\cdot e$. Clearly, $H\cdot e$ is also a k-uniform hypertree. Similarly, by $H\cdot e_1\cdot e_2$ we denote the graph obtained by contracting edges e_1,e_2 in H.

Lemma 7. *Let $e_1,e_2,\cdots,e_{t-1}$ be $t-1$ pendant edges of k-uniform hypertree H such that $\pi(e_i)=\{1^{(k-1)},t\}$ for $i=1,2,\cdots,t-1$. Let e be an edge with $\pi(e)=\{1^{(k-2)},t,d\}$ such that $e_1\cap e_2\cap\cdots\cap e_{t-1}\cap e=v$ with $d(v)=t$. Let t,d be integer number great than 1. If X and X' are an r-set of k-uniform hypertree H and $H\cdot e\cdot e_1\cdot e_2\cdot,\cdots,\cdot e_{t-1}$, respectively, then $X=X'\cup\{v\}$.*

Proof. Let X' be an r-set of $H\cdot e\cdot e_1\cdot e_2\cdot,\cdots,\cdot e_{t-1}$ and $e_1\cap e_2\cap\cdots\cap e_{t-1}\cap e=v$. It follows that $X'\cup\{v\}$ is an r-set of H. In fact, note that $H\cdot e\cdot e_1\cdot e_2\cdot,\cdots,\cdot e_{t-1}$ is also a k-uniform hypertree. Thus, by Lemma 1, we know that $\tau(H\cdot e\cdot e_1\cdot e_2\cdot,\cdots,\cdot e_{t-1}-X')=1$. It is not difficult to check that $X'\cup\{v\}$ is an r-set of H. Thus, we have $X=X'\cup\{v\}$. □

Lemma 8. *Let $e_1,e_2,\cdots,e_{t-1}$ be $t-1$ pendant edges of k-uniform hypertree H such that $\pi(e_i)=\{1^{(k-1)},t\}$ for $i=1,2,\cdots,t-1$. Let e be an edge with $\pi(e)=\{t,d_1,d_2,\ldots,d_{k-1}\}$ such that $e_1\cap e_2\cap\cdots\cap e_{t-1}\cap e=v$ with $d(v)=t\ge 2$. Let t,d_j be an integer number for $j=1,2,\cdots,k-1$ and at least two of $d_1,d_2,\ldots,d_{k-1}$ are great than one. If X and X'' are r-sets of k-uniform hypertree H and $H\cdot e_1\cdot e_2\cdot,\cdots,\cdot e_{t-1}$, respectively, then $X=X''\cup\{v\}$.*

Proof. Let X'' be an r-set of $H\cdot e_1\cdot e_2\cdot,\cdots,\cdot e_{t-1}$. Similar to Lemma 7, we know that $X''\cup\{v\}$ is an r-set of H. Thus, we have $X=X''\cup\{v\}$. □

Based on Lemmas 7 and 8, we provide a recursive algorithm for computing the cardinality $|X|$ of an r-set of k-uniform hypertree H. Let $d_i(\ge 2)$ be integer numbers for $i=1,2,3$.

Recursive algorithm:

Step 1: Set $|X|=0$.

Step 2: If H is a trivial hyperraph, go to step 10. Otherwise, go to step 3.

Step 3: For k-uniform hypertree $H=(V,E)$ and set

$$P=\{e: e\in E, \pi(e)=(1^{(k-1)},d_1)\};$$
$$Q=\{e: e\in E, \pi(e)=(1^{(k-2)},d_2,d_3)\};$$
$$U=\{e: e\in E\setminus(P\cup Q)\}.$$

Step 4: If $P\cup Q=\emptyset$, then let $|X|\leftarrow|X|+1$, go to step 10. Otherwise, go to step 5.

Step 5: If $Q\neq\emptyset$, go to step 6. Otherwise, go to step 8.

Step 6: For each edge $e_i\in Q$, if there exist $e_j\in P$ such that $e_i\cap e_j=v_{ij}$ and $d(v_{ij})=|\{e: v_{ij}\in e, e\in P\}|+1$, go to step 7. Otherwise, go to step 8.

Step 7: Let $|X|\leftarrow|X|+|\{v_{ij}: v_{ij}\in e, e\in Q\}|$ and set $H\leftarrow H\cdot N[e_i]$, where $N[e_i]$ denote the edge set of all edges which are incident to vertex v_{ij} in H; go to step 2.

Step 8: If $U\neq\emptyset$, go to step 9. Otherwise, let $|X|\leftarrow|X|+1$, and go to step 10.

Step 9: For each edge $e_s \in U$, if there exists $e_t \in P$ such that $e_s \cap e_t = v_{st}$ and $d(v_{st}) = |\{e : v_{st} \in e, e \in P\}| + 1$, then let $|X| \leftarrow |X| + |\{v_{st} : v_{st} \in e, e \in U\}|$ and $H \leftarrow H \cdot N[e_s]$, where $N[e_s]$ denote the edge set of all edges which incident to vertex v_{st} in H; go to step 2.

Step 10: Output the value of $|X|$.

Example 1. *Let T^4 be a four-uniform hypertree with $|E| = 25$ that underlies tree T. Using the above algorithm, we get the cardinality $|X|$ of the r-set X of T^4. The details of the algorithm execution are shown in Figure 4.*

1. $X \leftarrow \{u_1, u_2, u_6, u_9, u_{10}\}$;
2. $X \leftarrow \{u_1, u_2, u_6, u_9, u_{10}\} \cup \{u_3, u_8\}$;
3. $X \leftarrow \{u_1, u_2, u_6, u_9, u_{10}, u_3, u_8\} \cup \{u_4, u_7\}$;
4. $X \leftarrow \{u_1, u_2, u_6, u_9, u_{10}, u_3, u_8, u_4, u_7\} \cup \{u_5\}$.

Thus, $X = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$. By Lemma 4 and Theorem 3, we obtain that the rupture degree of four-uniform hypertree T^4 is $r(T^4) = 55$.

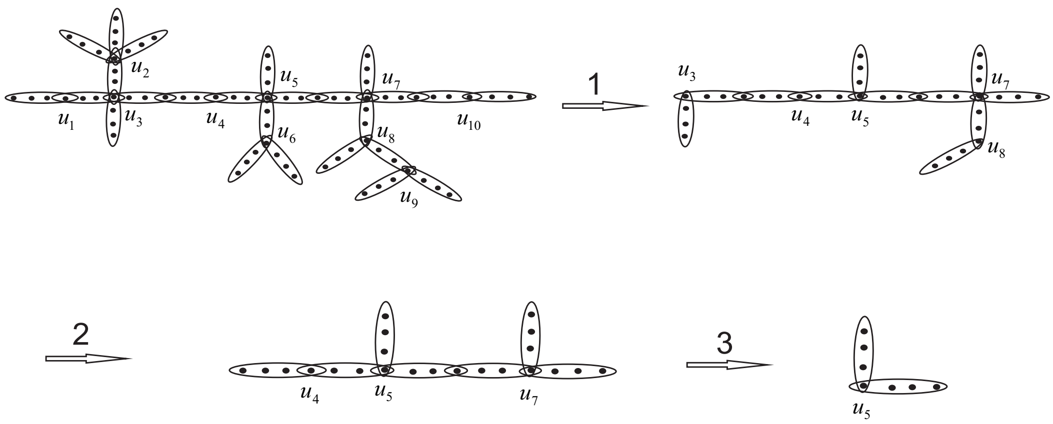

Figure 4. Computing the rupture degree of 4-uniform hypertree T^4 by our recursive algorithm.

Example 2. *Let H be a four-uniform hypertree with $|E| = 20$. Using the above algorithm, we obtain that the cardinality $|X|$ of the r-set X of H. The details of the algorithm execution are shown in Figure 5.*

(1) $X \leftarrow \{1, 9\}$;
(2) $X \leftarrow \{1, 9\} \cup \{2, 3, 4, 6, 7, 10, 11\}$;
(3) $X \leftarrow \{1, 9, 2, 3, 4, 6, 7, 10, 11\} \cup \{5, 8\}$.

Thus, we have $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. By Lemma 4 and Theorem 3, we obtain that the rupture degree of four-uniform hypertree H is $r(H) = 38$.

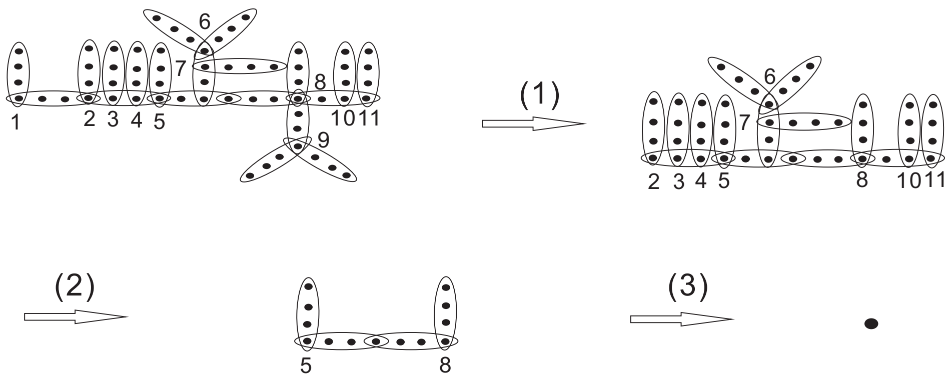

Figure 5. Rupture degree of 4-uniform hypertree H obtained by our recursive algorithm.

5. Conclusions

The rupture degree is an important parameter measuring the vulnerability of a network. However, there are few results on the vulnerability of hypernetworks. In this paper, the parameter rupture degree was used to measure the vulnerability of uniform linear hypergraphs. In fact, many parameters of vulnerability remain unexplored for hypergraphs, and they can be explored in nonuniform or nonlinear hypergraphs. Our work may stimulate more research in this field.

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Abbreviations

The following abbreviations are used in this manuscript:

MVC	Minimum vertex cover
NPC	NP-complete

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