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The Conchoidal Twisted Surfaces Constructed by Anti-Symmetric Rotation Matrix in Euclidean 3-Space

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Abstract: A twisted surface is a type of mathematical surface that has a nontrivial topology, meaning that it cannot be smoothly deformed into a flat surface without tearing or cutting. Twisted surfaces are often described as having a twisted or Möbius-like structure, which gives them their name. Twisted surfaces have many interesting mathematical properties and applications, and are studied in fields such as topology, geometry, and physics. In this study, a conchoidal twisted surface is formed by the synchronized anti-symmetric rotation matrix of a planar conchoidal curve in its support plane and this support plane is about an axis in Euclidean 3-space. In addition, some examples of the conchoidal twisted surface are given and the graphs of the surfaces are presented. The Gaussian and mean curvatures of this conchoidal twisted surface are calculated. Afterward, the conchoidal twisted surface formed by an involute curve and the conchoidal twisted surface formed by a Bertrand curve pair are given. Thanks to the results obtained in our study, we have added a new type of surface to the literature.

Keywords: conchoidal curve; twisted surfaces; involute curve; Bertrand curve; anti-symmetric rotation matrix

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1. Introduction

The theory of curves and surfaces forms the basis of differential geometry. With developing technology and increasing studies, the theory of surfaces creates a very wide field of study. The theory of surfaces has many applications, especially in physics, engineering, design, and computer modeling. Thus, the geometric structure of special surfaces in Euclidean space has become an important field of study for geometricians [1–4]. According to the Frenet equations of the null curves in the semi-Euclidean 4-space, the conditions of existence and geometric characterizations of the Bertrand curves of the null curves were examined [5]. Some classical results of Bertrand curves for timelike ruled and developable surfaces were examined using an E-study map. In addition, some new results and theorems have been obtained regarding the developability of Bertrand offsets of timelike ruled surfaces [6].

The invention of the conchoid has been attributed to the Greek geometer Nicomedes by Pappus and other classical writers in the 2nd century BC. The conchoid is visually mussel-shell shaped. A conchoid is a type of curve derived from a fixed point, another curve, and a fixed length. Conchoids can be used in magnetic research, building construction, optics, physics, etc. It has been used in many applications such as [7–9]. Oruç et al. studied the planar and space conchoid curves and surfaces in three-dimensional Euclidean space [10,11]. Dede (2013) computed the types of spacelike conchoid curves in the Minkowski plane [12]. Aslan and Şekerci (2021) examined the condition which is the conchoidal surface and the surface of revolution given with a conchoid curve to be a Bonnet surface in Euclidean

3-space. Additionally, the condition required for the conchoidal surface to be a Bonnet surface is that conchoidal surfaces of constant mean curvature are assumed to be infinitely isothermic Weingarten conchoidal surfaces are investigated in [13]. Ikawa (2000) found that Bour's theorem revealed the helicoid and the rotational surface with additional conditions having the same Gaussian map on it [14]. On the other hand, a twisted surface is obtained by rotating a planar curve about a line passing through its support plane, while simultaneously rotating the support plane about an axis. A twisted surface is a mathematical object that can be visualized as a surface that has been twisted or deformed in some way. Moreover, a twisted surface is a surface that cannot be flattened onto a plane without distorting or cutting it. One example of a twisted surface is the Möbius strip, which is a two-dimensional surface with only one side and one boundary. If you take a strip of paper, twist it once, and then join the ends together, you will get a Möbius strip. Another example of a twisted surface is the Klein bottle, which is a four-dimensional surface that cannot be embedded in three-dimensional space without self-intersections. Twisted surfaces have many interesting and useful properties in mathematics and physics, and they have been studied extensively by mathematicians and scientists. They are used in topology, differential geometry, and other areas of mathematics, as well as in physics, where they can be used to model various physical phenomena.

The resulting twisted surface can be viewed as a generalization of the rotating surface [15–18]. The twisted surface definition is given in the three-dimensional Euclidean and Minkowski spaces by Goemans and Ignace [16,17]. The concept of a twisted surface is studied in isotropic spaces as well. The twisted surfaces formed by the special planar curve in the three-dimensional Euclidean space have been examined in [18]. The support plane and twisted surfaces according to rotation types in pseudo-Galilean space are studied. In addition, the Gaussian and mean curvatures of these surfaces were calculated by Kazan and Karadağ in [19]. Dede et al. examined the flatness and minimality properties of the twisted surface in Galilean 3-space. In addition, three types were examined according to these characteristics [20]. In this research, a conchoidal twisted surface is achieved through the coordinated rotation of a planar conchoidal curve in its supporting plane and the support plane itself about an axis in Euclidean 3-space. Additionally, our paper provides several instances of the conchoidal twisted surface and showcases their graphical representations. The Gaussian and mean curvatures of the conchoidal twisted surface are computed. Moreover, the study presents the conchoidal twisted surface formed by an involute curve and the one created by a pair of Bertrand curves.

2. Materials and Methods

Let $E^3 = \left(IR^3, g\right)$ defined by the metric $\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3$ be called Euclidean 3-space. Here, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are the standard coordinates of E^3 space. Let $\alpha : I \to E^3$ be a unit speed curve. The Serret–Frenet frame $\{T, N, B\}$ of the curve α can be written by

$$T(s) = \alpha'(s),\ N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},\ B(s) = T(s) \wedge N(s),$$

where $\wedge$ is the vectoral product. The derivative formula of the Serret–Frenet frame in the matrix form can be calculated by equations

$$T'(s) = \kappa N(s),\ N(s) = -\kappa T(s) + \tau B(s),\ B'(s) = -\tau N(s),$$

where κ and τ are curvatures of the Serret–Frenet frame in Euclidean 3-space [3].

Let $\alpha : I \to E^3$ be a unit speed curve and $\alpha* : I \to E^3$ is given with the same interval. (α, M) and $(\alpha*, M)$ are the curve-surface pairs in E^3 and $I \subset E^3$. An involute of a curve in Euclidean plane E^2 is a curve to which all tangent lines of the initial curve at corresponding points are orthogonal. If $c = c(u)$ is a curve in E^2 parametrized by arc length, then the parametrizations of its involutes are

$$i(u) = c(u) + (-u + a)t(u), \tag{1}$$

where $a \in IR$ is a constant, and $t(u) = c'(u)$. For spatial curves in Euclidean 3-space, the situation is more complex (see discussion in [2]). The most classical definition can be found by Eisenhart [1], where an involute is defined by the identity (1) for curves parametrized by arc length [1,2]. The pair $(\alpha, \alpha*)$ is said to be a Bertrand pair if their principal normal vector fields are linearly dependent at each point. If the curve $\alpha*$ is a Bertrand partner of α, then we may write that

$$\alpha * (s) = \alpha(s) + \lambda N(s), \tag{2}$$

where λ is constant [3–6].

A conchoid curve $\alpha^d(t)$ is a curve derived from a fixed point O, another curve $\alpha(t)$, and a constant length d. In this case, Q is the set of points on the line OP where there is a moving point P such that the distance between P and Q is d. For an analytic representation, it is convenient to choose $O(0,0)$. Using a representation of a curve $\alpha(t)$ in terms of the polar coordinates $\alpha(t) = r(t)(\cos t, \sin t)$, its conchoid $\alpha^d(t)$ with respect to O and distance d is obtained as $\alpha^d(t) = (r(t) \pm d)(\cos t, \sin t)$. More generally, we can consider any parameterization $k(t)$ of the unit circle S^1. Then the curve $\alpha(t)$ and its conchoids $\alpha^d(t)$ are represented by

$$\alpha(t) = r(t)k(t) \text{ and } \alpha^d(t) = (r(t) \pm d)k(t), \tag{3}$$

respectively, where $\|k(t)\| = 1$, refs. [7–13]. Let M be a smooth surface in E^3 given with the patch $X(s,t)$ for $(s,t) \in D \subset E^2$. The tangent plane to M at an arbitrary point P of M is spanned by $X_s(P)$ and $X_t(P)$, where the vector fields $X_s(P)$ and $X_t(P)$ denote derivatives with respect to s and t, respectively. The unit normal vector field of the surface M is

$$N(s,t) = \frac{X_s(s,t) \wedge X_t(s,t)}{\|X_s(s,t) \wedge X_t(s,t)\|}, \tag{4}$$

where $\|X_s(s,t) \wedge X_t(s,t)\| = \sqrt{EG - F^2} = W$. The coefficients of the first and second fundamental forms on any T_pM plane of the surface M are

$$E = \langle X_s(s,t), X_s(s,t)\rangle,\ F = \langle X_s(s,t), X_t(s,t)\rangle,\ G = \langle X_t(s,t), X_t(s,t)\rangle, \tag{5}$$

$$e = \langle X_{ss}(s,t), N(s,t)\rangle,\ f = \langle X_{st}(s,t), N(s,t)\rangle,\ g = \langle X_{tt}(s,t), N(s,t)\rangle, \tag{6}$$

respectively The Gauss curvature K, mean curvature H of the surface M are calculated by

$$K = \frac{e\,g - f^2}{E\,G - F^2},\ H = \frac{E\,g - 2F\,f + G\,e}{2(E\,G - F^2)}, \tag{7}$$

respectively [3,4]. A surface of revolution is formed by revolving a plane curve about a line in E^3. For an open interval I, let $\gamma : I \to \Pi$ be a curve in a plane Π and let l be a straight line in Π which does not intersect the curve γ on the Euclidean space E^3. A rotation surface R is defined as a surface rotating the curve γ around l where they are called the profile curve and the axis, respectively. We may suppose that the axis l is the z -axis and the plane Π is the xz-plane, without loss of generality. Then the profile curve γ is given $\gamma(u) = (u, 0, \varphi(u))$. Hence a rotation surface R can be parametrized by

$$R(u,v) = (u\cos v,\ u\sin v,\ \varphi(u)) \tag{8}$$

Ref. [14]. In Euclidean 3-space, a twisted surface is obtained by rotating a plane curve about a line passing through its support plane, while the support plane is rotated about an axis. Let us assume that the profile curve α lies in the xz-plane. Thus, it can be parametrized as $\alpha(t) = (f(t), 0, g(t))$. If α is rotated about the straight line through the point $(a, 0, 0)$ parallel to the y-axis, then we obtain

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(bs) & 0 & -\sin(bs) \\ 0 & 1 & 0 \\ \sin(bs) & 0 & \cos(bs) \end{pmatrix} \begin{pmatrix} f(t) \\ 0 \\ g(t) \end{pmatrix} = \begin{pmatrix} a + f(t)\cos(bs) - g(t)\sin(bs) \\ 0 \\ f(t)\sin(bs) + g(t)\cos(bs) \end{pmatrix} \tag{9}$$

using the anti-symmetric rotation matrix. By rotating (9) about the z-axis, the following twisted surface is obtained in Euclidean 3-space.

$$\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a + f(t)\cos(bs) - g(t)\sin(bs) \\ 0 \\ f(t)\sin(bs) + g(t)\cos(bs) \end{pmatrix} = \begin{pmatrix} (a + f(t)\cos(bs) - g(t)\sin(bs))\cos s \\ (a + f(t)\cos(bs) - g(t)\sin(bs))\sin s \\ f(t)\sin(bs) + g(t)\cos(bs) \end{pmatrix}.$$

So, a parametrization of the twisted surface is defined by

$$X(s,t) = (a + f(t)\cos(bs) - g(t)\sin(bs))(\cos s, \sin s, 0) + (0, 0, f(t)\sin(bs) + g(t)\cos(bs)). \tag{10}$$

Note that since both rotations must be synchronized, they are expressed using the same parameter s. Here, the presence of the factor $b \in R$ allows for differences in the rotation speed of both rotations [15,16]. The coefficients E, F, G and e, f, g of the first and second fundamental forms of the twisted surface $X(s,t)$ in the Euclidean 3- space are

$$E = b^2\left(f^2(t) + g^2(t)\right) + \left(a + \frac{1}{b}h_s(s,t)\right)^2, \tag{11}$$

$$F = b\left(-g(t)f'(t) + f(t)g'(t)\right), \tag{12}$$

$$G = \left(f'(t)\right)^2 + \left(g'(t)\right)^2, \tag{13}$$

and

$$e = \frac{1}{W}\left(\left(a + \frac{1}{b}h_s(s,t)\right)\left(bh(s,t)h_{st}(s,t) - \left(a + \left(\frac{1}{b} + b\right)h_s(s,t)\right)h_t(s,t)\right) - 2b^2\left(f(t)f'(t) + g(t)g'(t)\right)h(s,t)\right), \tag{14}$$

$$f = \frac{1}{W}\left(\left(f(t)f'(t) + g'(t)g(t)\right)h_{st}(s,t) - b\left(\left(f'(t)\right)^2 + \left(g'(t)\right)^2\right)\left(a + \frac{1}{b}h_s(s,t)\right)\right), \tag{15}$$

$$g = \frac{1}{W}\left(\left(a + \frac{1}{b}h_s(s,t)\right)\left(-f'(t)g''(t) + f''(t)g'(t)\right)\right), \tag{16}$$

respectively [13–15]. Thus, from (7) the Gauss and mean curvature of the twisted surface $X(s,t)$ in Euclidean 3-space are

$$K = \frac{1}{W^4}\left(\left(mx - 2b^2h(s,t)y\right)(mz) - \left(h_{st}(s,t)y - bwm\right)^2\right), \tag{17}$$

$$H = \frac{1}{2W^3}\left(\left(mx - 2b^2h(s,t)y\right)w - 2\left(h_{st}(s,t)y - bmw\right)(bk) + (mz)\left(mx - 2b^2h(s,t)y\right)\right), \tag{18}$$

respectively, where, $x = \left(bh(s,t)h_{st}(s,t) - \left(a + \left(\frac{1}{b} + b\right)h_s(s,t)\right)h_t(s,t)\right)$

$$y = f(t)f'(t) + g(t)g'(t),\ z = \left(-f'(t)g''(t) + f''(t)g'(t)\right),$$

$$w = \left(f'(t)\right)^2 + \left(g'(t)\right)^2,\ k = -g(t)f'(t) + f(t)g'(t),\ m = \left(a + \frac{1}{b}h_s(s,t)\right).$$

Let α be a planar curve given with profile $\alpha(t) = (f(t),\ 0,\ g(t))$ and $\alpha*$ be the involute of the α curve. In that case, the involute curve is written as

$$\alpha* = \left(f(t) + (c-t)f'(t), 0, g(t) + (c-t)g'(t)\right), \tag{19}$$

The twisted surface formed by the involute of the curve α in Euclidean 3-space is given by

$$\begin{aligned}X_I(s,t) &= (a + (f(t) + (c-t)f'(t))\cos(bs) - (g(t) + (c-t)g'(t))\sin(bs))(\cos s, \sin s, 0)\\ &+(0,0,(f(t)+(c-t)f'(t))\sin(bs) + (g(t)+(c-t)g'(t))\cos(bs))\end{aligned} \tag{20}$$

Refs. [15–18]. Let α be a planar curve given with a profile curve $\alpha(t) = (f(t),\ 0,\ g(t))$ and β be the Bertrand curve pair of the α curve. In that case, the Bertrand curve is written as

$$\beta = \left(f(t) + \lambda \frac{f''(t)}{\sqrt{f''^2(t) + g''^2(t)}}, 0, g(t) + \lambda \frac{g''(t)}{\sqrt{f''^2(t) + g''^2(t)}} \right), \tag{21}$$

where λ is a real constant [18]. Thus, the twisted surface formed by the Bertrand pair of the curve α in Euclidean 3-space is

$$\begin{aligned}X_b(s,t) = &\left(a + \left(f(t) + \lambda \frac{f''(t)}{\sqrt{f''^2(t)+g''^2(t)}} \right)\cos(bs) - \left(g(t) + \lambda \frac{g''(t)}{\sqrt{f''^2(t)+g''^2(t)}} \right)\sin(bs) \right)(\cos s, \sin s, 0)\\ &+\left(0,0, \left(f(t) + \lambda \frac{f''(t)}{\sqrt{f''^2(t)+g''^2(t)}} \right)\sin(bs) + \left(g(t) + \lambda \frac{g''(t)}{\sqrt{f''^2(t)+g''^2(t)}} \right)\cos(bs) \right).\end{aligned} \tag{22}$$

3. Conchoidal Twisted Surfaces in Euclidean 3-Space

In this section, the conchoidal twisted surface $\Omega^d(s,t)$ has been introduced in the Euclidean 3- space. Additionally, the first and second fundamental forms of the conchoidal twisted surface were computed. Then, the Gaussian and mean curvature of this surface were found. The planar curve with real-valued functions can be parametrized as $\alpha(t) = r(t)(\cos t,\ 0,\ \sin t)$.

Theorem 1. *Let α and α^d planar curves parametrized as $\alpha(t) = r(t)(\cos t,\ 0,\ \sin t)$, $\alpha^d(t) = (r(t) \pm d)(\cos t,\ 0,\ \sin t)$ in Euclidean 3-space. Then, by rotating α and α^dabout the straight line through the point $(a,0,0)$ parallel with the y-axis and after by rotating this surface about the z-axis, twisted surface and conchoidal twisted surface in Euclidean 3-space are*

$$\Omega(s,t) = (a + r(t)\cos(t+bs))(\cos s, \sin s, 0) + (0,0,r(t)\sin(t+bs)), \tag{23}$$

and

$$\Omega^d(s,t) = (a + (r(t) \pm d)\cos(t+bs))(\cos s, \sin s, 0) + (0,0,(r(t) \pm d)\sin(t+bs)), \tag{24}$$

Proof. If we use an anti-symmetric rotation matrix for α about the straight line through the point $(a,0,0)$ parallel with the y-axis, then we obtain

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(bs) & 0 & -\sin(bs) \\ 0 & 1 & 0 \\ \sin(bs) & 0 & \cos(bs) \end{pmatrix} \begin{pmatrix} r(t)\cos t \\ 0 \\ r(t)\sin t \end{pmatrix} = \begin{pmatrix} a + r(t)\cos(t+bs) \\ 0 \\ r(t)\sin(t+bs) \end{pmatrix}, \tag{25}$$

By rotating (25) about the z-axis, we have the following twisted surface in Euclidean 3-space

$$\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a + r(t)\cos(t+bs) \\ 0 \\ r(t)\sin(t+bs) \end{pmatrix} = \begin{pmatrix} (a + r(t)\cos(t+bs))\cos s \\ (a + r(t)\cos(t+bs))\sin s \\ r(t)\sin(t+bs) \end{pmatrix},$$

Thus, a parametrization of the twisted surface is

$$\Omega(s,t) = ((a + r(t)\cos(t + bs))\cos s, (a + r(t)\cos(t + bs))\sin s, r(t)\sin(t + bs))$$

Similarly, a parametrization of the conchoidal twisted surface is

$$\Omega^d(s,t) = ((a + (r(t) \pm d)\cos(t + bs))\cos s, (a + (r(t) \pm d)\cos(t + bs))\sin s, (r(t) \pm d)\sin(t + bs)).$$

□

Corollary 1. *Let $\Omega(s,t)$ be a twisted surface given by (23). The coefficients E, F, G and e, f, g of the first and second fundamental forms of the twisted surface $\Omega(s,t)$ in Euclidean 3-space are*

$$E(s,t) = b^2 r^2(t) + \left(a + \frac{1}{b} h_S(s,t)\right)^2, \tag{26}$$

$$F(s,t) = b r^2(t), \tag{27}$$

$$G(s,t) = r'^2(t) + r^2(t), \tag{28}$$

and

$$e(s,t) = \frac{1}{W}\left(\left(a + \frac{1}{b} h_s(s,t)\right)\left(b h(s,t) h_{st}(s,t) - \left(a + \left(\frac{1}{b} + b\right) h_s(s,t)\right) h_t(s,t)\right) - 2b^2 r(t) r'(t) h(s,t)\right), \tag{29}$$

$$f(s,t) = \frac{1}{W}\left(r(t) r'(t) h_{st}(s,t) - b\left((r(t))^2 + (r'(t))^2\right)\left(a + \frac{1}{b} h_s(s,t)\right)\right), \tag{30}$$

$$g(s,t) = \frac{1}{W}\left(\left(a + \frac{1}{b} h_s(s,t)\right)\left(r''(t) r(t) - 2(r'(t))^2 - r^2(t)\right)\right), \tag{31}$$

respectively.

Corollary 2. *Let $\Omega^d(s,t)$ be a conchoidal twisted surface given by (24). The coefficients E, F, G and e, f, g of the first and second fundamental forms of the conchoidal twisted surface $\Omega^d(s,t)$ in Euclidean 3-space are*

$$E^d(s,t) = b^2 (r(t) \pm d)^2 + \left(a + \frac{1}{b} h_S(s,t)\right)^2, \tag{32}$$

$$F^d(s,t) = b(r(t) \pm d)^2, \tag{33}$$

$$G^d(s,t) = r'^2(t) + (r(t) \pm d)^2, \tag{34}$$

and

$$e^d(s,t) = \frac{1}{W^d}\left(\left(a + \frac{1}{b} h_s(s,t)\right)\left(b h(s,t) h_{st}(s,t) - \left(a + \left(\frac{1}{b} + b\right) h_s(s,t)\right) h_t(s,t)\right) - 2b^2 (r(t) \pm d) r'(t) h(s,t)\right), \tag{35}$$

$$f^d(s,t) = \frac{1}{W^d}\left((r(t) \pm d) r'(t) h_{st}(s,t) - b\left((r(t) \pm d)^2 + (r'(t))^2\right)\left(a + \frac{1}{b} h_s(s,t)\right)\right), \tag{36}$$

$$g^d(s,t) = \frac{1}{W^d}\left(\left(a + \frac{1}{b} h_s(s,t)\right)\left(r''(t)(r(t) \pm d) - 2(r'(t))^2 - (r(t) \pm d)^2\right)\right), \tag{37}$$

respectively.

Corollary 3. *The Gauss and mean curvatures of the twisted surface $\Omega(s,t)$ and the conchoidal twisted surface $\Omega^d(s,t)$ in Euclidean 3-space are*

$$K(s,t) = \frac{1}{W^4}\left(\begin{array}{c} \left(mx - 2b^2 r(t) r'(t) h(s,t)\right)\left(m\left(r''(t) r(t) - 2(r'(t))^2 - r^2(t)\right)\right) \\ -\left(r(t) r'(t) h_{st}(s,t) - b\left((r(t))^2 + (r'(t))^2\right)m\right)^2 \end{array}\right) \tag{38}$$

$$H(s,t) = \frac{1}{2W^3}\left(\begin{array}{c} \left(mx - 2b^2 r(t) r'(t) h(s,t)\right)\left((r(t))^2 + (r'(t))^2\right) \\ -2\left(r(t) r'(t) h_{st}(s,t) - b\left((r(t))^2 + (r'(t))^2\right)m\right)\left(b(r(t))^2\right) \\ +\left(m\left(r''(t)(r(t)) - 2(r'(t))^2 - (r(t))^2\right)\right)\left(mx - 2b^2 r(t) r'(t) h(s,t)\right) \end{array}\right) \tag{39}$$

and

$$K^d(s,t) = \frac{1}{\left(W^d\right)^4}\left(\begin{array}{c} \left(mx - 2b^2(r(t)\pm d) r'(t) h(s,t)\right)\left(m\left(r''(t)(r(t)\pm d) - 2(r'(t))^2 - (r(t)\pm d)^2\right)\right) \\ -\left((r(t)\pm d) r'(t) h_{st}(s,t) - b\left((r(t)\pm d)^2 + (r'(t))^2\right)m\right)^2 \end{array}\right) \tag{40}$$

$$H^d(s,t) = \frac{1}{2\left(W^d\right)^3}\left(\begin{array}{c} \left(mx - 2b^2(r(t)\pm d) r'(t) h(s,t)\right)\left((r(t)\pm d)^2 + (r'(t))^2\right) \\ -2\left((r(t)\pm d) r'(t) h_{st}(s,t) - b\left((r(t)\pm d)^2 + (r'(t))^2\right)m\right)\left(b(r(t)\pm d)^2\right) \\ +\left(m\left(r''(t)(r(t)\pm d) - 2(r'(t))^2 - (r(t)\pm d)^2\right)\right)\left(mx - 2b^2(r(t)\pm d) r'(t) h(s,t)\right) \end{array}\right) \tag{41}$$

Theorem 2. *If involute of the curve α and α^d rotated about the straight line through the point $(a,0,0)$ parallel with the y-axis and after rotating this surface about the z-axis, the twisted surface and conchoidal twisted surface in Euclidean 3-space are*

$$\Omega_I(s,t) = \begin{pmatrix} (a + \lambda_1 \cos(bs) - \lambda_2 \sin(bs)) \cos s \\ (a + \lambda_1 \cos(bs) - \lambda_2 \sin(bs)) \sin s \\ \lambda_1 \sin(bs) + \lambda_2 \cos(bs) \end{pmatrix} \tag{42}$$

and

$$\Omega_I^d(s,t) = \begin{pmatrix} (a + \lambda_3 \cos(bs) - \lambda_4 \sin(bs)) \cos s \\ (a + \lambda_3 \cos(bs) - \lambda_4 \sin(bs)) \sin s \\ \lambda_3 \sin(bs) + \lambda_4 \cos(bs) \end{pmatrix} \tag{43}$$

where

$$\lambda_1 = r(t)\cos t + (c-t)(r'(t)\cos t - r(t)\sin t)$$
$$\lambda_2 = r(t)\sin t + (c-t)(r'(t)\sin t + r(t)\cos t)$$

and

$$\lambda_3 = (r(t)\pm d)\cos t + (c-t)(r'(t)\cos t - (r(t)\pm d)\sin t)$$
$$\lambda_4 = (r(t)\pm d)\sin t + (c-t)(r'(t)\sin t + (r(t)\pm d)\cos t).$$

Proof. Let α be a planar curve given with a profile curve $\alpha(t) = r(t)(\cos t,\ 0,\ \sin t)$ and $\alpha*$ in Equation (19) be the involute of the curve α. If we rotate the involute of the curve α about the straight line through the point $(a,0,0)$ parallel with the y-axis, then we obtain

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(bs) & 0 & -\sin(bs) \\ 0 & 1 & 0 \\ \sin(bs) & 0 & \cos(bs) \end{pmatrix}\begin{pmatrix} \lambda_1 \\ 0 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a + \lambda_1 \cos(bs) - \lambda_2 \sin(bs) \\ 0 \\ \lambda_1 \sin(bs) + \lambda_2 \cos(bs) \end{pmatrix} \tag{44}$$

By rotating (44) about the z-axis, we have the following twisted surface

$$\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a+\lambda_1 \cos(bs) - \lambda_2 \sin(bs) \\ 0 \\ \lambda_1 \sin(bs) + \lambda_2 \cos(bs) \end{pmatrix} = \begin{pmatrix} (a+\lambda_1 \cos(bs) - \lambda_2 \sin(bs)) \cos s \\ (a+\lambda_1 \cos(bs) - \lambda_2 \sin(bs)) \sin s \\ \lambda_1 \sin(bs) + \lambda_2 \cos(bs) \end{pmatrix}$$

where the coefficients λ_1 and λ_2 are

$$\lambda_1 = r(t)\cos t + (c-t)(r'(t)\cos t - r(t)\sin t)$$
$$\lambda_2 = r(t)\sin t + (c-t)(r'(t)\sin t + r(t)\cos t).$$

Thus, a parametrization of the twisted surface formed by the involute of the curve α in (42) is obtained. Similarly, a parametrization of the conchoidal twisted surface formed by the involute of the curve α^d is given. □

Theorem 3. *If Bertrand curve pair of α and α^d rotated about the straight line through the point $(a, 0, 0)$ parallel with the y-axis and after rotating this surface about the z-axis, twisted surface, and conchoidal twisted surface in Euclidean 3-space are*

$$\begin{aligned} \Omega_b(s,t) = &\left(r(t)\cos t + \lambda \tfrac{\lambda_5}{\lambda_7}\right)(\cos(bs)\cos s, \cos(bs)\sin s, \sin(bs)) \\ &+\left(r(t)\sin t + \lambda \tfrac{\lambda_6}{\lambda_7}\right)(\sin(bs)\cos s, \sin(bs)\sin s, \cos(bs)) + (a\cos s, a\sin s, 0) \end{aligned} \tag{45}$$

and

$$\begin{aligned} \Omega_b^d(s,t) = &\left((r(t) \pm d)\cos t + \lambda \tfrac{\lambda_5 \mp d\cos t}{\lambda_8}\right)(\cos(bs)\cos s, \cos(bs)\sin s, 0) \\ &-\left((r(t) \pm d)\sin t + \lambda \tfrac{\lambda_6 \mp d\sin t}{\lambda_8}\right)(\sin(bs)\cos s, \sin(bs)\sin s, -\cos(bs)) \\ &+\left((r(t) \pm d)\sin t + \lambda \tfrac{\lambda_5 \pm d\cos t}{\lambda_8}\right)(0, 0, \sin(bs)) + (a\cos s, a\sin s, 0) \end{aligned} \tag{46}$$

where the coefficients are

$$\begin{aligned} \lambda_5 &= (r''(t) - r(t))\cos t - 2r'(t)\sin t \\ \lambda_6 &= (r''(t) - r(t))\sin t + 2r'(t)\cos t \\ \lambda_7 &= \sqrt{(r''(t) - r(t))^2 + 4r'^2(t)} \\ \lambda_8 &= \sqrt{(r''(t) - (r(t) \pm d))^2 + 4r'^2(t)}. \end{aligned}$$

Proof. Let α be a planar curve given with a profile curve $\alpha(t) = r(t)(\cos t,\ 0,\ \sin t)$ and β in Equation (21) be a Bertrand curve pair of the α curve, taking the coefficient with

$$\begin{aligned} \lambda_5 &= (r''(t) - r(t))\cos t - 2r'(t)\sin t, & \lambda_7 &= \sqrt{(r''(t) - r(t))^2 + 4r'^2(t)}, \\ \lambda_6 &= (r''(t) - r(t))\sin t + 2r'(t)\cos t, & \lambda_8 &= \sqrt{(r''(t) - (r(t) \pm d))^2 + 4r'^2(t)}. \end{aligned}$$

If the Bertrand curve pair β is rotated about the straight line through the point $(a, 0, 0)$ parallel with the y-axis, then we obtain

$$\begin{aligned}&\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(bs) & 0 & -\sin(bs) \\ 0 & 1 & 0 \\ \sin(bs) & 0 & \cos(bs) \end{pmatrix} \begin{pmatrix} r(t)\cos t + \lambda\frac{\lambda_5}{\lambda_7} \\ 0 \\ r(t)\sin t + \lambda\frac{\lambda_6}{\lambda_7} \end{pmatrix} \\ &= \begin{pmatrix} a + \left(\left(r(t)\cos t + \lambda\frac{\lambda_5}{\lambda_7} \right)\cos(bs) - \left(r(t)\sin t + \lambda\frac{\lambda_6}{\lambda_7} \right)\sin(bs) \right) \\ 0 \\ \left(r(t)\cos t + \lambda\frac{\lambda_5}{\lambda_7} \right)\sin(bs) + \left(r(t)\sin t + \lambda\frac{\lambda_6}{\lambda_7} \right)\cos(bs) \end{pmatrix}\end{aligned} \tag{47}$$

By rotating the vector in (47) about the z-axis, we have the twisted surface formula in (45). Thus, a parametrization of the twisted surface formed by the Bertrand curve pair of the curve α is obtained. Similarly, a parametrization of the conchoidal twisted surface formed by the Bertrand curve pair of the curve α^d is given. □

4. Numerical Examples

Example 1. *Taking the profile curves as* $\alpha(t) = r(t)(\cos t, 0, \sin t)$ *and* $\alpha^d(t) = (r(t) \pm d)(\cos t, 0, \sin t)$ *in Figures 1 and 2, the twisted surface (23) and the conchoidal twisted surface (24) which points are* $r(t) = t^2$ *for* $t \in [0, 1]$ *and* $a = 1$, $b = \frac{1}{2}$, $d = \frac{1}{2}$ *becomes* $\Omega(s,t) = \left(\left(1 + t^2\cos\left(t + \frac{s}{2}\right)\right)\cos s, \left(1 + t^2\cos\left(t + \frac{s}{2}\right)\right)\sin s, t^2\sin\left(t + \frac{s}{2}\right)\right)$, *and*

$$\Omega^d(s,t) = \left(\left(1 + \left(t^2 \pm \frac{1}{2} \right)\cos\left(t + \frac{s}{2} \right) \right)\cos s, \left(1 + \left(t^2 \pm \frac{1}{2} \right)\cos\left(t + \frac{s}{2} \right) \right)\sin s, \left(t^2 \pm \frac{1}{2} \right)\sin\left(t + \frac{s}{2} \right) \right)$$

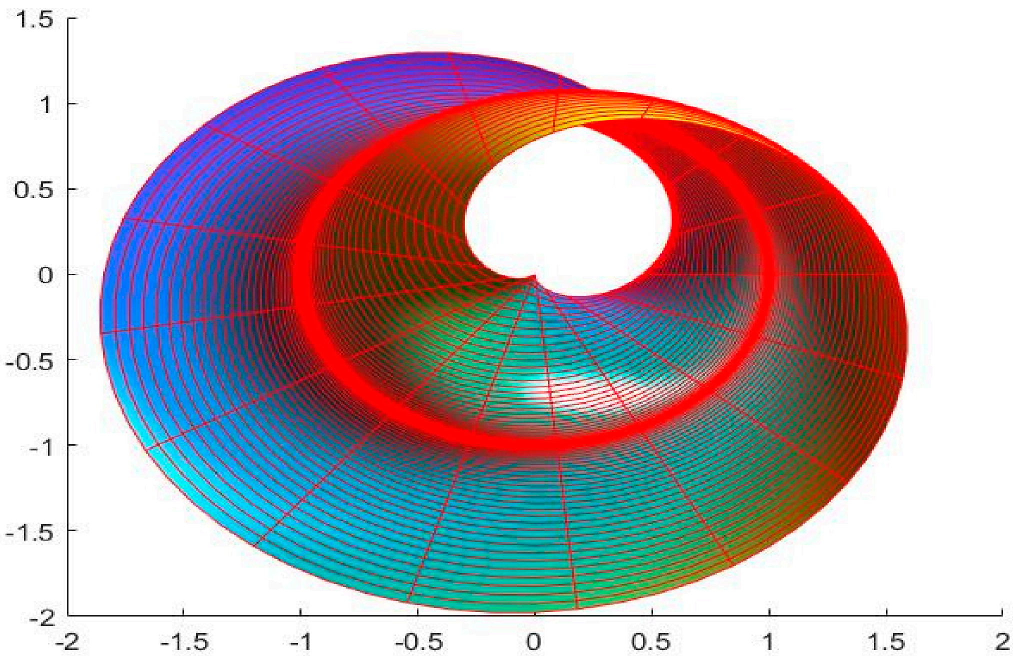

Figure 1. Twisted surface $\Omega(s,t)$.

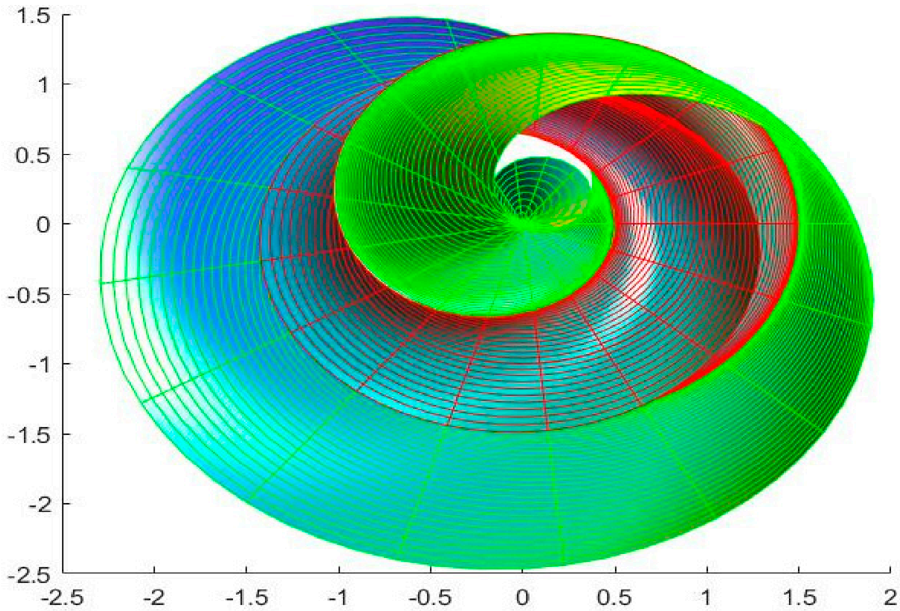

Figure 2. Conchoidal Twisted surface $\Omega^d(s,t)$.

Example 2. *Let us take the profile curves as* $\alpha(t) = r(t)(\cos t,\, 0,\, \sin t)$ *and* $\alpha^d(t) = (r(t) \pm d)(\cos t,\, 0,\, \sin t)$*, the twisted surface in (42) and the conchoidal twisted surface in (43) formed by the involute of the curve* α *and* α^d *which points are* $r(t) = t$ *for* $t \in [0,1]$ *and* $a = 1$*,* $b = 2$*,* $c = 1$*,* $d = 1$ *becomes*

$$\Omega_I(s,t) = \begin{pmatrix} (1 + \lambda_1 \cos(2s) - \lambda_2 \sin(2s)) \cos s \\ (1 + \lambda_1 \cos(2s) - \lambda_2 \sin(2s)) \sin s \\ \lambda_1 \sin(2s) + \lambda_2 \cos(2s) \end{pmatrix}$$

where the coefficients $\lambda_1 = t \cos t + (1-t)(\cos t - t \sin t)$ *and* $\lambda_2 = t \sin t + (1-t)(\sin t + t \cos t)$*, see in Figures 3 and 4.*

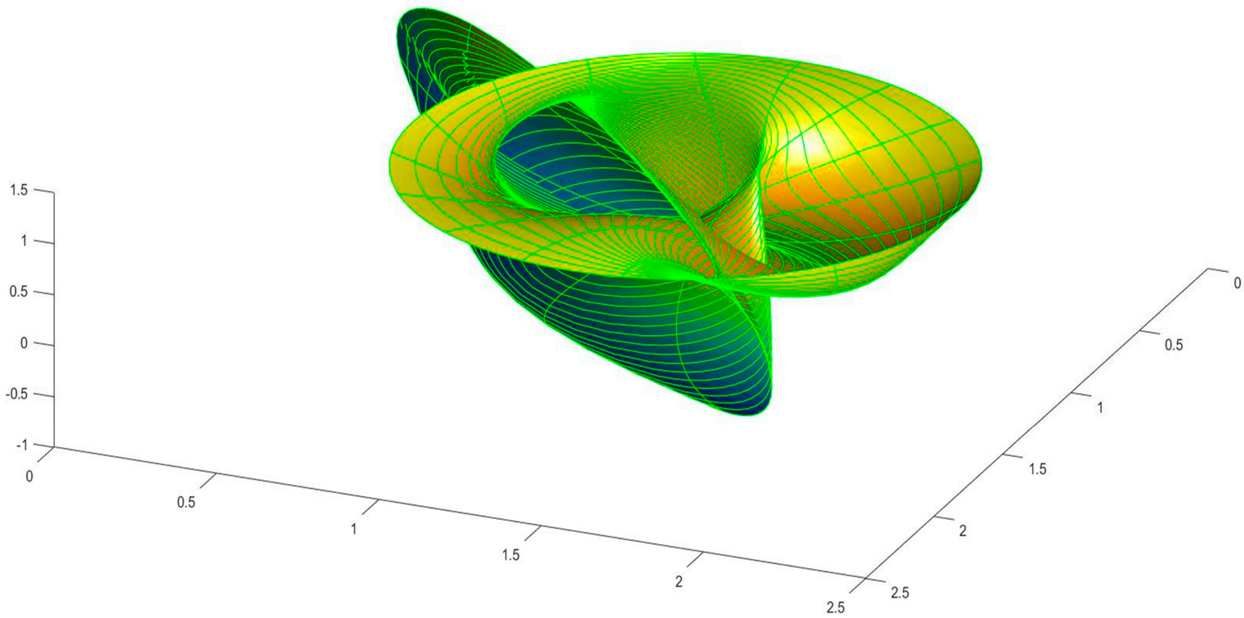

Figure 3. The twisted surface $\Omega_I(s,t)$ formed by the involute of the curve α.

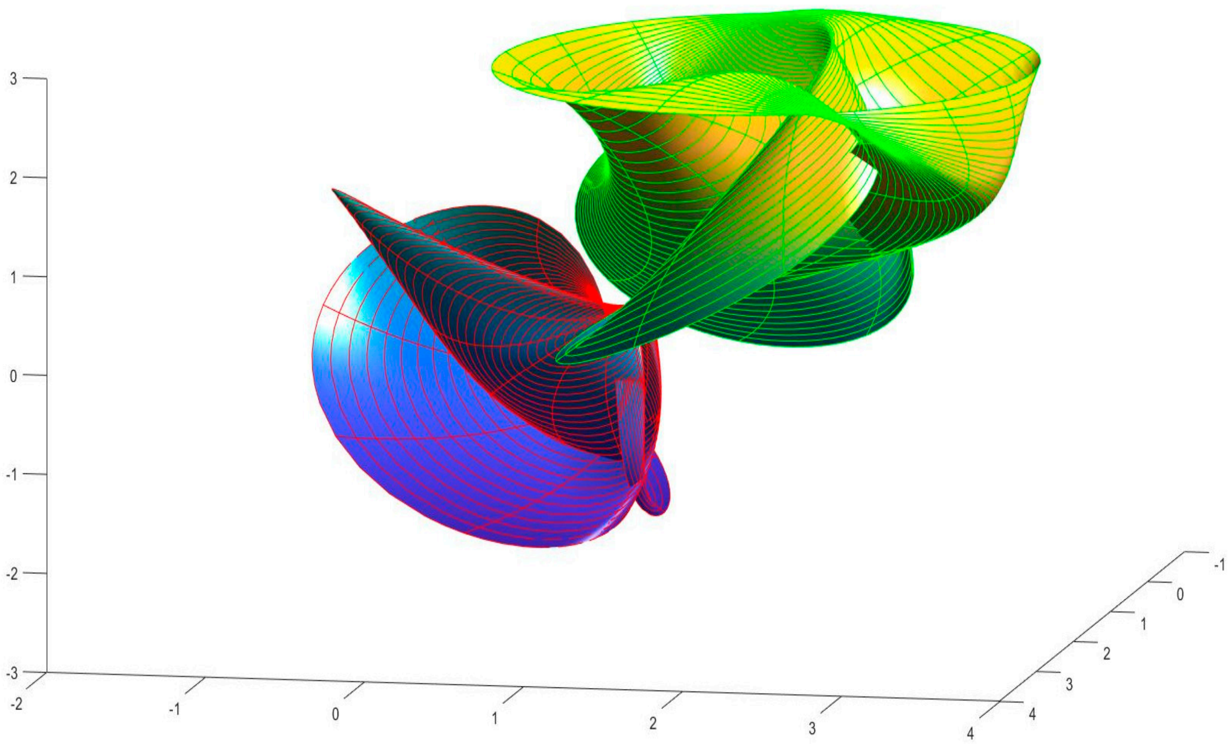

Figure 4. The conchoidal twisted surface $\Omega_I^d(s,t)$ formed by the involute of the curve α^d.

Additionally, taking the coefficients $\lambda_3 = (t \pm 1)\cos t + (1-t)(\cos t - (t \pm 1)\sin t)$ and $\lambda_4 = (t \pm 1)\sin t + (1-t)(\sin t + (t \pm 1)\cos t)$ the conchoidal twisted surface formed by the involute of the curve α^d is obtained by

$$\Omega_I^d(s,t) = \begin{pmatrix} (1 + \lambda_3 \cos(2s) - \lambda_4 \sin(2s))\cos s \\ (1 + \lambda_3 \cos(2s) - \lambda_4 \sin(2s))\sin s \\ \lambda_3 \sin(2s) + \lambda_4 \cos(2s) \end{pmatrix}.$$

Example 3. *Let us take the profile curves as $\alpha(t) = r(t)(\cos t,\ 0,\ \sin t)$ and $\alpha^d(t) = (r(t) \pm d)(\cos t,\ 0,\ \sin t)$, the twisted surface in (45) formed by the Bertrand curve pair of α which points* in Figure 5 *are $r(t) = t$ for $t \in [0,1]$, $a = 1$, $b = 2$, $\lambda = 1$, $d = 1$ becomes*

$$\begin{aligned} \Omega_b(s,t) = {} & \left(t\cos t + \lambda \frac{-t\cos t - 2\sin t}{\sqrt{t^2+4}}\right)(\cos(2s)\cos s, \cos(2s)\sin s, \sin(2s)) \\ & + \left(t\sin t + \lambda \frac{-t\sin t + 2\cos t}{\sqrt{t^2+4}}\right)(\sin(2s)\cos s, \sin(2s)\sin s, \cos(2s)) + (\cos s, \sin s, 0) \end{aligned}$$

where $\lambda_5 = -t\cos t - 2\sin t\lambda_6 = -t\sin t + 2\cos t$, $\lambda_7 = \sqrt{t^2+4}$ in Theorem 3.

The conchoidal twisted surface in (46) formed by the Bertrand curve pair of α^d which points are $r(t) = t$ for $t \in [0,1]$, $a = 1$, $b = 2$, $\lambda = 1$, $d = 1$ becomes

$$\begin{aligned} \Omega_b^d(s,t) = {} & \left(1 + \left((t \pm 1)\cos t - \left(\frac{(t\pm1)\cos t + 2\sin t}{\sqrt{(t\pm1)^2+4}}\right)\right)\cos(2s) - \left((t \pm 1)\sin t - \left(\frac{(t\pm1)\sin t - 2\cos t}{\sqrt{(t\pm1)^2+4}}\right)\right)\sin(2s)\right)(\cos s, \sin s, 0) \\ & + \left(0, 0, \left((t \pm 1)\sin t + \left(\frac{(t\pm1)\cos t + 2\sin t}{\sqrt{(t\pm1)^2+4}}\right)\right)\sin(2s) + \left((t \pm 1)\sin t - \left(\frac{(t\pm1)\sin t - 2\cos t}{\sqrt{(t\pm1)^2+4}}\right)\right)\cos(2s)\right) \end{aligned}$$

where the coefficients in Theorem 3 are $\lambda_5 = -t\cos t - 2\sin t$, $\lambda_6 = -t\sin t + 2\cos t$, $\lambda_8 = \sqrt{(t \pm 1)^2 + 4}$, see in Figure 6.

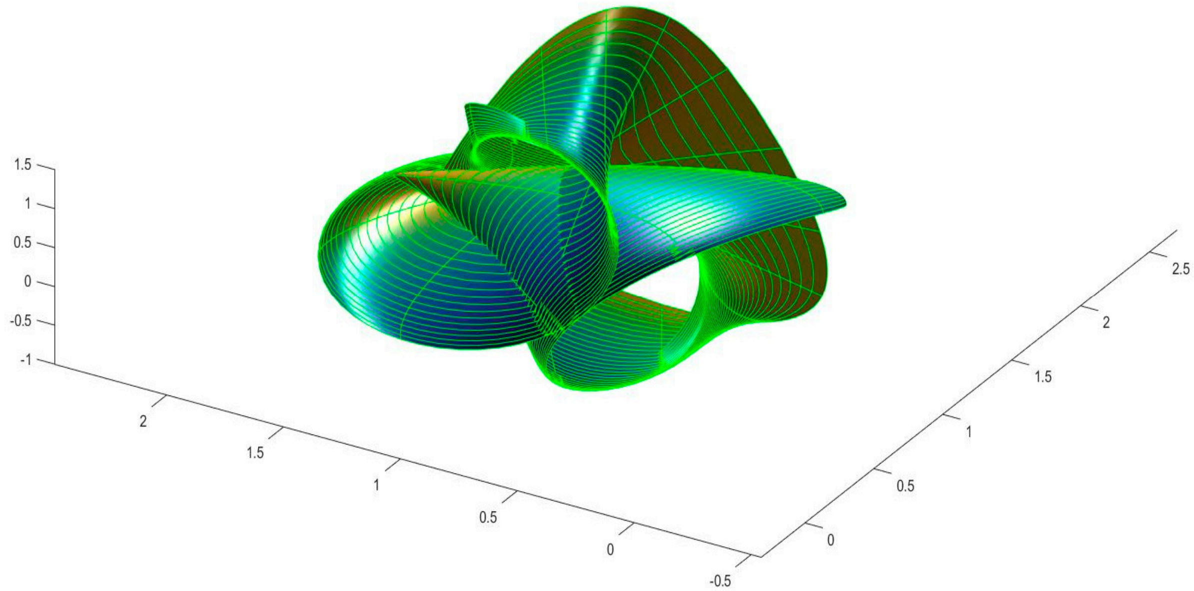

Figure 5. The twisted surface $\Omega_b(s,t)$ with Bertrand curve pair β.

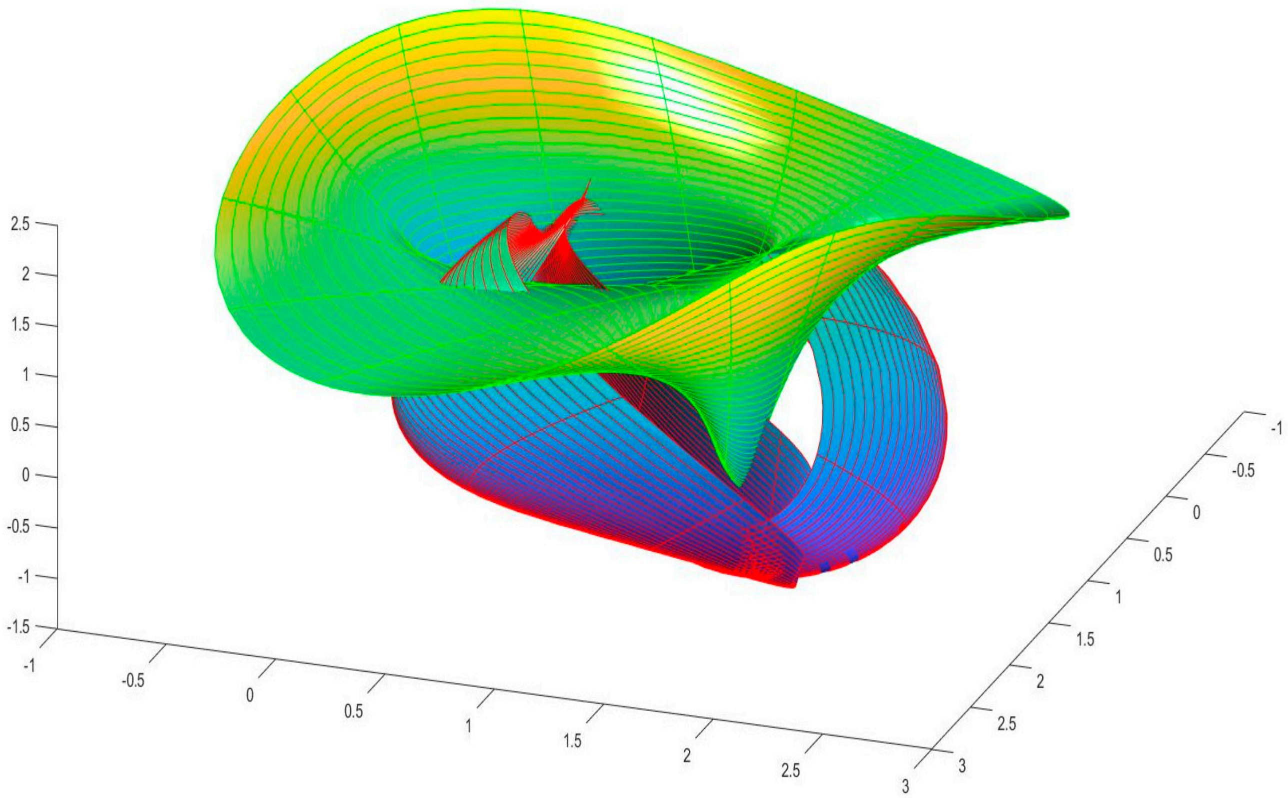

Figure 6. Conchoidal twisted surface $\Omega_b^d(s,t)$ with Bertrand curve pair β^d.

5. Conclusions

In this study, some basic definitions were given for creating twisted and conchoidal twisted surfaces. The first and second fundamental forms of the conchoidal twisted surface were computed. Then, the Gaussian and mean curvature of the conchoidal twisted surface were calculated. Additionally, the conchoidal twisted surfaces formed by an involute curve and a Bertrand curve pair were defined.

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